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Answer 1

We should select 1 star from 10 distinct stars, 2 habitable planets from 20 distinct habitable planets and lastly 8 non-habitable planets from 80 distinct non-habitable planets. We can choose it by $C(10,1) \cdot C(20,2) \cdot C(80,8)$. Then, since all these stars and planets are distinct we will consider three cases:

First, there could be six non-habitable planets between the two habitable ones. And we have two non-habitable planets out. We can choose that six by $C(8,6)$ and order them with 6!. If we behave the two habitable ones and the six non-habitable ones as a package. We can think of we have 3 elements (2 non-habitable planets are 2 elements and the other eight as one element) to order. (We handle two habitables below.) The number of orderings of them is 3!. So, in total we have $C(8,6) \cdot 6! \cdot 3!$ different options for this part.

Second, there could be seven non-habitable planets between the two habitable ones. We have $C(8,7) \cdot 7! \cdot 2!$ different options for this part (similar to first one).

Third, there could be eight non-habitable planets between the two habitable ones. We have $C(8,8) \cdot 8!$ different options for this part (similar to first one).

For all these three parts, we have 2! different options for the two habitable planets.

Hence, if we combine all the options that we calculated above, there will be
$C(10,1) \cdot C(20,2) \cdot C(80,8) \cdot 2! \cdot (C(8,6) \cdot 6! \cdot 3! + C(8,7) \cdot 7! \cdot 2! + C(8,8) \cdot 8!)$ different ways to form a galaxy.

Answer 2

This is a linear non-homogeneous recurrence relation. Its associated homogeneous recurrence relation is:
$a_n = 2a_{n-1} + 15a_{n-2} - 36a_{n-3}$

Characteristic equation is:
$r^3 - 2r^2 - 15r + 36 = (r-3)^2(r+4)$

The characteristic roots are r=3 of multiplicity two and r=-4

So the homogeneous solution is:
$a_n^{(h)} = (A + Bn)3^n + C(-4)^n$, where A, B and C are constant numbers.

We now find a particular solution. Since $F(n) = 2^n$, a reasonable solution might be $a_n^{(p)} = X \cdot 2^n$,

where X is a constant.
If we write down the recurrence relation according to $a_n^{(p)}$:
$X \cdot 2^n = 2X \cdot 2^{n-1} + 15X \cdot 2^{n-2} - 36X \cdot 2^{n-3} + 2^n$
Dividing both sides to 2^{n-3}, we get:
$8X = 8X + 30X - 36X + 8$, $x = 4/3$

So the particular solution is:
$a_n^{(p)} = (4/3)2^n$

By the theorem 5(from book, page 521) all solutions are of the form:
$a_n = a_n^{(p)} + a_n^{(h)} = (4/3)2^n + (A + Bn)3^n + C(-4)^n$

Answer 3

$a_1 = 5$ since only 1, 3, 5, 7 and 9 can be valid one-digit activation code. A recurrence relation can be derived for this sequence by considering how a valid n-digit code can be obtained from code of n - 1 digits. There are two ways to form a valid code with n digits from a code with one fewer digit.

First, a valid code of n digits can be obtained by appending an even digit to a valid code of n-1 digits. This appending could be done in five ways because there are only five even digits. Thus, valid code with n digits can be created in this manner in $5a_{n-1}$ ways.

Second, a valid code of n digits can be obtained by appending an odd digit to an invalid code of n-1 digits. There are 10^{n-1} code with n-1 digits and a_{n-1} of them are valid. So, there are $(10^{n-1} - a_{n-1})$ invalid code with n-1 digits and this appending could be done in five ways because there are only five odd digits. Thus, valid code with n digits can be created in this manner in $5 \cdot (10^{n-1} - a_{n-1})$ ways.

$a_n = 5a_{n-1} + 5 \cdot (10^{n-1} - a_{n-1}) = 5 \cdot 10^{n-1}$

$a_n = 5 \cdot 10^{n-1}$
$a_{n-1} = 5 \cdot 10^{n-2} = a_n/10$
$a_{n-2} = 5 \cdot 10^{n-3} = a_{n-1}/10$

From equations above, we clearly see that:
$a_n/10 = a_{n-1}$
Hence, our recurrence relation for a_n is:
$a_n = 10a_{n-1}$ with $a_1 = 5$ and $n \geq 1$

Answer 4

Let $G(x)$ be the generating function for the sequence a_k.

$$
\begin{aligned}
G(x) &= \sum_{k=0}^{\infty} a_k \cdot x^k \\
&= a_0 + a_1 \cdot x + a_2 \cdot x^2 + \sum_{k=3}^{\infty} a_k \cdot x^k
\end{aligned}
$$

Now, we substitute $3a_{k-1} - 3a_{k-2} + a_{k-3}$ into a_k.

$$
\begin{aligned}
G(x) &= a_0 + a_1 \cdot x + a_2 \cdot x^2 + \sum_{k=3}^{\infty} (3a_{k-1} - 3a_{k-2} + a_{k-3}) \cdot x^k \\
&= a_0 + a_1 \cdot x + a_2 \cdot x^2 + 3 \cdot \sum_{k=3}^{\infty} a_{k-1} \cdot x^k - 3 \cdot \sum_{k=3}^{\infty} a_{k-2} \cdot x^k + \sum_{k=3}^{\infty} a_{k-3} \cdot x^k \\
&= a_0 + a_1 \cdot x + a_2 \cdot x^2 + 3x \cdot \sum_{k=3}^{\infty} a_{k-1} \cdot x^{k-1} - 3x^2 \cdot \sum_{k=3}^{\infty} a_{k-2} \cdot x^{k-2} + x^3 \cdot \sum_{k=3}^{\infty} a_{k-3} \cdot x^{k-3} \\
&= a_0 + a_1 \cdot x + a_2 \cdot x^2 + 3x \cdot \sum_{k=2}^{\infty} a_k \cdot x^k - 3x^2 \cdot \sum_{k=1}^{\infty} a_k \cdot x^k + x^3 \cdot \sum_{k=0}^{\infty} a_k \cdot x^k \qquad (1)
\end{aligned}
$$

Then, we need to find corresponding generating function equations for these summations.

$$
\begin{aligned}
\sum_{k=2}^{\infty} a_k \cdot x^k &= G(x) - (a_0 + a_1 \cdot x) \\
\sum_{k=1}^{\infty} a_k \cdot x^k &= G(x) - a_0 \\
\sum_{k=0}^{\infty} a_k \cdot x^k &= G(x)
\end{aligned}
$$

Again, substitute these into equation (1).

$$G(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + 3x \cdot [G(x) - (a_0 + a_1 \cdot x)] - 3x^2 \cdot [G(x) - a_0] + x^3 \cdot G(x)$$

Rearranging the terms,

$$G(x) = \frac{(-1) \cdot (a_0 + (a_1 - 3a_0) \cdot x + (a_2 - 3a_1 + 3) \cdot x^2)}{(x-1)^3} \tag{2}$$

Because $a_0 = 1$, $a_1 = 3$ and $a_2 = 6$, putting their values into the equation (2),

$$G(x) = \frac{-1}{(x-1)^3} = \frac{-1}{(-1)^3 \cdot (1-x)^3} = \frac{1}{(1-x)^3}$$

As the identity from Useful Generating Functions Table in the textbook claims that $\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty}(C(n+k-1,k)x^k$, we have $\frac{1}{(1-x)^3} = \sum_{k=0}^{\infty}(C(k+2,k)x^k$.

Hence, $a_k = C(k+2,k) = ((k+2)(k+1))/2$

Answer 5

a. If relation is an equivalence relation; it must be reflexive, symmetric and transitive.

R is a relation on the ordered pairs set of positive integers such that (a,b)R(c,d) if and only if $a + d = b + c$

1) Reflexive;

$(a,b)R(a,b) : a + b = b + a$
Therefore it is reflexive.

2) Symmetric;

$(a,b)R(c,d) : a + d = b + c$
$(c,d)R(a,b) : c + b = d + a$
$(a,b)R(c,d) \rightarrow (c,d)R(a,b)$
Therefore it is symmetric.

3) Transitive;

$(a,b)R(c,d) \wedge (c,d)R(e,f)$
$a + d = b + c \wedge c + f = d + e$
If we sum, we can get

$a + f = b + e$
And it shows $(a,b)R(e,f)$
Therefore it is transitive

Since this equation satisfy with reflexivity, symmetry and transition; it is an equivalence relation.

b. $(1, 2)R(x, y) \rightarrow y = x + 1$

If we say $x = n$, we get $y = n + 1$

To find equivalence class of (1,2) we should check for, $(a, b)R(x, y)$

$a + y = b + x$

$a + n + 1 = b + n \rightarrow b = a + 1$

So, equivalence class of (1,2) is $(n, n + 1)$