CENG 384 - Signals and Systems for Computer Engineers
Spring 2022
Homework 3

Çetin, Berkay Bartuğ
e2309839@ceng.metu.edu.tr

Ceylan, Anıl Berdan
e2304277@ceng.metu.edu.tr

May 15, 2022

---

1. (a) $x(t) = \frac{1}{2j}\left(e^{\frac{\pi}{5}jt} - e^{-\frac{\pi}{5}jt}\right) + \frac{1}{2}\left(e^{\frac{\pi}{4}jt} + e^{-\frac{\pi}{4}jt}\right)$
   $x(t) = \frac{e^{\frac{\pi}{5}jt}}{2j} - \frac{e^{-\frac{\pi}{5}jt}}{2j} + \frac{1}{2}e^{\frac{\pi}{4}jt} + \frac{1}{2}e^{\frac{-\pi}{4}jt}$
   $x(t) = \frac{e^{4\omega_0 jt}}{2j} - \frac{e^{-4\omega_0 jt}}{2j} + \frac{1}{2}e^{5\omega_0 jt} + \frac{1}{2}e^{-5j\omega_0 t}$
   $a_4 = \frac{1}{2j},\ a_{-4} = -\frac{1}{2j},\ a_5 = \frac{1}{2},\ a_{-5} = \frac{1}{2}$

   (b) We know that $sin(4\pi n) = 0\ and\ cos(2\pi n) = 1$ as n is an integer.
   $x[n] = \frac{3}{2} + e^{j\pi n}\ and\ x[n+N] = \frac{3}{2} + e^{j\pi n} + e^{j\pi N}$
   $We\ need\ to\ make\ e^{j\pi N} = 1,\ So\ N = 2.$
   $x[n] = \sum_{k=0}^{N-1} a_k e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{N-1} a_k e^{\frac{j2\pi kn}{2}}$
   Hence, $a_0 = \frac{3}{2}\ and\ a_1 = 1$.
   We know that spectral coefficients are periodic, so;
   $a_{2k} = \frac{3}{2}\ and\ a_{2k+1} = 1$.

2. $Fundamental\ Frenquency\ = \frac{2\pi}{N} = \frac{2\pi}{7}$
   $a_{-3} = 2e^{-\frac{j\pi}{2}},\ a_3 = 2e^{\frac{j\pi}{2}},\ a_{-2} = a_2 = 2,\ a_{-1} = 2e^{-\frac{j\pi}{2}},\ a_1 = 2e^{\frac{j\pi}{2}}$
   $x[n] = 2e^{-\frac{j\pi}{2}} \cdot e^{-\frac{j2\pi}{7}3n} + 2e^{-\frac{j2\pi}{7}2n} + 2e^{-\frac{j\pi}{2}} \cdot e^{-\frac{j2\pi}{7}n} + 2e^{\frac{j\pi}{2}} \cdot e^{\frac{j2\pi}{7}n} + 2e^{\frac{j2\pi}{7}2n} + 2e^{\frac{j\pi}{2}} \cdot e^{\frac{j2\pi}{7}3n}$
   $x[n] = 2e^{-j\left(\frac{\pi}{2}+\frac{6\pi n}{7}\right)} + 2e^{j\left(\frac{\pi}{2}+\frac{6\pi n}{7}\right)} + 2e^{-j\left(\frac{\pi}{2}+\frac{2\pi n}{7}\right)} + 2e^{j\left(\frac{\pi}{2}+\frac{2\pi n}{7}\right)} + 2e^{-j\left(\frac{4\pi n}{7}\right)} + 2e^{j\left(\frac{4\pi n}{7}\right)}$
   $$x[n] = 4\left(\frac{\left(e^{-j\left(\frac{\pi}{2}+\frac{6\pi n}{7}\right)} + e^{j\left(\frac{\pi}{2}+\frac{6\pi n}{7}\right)}\right)}{2} + \frac{\left(e^{-j\left(\frac{\pi}{2}+\frac{2\pi n}{7}\right)} + e^{j\left(\frac{\pi}{2}+\frac{2\pi n}{7}\right)}\right)}{2} + \frac{\left(e^{-j\left(\frac{4\pi n}{7}\right)} + e^{j\left(\frac{4\pi n}{7}\right)}\right)}{2}\right)$$
   $x[n] = 4cos\left(\frac{\pi}{2} + \frac{6\pi n}{7}\right) + 4cos\left(\frac{4\pi n}{7}\right) + 4cos\left(\frac{\pi}{2} + \frac{2\pi n}{7}\right)$
   We know that $cos\theta = sin\left(\frac{\pi}{2} + \theta\right)$. So,
   $x[n] = -4sin\left(\frac{6\pi n}{7}\right) + 4sin\left(\frac{4\pi n}{7} + \frac{\pi}{2}\right) - 4sin\left(\frac{2\pi n}{7}\right)$

3. (a) $x(t) = sin\left(\frac{\pi}{8}t\right) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t},\ Hence\ a_0 = 0,\ a_{-1} = -\frac{1}{2j},\ a_1 = \frac{1}{2j}$

   (b) $y(t) = cos\left(\frac{\pi}{8}t\right) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t},\ Hence\ a_0 = 0,\ a_{-1} = a_1 = \frac{1}{2}$

   (c) $z(t) = x(t)\,y(t) = \sum_{t=-\infty}^{\infty} a_L b_{K-L}$
   $c_k = a_o b_k + a_1 b_{k-1} + a_{-1} b_{k+1}$
   $c_0 = 0,\ c_{-1} = 0,\ c_1 = 0,\ c_2 = \frac{1}{4j},\ c_{-2} = -\frac{1}{4j}$

4. Since x(t) is real and odd, we can conclude that $x(t) = -x(-t)$ and $a_k = -a_{-k}$. This signal must be symmetric to the origin so $a_{0=0}$. Moreover, since $a_k = 0\ for\ |k| > 2$, only $a_{-2},\ a_{-1},\ a_1\ and\ a_2$ might be nonzero.
From $a_2 = 3j$, we conclude that $a_{-2} = -3j$.
$\frac{1}{4}\int_0^4 |x(t)|^2 = \sum_{-\infty}^{\infty} |a_k|^2 = |a_{-2}|^2 + |a_{-1}|^2 + |a_1|^2 + |a_2|^2 = 18$
$9 + |a_{-1}|^2 + |a_1|^2 + 9 = 18$
Hence $a_1 = a_{-1} = 0$.

5. (a) We know that $a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-\frac{jk2\pi n}{N}} = \frac{1}{9}\sum\ _{n=0}^{8} x[n]\, e^{-\frac{jk2\pi\ n}{9}} = \frac{1}{9}\sum\ _{n=0}^{4} x[n]\, e^{-\frac{jk2\pi\ n}{9}}$
$a_0 = \frac{1}{9}\sum_{n=0}^{4} 1 = \frac{5}{9}$
$a_1 = \frac{1}{9}\sum_{n=0}^{4} e^{-\frac{j2\pi n}{9}} = \frac{1}{9}\left(1 + e^{-\frac{j2\pi}{9}} + e^{-\frac{j4\pi}{9}} + e^{-\frac{j6\pi}{9}} + e^{-\frac{j8\pi}{9}}\right)$
$a_2 = \frac{1}{9}\sum_{n=0}^{4} e^{-\frac{j4\pi n}{9}} = \frac{1}{9}\left(1 + e^{-\frac{j4\pi}{9}} + e^{-\frac{j8\pi}{9}} + e^{-\frac{j12\pi}{9}} + e^{-\frac{j16\pi}{9}}\right)$
$a_3 = \frac{1}{9}\sum_{n=0}^{4} e^{-\frac{j6\pi n}{9}} = \frac{1}{9}\left(1 + e^{-\frac{j6\pi}{9}} + e^{-\frac{j12\pi}{9}} + e^{-\frac{j18\pi}{9}} + e^{-\frac{j24\pi}{9}}\right)$
$a_4 = \frac{1}{9}\sum_{n=0}^{4} e^{-\frac{j8\pi n}{9}} = \frac{1}{9}\left(1 + e^{-\frac{j8\pi}{9}} + e^{-\frac{j16\pi}{9}} + e^{-\frac{j24\pi}{9}} + e^{-\frac{j32\pi}{9}}\right)$

(b) We know that $a_k = \frac{1}{N}\sum\ _{n=0}^{N-1} y[n]\, e^{-\frac{jk2\pi n}{N}} = \frac{1}{9}\sum\ _{n=0}^{8} y[n]\, e^{-\frac{jk2\pi n}{9}} = \frac{1}{9}\sum\ _{n=0}^{3} y[n]\, e^{-\frac{jk2\pi n}{9}}$
$a_0 = \frac{1}{9}\sum\ _{n=0}^{3} 1 = \frac{4}{9}$
$a_1 = \frac{1}{9}\sum\ _{n=0}^{3} e^{-\frac{j2\pi n}{9}} = \frac{1}{9}\left(1 + e^{-\frac{j2\pi}{9}} + e^{-\frac{j4\pi}{9}} + e^{-\frac{j6\pi}{9}}\right)$
$a_2 = \frac{1}{9}\sum\ _{n=0}^{3} e^{-\frac{j4\pi n}{9}} = \frac{1}{9}\left(1 + e^{-\frac{j4\pi}{9}} + e^{-\frac{j8\pi}{9}} + e^{-\frac{j12\pi}{9}}\right)$
$a_3 = \frac{1}{9}\sum\ _{n=0}^{3} e^{-\frac{j6\pi n}{9}} = \frac{1}{9}\left(1 + e^{-\frac{j6\pi}{9}} + e^{-\frac{j12\pi}{9}} + e^{-\frac{j18\pi}{9}}\right)$

(c) $H(\omega) = \frac{Y(\omega)}{X(\omega)}$
Where $Y(\omega\,) = \frac{\left(1+e^{-\frac{j2\pi}{9}}+e^{-\frac{j4\pi}{9}}+e^{-\frac{j6\pi}{9}}\right)+\left(1+e^{-\frac{j4\pi}{9}}+e^{-\frac{j8\pi}{9}}+e^{-\frac{j12\pi}{9}}\right)+\left(1+e^{-\frac{j6\pi}{9}}+e^{-\frac{j12\pi}{9}}+e^{-\frac{j18\pi}{9}}\right)}{9}$ and
$X(\omega\,) = \frac{\left(1+e^{-\frac{j2\pi}{9}}+e^{-\frac{j4\pi}{9}}+e^{-\frac{j6\pi}{9}}+e^{-\frac{j8\pi}{9}}\right)+\left(1+e^{-\frac{j4\pi}{9}}+e^{-\frac{j8\pi}{9}}+e^{-\frac{j12\pi}{9}}+e^{-\frac{j16\pi}{9}}\right)}{9}$
$+\frac{\left(1+e^{-\frac{j6\pi}{9}}+e^{-\frac{j12\pi}{9}}+e^{-\frac{j18\pi}{9}}+e^{-\frac{j24\pi}{9}}\right)+\left(1+e^{-\frac{j8\pi}{9}}+e^{-\frac{j16\pi}{9}}+e^{-\frac{j24\pi}{9}}+e^{-\frac{j32\pi}{9}}\right)}{9}$
If we define a variable $p = \left(e^{-\frac{j2\pi}{9}}\right)$ the result will be;
$H(\omega) = \frac{3+p+2p^2+2p^3+p^4+2p^6+p^9}{4+p+2p^2+2p^3+3p^4+2p^6+2p^8+p^9+2p^{12}+p^{16}}$