4

a

$$ax^3 + bx^2 + cx + d = 0$$

Let $x = y + h$. Then

$$a(y+h)^3 + b(y+h)^2 + c(y+h) + d = 0$$
$$\Rightarrow a(y^3 + 3y^2h + 3yh^2 + h^3) + b(y^2 + 2yh + h^2) + c(y+h) + d = 0$$
$$\Rightarrow ay^3 + 3ay^2h + 3ayh^2 + ah^3 + by^2 + 2byh + bh^2 + cy + ch + d = 0$$
$$\Rightarrow ay^3 + (3ah + b)y^2 + (3ah^2 + 2bh + c)y + ah^3 + bh^2 + ch + d = 0$$

To get to the depressed cubic Let $3ah + b = 0$. Therefore

$$h = -\frac{b}{3a}$$

Then

$$ay^3 + \left(3a\frac{-b}{3a} + b\right)y^2 + \left(3a\left(\frac{-b}{3a}\right)^2 + 2b\frac{-b}{3a} + c\right)y + a\left(\frac{-b}{3a}\right)^3 + b\left(\frac{-b}{3a}\right)^2 + c\frac{-b}{3a} + d = 0$$

$$\Rightarrow ay^3 + \left(\frac{b^2}{3a} - \frac{2b^2}{3a} + c\right)y - \frac{b^3}{27a^2} + \frac{b^3}{9a^2} - \frac{bc}{3a} + d = 0$$
$$\Rightarrow y^3 + \left(\frac{\frac{b^2}{3a} - \frac{2b^2}{3a} + c}{a}\right)y + \frac{\frac{b^3}{9a^2} - \frac{b^3}{27a^2} - \frac{bc}{3a} + d}{a} = 0$$
$$\Rightarrow y^3 + \left(\frac{c}{a} - \frac{b^2}{3a^2}\right)y + \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} = 0$$

b

The solution for the depressed cubic $y^3 + cy + d = 0$ is

$$u + v$$

Where

$$u = \sqrt[3]{\left(\frac{-d}{2}\right) - \sqrt{R}}$$

,

$$v = \sqrt[3]{\left(\frac{-d}{2}\right) + \sqrt{R}}$$

and

$$R = \left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3$$

So with the depressed cubic from part a we get

$$u = \sqrt[3]{-\frac{1}{2}\left(\frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}\right) - \sqrt{R}}$$
$$= \sqrt[3]{\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} - \sqrt{R}}$$

,

$$v = \sqrt[3]{-\frac{1}{2}\left(\frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}\right) - \sqrt{R}}$$
$$= \sqrt[3]{\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} + \sqrt{R}}$$

and

$$R = \left(\frac{1}{2}\left(\frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}\right)\right)^2 + \left(\frac{1}{3}\left(\frac{c}{a} - \frac{b^2}{3a^2}\right)\right)^3$$
$$= \frac{b^3d}{27a^4} - \frac{b^2c^2}{108a^4} - \frac{bcd}{6a^3} + \frac{c^3}{27a^3} + \frac{d^2}{4a^2}$$

The other two roots will be

$$\omega u + \omega^2 v$$

and

$$\omega^u + \omega v$$

where ω is the cube root of unity.

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$$

Now we replace x back in for $y + h$. So because $y = u + v$ now $x - h = u + v$ which leads to $x = h + u + v$. Now our roots are

$$r_1 = \frac{-b}{3a} + \sqrt[3]{\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} - \sqrt{R}} + \sqrt[3]{\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} + \sqrt{R}}$$
$$r_2 = \frac{-b}{3a} + \omega\sqrt[3]{\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} - \sqrt{R}} + \omega^2\sqrt[3]{\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} + \sqrt{R}}$$
$$r_3 = \frac{-b}{3a} + \omega^2\sqrt[3]{\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} - \sqrt{R}} + \omega\sqrt[3]{\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a} + \sqrt{R}}$$

c

$$x^3 - 10x^2 + 31x - 28 = 0$$
$$a = 1, b = -10, c = 31, d = -28$$
$$R = (-10)^3(-28) - (-10)^2 31^2 - (-10)31(-28) + 31^3 + (-28)^2$$
$$= -46205$$
$$u = \sqrt[3]{(-10)31 - (-10)^3 - \sqrt{-46205}}$$
$$= \sqrt[3]{690-}$$

Lard