Simulation of Stochastic Processes

A Review Paper

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1. Introduction

Many physical systems have attributes varying across space or domain and time. The variations in the input attributes like geometrical, structural, and material properties can change the system response significantly. This can also affect the predictive capabilities of the system and the way we understand the system behavior. Hence the degree of variation or randomness in the input and output has to be quantified using laws of probability. The response, either discrete or continuous, can be modeled using random fields or stochastic processes that contain probabilistic dependence among responses observed at various temporal and spatial locations. Markovian dependence is widely used in random fields due to its simplicity and utilizes the concept that the future response depends only on the immediately previous response. All the other past responses do not affect the future response.

A large number of methods have been developed to solve problems involving stochastic processes using approximate simulation techniques for both stationary and non-stationary processes. For example, earthquake induced motion, noise in electronic devices are non stationary since their statistical behavior depends on the time of origin whereas ocean surface, engineering properties like thickness, modulus of elasticity occur in nature and do not depend on the time of origin and hence termed as stationary. The methods also depend on whether the underlying random field is Gaussian or non-Gaussian. Monte Carlo simulation appears to be the only method that can solve the problems involving randomness described by stochastic processes. This report therefore reviews various simulation techniques available in literature. Definitions for several keywords used in this report can be found in any standard text book on random processes (Papoulis, 1991; Soong & Grigoriu, 1992; Vanmarcke, 1983).

2.0 Review of Simulation Methods

Most available techniques for generating sample functions of random processes focus on stationary Gaussian processes. Several such methods are discussed and some of them can be generalized to a class of non-stationary processes and /or non-Gaussian processes. In each case, equations for only spectral representation, Karhunen - Loeve (K-L) and Polynomial Chaos expansion-based methods for finite variances are provided since they are more efficient, simple and less cumbersome. Also the discussions are limited to continuous stochastic processes with underlying parametric marginal probability distributions.

2.1. Stationary Gaussian Process

2.1.1 ARMA methods

Earlier studies on auto-regressive moving average ARMA methods (Mignolet *et al*, 1987; Samaras *et al*, 1985) were more mathematically motivated. Gersch & Yonemoto (1977) devised a two-stage least square procedure in order to estimate the parameters of ARMA models, following prescribed covariance functions. In the ARMA approach, coefficient matrices are first estimated such that the generated multivariate time series reproduces the prescribed correlation function matrix. The digital generation of sample functions of multivariate stationary Gaussian processes then involves a straightforward recursive numerical operation that makes use of the coefficient matrix. In principle, the ARMA representation allows generation of sample functions of infinite length and with such speed and computational mode that real time generations of sample function can be easily achieved. The sample functions are found to accurately reproduce the prescribed correlation function matrix even with very long sample functions. The ARMA method can produce longer sample functions compared to the spectral representation method. Naganuma *et al* (1987) extended the method to simulate two-dimensional homogenous Gaussian

processes. However, the selection of the model order and the calibration of its parameters can be complex.

2.1.2 Local average sub-division (LAS) method

The concept behind the LAS approach arose out of the stochastic sub-division algorithm described by Fournier *et al* (1982). This method was limited to modeling of processes with a specific spectrum and suffered from the problem of aliasing. Lewis (1987) generalized the approach to allow for the modeling of any arbitrary spectrum by utilizing a midpoint displacement algorithm. However, the problem of aliasing remains unresolved as the power spectral density is not modified at each stage to reflect the increase in the Nyquist frequency associated with each increase in resolution. Fenton and Vanmarcke (1990) reviewed these earlier methods and devised the LAS method that resolves the problem of aliasing while preserving internal consistency within each sub-division. The generated discrete points in the process represent local averages of a homogenous random function defined by its mean and covariance function, with averaging performed over incremental domains formed by different levels of discretization. This method is of practical significance as most engineering measurements are only defined over some finite domain and thus represents a local average of the property. The method allows the conditioning of the realization to incorporate known data or change the resolution within sub-regions. Regions of interest can be modeled with higher resolution. This is an important contribution to finite element modeling of random processes.

2.1.3 Sampling theorem

The sampling theorem approach in simulation is a more recent development (Grigoriu, 1993b). It is developed for simulating realizations of real valued stationary Gaussian vector processes and vector fields. The method is based on parametric model that can be derived from a

sampling theorem, and consists of a superposition of a finite set of deterministic functions of time or space with random amplitudes.

2.1.4 Spectral representation method

The spectral method is the most widely used method for its versatility and robustness. By this method, the sample functions of the random processes are generated according to prescribed power spectral density function. The generation can be performed with great efficiency by taking advantage of the Fast Fourier Transform (FFT). Spectral representation based algorithm generate ergodic sample functions, in the sense of that the temporal power spectra of each and every generated sample function are identical to the corresponding prescribed power spectra, when the length of the process is equal to one period. Another major advantage of the method is its versatility. It is the only method available for simulating stationary and non-stationary Gaussian and non-Gaussian processes. Consider a univariate Gaussian stochastic process $g(t)$ with zero mean and unit variance and whose spectral density is given by $S_{gg}(\omega)$ or autocorrelation function $R_{gg}(\tau)$ positive ω. Autocorrelation $R_{gg}(\tau)$ function can be used to generate the spectral density function using Wiener-Khintchine formula

$$S_{gg}(\omega) = \frac{1}{\pi}\int_0^{\infty} R_{gg}(\tau)\cos(\omega\tau)d\tau \tag{1}$$

or when spectral density function is given, autocorrelation function can be derived using

$$R_{gg}(\tau) = \int_0^{\infty} S_{gg}(\omega)\cos(\omega\tau)d\omega \tag{2}$$

The stochastic process then can be generated (Shinozuka & Deodatis, 1991; 1996) using the expression

$$g(t) = \sqrt{2}\sum_{k=0}^{N-1}\sqrt{2S_{gg}(\omega_k)\Delta\omega}\cos(\omega_k t + \phi_k) \tag{3}$$

where $\Delta\omega = \omega_u/N$ and ω_u is a value beyond which the spectral density is negligibly small. Also, $\omega_k = k\,\Delta\omega$, where k = 0, 1, 2, …, N-1 and ϕ_k is the phase angle uniformly distributed between 0 and 2π. Alternatively, a sine-cosine representation of a Gaussian process can be simulated using

$$g(t) = \sum_{k=0}^{N-1} \sqrt{2S_{gg}(\omega_k)\Delta\omega}\left[U_k \cos(\omega_k t) + V_k \sin(\omega_k t)\right] \tag{4}$$

where U_k and V_k are independent standard normal variables. Note that a cosine-only series is more efficient since it requires less number of variables. A stationary Gaussian process with mean μ and standard deviation σ can be simulated using equation

$$g(t) = \mu + \sigma\sqrt{2}\sum_{k=0}^{N-1} \sqrt{2S_{gg}(\omega_k)\Delta\omega}\cos(\omega_k t + \phi_k) \tag{5}$$

Recently a new model was proposed for generating samples of real-valued stationary Gaussian processes (Grigoriu, 2000). The model is based on the spectral representation theorem stating that a weakly stationary process can be viewed as a superposition of harmonics with random amplitudes, frequencies and phase. This method is different from the classical method shown in Eq, (5) where the frequencies are not random. Let $g(t)$ be a stationary Gaussian process with zero mean and unit variance and spectral density $S_{gg}(\omega)$, then the process can be approximated by a series $\boldsymbol{y}(t)$ as

$$y(t) = \sum_{k=1}^{N_1(\varpi_u)} h\left(\Gamma_{1,k}\right)\cos\left(\Gamma_{1,k}t\right)y_{1,k} + \sum_{l=1}^{N_2(\varpi_u)} h\left(\Gamma_{2,l}\right)\sin\left(\Gamma_{2,l}t\right)y_{2,l} \tag{6}$$

where ω_u is a value beyond which the spectral density is negligibly small, $\Gamma_{1,k}$ and $\Gamma_{2,l}$ are independent uniformly distributed variables in [0, ω_u] and $y_{1,k}$ and $y_{2,l}$ are independent zero mean random variables from any distribution F. The function $h(\omega)$ is derived from the spectral density function and finite second moment of $y_{1,1}$ as

$$h(\varpi) = \sqrt{\frac{S_{gg}(\omega)}{v\mu_2}} \tag{7}$$

where the intensity parameter $v = \frac{1}{\Delta\varpi}$ with $\Delta\varpi$ being the average spacing between the frequencies $\Gamma_{1,k}$ and $\Gamma_{2,l}$. Also, the number of samples can be obtained approximately using the relation

$$N_p(\varpi_u) = \max(0,[v,\ \omega_u + \sqrt{G}\ v\omega_u - 0.5]) \text{ for } p = 1, 2 \tag{8}$$

where G is a positive standard Gaussian variable and $[x]$ denotes the integer part of x.

2.1.5 Karhunen-Loeve Expansion Method

By representing a given covariance function as a K-L series expansion, efficient simulation of Gaussian stochastic fields can theoretically be achieved. This arises from the condensation of a substantial amount of computational effort into an analytical pre-processing step, thereby drastically reducing the requisite computational effort while safeguarding accuracy (Ghanem & Spanos, 1991). To achieve this, the eigenvalues and eigenfunctions of the covariance function must be determined, which involves the solution of an integral equation and can pose some difficulty. The advantages of K-L expansion and the difficulty involved in using it have been discussed by a number of researchers (Ghanem & Spanos, 1991; Li & Der Kiureghian, 1993; Zhang & Ellingwood, 1994). Gutierrez *et al* (1992) have also studied the numerical solution of K-L expansion. K-L expansion has an edge over the spectral method for highly correlated processes.

Consider a univariate stationary Gaussian process $g(t)$ with zero mean and unit variance with autocorrelation function $R(\tau)$. The stochastic process can be simulated using the expression (Ghanem & Spanos, 1991)

$$g(t) = \overline{g} + \sum_{i=1}^{n} \sqrt{\lambda_i}\, f_i(t)\xi_i \tag{9}$$

where λ_i and $f_i(t)$ are the eigenvalues and eigenfunctions of the covariance function $C(t_1, t_2)$ and ξ_i is standard normal random variable. The accuracy of the simulation depends on the number of terms n. The eigenvalues and eigenfunctions can be solved from the integral equation

$$\int_D C(t_1, t_2)\, f_i(t_2) dt_2 = \lambda_i\, f_i(t_1) \tag{10}$$

The above equation can be solved either analytically or numerically. Numerical methods are more flexible and can be applied when the covariance function is complicated. In such a method, the covariance function is approximated as

$$f_i(t) = \sum_{j=0}^{M-1} d^i_j \psi_j(t) \tag{11}$$

where $\psi_j(t)$ is jth basic function and d^i_j is the jth element of the ith column of the eigenvector vector matrix D^i obtained by solving the equation of form

$$AD_i = \lambda_i B D^i \tag{12}$$

in which, the components in all the M x M matrices are

$$A_{ij} = \iint C(t_1, t_2)\psi_i(t_1)\psi_j(t_2) dt_1 dt_2 \tag{13}$$

$$\text{and } B_{ij} = \int \psi_i(t_2)\psi_j(t_2) dt_2$$

2.1.6 Wavelet representation

In the wavelet representation method (Gurley & Kareem, 1994; Zeldin & Spanos, 1996), the series is a combination of a set of deterministic functions which form an orthogonal wavelet basis and corresponding random coefficients with weak correlation. The procedure of using wavelet is

quite complex and it is currently not obvious that wavelets possess overwhelming advantages over traditional methods such as spectral representation and ARMA models.

2.2 Non-Stationary Gaussian Process

2.2.1 ARMA Method

The ARMA method can also be extended to simulate non-stationary Gaussian processes (Deodatis & Shinozuka, 1988; Sharma & Shah, 1985) in this case, the parameters are functions of time. Simulation based on non-stationary ARMA models follows the steps for stationary processes. However, the determination of the model parameters from the condition that the differences between the model and the actual process be minimized can be complex since it involves solutions of the equalities at every time step (Soong & Grigoriu, 1993). Besides the autoregressive series with time dependent coefficients, inhomogeneous filtered Poisson processes, oscillatory processes, and processes with modulated amplitude and frequency (Deodatis & Shinozuka, 1988; Grigoriu *et al*, 1988; Shinozuka & Sato, 1967; Shinozuka & Jan, 1972) are also commonly used. The calibration of these models to a target non-stationary process can be complex because it requires finding one or more functions of time and may encounter theoretical difficulties because the models cannot represent arbitrary non-stationary processes. An alternative model could be spectral or parametric representation.

LAS method provides an efficient procedure for simulating non-stationary processes using non-uniform meshes which spectral and ARMA approaches are not well suited. Non-stationary processes encountered in stochastic mechanics are often modeled by introducing the concept of evolutionary spectrum (Priestley, 1981) i.e., spectral density function as a function of time. The evolutionary spectral density and cross-correlation structure of nonstationary random fields can be analyzed in terms of the envelope and frequency modulation functions. A method

for the conditional simulation of amplitude frequency modulated space-time fields is developed to map nonstationary fields to a domain where the conditional simulation is performed as for stationary, space-time fields (Zavoni & Santa-Cruz, 2000).

2.2.2 Trigonometric series representation

Consider a nonstationary Gaussian process $g(t)$ with covariance function $C(t_1, t_2)$. For a zero mean process and when $C(0, 0) = 1$, the covariance will be same as autocorrelation function $R(t_1, t_2)$. The process can be represented using a finite trigonometric series (Grigoriu, 1993a) as

$$g_n(t) = \frac{A_0}{2} + \sum_{k=1}^{n} \left(A_k \cos \omega_k t + B_k \sin \omega_k t \right) \tag{14}$$

defined on $(0, T)$ in which $\omega_k = 2\pi k/T$ and $k = 0, 1, \ldots, n$. The random coefficients $A_0, A_1, \ldots, A_n, B_1, \ldots, B_n$ are Gaussian with zero mean and covariance given by

$$E(A_k A_l) = \frac{4}{T^2} \int_0^T \int_0^T R(t_1, t_2) \cos \omega_k t_1 \cos \omega_l t_2 \, dt_1 dt_2 \quad \text{for } k, l = 0, 1, \ldots, n \tag{15}$$

$$E(A_k B_l) = \frac{4}{T^2} \int_0^T \int_0^T R(t_1, t_2) \cos \omega_k t_1 \sin \omega_l t_2 \, dt_1 dt_2 \quad \text{for } k = 0, 1, \ldots, n \text{ and } l = 0, 1, \ldots, n$$

$$E(B_k B_l) = \frac{4}{T^2} \int_0^T \int_0^T R(t_1, t_2) \sin \omega_k t_1 \sin \omega_l t_2 \, dt_1 dt_2 \quad \text{for } k, l = 1, 2, \ldots, n$$

The random coefficients ($2n+1$ in number) may be obtained first by generating independent standard normal variables and pre-multiplying with lower triangular matrix derived from Cholesky decomposition of the covariance matrix of Eq. (15). Since the concept of classical spectral density function is not applicable for a non-stationary process, simulation is based on covariance function only.

2.2.3 Parametric models

The non-stationary Gaussian process shown in Eq. (14) can also be generated using non-trigonometric series such as Bernstein and other interpolation polynomials and spline functions (Grigoriu, 1995). However, the convergence rates of parametric models may differ. The general parametric representation of the process can be

$$g_n(t) = \sum_{i=1}^{n} g_i \varphi_i(t) \tag{16}$$

where $\varphi_i(t)$ is a deterministic function of time t and g_i is a random coefficient. A very slow rate of convergence of these polynomials is a drawback and hence less widely used than simple trigonometric series. Recently, alternative models for nonstationary Gaussian process have been proposed (Grigoriu, 2003) depending on whether the process is Markovian or not. A process $\boldsymbol{X}(t)$ can be called as Markovian if for its covariance function $c(t, s)$, there exists $c(t, s)^{-1}$, then

$$c(t, s) = c(t, u)c(u, u)^{-1}c(u, s) \;\forall\; t < u < s \tag{17}$$

Then the Gaussian random variable $\boldsymbol{X}(t)|\boldsymbol{X}(s)$ will have a conditional density function

$$\boldsymbol{X}(t)|\,(\boldsymbol{X}(s) = \xi) \sim N(c(t, s)c(s, s)^{-1}\xi,\; c(t, t) - c(t, s)c(s, s)c(s, t)) \tag{18}$$

Now, samples of process $\boldsymbol{X}(t)$ can be generated in $[t_0, t_1]$ using a two-step algorithm.

1. Select a partition $t_0 = s_0 < s_1 < \ldots < s_m = t_1$ of time interval $[t_0, t_1]$
2. Apply the recurrence formula $\boldsymbol{X}(s_k, w) = \boldsymbol{X}(s_{k-1}, w) + \boldsymbol{G}_{k-1}(w)$ for $k = 1, \ldots., m$

to generate a sample $\boldsymbol{X}(s_k, w)$ of $\boldsymbol{X}$ at time s_k, where $\boldsymbol{X}(s_{k-1}, w)$ is the value of the sample w of X at time s_k-1, $\boldsymbol{G}_{k-1} = \boldsymbol{X}(s_k)|\,\boldsymbol{X}(s_{k-1})$. The use of this algorithm is very inefficient since it involves inversion of covariance matrix for each time step. If the Gaussian process is not necessarily Markovian, then a trigonometric representation method described earlier can be used to simulate the nonstationary process. An alternative method by Grigoriu (2003) based on augmented conditional simulation has proven to be more complicated and inefficient than simulation

technique using Fourier series representation. Thus efficient algorithms can be used when Gaussian process is Markovian but not when it is non-Markovian.

2.2.4 Karhunen-Loeve Expansion

The procedure for simulating non-stationary Gaussian process is no different from the stationary process. The covariance function is decomposed as described in Eq.(9). The advantage of K-L expansion over the spectral or parametric representation lies in its ease of extension to the simulation of general non-stationary Gaussian processes.

2.2.5 Wavelet representation

As discussed in an earlier section, the wavelet transform provides an improved time-frequency representation of the signal. The estimation of time-varying spectral can be reliable by employing the time-scale analysis associated with the wavelet transform. In this regard, wavelet representation has an advantage in simulating non-stationary processes (Gurley & Kareem, 1994). Furthermore, like LAS method, wavelet representation is also the scale type method, which supports non-uniform meshing.

2.3 Stationary Non-Gaussian Process

Simulation methods for Gaussian processes are quite well established. However, research on the simulation of non-Gaussian processes is comparatively recent and limited. With the exception of spectral representation and Karhunen-Loeve expansion methods, the techniques like ARMA and LAS methods, Sampling theorem based simulation techniques, etc are currently restricted to simulation of Gaussian processes only. The class of methods that seek to generate sample functions of non-Gaussian process according to their prescribed marginal probability distributions and prescribed spectral density function will be discussed in this section. Grigoriu (1984) proposed one of the earlier non-Gaussian translation processes. Translation processes are

processes obtainable from nonlinear transformations of Gaussian processes. Most of the methods for generating a non-Gaussian process involve simulation of stationary Gaussian process using a calibrated covariance or spectral density function and then transformed suitably (translation process) to a required marginal probability distribution.

2.3.1 Spectral Representation Method

Consider a univariate non-Gaussian stochastic process $f(t)$ (say F distributed) with an autocorrelation function given by $R_{ff}(\tau)$. The stochastic process then can be generated using the following steps (Grigoriu, 1984):

1. Construct the autocorrelation function $R_{gg}(\tau)$ for underlying Gaussian process by solving the equation below for different values of τ.

$$R_{ff}(\tau) = \frac{1}{2\pi\sqrt{1-R_{gg}^2(\tau)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{F^{-1}[\Phi(g_1)]-\mu_f}{\sigma_f}\right)\left(\frac{F^{-1}[\Phi(g_2)]-\mu_f}{\sigma_f}\right) \exp\left(\frac{-g_1^2-g_2^2+2R_{gg}(\tau)g_1g_2}{2(1-R_{gg}^2(\tau))}\right) dg_1 dg_2 \quad (19)$$

where F^{-1} is the inverse cumulative distribution of the underlying non-Gaussian process. This method is useless if $R_{ff}(\tau)$ does not lie in the region (-1, 1) and $R_{gg}(\tau)$ cannot be solved.

2. Construct the new spectral density $S_{gg}(\tau)$ from the transformed autocorrelation function $R_{gg}(\tau)$ and use it to generate a Gaussian process of zero mean and unit variance

$$g(t) = \sqrt{2}\sum_{k=0}^{N-1} \sqrt{2S_{gg}(\omega_k)\Delta\omega}\cos(\omega_k t + \phi_k) \quad (20)$$

where $\Delta\omega = \omega_u/N$ and ω_u is a value beyond which the spectral density is negligibly small. Also, $\omega_k = k\,\Delta\omega$, where $k = 0, 1, 2, \ldots, N\text{-}1$ and ϕ_k is the phase angle uniformly distributed between 0 and 2π.

3. Obtain non-Gaussian sample by the inverse transformation of the above Gaussian process

$$f(t) = F^{-1}[\Phi(g(t))] \quad (21)$$

This method is very accurate and reliable for any non-Gaussian distribution. However, the calibration of covariance function for underlying Gaussian process does not ensure that $R_{gg}(\tau)$ is a positive definite function (Grigoriu, 1995). For some translation processes, it may not be possible to solve the underlying covariance function mathematically. Yamazaki & Shinozuka (1988) proposed an iterative method involving repeated updating of spectral density function $S_{gg}(\omega)$ of the underlying Gaussian process in such a way that eventually when a transformed Gaussian realization would produce the specified marginal probability distribution and spectral density function.

$$S_{gg}^{j+1}(\omega) = \frac{S_{gg}^{j}(\omega)}{S_{ff}^{j}(\omega)} S_{ff}^{T}(\omega) \tag{22}$$

where $S^{j}_{gg}(\omega)$ is the spectral density function used to generate the Gaussian sample function $g(t)$ at jth iteration and $S^{T}_{ff}(\omega)$ is the target spectral density function. For highly Non-Gaussian processes, this method does not seem to work well. Also the underlying process $g(t)$ in each iteration tends to behave non-stationary and also deviate from Gaussianity. Grigoriu (1998) provides a means for checking if the covariance function is admissible for calibration. First calculate R^{*} from the equation

$$R^{*} = \frac{E[F^{-1}(\Phi(y))F^{-1}(\Phi(-y))] - (E[F^{-1}(\Phi(y))])^{2}}{E[(F^{-1}(\Phi(y)))^{2}] - (E[F^{-1}(\Phi(y))])^{2}} \tag{23}$$

and if $R_{ff}(\tau) \in (R^{*}, 1)$, for every τ, then calculate $R_{gg}(\tau)$ according to Eq. (19) and/or $R_{gg}(t)$ is not positive definite, alternative approximation schemes have to be explored depending on the problem (Grigoriu, 1998). One can convert $R_{gg}(\tau)$ into a positive definite function by increasing the time step τ, or by dropping some terms in the correlation matrix using principal component analysis (PCA) concepts or singular value decomposition method (SVD) or when the spectral

density function is negative, such values may be set to zero. Another option is to select a more general non-Gaussian process with a different distribution function F or $R_{ff}(\tau)$. Deodatis and Micaletti (2001) proposed an alternative iterative algorithm modifying Yamazaki and Shinozuka's method (1988). The generation of non-Gaussian process is as follows:

1. Generate the sample function of underlying Gaussian stochastic process using

$$\hat{g}(j\Delta t) = 2\sum_{k=0}^{M-1}\sqrt{S_{gg}\left(k\Delta\omega+\frac{\Delta\omega}{2}\right)\Delta\omega}\exp\left(i\phi_k\right)\exp\left(i\left(k\Delta\omega\right)\right)\left(j\Delta t\right) \text{ for } j = 0, 1, \ldots, M\text{-}1 \quad (24)$$

with $S_{gg}(\omega)$ being derived from a corrective step to Eq. (22) as

$$S_{gg}^{j+1}(\omega) = S_{gg}^{j}(\omega)\left[\frac{S_{ff}^{T}(\omega)}{S_{ff}^{j}(\omega)}\right]^{\beta} \quad (25)$$

and β is normally chosen subjectively to be a positive value close to zero.

2. The sample function is then shifted to $g_o(t)$ that is defined as

$$g_0(t) = \exp\left(i\frac{\Delta\omega}{2}t\right)\hat{g}(t) \; = g_R(t) + ig_I(t) \quad (26)$$

where $g_R(t)$ is the real part and $g_I(t)$ is the imaginary of the complex process $g_0(t)$. Compute the empirical marginal cumulative distribution function of $g_R(t)$ as G.

3. Using G and target distribution F, the non-Gaussian process can be generated using the transformation

$$f(t) = F^{-1}[G(g_R(t))] \quad (27)$$

The above process is repeated until target spectral densities and marginal distributions match with simulated functions. This correlation distortion concept utilizes mapping of CDF in Eq. (27) to generate non-Gaussian processes. Gurley and Kareem (1997) presented an alternative method using Hermite polynomial expansion-based spectral correction process that uses only the first

four moments of the target marginal probability distribution function. It also requires an optimization routine to calculate the Hermite polynomial coefficients.

Masters and Gurley (2003) proposed an empirical generalized mapping scheme of form with to simulate non-Gaussian processes. The algorithm is given in various steps below:

1. Initially, generate an underlying Gaussian process $y(t)$ with the target spectral density $S^T(\omega)$ using spectral representation method. The iterative procedure begins in next step.
2. Correct the probability content of $y(t)$ in current iteration using the mapping function

$$x(t) = F^{-1}[H(y(t))] \tag{28}$$

where F is the target density function and H is the empirical distribution of $y(t)$.

3. In the frequency domain, maintain the phase angle but replace Fourier amplitude repeatedly such that sample function matches $S^T(\omega)$.

Methods to generate stationary non-Gaussian processes with infinite variance can be found in Grigoriu (2000).

2.3.2 Karhunen-Loeve expansion

Consider a univariate non-Gaussian stochastic process $f(t)$ (say F distributed) with an autocorrelation function $R_{ff}(\tau)$. The stochastic process then can be generated using the following steps:

1. Construct the autocorrelation function $R_{gg}(\tau)$ for underlying Gaussian process by solving the equation below for different values of τ.

$$R_{ff}(\tau) = \frac{1}{2\pi\sqrt{1-R_{gg}^2(\tau)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{F^{-1}[\Phi(g_1)]-\mu_f}{\sigma_f}\right)\left(\frac{F^{-1}[\Phi(g_2)]-\mu_f}{\sigma_f}\right) \exp\left(\frac{-g_1^2-g_2^2+2R_{gg}(\tau)g_1g_2}{2(1-R_{gg}^2(\tau))}\right) dg_1 dg_2 \tag{29}$$

where F^{-1} is the inverse cumulative distribution of the underlying non-Gaussian process.

2. Derive Eigenfunctions and Eigenvalues from the transformed autocorrelation function $R_{gg}(\tau)$ and use it to generate a Gaussian process of zero mean and unit variance similar to Eq. (12).

$$g(t) = \bar{g}(t) + \sum_{i=1}^{n} \sqrt{\lambda_i}\, f_i(t)\xi_i \tag{30}$$

3. Obtain non-Gaussian sample by the inverse transformation of the above Gaussian process

$$f(t) = F^{-1}[\Phi(g(t))] \tag{31}$$

Again, this method may not work if $R_{gg}(\tau)$ turns out to be a non-positive function. Huang (2000) and Phoon *et al* (2002) proposed a robust but complicated iterative scheme for simulating non-Gaussian processes using K-L expansion technique. In their method the random variable in Eq. (30) is generated using target distribution F and the Eigen decomposition of target covariance function. The generated samples using the expansion will have prescribed covariance function but not target PDF. So, the probability content is adjusted using the relation

$$\xi_m(t) = F^{-1}[G(g(t))] \tag{32}$$

where G is an empirical distribution (generally non-Gaussian) of $g(t)$. The samples $\xi_m(t)$ are then rearranged so that its correlation function matches with the target function. Thus a new generation of ξ_i's are obtained. The process is repeated until the simulated and target covariance function, PDFs match. Detailed description of the algorithm is available in the PhD dissertation by Huang (2000).

2.3.3 Polynomial Chaos

Sakamoto and Ghanem (2002a) proposed Polynomial Chaos decomposition for stationary Non-Gaussian process. An arbitrary random variable F can be represented as a series of Hermite polynomials Γ_i in a Gaussian variable g as

$$f = \sum_{i=0}^{N_0} U_i \Gamma_i \tag{33}$$

$$= F_0 + F_1 g + F_2\left(g^2 - 1\right) + F_3\left(g^3 - 3g\right) + \ldots$$

$$+ F_4\left(g^4 - 6g^2 + 3\right) + \ldots.$$

The coefficients in the expansion can be evaluated numerically using the expression

$$F_i = \frac{\left\langle f\Gamma_i \right\rangle}{\left\langle \Gamma_i \right\rangle^2} \tag{34}$$

The representation given in Eq. (33) for a random variable can be extended to a random field using an expression

$$\begin{aligned} f(t) = F_0(t) + F_1(t)g(t) + F_2(t)\left(g(t)^2 - 1\right) + ... \\ .... + F_3(t)\left(g(t)^3 - 3g(t)\right) + F_4(t)\left(g(t)^4 - 6g(t)^2 + 3\right) + ... \end{aligned} \tag{35}$$

where the stochastic process g(t) is Gaussian with zero mean and unit variance. For a stationary process, the coefficients are constants (F_i) and hence can simply be derived from Eq. (34) for a target marginal probability distribution F. The underlying correlation function $\rho_g(t_1, t_2)$ for the Gaussian process $g(t)$ can be determined from the correlation function $\rho_f(t_1, t_2)$ of target distribution, by solving the equation

$$\rho_f(t_1, t_2) = \sum_{i=1}^{N_N} F_i^2 i! \rho_g(t_1, t_2)^i \tag{36}$$

Further, the stationary Gaussian process $g(t)$ can be generated from the correlation function using Karhunen-Loeve expansion method discussed in previous subsections.

$$g(t) = \sum_{i=0}^{N_D} \sqrt{\lambda_i}\, f_i(t) \xi_i \tag{37}$$

where λ_i and $f_i(t)$ are eigenvalues and eigenfunctions obtained from decomposition of the correlation function $\rho_g(t_1, t_2)$. The accuracy of this method depends on three numbers, N_N -

controlling the accuracy of solution in obtaining $\rho_g(t_1, t_2)$ i.e., how well the simulated covariance function agrees with the target function, N_O – effecting how well the target CDF matches with the simulated one, and finally N_D – effecting the accuracy of underlying Gaussian process.

2.4 Non-Stationary Non-Gaussian Process

Compared to the research done in the simulation of Gaussian processes, non-stationary non-Gaussian process (NSNG) simulation is not so popular in the literature. The methods thus far discussed can simply be extended to NSNG random fields also.

2.4.1 Trigonometric Series representation

Consider a Non-stationary, Non-Gaussian process $f(t)$ with covariance function $C_{ff}(t_1, t_2)$ and auto-correlation function $R_{ff}(t_1, t_2)$. The stochastic process then can be generated using the following steps:

1. Construct the autocorrelation function $R_{gg}(t_1, t_2)$ for underlying Gaussian process by solving the equation below for different values of t_1, t_2.

$$R_{ff}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{F^{-1}\left[\Phi\left(\frac{g_1-\mu_{gg}(t)}{\sigma_{gg}(t)}\right)\right]-\mu_{ff}(t)}{\sigma_{ff}(t)} \right) \left(\frac{F^{-1}\left[\Phi\left(\frac{g_2-\mu_{gg}(t)}{\sigma_{gg}(t)}\right)\right]-\mu_{ff}(t)}{\sigma_{ff}(t)} \right) f(g_1,g_2,R_{gg}(t_1,t_2))dg_1dg_2 \quad (38)$$

$$\text{with } f(g_1,g_2,R_{gg}(t_1,t_2)) = \frac{1}{2\pi\sqrt{1-R_{gg}^2(t_1,t_2)}} \exp\left(\frac{-g_1^2-g_2^2+2R_{gg}(t_1,t_2)g_1g_2}{2(1-R_{gg}^2(t_1,t_2))} \right)$$

where F^{-1} is the inverse cumulative distribution of the underlying non-Gaussian process. Also, the equivalent mean and standard deviation of underlying Gaussian process can be obtained from solving the equations below.

$$\sigma_{ff}^2(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_{gg}(t)} \left[F^{-1}\left(\Phi\left(\frac{x-\mu_{gg}(t)}{\sigma_{gg}(t)} \right) \right) - \mu_{ff}(t) \right]^2 \exp\left(-\frac{1}{2}\left(\frac{x-\mu_{gg}(t)}{\sigma_{gg}(t)} \right)^2 \right) dx \quad (39)$$

$$\text{and } \mu_{ff}(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_{gg}(t)} \left[F^{-1}\left(\Phi\left(\frac{x-\mu_{gg}(t)}{\sigma_{gg}(t)} \right) \right) \right] \exp\left(-\frac{1}{2}\left(\frac{x-\mu_{gg}(t)}{\sigma_{gg}(t)} \right)^2 \right) dx$$

This method is useless if $R_{ff}(t_1, t_2)$ does not lie in the region (-1, 1) and $R_{gg}(t_1, t_2)$ cannot be solved.

2. Use the transformed autocorrelation function $R_{gg}(t_1, t_2)$ to generate a Gaussian process of zero mean and unit variance

$$g_n(t) = \frac{A_0}{2} + \sum_{k=1}^{n} \left(A_k \cos \omega_k t + B_k \sin \omega_k t \right) \quad (40)$$

defined on [0, T] in which $\omega_k = 2\pi k/T$ and $k = 0, 1, \ldots, n$. The random coefficients $A_0, A_1, \ldots, A_n$, $B_1, \ldots, B_n$ are Gaussian with zero mean and covariance given by Eq. (15).

3. Obtain non-Gaussian sample by the inverse transformation of the above Gaussian process

$$f(t) = F^{-1}\left(\Phi\left(\frac{g_n(t)-\mu_{gg}(t)}{\sigma_{gg}(t)} \right) \right) \quad (41)$$

Again, the limitations of using Eq. (38) can be overcome by the algorithms described in Section 2.3.2.

2.4.2 Karhunen-Loeve expansion

The procedure for simulating a Non-Stationary Non-Gaussian process is similar to the one described in Section 2.4.1 with $R_{gg}(t_1, t_2)$ being solved using Eq. (38). $\mu_{gg}(t)$ and $\sigma_{gg}(t)$ are also estimated as described in Section 2.4.1. A Gaussian process $g_n(t)$ is simulated using Eq. (30) first and a non-Gaussian sample is obtained by the inverse transformation of the above Gaussian process

$$f(t) = F^{-1}\left(\Phi\left(\frac{g_n(t) - \mu_{gg}(t)}{\sigma_{gg}(t)} \right) \right) \tag{42}$$

Again, the limitations of using Eq. (38) can be overcome by the algorithms described in Section 2.3.2 since simulation of stationary and nonstationary Gaussian fields is similar when using K-L expansion.

2.4.3 Polynomial Chaos

Polynomial Chaos based representation discussed in Section (2.3.3) can be extended to Non-Stationary process as well with an exception that the coefficients in the expansion shown in Eq. (35) are no longer constants and depend on time *t*. For the sake of simplicity, it can be assumed that $\rho_g(t_1, t_2)$ is equal to $\rho_f(t_1, t_2)$ and still achieve required accuracy (Sakamoto & Ghanem, 2002a).

3.0 Stochastic Vector fields

While methods for simulating a univariate processes were described so far, multivariate processes or vector random fields can also be simulated similarly. A numerical simulation scheme is presented in Li & Kareem (1997) that combines the advantages of the discrete Fourier transform algorithm and a digital filtering scheme to generate nonstationary multivariate random processes. The resulting time histories provide piecewise continuous evolutionary spectra and the simulation technique offers significant computational efficiency. A spectral representation-based simulation methodology is proposed by Popescu *et al* (1998) and Deodatis (1996) to generate sample functions of a multi-variate, multi-dimensional, non-Gaussian stochastic vector field, according to a prescribed cross-spectral density matrix and prescribed (non-Gaussian) marginal probability distribution functions. Polynomial Chaos based expansions may be used to simulate multidimensional non-stationary non-Gaussian processes (Sakamoto & Ghanem, 2002a; 2002b).

Remarks

It is very clear from the discussion on various simulation techniques; the researcher should select a method that is suitable to the particular problem of interest, his/ her computing capabilities, available time etc. All methods work well when large enough samples are generated and so issues like efficiency and comfort levels play a role in selecting a simulation method. The tutorial did not discuss vector random fields in detail and also miscellaneous methods that are available to simulate stochastic processes might have been omitted. Thus the report needs to be updated periodically as and when new efficient techniques are available. Numerical examples have been provided in the Appendix for the readers wishing to reproduce their simulations and compare the results for accuracy. In all the examples considered, the underlying calibrated correlated functions turned out to be positive definite.

References

1. Fenton, G. A., and VanMarcke, E. H., "Simulation of Random Fields via Local Average Subdivision, *J. of Engg Mech,* Vol. 116 (8), pp. 1733-1749, 1990.
2. Fournier, A., Fussel, D., and Carpenter, L., "Computer Rendering of Stochastic Models," *Communications of ACCM,* Vol. 25 (6), pp. 371-384, 1982.
3. Deodatis G., and Shinozuka, M., "Auto-regressive Model for Nonstationary Stochastic Processes," *ASCE J. of Engg Mech,* Vol. 114 (11), 1995-2012, 1988.
4. Deodatis, G., "Non-stationary Stochastic Vector Processes: Seismic Ground Motion Applications," *Prob. Engg Mech,* Vol. 11 (3), pp. 149-167, 1996.
5. Deodatis, G., and Micaletti, R. C., "Simulation of Highly Skewed Non-Gaussian Stochastic Processes," *ASCE J. of Engg Mech,* Vol. 127 (12), pp. 1284- 1295, 2001.

6. Gersch, W., and Yonmentoto, J., "Synthesis of Multivariate Random Vibration Systems: A Two Stage Least Squares ARMA Model Approach," *J. of Sound and Vibration*, Vol. 52 (4), pp. 553- 565, 1977.
7. Ghanem, R, and Spanos P. *Stochastic finite elements: a spectral approach*, New York: Springer, 1991.
8. Grigoriu, M., "Crossings of Non-Gaussian translation Processes," *ASCE J. of Engg Mech*, Vol. 110 (4), pp. 610-620, 1984.
9. Grigoriu, M., Ruiz, S., and Rosenblueth, E., "Nonstationary Models of Seismic Ground Acceleration," *Earthquake Spectra*, Vol. 4 (3), pp. 551-568, 1988
10. Grigoriu, M., "Simulation of Nonstationary Gaussian Process by Random Trigonometric Polynomials," *ASCE J. Engg Mech*, Vol. 119 (2), pp. 328 – 343, 1993a.
11. Grigoriu, M., "Simulation of Stationary Processes via Sampling Theorem," *J. of Sound and Vibration*, Vol. (166), pp. 301, 313, 1993b
12. Grigoriu, M., *Applied Non-Gaussian Processes*, Prentice-Hall, Englewood Cliffs, N. J., 1995.
13. Grigoriu, M., "Parametric Models of Nonstationary Gaussian Processes," *Probabilistic Engg. Mech*, Vol. 10 (2), pp. 95-102, 1995.
14. Grigoriu, M., "Simulation of Stationary Non-Gaussian Translation Processes," *J. of Engg Mech*, Vol. 124 (2), pp. 121-126, 1998.
15. Grigoriu, M., "On Spectral Representation based Model for Monte Carlo Simulation," *Probabilistic Engg. Mech*, Vol. 15 (4), pp. 365-370, 2000.
16. Grigoriu, M., "A Class of Models for Nonstationary Gaussian Processes," *Probabilistic Engg Mech*, Vol. 18 (1), pp. 203-213, 2003.

17. Gurley, K., and Kareem, A., "On the Analysis and Simulation of Random Process Utilizing Higher Spectra and Wavelet Transforms," *Proceedings of 2nd International Conference on Computational Stochastic Mechanics,* Athens, Greece Balkema, Roterdam, 1994.
18. Gurley, K., and Kareem, A., "Analysis, Interpretation, Modeling and Simulation of Unsteady Wind and Pressure Data," *J. Wind. Eng. Ind. Aerodyn.,* Vol.69 (71), pp. 657–669, 1997.
19. Gutierrez, R., Ruiz, J. C., and Valderrama, M. J., "On the Numerical Exapnsion of a Second Order Stochastic Process," *Applied Stochastic Models and Data Analysis,* Vol. 8, pp. 67-77, 1992.
20. Huang, S., "Simulation of Random Processes Using Karhunen-Loeve Expansion," PhD Dissertation, National University of Singapore, Singapore, 2001.
21. Lewis, J. P., "Generalized Stochastic Subdivision," *Transactions on Graphics,* Vol. 6(3), pp. 167-190, 1987.
22. Li, C. C., and Der Kiureghian, A.,, "Optimal Discretization of Random Process, *ASCE J. of Engg Mech,* Vol. 119 (6), pp. 1136-1154, 1993.
23. Li, Y., and Kareem, A., "Simulation of Multivariate Nonstationary Random Processes: Hybrid DFT and Digital Filtering Approach," *J. of Engg Mech,* Vol 123(12), pp. 1302-1310, 1997.
24. Mignolet, M. P., and Spanos, P. D., "Recursive Simulation of Stationary Multivariate Random Processes: Part II," *ASME J. of App. Mech,* Vol. 54 (3), pp. 681-687, 1987.
25. Papoulis, A., *Probability, Random Variables, and Stochastic Processes,* 3rd edition, McGraw-Hill International Editions, Singapore, 1991.

26. Phoon, K. K., Huang, S. P., and Quek, S. T., "Simulation of Second-order Processes Using Karhunen-Loeve Expansion," *Comput. Struct.*, Vol. 80 (12), pp. 1049–1060, 2002.
27. Popescu, R., Deodatis, G., and Prevost, J. H., "Simulation of Homogeneous Non-Gaussian Stochastic Vector Fields," *Prob Engg Mech,* Vol. 13 (1), pp. 1-13, 1998.
28. Priestly, M. D., *Spectral Analysis and Time Series*, Academic Press, San Diego, 1981.
29. Sakamoto, S., and Ghanem, R., "Polynomial Chaos Decomposition for the Simulatio of Non-Gaussian Nonstationary Stochastic Processes," *ASCE J. of Engg Mech,* Vol. 128 (2), pp. 190-201, 2002a.
30. Sakamoto, S., and Ghanem, R., "Simulation of Multi-dimensional non-Gaussian Nonstationary Random Fields," *Prob. Engg Mech,* Vol. 17 (2), pp. 167-176, 2002b.
31. Samaras, E., Shinozuka, M., and Tsurui, A., "ARMA Representation of Random Processes," *ASCE J. Of Engg Mech,* Vol. 111 (3), pp. 449-461, 1985.
32. Sharma, M. P., and Shah, H. C., "Representation of Nonstationary Ground Motion Acceleration using Time-varying Model," *Proceedings of ICOSSAR'85,* pp. 521-525, 1985.
33. Shinozuka, M., and Sato, Y., "Simulation of Nonstationary Random Processes, *ASCE J. of Engg Mech Div,* Vol. 93 (1), pp. 11-40, 1967.
34. Shinozuka, M., and Jan, C. M., "Random Processes with Evolutionary Power," *ASCE J. of Engg Mech Div*, Vol. 96 (EM4), pp. 29-53, 1970.
35. Shinozuka, M., and Deodatis, G., "Simulation of stochastic processes by spectral representation," *Appl. Mech. Rev.,* Vol. 44(4), pp. 191–204, 1991.

36. Shinozuka, M., and Deodatis, G., "Simulation of multi-dimensional Gaussian stochastic fields by spectral representation," *Appl. Mech. Rev.*, Vol. 49(1), pp. 29–53, 1996.

37. Soong, T. T., and Grigoriu, M., *Random Vibration of Mechanical and Structural Systems*, Prentice-Hall, Upper Saddle River, NJ, 1992.

38. VanMarcke, E., *Random Fields: Analysis and Synthesis*¸ MIT Press, Cambridge, MA, 1983.

39. Yamazaki, F., and Shinozuka, M., "Digital Simulation of Non-Gaussian Stochastic Fields," *ASCE J. of Engg Mech.,* Vol. 114 (7), pp. 1183-1197, 1988.

40. Zavoni, E. H., and Santa-Cruz, S, "Conditional Simulation of a Class of Nonstationary Space-Time Random Fields," *J. of Engg Mech,* Vol. 126 (4), pp. 398-404, 2000.

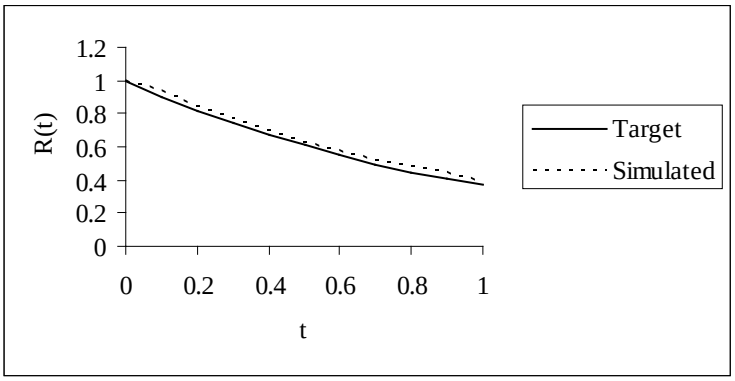
41. Zeldin, B. A., and Spanos, P. D.," Random Field Simulation using Wavelet Base," *ASME J. of App Mech*, Vol. 63 (4), pp. 946-952, 1996.

42. Zhang, J., and Ellingwood, B., "Orthogonal Series Expansions of Random Processes in Reliability Analysis," *ASCE J. of Engg Mech,* Vol. 120 (12), pp. 2660-2677, 1994.

APPENDIX

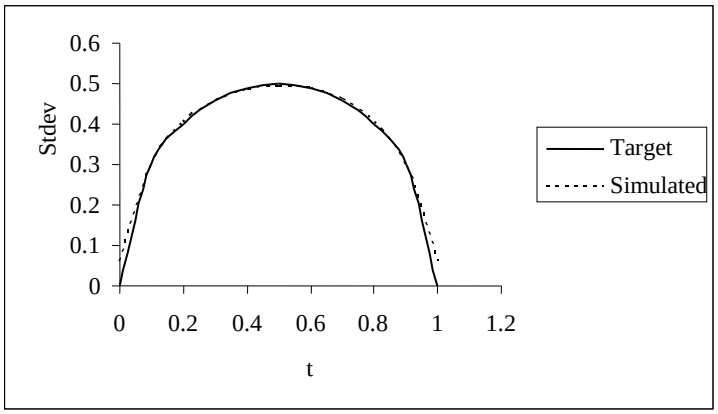
Example 1: Simulate a standard Gaussian process using spectral method with an autocorrelation function, $R(\tau) = \exp|-\tau|$.

Solution: Using Eq. (3) the spectral density for the autocorrelation function is found to be $\frac{1}{\pi(1+\varpi^2)}$. Also, choosing $N = 500$, $\omega_u = 8\pi$ and hence $\Delta\omega = 0.016\pi$. The following figure show the target and simulated autocorrelation function.

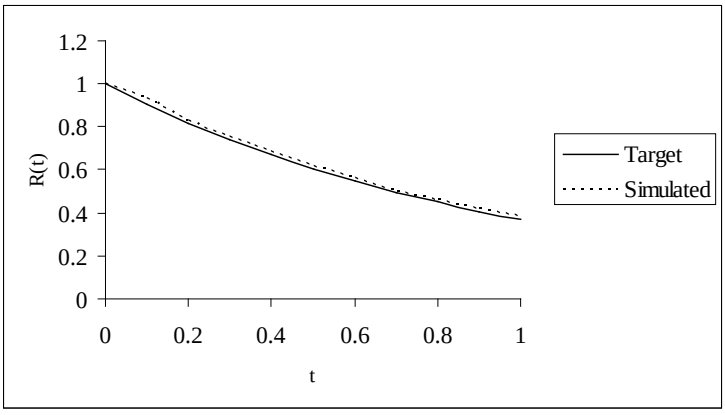


Example 2: Simulate a Gaussian process using spectral method with zero mean and correlation function (Brown –Bridge process) $R(t_1, t_2) = \min(t_1, t_2) - t_1 t_2/a$

Choosing $a = 1.0$, $T = 1$ and $n = 15$, the simulated standard deviation is plotted against the target standard deviation from square root of $R(t, t)$ i.e, $\sqrt{t - t^2}$ and shown in the figure below. Gauss quadrature method was used to estimate the elements of covariance matrix.

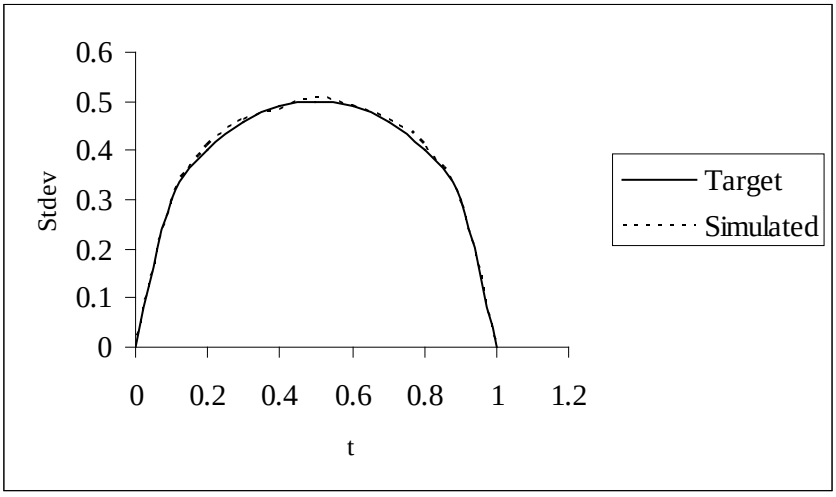


Example 3: Simulate a Gamma process $G(2,1)$ using spectral method with an autocorrelation function, $R_{ff}(\tau) = \exp|-\tau|$. Solving Eq. (5), the equivalent $R_{gg}(\tau)$ was found to be $\exp(-0.928\tau)$ and the corresponding spectral density function was determined as $\frac{0.928}{\pi\left(0.86118+\varpi^2\right)}$. Also, choosing $N = 500$, $\omega_u = 8\pi$ and hence $\Delta\omega = 0.016\pi$. The following figure show the target and simulated autocorrelation function.



Example 4: Simulate a Gamma process defined in $[0, T]$ using spectral method with mean $\mu_{ff}(t) = 0.3162$ and covariance function, $C_{ff}(t_1, t_2) = \min(t_1, t_2) - t_1 t_2/a$.

Choosing a = 1.0, T = 1 and n = 15, solving Eq. (8), the equivalent $R_{gg}(t_1, t_2)$ was found to be $1.08837\min(t_1, t_2) - t_1 t_2/0.91937$. The simulated standard deviation is plotted against the target standard deviation from square root of $\sigma_{ff}(t, t)$ i.e, $\sqrt{t - t^2}$ and shown in the figure below.



Example 5: Simulate a standard Gaussian process using K-L expansion method with an autocorrelation function, $R(\tau) = \exp|-\tau|$.

The eigen decomposition of the autocorrelation function yielded the following eigenvalues and eigenfunctions:

$$\lambda_i = \frac{2}{1+\varpi_i^2} \text{ and } f_i(t) = \frac{\cos(\varpi_i t)}{\sqrt{1+\dfrac{\sin(2\varpi_i)}{2\varpi_i}}} \text{ for } i = 0, 2, 4, 6, \ldots. \tag{14}$$

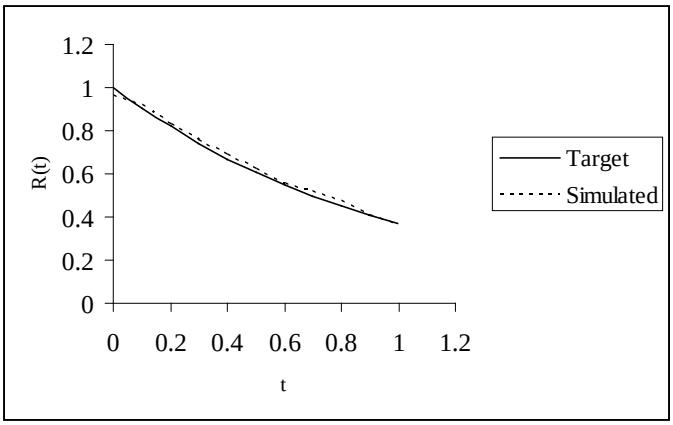
$$\lambda_i = \frac{2}{1+\varpi^{*2}_i} \text{ and } f_i(t) = \frac{\sin(\varpi^*_i t)}{\sqrt{1-\dfrac{\sin(2\varpi^*_i)}{2\varpi^*_i}}} \text{ for } i = 1, 3, 5, 7, \ldots.$$

in which ω_i and ω^*_i are obtained by solving

$$\omega \tan(\omega) - 1 = 0 \text{ for } i = 0, 2, 4, 6, \ldots. \tag{15}$$

$$\omega^* + \tan(\omega^*) = 0 \text{ for } i = 1, 3, 5, 7, \ldots.$$

Choosing $n = 9$, the following figure shows the target and simulated autocorrelation function



Example 6: Simulate a Gaussian process using K-L expansion method with zero mean and correlation function (Brown –Bridge process)

$$R(t_1, t_2) = \min(t_1, t_2) - t_1 t_2 / a$$

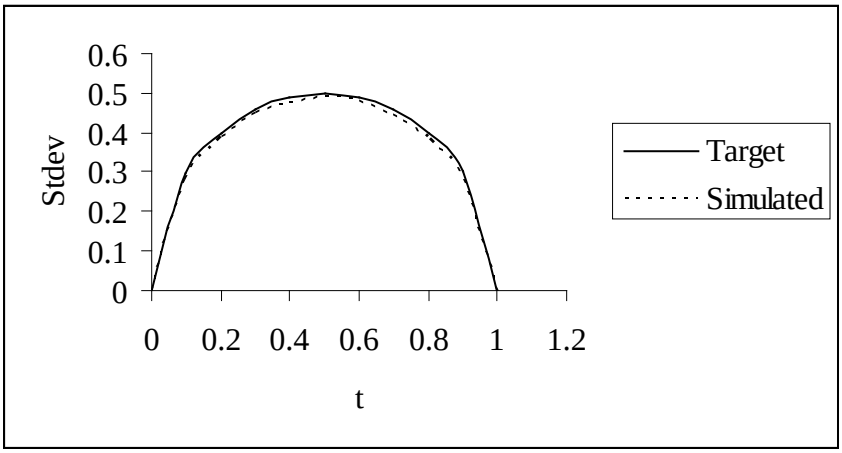
Eigen decomposition of the autocorrelation function yielded the following eigenvalues and eigenfunctions:

$$\lambda_i = \frac{a^2}{i^2 \pi^2} \text{ and } f_i(t) = \sqrt{\frac{2}{a}} \sin\left(\frac{t}{\sqrt{\lambda_i}}\right) \text{ for } i = 1, 2, 3, \ldots \quad (16)$$

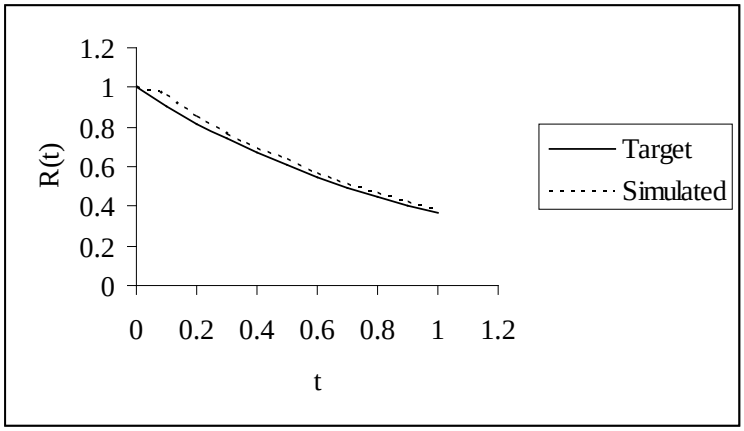
The stochastic process can be simulated using the expression

$$g(t) = \bar{g}(t) + \sum_{i=1}^{n} \sqrt{\lambda_i}\, f_i(t) \xi_i \quad (17)$$

Here $\bar{g}(t) = 0$, $a = 1$, and $n = 10$. The simulated standard deviation is plotted against the target standard deviation from square root of $R(t, t)$ i.e, $\sqrt{t - t^2}$ and shown in the figure below.



Example 7: Simulate a Gamma process $G(2,1)$ using K-L expansion method with an autocorrelation function, $R_{ff}(\tau) = \exp|-\tau|$. Solving Eq. (18), the equivalent $R_{gg}(\tau)$ was found to be $\exp(-0.928\tau)$. The eigenvalues and eigenfunctions are derived as shown in Example 5. The simulated autocorrelation function is plotted against the target autocorrelation function $\exp(-\tau)$ for 10 terms in K-L expansion and shown in the figure below.



Example 8: Simulate a Gamma process defined in $[0, T]$ using K-L expansion method with mean $\mu_{ff}(t) = 0.3162$ and covariance function, $C_{ff}(t_1, t_2) = \min(t_1, t_2) - t_1 t_2/a$.

Choosing $a = 1.0$, $T = 1$ and $n = 10$, solving Eq. (8), the equivalent $R_{gg}(t_1, t_2)$ was found to be $1.08837\min(t_1, t_2) - t_1 t_2/0.91937$. This function is decomposed for eigenvalues and

Eigenfunctions as in Example 6. The simulated standard deviation is plotted against the target standard deviation from square root of $\sigma_{ff}(t, t)$ i.e, $\sqrt{t - t^2}$ and shown in the figure below.