CENG 223

Discrete Computational Structures

Fall 2020-2021

Take Home Exam 2 - Solutions

Answer 1

a.

i) This is a topology.

ii) This is not a topology since it misses the unions $\{a, b\}$ and such.

iii) This is a topology.

iv) This is not a topology since it misses the unions $\{a, b, c\}$ and $\{b, c, d\}$.

b. First note that for all of the sets A and $\emptyset$ are in the given sets. Hence we will check for the second and third criteria. We will do this by considering arbitrary elements U_1 and U_2 of the given sets.

i) This is a topology.

We have that $A-U_1$ and $A-U_2$ are finite or all of A. **Intersection:** $A-(U_1 \cap U_2) = (A-U_1) \cup (A-U_2)$. As by taking the union of two finite sets, we end up again with such sets, $A-(U_1 \cap U_2)$ is in the given set as well.

Union: $A-(U_1 \cup U_2) = (A-U_1) \cap (A-U_2)$. Similar reasoning.

ii) This is a topology.

Similar reasoning as in i).

Since sets are countable, we have injective functions $f_1 : (A-U_1) \to \mathbb{Z}^+$ and $f_2 : (A-U_2) \to \mathbb{Z}^+$. **Intersection:** $A-(U_1 \cap U_2) = (A-U_1) \cup (A-U_2)$. In this case, we can construct a new injection using $f_1 : (A-U_1) \to \mathbb{Z}^+$ and $f_2 : (A-U_2) \to \mathbb{Z}^+$ where each element $a \in A-(U_1 \cap U_2)$ is mapped to $2\,f_1(a)$ if $a \in (A-U_1)$ and $2\,f_2(a)+1$ if $a \in (A-U_2)-(A-U_1)$.

Union: $A-(U_1 \cup U_2) = (A-U_1) \cap (A-U_2)$. we can choose either of the functions as the new injective function $f_i\Big|_{A-(U_1 \cup U_2)} : A-(U_1 \cup U_2) \to \mathbb{Z}^+$ (such acquired functions are called the **restriction of** f_i).

iii) This is not a topology.

Consider $A = \mathbb{Z}^+$, $U_1 = \{2x+1 \in \mathbb{Z}^+ \,|\, x \in \mathbb{Z}^+\}$ and $U_2 = \{2x \in \mathbb{Z}^+ \,|\, x \in \mathbb{Z}^+\}$. Note that since $(A - U_1)$ and $(A - U_2)$ are both infinite they are in the given set. However, $A - (U_1 \cup U_2) = \{1\}$ is not in the given set because it is neither infinite nor is $\emptyset$ or A. So this set does not qualify to be a topology.

Answer 2

a. Let (n, p) and (m, q) be elements of $A \times (0, 1)$. Assume that $f(n, p) = f(m, q)$. Then we have that $n + p = m + q$ and so $n - m = q - p$. Since n and m are integers, so is $n - m$. Since $0 < p, q < 1$, we have $-1 < q - p = n - m < 1$ which implies $n - m = 0$ and hence $n = m$. Consequently, $p = q$ and $(n, p) = (m, q)$. Hence f is injective.

b. For all (n, p) in $A \times (0, 1)$, we have $n \geq 0$ and $p > 0$ which implies $f(n, p) = n + p > 0$. Hence 0 is not mapped to by any input to the function f. Thus f is not surjective.

c Since we have an injection from $A \times (0, 1)$ to $[0, \infty)$ and an injection from $[0, \infty)$ to $A \times (0, 1)$ to $[0, \infty)$, by the Schröder-Bernstein theorem there is a one-to-one correspondence between these two sets. As a result the two sets are of the same cardinality.

Answer 3

a. f is a relation given by $f = \{(0, n), (1, m)\}$ for $n, m \in \mathbb{Z}^+$. Hence, we can represent f as the ordered pair $(n, m) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. As a result we acquire a bijection A and $\mathbb{Z}^+ \times \mathbb{Z}^+$. Since the latter is a finite Cartesian product of two countable sets it is countable. Therefore, A is countable as well.

b. Similar to the case in **a.**, where instead of $\mathbb{Z}^+ \times \mathbb{Z}^+$ we now have $\mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+$, which is again a finite Cartesian product of countable sets. As a result B is countable.

d. Let us consider $[0, 1] = \{x \in \mathbb{R} \,|\, 0 \leq x \leq 1\}$ which we know is uncountably infinite. We can represent all $x \in [0, 1]$ in binary as (d, i) where $d \in \mathbb{Z}$ indicates the dth digit after the decimal point and $i \in \{0, 1\}$ is the value of that digit. Note that the same representation (d, i) defines a specific $f \in D$. Thus, we have a bijection between the set D and $[0, 1]$. Since the latter is uncountable, so is the former.

c. Since $D \subset C$ and D is uncountable, so is C.

e. Let $f_n : \mathbb{Z}^+ \to \{0, 1\}$ be the functions such that f_n is zero after the nth digit. Then we can define a bijection between $g : \{1, \ldots, n\} \to \{0, 1\}$ and f_n. Since the set of all such gs are countable (it is a subset of B) the set E_n of all f_n are countable. Finally, remark that $E = \bigcup_{n \in \mathbb{Z}^+} E_n$ which is a countable union of countable sets. So E is countable as well.

Answer 4

a. By Stirling's approximation we have $n! \approx \dfrac{n^n\sqrt{2\pi n}}{e^n}$. Then we can calculate the limit $lim_{n\to\infty}\dfrac{n!}{n^n}$ as

$$lim_{n\to\infty}\frac{n!}{n^n} = \frac{n^n\sqrt{2\pi n}}{n^n e^n} = \frac{\sqrt{2\pi n}}{e^n} = 0.$$

Hence $n!$ is not $\Theta(n^n)$ — it grows slower than $\Theta(n^n)$

b. By the Binomial theorem we have

$$(n+a)^b = \sum_{k=0}^{b} C(n,k)n^k a^{n-k},$$

where $C(n,k)$ is the $(k+1)$th entry of the nth row of Pascal's triangle given by $C(n,k) = \dfrac{n!}{(n-k)!k!}$. The limit $lim_{n\to\infty}\dfrac{(n+a)^b}{n^b}$ converges to a non-zero real number. Hence, we conclude that $(n+a)^b = \Theta(n^b)$.

Answer 5

a. Let x **mod** $y = a$. Then $x = ky + a$. So

$$(2^x - 1) \textbf{ mod } (2^y - 1) = ((2^y)^k\, 2^a - 1) \textbf{ mod } (2^y - 1) = (2^a - 1) \textbf{ mod } (2^y - 1)$$

In the last step we applied 2^y **mod** $2^y - 1 = 1$. Since a is defined as some number modulo y it is lesser than y and greater than 0. Hence we can write $(2^a - 1)$ **mod** $(2^y - 1)$ as simply $(2^a - 1)$. Yet, this is nothing but $2^{x \textbf{ mod } y} - 1$

b. Note that the given equality in **a.** can be applied recursively. Starting by x and y in the LHS we have

$$\begin{aligned}
(2^x - 1) \textbf{ mod } (2^y - 1) &= 2^{x \textbf{ mod } y} - 1 = 2^{r_0} - 1\\
(2^y - 1) \textbf{ mod } (2^{r_0} - 1) &= 2^{y \textbf{ mod } r_0} - 1 = 2^{r_1} - 1\\
&\vdots\\
(2^{r_{n-1}} - 1) \textbf{ mod } (2^{r_n} - 1) &= 2^{r_{n-1} \textbf{ mod } r_n} - 1 = 2^0 - 1 = 0
\end{aligned}$$

Now we are ready to compute $\textbf{gcd}\,(2^x - 1, 2^y - 1)$.

$$\begin{aligned}
\textbf{gcd}\,(2^x - 1, 2^y - 1) &= \textbf{gcd}\,(2^y - 1, 2^{r_0} - 1) = \cdots = \textbf{gcd}\,(2^{r_{n-1}} - 1, 2^{r_n} - 1) = \textbf{gcd}\,(2^{r_n} - 1, 0) = 2^{r_n} - 1\\
&= 2^{\textbf{gcd}\,(r_n, r_n)} - 1 = 2^{\textbf{gcd}\,(r_{n-1}, r_n)} - 1 = \cdots = 2^{\textbf{gcd}\,(y, r_0)} - 1 = 2^{\textbf{gcd}\,(x,y)} - 1
\end{aligned}$$