CENG 384 - Signals and Systems for Computer Engineers
20212
Written Assignment 4 Solutions

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1. (a) The D.E. of the system can be derived from the diagram as follows:

$$y'' + 2y' + y = x'' + x' - x.$$

(b) The frequency response, $H(j\omega)$, can be calculated by taking the transforms of both sides and eliminating $X(j\omega)$ on both sides:

$$H(j\omega) = \frac{(j\omega)^2 + (j\omega) - 1}{(j\omega)^2 + 2(j\omega) + 1} = \frac{(j\omega)^2 + (j\omega) - 1}{(j\omega + 1)^2}.$$

(c) The F.R. has double roots, therefore, it should be expanded as follows:

$$H(j\omega) = 1 + \frac{-j\omega - 2}{(j\omega + 1)^2} = 1 + \frac{A}{j\omega + 1} + \frac{B}{(j\omega + 1)^2}.$$

Solving for A and B, we obtain:

$$H(j\omega) = 1 + \frac{-1}{j\omega + 1} + \frac{-1}{(j\omega + 1)^2}.$$

The inverse of $H(j\omega)$ gives us:

$$h(t) = \delta(t) - e^{-t}u(t) - te^{-t}u(t).$$

(d) The transform of $x(t)$ is $X(j\omega) = 1/(j\omega + 1)$. Multiplying this with $H(j\omega)$ gives:

$$Y(j\omega) = \frac{1}{j\omega + 1} + \frac{-1}{(j\omega + 1)^2} + \frac{-1}{(j\omega + 1)^3},$$

whose inverse is then:

$$y(t) = e^{-t}u(t) - te^{-t}u(t) - \frac{1}{2}t^2e^{-t}u(t).$$

2. (a) If we use $\delta(t)$ as the input, then we get $h(t)$ as the output.

$\frac{dh(t)}{dt} = \delta(t+1) - \delta(t-1)$, taking integral of both sides we get:

$$h(t) = u(t+1) - u(t-1).$$

(b) The frequency response, $H(j\omega)$, can be calculated by taking the transforms of both sides of the differential equation and eliminating $X(j\omega)$ on both sides:

$$H(j\omega) = \frac{e^{j\omega} - e^{-j\omega}}{j\omega} = \frac{\cos(\omega) + j\sin(\omega) - (\cos(\omega) - j\sin(\omega))}{j\omega},$$

$$H(j\omega) = \frac{2j\sin(\omega)}{j\omega} = \frac{2\sin(\omega)}{\omega}.$$

3. (a) $H_1(e^{j\omega}) = H_2(e^{j\omega}) = \frac{1}{1-0.5e^{-j\omega}}$, so we get $H(e^{j\omega})$ as follows:

$$H(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega}) = \frac{1}{(1-0.5e^{-j\omega})^2}.$$

(b) From Euler's Formula we can write $x[n]$ as the following:

$$\sin\left(\frac{\pi}{3}n + \frac{\pi}{4}\right) = \frac{e^{j\frac{\pi}{3}n}e^{j\frac{\pi}{4}} - e^{-j\frac{\pi}{3}n}e^{-j\frac{\pi}{4}}}{2j}.$$

From table 5.2 in the textbook, we get the Fourier Transform of $x[n]$ as follows:

$$X(e^{j\omega}) = \frac{\pi}{j}\left(e^{j\frac{\pi}{4}}\delta\left(\omega - \frac{\pi}{3}\right) - e^{-j\frac{\pi}{4}}\delta\left(\omega + \frac{\pi}{3}\right)\right).$$

(c)

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}).$$

$$Y(e^{j\omega}) = \frac{\pi}{j}\left(e^{j\frac{\pi}{4}}\delta\left(\omega - \frac{\pi}{3}\right) - e^{-j\frac{\pi}{4}}\delta\left(\omega + \frac{\pi}{3}\right)\right)\frac{1}{(1-0.5e^{-j\omega})^2}.$$

$$\omega = \frac{\pi}{3} : \frac{\pi}{j}e^{j\frac{\pi}{4}}\frac{1}{(1-\frac{1}{2}e^{-j\frac{\pi}{3}})^2}$$

$$\omega = -\frac{\pi}{3} : \frac{\pi}{j}(-e^{-j\frac{\pi}{4}})\frac{1}{(1-\frac{1}{2}e^{j\frac{\pi}{3}})^2}$$

4. (a) We can use the known transforms $\delta[n] \longleftrightarrow 1$ and $a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}}$ to obtain the following ($a = 2^{-1} = 1/2$ in our case):

$$H(e^{j\omega}) = 2 + \frac{1}{1-(1/2)e^{-j\omega}} = \frac{6-2e^{-j\omega}}{2-e^{-j\omega}}.$$

(b) From the frequency response, we can obtain the Fourier Transform of the difference equation as follows:

$$\begin{aligned}
&H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{6-2e^{-j\omega}}{2-e^{-j\omega}} \\
&\Longrightarrow 2Y(e^{j\omega}) - e^{-j\omega}Y(e^{j\omega}) = 6X(e^{j\omega}) - 2e^{-j\omega}X(e^{j\omega}) \\
&\Longrightarrow 2y[n] - y[n-1] = 6x[n] - 2x[n-1].
\end{aligned}$$

(c) Rewriting $x[n]$ as $x[n] = (-1)^n = e^{j\pi n}$ reveals a very simple periodic signal for which we can easily identify the Fourier Series coefficients:

$$x[n] = e^{j\pi n} = e^{j(2\pi/2)n},$$

from which we discover that $\omega_0 = \pi$ and $x[n]$ has a single non-zero Fourier Series coefficient: $a_1 = 1$. Therefore:

$$X(e^{j\omega}) = 2\pi\delta(\omega - \omega_0) = 2\pi\delta(\omega - \pi), \text{for a period of } 2\pi, \text{ e.g. } 0 < \omega \le 2\pi.$$

From this, we can calculate $Y(e^{j\omega})$ as follows (for $0 < \omega \le 2\pi$):

$$\begin{aligned}
Y(e^{j\omega}) &= X(e^{j\omega})H(e^{j\omega}) = 2\pi\delta(\omega - \pi)\left(2 + \frac{1}{1-(1/2)e^{-j\omega}}\right) \\
&= 2\pi\delta(w - \pi)\left(2 + \frac{1}{1-(1/2)e^{-j\pi}}\right) = 2\pi\delta(\omega - \pi)\left(2 + \frac{1}{1-(1/2)(-1)}\right) \\
&= 2\pi\delta(\omega - \pi)\frac{8}{3} = \frac{16\pi}{3}\delta(\omega - \pi)
\end{aligned}$$