ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

Department of Computer Engineering

CENG 384 Homework Exam 1 Key

22 April 2022

METU Honor Code and Pledge

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*"I have read and understood the implications of the METU Honor Code. To be precise, I understand that this is a **open** formula-sheet exam, and I am forbidden to access any other source of information other than provided within the exam. I will TURN OFF all my electronic equipment (phones, smart watches, etc.) and put it off the table along with other notes and materials that I may have with me. I understand that leaving electronic devices on during the exam is strictly forbidden. I understand and accept to obey all the rules announced by the course staff, and that failure to obey these will result in disciplinary action."*

Name, SURNAME: **Signature:**
ID:

Specifications About the Exam

- **Duration:** **X** minutes.
- **Notes:**
  - One A4-size double-sided formula-sheet is allowed.
  - Each answer should contain a justifying explanation. However, give shortest possible answers. Unnecessary or irrelevant wordings will reduce your grades.
  - If you use the back of a page for extra space, please mark down a proper notice on the relevant pages.

Grade

Q1:
Q2:
Q3:
Q4:
Q5:

Total:

$\boxed{1}$ (10 pts, 2 pts each) Determine whether the following statements are *True* or *False*. Circle the correct answer for each question.

a) [T / F] $x[n] * h[n] * x[n] = h[n] * (x[n])^2$.

b) [T / F] $y(t) = x(2t)$ is not linear.

c) [T / F] Linearity is a necessary condition for time invariance.

d) [T / F] $x[n] = Re\{(1+j)^n\}$ is not periodic.

e) [T / F] Let $z = \frac{(1+j)(5-5j)(\sqrt{3}+j)}{10j(5+j5\sqrt{3})}$; z evaluates to $\frac{1}{5}e^{\frac{-j2\pi}{3}}$.

a) False

b) False

c) False

d) True

e) True

$\boxed{2}$ (24 pts, 12 pts each) Which of the following properties do the systems below have?

a) $y(t) = x(t+2)$

b) $y(t) = \int_{-\infty}^{2t} x(3\tau)d\tau$

Memory, stability, causality, linearity, invertibility and time-invariance. Clearly explain your reasoning to get full credit.

a) $y(t) = x(t+2)$

- **Memory:** Has memory, $y(0) = x(2)$
- **Stability:** Stable, all bounded inputs result in bounded outputs
- **Causality:** Not Causal $y(2) = x(4)$, y depends on inputs in the future
- **Linearity:** Superposition property holds, therefore linear
- **Invertibility:** Invertible, $x(t) = y(t-2)$
- **Time Invariance:** Time invariant, shift in time results in identical time shift

b) $y(t) = \int_{-\infty}^{2t} x(3\tau)d\tau$

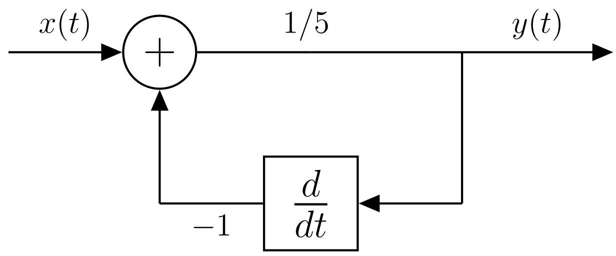
- **Memory:** Has memory, output depends on past values of input
- **Stability:** Not stable, even we bound input, we cannot bound the response.
- **Causality:** Not Causal, system uses inputs from the future
- **Linearity:** Superposition property holds, therefore linear
- **Invertibility:** Invertible, we can invert by differentiating
- **Time Invariance:** Time varying, output is not shifted with same value if we shift time in the input.

$\boxed{\mathbf{3}}$ (10 pts) Given that $x_1[n]$ is an even signal and $x_2[n]$ is an odd signal, show whether $x_1[n] \cdot x_2[n]$ is an odd signal or an even signal. Show your work.

Since $x_1[n]$ is even, $x_1[n] = x_1[-n]$.
Similarly, since $x_2[n]$ is odd, $x_2[n] = -x_2[-n]$.

$$\begin{aligned} y[n] &= x_1[n] \cdot x_2[n] \\ y[-n] &= x_1[-n]x_2[-n] \\ &= x_1[n](-x_2[n]) \\ &= -y[n] \implies y[n] = x_1[n] \cdot x_2[n] \text{ is odd.} \end{aligned}$$

4 (30 pts, 5-15-10 pts respectively) Consider an LTI system given by the following block diagram:



a) Find the differential equation which represents this system.

b) Find the output $y(t)$, when the input $x(t) = (e^{-t} + e^{-3t})u(t)$. Assume that the system is initially at rest.

c) Find the impulse response of this system. Show your work.

a)

$$y'(t) + 5y(t) = x(t)$$

b) char eqn. : $r + 5 = 0 \quad \Rightarrow \quad r = -5 \quad \Rightarrow \quad y_h(t) = K \cdot e^{-5t}$
$y_p(t) = Ae^{-t} + Be^{-3t}$
$y_p'(t) = -Ae^{-t} - 3Be^{-3t}$

$$-Ae^{-t} - 3Be^{-3t} + 5(Ae^{-t} + Be^{-3t}) = e^{-t} + e^{-3t}$$
$$4Ae^{-t} + 2Be^{-3t} = e^{-t} + e^{-3t}$$

$4A = 1 \quad \Rightarrow \quad A = \frac{1}{4},\ 2B = 1 \quad \Rightarrow \quad B = \frac{1}{2}$

$$y(t) = y_h(t) + y_p(t) = (Ke^{-5t} + \frac{1}{4}e^{-t} + \frac{1}{2}e^{-3t})u(t)$$
$$y(0) = K + \frac{1}{4} + \frac{1}{2} = 0 \quad \Rightarrow \quad K = -\frac{3}{4}$$
$$y(t) = (\frac{1}{4}e^{-t} + \frac{1}{2}e^{-3t} - \frac{3}{4}e^{-5t})u(t)$$

c)

$$y'(t) + 5y(t) = x(t)$$
$$h'(t) + 5h(t) = \delta(t) \quad (*)$$

$$h_h(t) = Ce^{\alpha t}$$
$$h_h'(t) = \alpha Ce^{\alpha t}$$
$$\alpha Ce^{\alpha t} + 5Ce^{\alpha t} = 0 \implies \alpha = -5 \implies h_h(t) = Ce^{-5t}$$

For $h_p(t)$, multiply both sides with dt and integrate between 0^- and 0^+.

$$\int_{0^-}^{0^+} dh(t) + \cancelto{0 \text{ (continuity)}}{5\int_{0^-}^{0^+} h(t)dt} = \int_{0^-}^{0^+} \delta(t)dt = 1$$

$h(0^-) = 0$ since the system is initially at rest. $\implies h(0^+) = 1$. Plug this into $(*)$
$h(0^+) = Ce^{-5 \cdot 0} = C = 1 \implies h(t) = e^{-5t}u(t)$.

$\boxed{5}$ (26 pts, 13 pts each) Consider the following linear system:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]g[n-3k],$$

where $g[n] = u[n-1] - u[n-3]$.

a) Find and plot $y[n]$ if $x[n] = \delta[n-1]$.

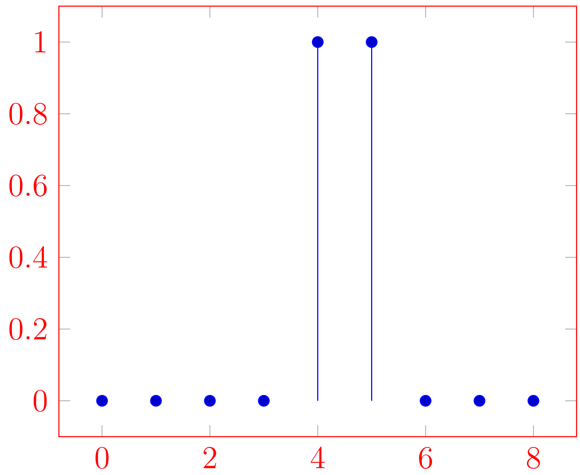
b) Find and plot $y[n]$ if $x[n] = u[n-1]$.

a) Find y[n] if $x[n] = \delta[n-1]$.

When you plug $x[n]$ into the equation above you will get a result which is in terms of $g[n]$. This comes from the sifting property.

$$y[n] = \sum_{k=-\infty}^{\infty} g[n-3k]\delta[k-1]$$

As you can see this equation is defined only when $k = 1$. Therefore we get the result $g[n-3]$ which is equal to $u[n-4] - u[n-6]$.



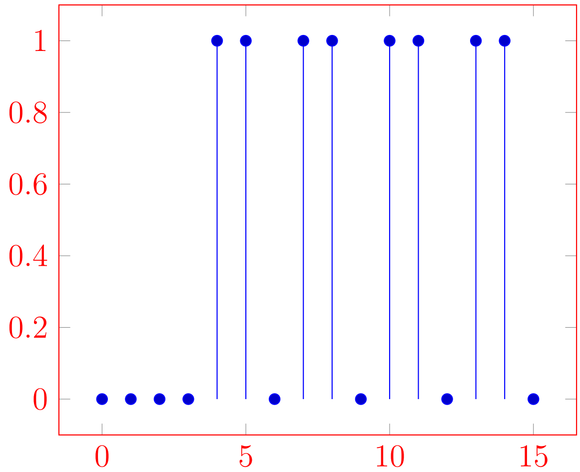
b) Find y[n] if $x[n] = u[n-1]$.

When you plug $x[n]$ into the equation above you will get the following equation:

$$y[n] = \sum_{k=-\infty}^{\infty} g[n-3k]u[k-1]$$

Since $u[n-1]$ is defined on the interval of $[1, \infty)$ the formula simply becomes the following:

$$y[n] = \sum_{k=1}^{\infty} g[n-3k]$$



In the figure above you can see the values of $y[n]$ for the first 15 n values. From the figure we see that y[n] will be $u[n-4] - \sum_{k=2}^{\infty} \delta[n-3k]$.