Spectral Stochastic Finite-Element Formulation for Reliability Analysis

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Abstract: An approach for the solution of problems of structural mechanics involving material variability is proposed. The material property is modeled as a stochastic process. The Karhunen-Loeve expansion is used to represent this process in a computationally expedient manner by means of a set of random variables. Further, the well-established deterministic finite-element method is used to discretize the differential equations governing the structural response. A spectral expansion of the nodal random variables is introduced involving a basis in the space of random variables. The basis consists of the polynomial chaoses that are polynomials orthogonal with respect to the Gaussian probability measure. The new formulation allows the computation of the probability distribution functions of the response variables in an expeditious manner. Two problems from structural mechanics are investigated using the proposed approach. The derived results are found in good agreement with data obtained by a Monte Carlo simulation solution of these problems.

Introduction

Modern engineering structures are becoming increasingly complex and involve state-of-the-art technology from both material sciences and control engineering. This fact, along with the recent availability of enhanced computational resources, has revived the interest in the analysis of systems where properties exhibit random fluctuations. Clearly, providing for these random effects at the modeling level can introduce more robustness and realism in the design. The added mathematical complications, however, have long hindered the development of a rational formalism for analyzing the effects of random material properties on the response of engineering systems. Current analytical methods consist of using either the Monte Carlo simulation approach (Shinozuka and Astill 1972; Shinozuka and Lenoe 1976; *Shinozuka* 1987), the perturbation method (Nakagiri and Hisada 1982; Liu et al. 1988), or, finally, the Neumann expansion method (Shinozuka and Nomoto 1980; Adomian and Malakian 1980; Benaroya and Rehak 1987; *Shinozuka* 1987). Except for the Monte Carlo simulation approach, all of these methods are best suited for small random fluctuations in the material properties. Furthermore, typical results of a probabilistic analysis of relevant systems have been restricted to second-order statistics of the response quantities. Obviously, this information does not suffice for a complete structural reliability analysis. The Monte Carlo simulation on the other hand can be viewed as a brute-force approach that may sacrifice computational efficiency for the sake of a simplified analytical formulation, and is usually used as a means of validating competing alternatives. New techniques have emerged

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that rely on the Karhunen-Loeve decomposition of a random process coupled with either a Neumann expansion scheme or a polynomial chaos expansion in conjunction with a Galerkin projection (Ghanem 1988). In this manner, a more efficient implementation of the randomness into the solution procedure was achieved. Results from both methods relating to the second-order properties of the response process were found in good agreement with simulation results over a wide range of random fluctuations. The present paper provides an extension of these methods to be applicable for structural reliability calculations. The stochastic finite-element formalism is presented whereby use is made of the concept of spectral approximation as employed in the deterministic setting. In this context, the spectrum defined by nonlinear transformation of the Brownian motion is used as a basis to expand the structural response process. Results pertaining to the failure probability in two structural mechanics problems are obtained and found in good agreement with pertinent Monte Carlo simulation data.

Mathematical Preliminaries

A key concept in the subsequent development is the realization that the random aspect of the problem becomes amenable to analysis through the addition to the problem of a new independent variable θ. This variable represents the random dimension. In this context, random variables can be viewed as functions of this new variable. The treatment of random variables as measurable functions is well established and can be found in a number of references (Doob 1956; Loeve 1977; Ghanem 1988). It may then be expected that a rational extension of the deterministic finite-element method to the case involving random material properties would involve an approximation along this random dimension. The development and implementation of such an approximation is the subject of this paper.

As a typical example of the class of problems amenable to the subsequent treatment, consider an operator equation of the form

$$Q(\mathbf{x},\theta)\,[u(\mathbf{x},\theta)] = f(\mathbf{x},\theta) \quad \ldots\ldots (1)$$

where $\mathbf{x}$ = vector of spatial coordinates; and θ = random character of the quantities involved. In general $Q(\mathbf{x},\theta)$ = a differential operator; and $u(\mathbf{x},\theta)$ = response to an input $f(\mathbf{x},\theta)$. Conceptually, given the joint probabilistic information about $Q(\mathbf{x},\theta)$ and $f(\mathbf{x},\theta)$, the solution is completely determined when the joint probabilistic information about $Q(\mathbf{x},\theta)$, $f(\mathbf{x},\theta)$, and $u(\mathbf{x},\theta)$ is obtained. Traditionally, a number of simplifying assumptions are introduced, and a partial solution to the problem, usually restricted to the second-order statistics of the response, is sought. In the following, a formulation is presented that yields not only second-order statistics of the response, but its probability distribution function as well.

Eq. (1) can be rewritten

$$\{\mathbf{L}(\mathbf{x}) + \mathbf{R}[\mathbf{x},\alpha_x(\theta)]\}\,[u(\mathbf{x},\theta)] = f(\mathbf{x},\theta) \quad \ldots\ldots (2)$$

where $\mathbf{L}(\mathbf{x})$ = average operator and is obtained from $Q(\mathbf{x},\theta)$ by statistical averaging; and $\mathbf{R}[\mathbf{x},\alpha_x(\theta)]$ = fluctuations of the operator about its mean. Note that in (2), the dependence of $\mathbf{R}[\mathbf{x},\alpha_x(\theta)]$ on the randomness parameter θ through a zero-mean random process $\alpha_x(\theta)$ is made explicit. The process $\alpha_x(\theta)$ can be thought of as representing the stochastic fluctuations about its mean of some property of the system described by the operator $Q(\mathbf{x},\theta)$. In general, $\alpha_x(\theta)$ is continuous with respect to its index $\mathbf{x}$, although the cor-

responding functional dependence is not explicitly known. In view of that, the traditional spatial discretization of a random process cannot be expected to perform adequately. Instead, the random process $\alpha_x(\theta)$ is discretized using a spectral measure defined with respect to the random dimension. This procedure effectively replaces the process by a set of deterministic functions of the spatial variable x with random magnitude. One such approximation can be obtained via the Karhunen-Loeve expansion whereby a random process $\alpha_x(\theta)$ is expressed

$$\alpha_x(\theta) = \sum_{m=1}^{\infty} \sqrt{\lambda_m}\, \xi_m(\theta) a_m(\mathbf{x}) \quad \ldots\ldots (3)$$

where $\xi_m(\theta)$ = a set of orthogonal random variables; and λ_m and $a_m(\mathbf{x})$ = deterministic quantities representing, respectively, the eigenvalues and eigenfunctions of the covariance kernel of $\alpha_x(\theta)$. Obviously, the joint distribution of $\xi_m(\theta)$ depends on that of $\alpha_x(\theta)$. In the case that this latter is a Gaussian process, the set $\boldsymbol{\xi}(\theta) \equiv \{\xi_m(\theta)\}$ forms a Gaussian vector. Eq. (2) is then equivalent to

$$\{\mathbf{L}(\mathbf{x}) + \mathbf{R}[\mathbf{x}, \boldsymbol{\xi}(\theta)]\}\, [u(\mathbf{x},\theta)] = f(\mathbf{x},\theta) \quad \ldots\ldots (4)$$

where the $\alpha_x(\theta)$ is replaced by the random variables from its Karhunen-Loeve expansion. Examining (4), it is seen that the response at any point in the domain can be treated as a random variable that depends on the set of uncorrelated random variables $\{\xi_m(\theta)\}$. This dependence, although not known a priori, can be formally expressed

$$u(\mathbf{x},\theta) = s[\{\xi_m(\theta)\}], \quad \ldots\ldots (5)$$

where $s[\ ]$ = some nonlinear functional of its arguments. Eq. (5) expresses the response at a point $\mathbf{x}$ in the domain as a nonlinear surface in the space spanned by the stochasticity of the system. Then the problem can be reformulated as that of evaluating accurate estimates to this response surface. Traditionally, such approximations have been obtained either by point fitting or curvature fitting a tangent plane or parabola or piecewise polynomials to the true response surface, at some optimal point (Grigoriu 1982; Der-Kiureghian 1987). Having used the Karhunen-Loeve expansion to minimize the number of random variables, it may be expected that higher-order curve fitting may be achieved. Furthermore, the spectral stochastic finite-element approach, as developed in this paper, is shown to readily provide a higher-order expansion in terms of orthogonal polynomials. Thus, it permits a quite accurate representation of the response process.

Deterministic Variational Finite-Element Analysis

In this section the variational formulation of the finite-element method is adopted to discretize in space the governing differential equations. Let $\boldsymbol{\Omega}$ denote the domain over which the structure is defined. Following Ghanem (1988), let this domain be subdivided into a number NE of finite elements, with a corresponding number N of degrees of freedom. Assuming the medium under consideration to behave elastically, the strain energy V^e within an element e can be represented in terms of the strain vector $\boldsymbol{\varepsilon}^e$ and the stress vector $\boldsymbol{\sigma}^e$ as

$$V^e = \frac{1}{2} \int_{\boldsymbol{\Omega}^e} \boldsymbol{\sigma}^{eT}(\mathbf{x}) \boldsymbol{\varepsilon}^e(\mathbf{x}) d\mathbf{x} \quad \ldots\ldots (6)$$

where T = vector transposition; and Ω^e = the domain of element e. Using the constitutive relations associated with the problem at hand, the stresses within each element can be expressed in terms of the strains in that element according to the following equation:

$$\boldsymbol{\sigma}^e(\mathbf{x}) = [\mathbf{D}^e(\mathbf{x},\theta)]\boldsymbol{\varepsilon}^e(\mathbf{x}) \quad \ldots\ldots (7)$$

The symbol $[\mathbf{D}^e(\mathbf{x},\theta)]$ = constitutive matrix for the specific problem at hand; and the arguments $\mathbf{x}$ and θ indicate that this matrix involves random material properties with spatial variability. Furthermore, interpolation functions can be used to express the response quantities within an element in terms of the nodal degrees of freedom of that element. This procedure leads to an equation of the form

$$\mathbf{u}^e(\mathbf{x},\theta) = [\mathbf{H}^e(\mathbf{x})]\boldsymbol{\mu}^e(\theta) \quad \ldots\ldots (8)$$

In (8), $\mathbf{u}^e(\mathbf{x},\theta)$ = vector of the degrees of freedom at a point $\mathbf{x}$ in element e; $[\mathbf{H}^e(\mathbf{x})]$ = an interpolation matrix consisting of appropriate shape functions; and $\boldsymbol{\mu}^e(\theta)$ = a vector representing the degrees of freedom asssociated with the nodes of the element. Eq. (8) can be used in conjunction with the strain compatibility relations relating the strain vector at a point to the degrees of freedom at that point, to express the strain energy within an element in terms of the nodal degrees of freedom. Specifically

$$V^e = \frac{1}{2}\boldsymbol{\mu}^{eT}(\theta)\int_{\Omega^e}[\mathbf{B}^e(\mathbf{x})]^T[\mathbf{D}^e(\mathbf{x},\theta)][\mathbf{B}^e(\mathbf{x})]d\mathbf{x}\boldsymbol{\mu}^e(\theta) \quad \ldots\ldots (9)$$

where $[\mathbf{B}^e(\mathbf{x})]$ = matrix that relates strains to nodal degrees of freedom, and can be obtained from the interpolation matrix $[\mathbf{H}^e(\mathbf{x})]$ by appropriate differentiation; its numerical form depends on the specific problem being considered. The total strain energy in the structural system is obtained by adding the contribution from all elements. This procedure leads to an equation involving all N degrees of freedom in the system, having the form

$$V = \boldsymbol{\mu}^T(\theta)[\mathbf{K}(\theta)]\boldsymbol{\mu}(\theta) \quad \ldots\ldots (10)$$

In (10), $\boldsymbol{\mu}(\theta)$ = N-dimensional vector of degrees of freedom for the entire system; and $\mathbf{K}(\theta)$ = $N \times N$ global stiffness matrix obtained by assembling the elemental stiffness matrices (Ghanem 1988). The total potential energy in the system is obtained upon providing for the contribution of the externally applied loads. The solution to the discrete problem is then found by setting to zero the variation of the total energy with respect to the degrees of freedom of the system. This leads to the algebraic system of equations

$$[\mathbf{K}(\theta)]\boldsymbol{\mu}(\theta) = \mathbf{p} \quad \ldots\ldots (11)$$

where $\mathbf{p}$ = vector of consistent nodal forces, and is obtained using standard finite-element techniques. It is assumed in what follows that the random material property appears as a multiplicative factor in the expression for the constitutive matrix. That is

$$[\mathbf{D}^e(\mathbf{x},\theta)] = \alpha_x(\theta)\mathbf{p}^e \quad \ldots\ldots (12)$$

where $\alpha_x(\theta)$ = a random process; and $\mathbf{p}^e$ = a deterministic matrix assumed, without loss of generality, to be constant over element e. Truncating the Karhunen-Loeve expansion for the process $\alpha_x(\theta)$ at the Mth term and substituting back into (9), it can be shown (Ghanem 1988) that (11) can be rewritten

$$\left(\overline{\mathbf{K}} + \sum_{m=1}^{M} \xi_m(\theta)\, \mathbf{K}_m\right)\boldsymbol{\mu}(\theta) = \mathbf{p} \quad \text{(13)}$$

The finite element discretization procedure of (13) eliminates the dependence of the governing equations on the independent variable x. Eq. (13) can be viewed as an operator equation with respect to the independent variable θ, and the problem remains of computing an approximation for the solution to this equation.

Spectral Representation of Random Functions

Any element $\mu(\theta)$ from the space Θ of random variables can be expanded in a uniformly convergent series

$$\mu(\theta) = \sum_{i=0}^{\infty} c_i \gamma_i(\theta) \quad \text{(14)}$$

where c_i = some constant independent of θ; and $\{\gamma_i(\theta)\}_{i=1}^{\infty}$ = a basis in Θ. Polynomial chaos was introduced (Wiener 1932) as a means for expanding random variables in terms of polynomials that are orthogonal with respect to the Gaussian measure. These polynomials were adapted for implementation with the stochastic finite-element method (Ghanem 1988). It can be shown that any element $\mu(\theta)$ of Θ admits the mean-square convergent expansion

$$\mu(\theta) = \sum_{p\geq 0} \sum_{n_1+\ldots+n_r=p} \sum_{l_1,\ldots,l_r} a_{l_1\ldots l_r}^{n_1\ldots n_r}\, \Gamma_p[\xi_{l_1}(\theta), \ldots, \xi_{l_r}(\theta)] \quad \text{(15)}$$

where $\Gamma_p(\ )$ = polynomial chaos of order p. The superscript n_i on the deterministic coefficient a refers to the number of occurrences of $\xi_{l_i}(\theta)$ in the argument list for $\Gamma_p(\ )$. It is noted that (15) is a variation of the Cameron-Martin theorem (Cameron and Martin 1947). Expanding (15), any element $\mu(\theta)$ of Θ can then be written

$$\mu(\theta) = a_0\Gamma_0 + \sum_{i_1=1}^{\infty} a_{i_1}\Gamma_1[\xi_{i_1}(\theta)] + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} a_{i_1i_2}\Gamma_2[\xi_{i_1}(\theta)\xi_{i_2}(\theta)]$$

$$+ \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1i_2i_3}\Gamma_3[\xi_{i_1}(\theta)\xi_{i_2}(\theta)\xi_{i_3}(\theta)]$$

$$+ \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2}\sum_{i_4=1}^{i_3} a_{i_1i_2i_3i_4}\Gamma_4[\xi_{i_1}(\theta)\xi_{i_2}(\theta)\xi_{i_3}(\theta)\xi_{i_4}(\theta)] + \ldots, \quad \text{(16)}$$

where $\Gamma_p(\ )$ = successive polynomial chaoses of their arguments, the expansion being convergent in the mean square. The polynomial chaoses of order greater than one have mean zero, and polynomials of different order are orthogonal to each other; so are polynomials of the same order with a different argument list. Up to the fourth order, the polynomial chaoses are given by the expressions

$$\Gamma_0 = 1 \quad \text{(17)}$$

$$\Gamma_1[\xi_{i_1}(\theta)] = \xi_{i_1}(\theta) \quad \text{(18)}$$

$$\Gamma_2(\xi_{i_1}(\theta),\xi_{i_2}(\theta)] = \xi_{i_1}(\theta)\xi_{i_2}(\theta) - \delta_{i_1i_2} \quad \text{(19)}$$

$$\Gamma_3[\xi_{i_1}(\theta),\xi_{i_2}(\theta),\xi_{i_3}(\theta)] = \xi_{i_1}(\theta)\xi_{i_2}(\theta)\xi_{i_3}(\theta) - \xi_{i_1}(\theta)\delta_{i_2i_3} - \xi_{i_2}(\theta)\delta_{i_1i_3} - \xi_{i_3}(\theta)\delta_{i_1i_2} \quad \ldots\ldots (20)$$

$$\Gamma_4[\xi_{i_1}(\theta),\xi_{i_2}(\theta),\xi_{i_3}(\theta),\xi_{i_4}(\theta)] = \xi_{i_1}(\theta)\xi_{i_2}(\theta)\xi_{i_3}(\theta)\xi_{i_4}(\theta) - \xi_{i_1}(\theta)\xi_{i_2}(\theta)\delta_{i_3i_4} - \xi_{i_1}(\theta)\xi_{i_3}(\theta)\delta_{i_2i_4} - \xi_{i_1}(\theta)\xi_{i_4}(\theta)\delta_{i_2i_3} - \xi_{i_2}(\theta)\xi_{i_3}(\theta)\delta_{i_1i_4} - \xi_{i_2}(\theta)\xi_{i_4}(\theta)\delta_{i_1i_3} - \xi_{i_3}(\theta)\xi_{i_4}(\theta)\delta_{i_1i_2} + \delta_{i_1i_2}\delta_{i_3i_4} + \delta_{i_1i_3}\delta_{i_2i_4} + \delta_{i_1i_4}\delta_{i_2i_3} \quad \ldots\ldots (21)$$

where δ_{ij} = Kronecker delta. In general, the pth-order polynomial chaos can be expressed as

$$\Gamma_p(\xi_{i_1}, \ldots, \xi_{i_n}) = \sum_{\substack{r=n \\ r \text{ even}}}^{0} (-1)^r \sum_{\pi(i1,\ldots,in)} \prod_{k=1}^{r} \xi_{ik} \left\langle \prod_{l=r+1}^{n} \xi_{i_1} \right\rangle \qquad n \text{ even}$$

$$e\Gamma_p(\xi_{i_1}, \ldots, \eta_{i_n}) = \sum_{\substack{r=n \\ r \text{ even}}}^{0} (-1)^{r-1} \sum_{\pi(i1,\ldots,in} \prod_{k=1}^{r} \xi_{ik} \left\langle \prod_{l=r+1}^{n} \xi_{il} \right\rangle \qquad n \text{ odd} \quad \ldots\ldots (22)$$

where $\langle\ \rangle$ denotes the operation of mathematical expectation; $\pi(\ )$ denotes a permutation of its arguments; and the summation extends over all permutations such that the sets $\{\xi_{i_1}, \ldots, \xi_{i_r}\}$ is modified by the permutation. Note that upon identifying the polynomial chaoses with the multidimensional Hermite polynomials, an alternative expression for deriving them is given by the equation

$$\Gamma_p[\xi_{i_1}(\theta), \ldots, \xi_{i_n}(\theta)] = (-1)^n \frac{\partial^n}{\partial\xi_{i_1}, \ldots, \partial\xi_{i_n}} e^{(-1/2)\boldsymbol{\xi}^T\boldsymbol{\xi}} \quad \ldots\ldots (23)$$

where $\boldsymbol{\xi}$ = vector of random variables $\{\xi_{i_1}, \ldots, \xi_{i_n}\}$. In the implementation of the foregoing formulation, (22) was used in conjunction with numerical computer codes, whereas (23) was found to be more suitable for implementation with symbolic-manipulation codes such as MACSYMA (1986). Indeed, a MACSYMA code was developed to explicitly derive the polynomial chaoses of arbitrary order and to compute the averages of their products. Note that the first two terms in (16) represent the mean and the Gaussian component of the function $\mu(\theta)$, respectively. Thus, for a Gaussian process, the expansion [(16)] reduces to a single summation, the coefficients a_{i_1} being the coefficients in the Karhunen-Loeve expansion of the process (Ghanem and Spanos, 1988). Eq. (16) is an expansion in θ, so that any other dependence of the function $\mu(\theta)$ will be carried over to the expansion coefficients a_i. In other words, if $\mu(\theta)$ is an element of R^N, so are these coefficients. Symbolically, the random-response vector may be expressed as in (5), which is rewritten here for the vector case as

$$\boldsymbol{\mu}(\theta) = \mathbf{s}[\{\xi_m(\theta)\}] \quad \ldots\ldots (24)$$

with $\mathbf{s}[\ ]$ = nonlinear vector functional of its argument. Expressed in this form, the response can be viewed as the output of a nonlinear filter to Gaussian white-noise input. The problem can then be restated as one of finding a convergent representation, optimal in some sense, of the functional

s[ ]. For notational expediency, the process $\alpha_s(\theta)$ is subsequently assumed to be Gaussian. Using the notation

$$\mathbf{K}_0 \equiv \overline{\mathbf{K}} \quad \ldots\ldots (25a)$$

and

$$\xi_0(\theta) \equiv 1 \quad \ldots\ldots (25b)$$

(13) can be rewritten in a more compact form as

$$\left[\sum_{m=0}^{M} \xi_m(\theta)\, \mathbf{K}_m\right]\boldsymbol{\mu}(\theta) = \mathbf{f} \quad \ldots\ldots (26)$$

Spectral Stochastic Finite-Element Analysis

Preliminary Remarks

Truncating (16) at the pth-order polynomial and substituting for $\boldsymbol{\mu}(\theta)$, (26) yields an expression for the resulting error

$$\boldsymbol{\varepsilon} = \mathbf{f} - \left\{\sum_{m=0}^{M} \mathbf{K}_m\, \mathbf{a}_0 \Gamma_0 \xi_m(\theta) + \sum_{m=0}^{M} \mathbf{K}_m \sum_{i_1=1}^{M} \mathbf{a}_{i_1} \Gamma_1[\xi_{i_1}(\theta)]\xi_m(\theta) + \ldots\right.$$

$$\left. + \sum_{m=0}^{M} \mathbf{K}_m \sum_{i_1=1}^{M} \ldots \sum_{i_p=1}^{i_{p-1}} \mathbf{a}_{i_1 \ldots i_p} \Gamma_p[\xi_{i_1}(\theta), \ldots, \xi_{i_p}(\theta)]\xi_m(\theta)\right\} \quad \ldots\ldots (27)$$

This error results from truncating the series in (16) after a finite number of terms, as well as from using a finite number of elements from the infinite set $\{\xi_m(\theta)\}_{m=1}^{\infty}$. The error, as expressed by (27), is minimized by requiring it to be orthogonal to the solution space with respect to the inner product defined by the statistical-averaging operator. Indeed, a procedure similar to the one used in the Galerkin-based deterministic finite-element method can be used here. Mathematically, this can be stated as

$$\langle \boldsymbol{\varepsilon}\, \Gamma_s[\pi_s^M(\xi_{i_1}, \ldots, \xi_{i_s})]\rangle = 0, \qquad s \le p \quad \ldots\ldots (28)$$

where $\pi_s^M(\xi_{i_1}, \ldots, \xi_{i_s})$ = permutation operator defined in the previous section. This orthogonality constraint results in a set of algebraic equations that can be solved for the vector coefficients $\mathbf{a}_{i_1 \ldots i_k}$, $(i_1 + \ldots + i_k \le p)$. From these coefficients, the spectral expansion of the response vector is determined with respect to the polynomial chaos basis. From (16) it can be seen that all the probabilistic information concerning the random response vector $\boldsymbol{\mu}(\theta)$ is contained in the expansion coefficients. In particular, the average response is equal to $\mathbf{a}_0$. Thus, at least theoretically at this point, once these coefficients have been computed, the joint probability distribution of the nodal response vector $\boldsymbol{\mu}(\theta)$ can be determined.

Numerical Solution of Integral Equation

The first step in implementing the proposed method consists of expanding the random process representing the randomness of the system parameters. When dealing with non-Gaussian processes, (16) can be used to that effect. However, since in this case the process is assumed to have a Gaussian distribution, its Karhunen-Loeve expansion is most efficient for this purpose. Then, the computations leading to the expansion consist of solving an integral eigenvalue equation for the covariance kernel. For most useful

covariance models, the solution for such an equation can be obtained analytically. In general, however, a numerical solution may have to be implemented (Delves and Mohamed 1985). In the next section, the analytical solution of the integral equation for a two-dimensional domain will be demonstrated. In this section, a Galerkin-type procedure is described for the solution of the associated Fredholm equation given by the equation

$$\int_{\Omega} C(\mathbf{x}_1,\mathbf{x}_2) f_k(\mathbf{x}_2) d\mathbf{x}_2 = \lambda_k f_k(\mathbf{x}_1) \quad \ldots\ldots (29)$$

Let $h_i(\mathbf{x})$ be a complete set of functions in the Hilbert space $\mathbf{L}_2$. Each eigenfunction of the kernel $C(\mathbf{x}_1,\mathbf{x}_2)$ may be represented as

$$f_k(\mathbf{x}) = \sum_{i=1}^{N} d_i^{(k)} h_i(\mathbf{x}) \quad \ldots\ldots (30)$$

with an error ε_N resulting from truncating the summation at the Nth term. Substituting (30) into (29) yields

$$\varepsilon_N = \sum_{i=1}^{N} d_i^{(k)} \left[\int_{\Omega} C(\mathbf{x}_1,\mathbf{x}_2) h_i(\mathbf{x}_2) d\mathbf{x}_2 - \lambda_n h_i(\mathbf{x}_1) \right] \quad \ldots\ldots (31)$$

Requiring the error to be orthogonal to the approximating space yields equations of the following form:

$$[\varepsilon_N, h_j(\mathbf{x})] = 0, \qquad j = 1, \ldots, N \quad \ldots\ldots (32)$$

Equivalently, (32) can be expanded as

$$\sum_{i=1}^{N} d_i^{(k)} \left\{ \int_{\Omega} \left[\int_{\Omega} C(\mathbf{x}_1,\mathbf{x}_2) h_i(\mathbf{x}_2) d\mathbf{x}_2 \right] h_j(\mathbf{x}_1) d\mathbf{x}_1 - \lambda_n \int_{\Omega} h_i(\mathbf{x}) h_j(\mathbf{x}) d\mathbf{x} \right\} = 0 \quad \ldots\ldots (33)$$

Introducing

$$\mathbf{C}_{ij} = \int_{\Omega} \int_{\Omega} C(\mathbf{x}_1,\mathbf{x}_2) h_i(\mathbf{x}_2) d\mathbf{x}_2 h_j(\mathbf{x}_1) d\mathbf{x}_1 \quad \ldots\ldots (34)$$

$$\mathbf{B}_{ij} = \int_{\Omega} h_i(\mathbf{x}) h_j(\mathbf{x}) d\mathbf{x} \quad \ldots\ldots (35)$$

$$\mathbf{D}_{ij} = d_i^{(j)} \quad \ldots\ldots (36)$$

and

$$\mathbf{\Lambda}_{ij} = \delta_{ij} \lambda_i \quad \ldots\ldots (37)$$

(33) can be written

$$\mathbf{CD} = \mathbf{\Lambda BD} \quad \ldots\ldots (38)$$

where $\mathbf{C}$, $\mathbf{B}$, and $\mathbf{D}$ = three N-dimensional matrices whose elements are given by (34)–(36), respectively. Eq. (38) represents a generalized algebraic eigenvalue problem that may be solved for the matrix $\mathbf{D}$ and the eigenvalues λ_k. Then, substituting $d_i^{(k)}$ into (30) yields the eigenfunctions of the covar-

iance kernel. The preceding procedure can be implemented using piecewise polynomials as the basis for the expansion. With this choice of basis, the columns of matrix **D** become the eigenfunctions computed at the respective nodal points of the induced mesh, and the ijth element of the matrix **C** becomes the weighted correlation between the process at nodes i and j. Note that both matrices **C** and **B** are symmetric and positive definite; this fact simplifies the numerical solution substantially.

Implementation of Polynomial Chaos Expansion

The next step in the solution procedure consists of implementing the expansion provided by (16) for the solution process. The size of the resulting system of equations is proportional to the number of polynomial chaoses used. It can be seen that the set of pth-order polynomial chaoses is given by $\Gamma_p\{\pi_p^M[\xi_{i_1}(\theta), \ldots, \xi_{i_p}(\theta)]\}$. Specifically, assuming M to be equal to three, the set of first-order polynomial chaoses is $\{\xi_{i_1}(\theta), \xi_{i_2}(\theta), \xi_{i_3}(\theta)\}$, and that of the second-order polynomials is $\{\Gamma_2[\xi_{i_1}(\theta), \xi_{i_1}(\theta)], \Gamma_2[\xi_{i_1}(\theta), \xi_{i_2}(\theta)], \Gamma_2[\xi_{i_1}(\theta), \xi_{i_3}(\theta)], \Gamma_2[\xi_{i_2}(\theta), \xi_{i_2}(\theta)], \Gamma_2[\xi_{i_2}(\theta), \xi_{i_3}(\theta)], \Gamma_2[\xi_{i_3}(\theta), \xi_{i_3}(\theta)]\}$, and so on. In general, for the pth-order chaos, using M elements from the set $\{\xi_m(\theta)\}_{m=1}^{\infty}$, there are $(1/p!)\ \Pi_{k=0}^{p-1}\ (M + k)$ elements in the set of the basis vector spanning the corresponding subspace. The total number of basis vectors, therefore, is

$$P = 1 + \sum_{p=1}^{p} \frac{1}{p!} \prod_{k=0}^{p-1} (M + k) \quad \ldots\ldots (39)$$

To prevent further complicating the notation, let the resulting set of P basis vectors be mapped in a one-to-one mapping to a set with ordered indices denoted by $\{\gamma_i(\theta)\}_{i=1}^{P}$. Then, (16) may be rewritten

$$\boldsymbol{\mu}(\theta) = \sum_{i=0}^{P} \mathbf{c}_i \gamma_i(\theta) \quad \ldots\ldots (40)$$

where now $\boldsymbol{\mu}(\theta)$ and $\mathbf{c}_i$ = N-dimensional vectors. Table 1 shows explicit expressions for the first 10 two-dimensional polynomial chaoses.

Repetitive application of the orthogonality condition given by (28) for successive polynomial chaoses leads to the matrix equation

$$\mathbf{Kc} = \mathrm{F} \quad \ldots\ldots (41)$$

where $\mathbf{c}$ = PN-dimensional vector of coefficients; $\mathbf{K}$ = a $PN \times PN$ matrix involving block submatrices. The (i,j) block is an $N \times N$ matrix given by the equation

$$\mathbf{K}_{ij} = \sum_{m=0}^{M} \mathbf{K}_m \langle \gamma_i(\theta)\gamma_j(\theta)\xi_m(\theta)\rangle, \qquad i, j = 0, \ldots, P - 1 \quad \ldots\ldots (42)$$

Further, $\mathbf{F}$ is a PN-dimensional vector whose ith N-dimensional block is given by the equation

$$\mathbf{F}_i = \langle \mathbf{f}\gamma_i(\theta)\rangle, \qquad i = 0, \ldots, P - 1 \quad \ldots\ldots (43)$$

Thus, if $\mathbf{f}$ is statistically independent of the operator-induced randomness, (43) reduces to

$$\mathbf{F}_i = \langle \mathbf{f}\rangle \qquad \text{if } i = 0 \quad \ldots\ldots (44a)$$

TABLE 1. Two-Dimensional Polynomial Chaoses and Their Variances

j (1)	p (order of homogeneous chaos) (2)	jth polynomial chaos (γ_j) (3)	$\langle\gamma_j^2\rangle$ (4)
0	$p = 0$	1	1
1	$p = 1$	ξ_1	1
2	$p = 1$	ξ_2	1
3	$p = 2$	$\xi_1^2 - 1$	2
4	$p = 2$	$\xi_1\xi_2$	1
5	$p = 2$	$\xi_2^2 - 1$	2
6	$p = 3$	$\xi_1^3 - 3\xi_1$	6
7	$p = 3$	$\xi_1^2\xi_2 - \xi_2$	2
8	$p = 3$	$\xi_1\xi_2^2 - \xi_1$	2
9	$p = 3$	$\xi_2^3 - 3\xi_2$	6
10	$p = 4$	$\xi_1^4 - 6\xi_1^2 + 3$	24
11	$p = 4$	$\xi_1^3\xi_2 - 3\xi_1\xi_2$	6
12	$p = 4$	$\xi_1^2\xi_2^2 + \xi_1^2 - \xi_2^2 + 1$	4
13	$p = 4$	$\xi_1\xi_2^3 - 3\xi_1\xi_2$	6
14	$p = 4$	$\xi_2^4 - 6\xi_2^2 + 3$	24

TABLE 2. Coefficient P for the size of the extended system

M[b] (1)	p[a] 0 (2)	1 (3)	2 (4)	3 (5)	4 (6)
2	1	3	6	10	15
4	1	5	15	35	70
6	1	7	28	83	210

[a]p = order of homogeneous chaos expansion.
[b]M = number of terms in Karhunen-Loeve expansion.

$$\mathbf{F}_i = 0 \quad \text{if } i \neq 0 \qquad (44b)$$

The right-hand side of (42) involves computing the sums of averages of products of independent Gaussian random variables. In view of that, the matrices $\mathbf{K}_{ij}$ in (42) can be easily generated numerically. Table 2 shows values of P for various combinations of p and M.

Although the figures in Table 2 suggest large values for the parameter P, it has been found (Ghanem and Spanos, 1988) that values for M of two or four are enough for an adequate representation of fairly large levels of random fluctuations. It is worth noting that the polynomial chaoses form a hierarchical orthogonal basis. This fact suggests a number of efficient schemes for solving the resulting system of equations (Babuska et al., 1986). One scheme may be based on the fact that the solutions resulting from using successively higher-order expansions provides a converging approximation of the true solution. Therefore, the solution from any lower-order expansion

can be used as a first approximation in an iterative scheme for finding the coefficients of the higher-order expansions. Another computationally efficient scheme can be based on the observation that the coefficients $\langle \gamma_i(\theta)\gamma_j(\theta)\xi_m(\theta)\rangle$ in (42) vanish often enough to warrant the tabulation of these coefficients in a preprocessing stage. Thus, the time-consuming computation of zero coefficients can be avoided. Table 3 shows the value of these coefficients for $m = 1$ and $m = 2$, respectively. In Table 3, the order M of the Karhunen-Loeve expansion is equal to 2, and the order p of the polynomial chaos is equal to 3.

Reliability Calculations

Second-order properties of the response process can be readily obtained once the coefficients in the polynomial chaos expansion have been computed. The covariance matrix of the response has a concise representation given by the equation

$$\mathbf{R}_{\mu\mu} = \sum_{m=0}^{M} \mathbf{c}_m \langle \gamma_m(\theta)^2 \rangle \quad \ldots\ldots (45)$$

where $\langle \gamma_m(\theta)^2 \rangle$ = variance of the **m**th polynomial chaos, and is given in Table 1. Additional probabilistic information about the response process can be obtained by noting that the response at any given point on the structure can be written symbolically as

$$u(\mathbf{x},\theta) = h[\alpha_x(\theta)] \quad \ldots\ldots (46)$$

TABLE 3. Two-Dimensional Polynomial Chaoses

Indices of the Polynomial Chaoses			
i (1)	j (2)	k (3)	$c_{ijk}^{(2)}$[a] (4)
$i = 1$	1	3	2
$i = 1$	2	4	1
$i = 1$	3	6	6
$i = 1$	4	7	2
$i = 1$	5	8	2
$i = 1$	6	10	24
$i = 1$	7	11	6
$i = 1$	8	12	4
$i = 1$	9	13	6
$i = 2$	1	4	1
$i = 2$	2	5	2
$i = 2$	3	7	2
$i = 2$	4	8	2
$i = 2$	5	9	6
$i = 2$	6	11	6
$i = 2$	7	12	4
$i = 2$	8	13	6
$i = 2$	9	14	24

[a] $c_{ijk}^{(2)} = \langle \xi_i \gamma_j \gamma_k \rangle$ and $c_{ijk}^{(2)} = c_{ikj}^{(2)}$.

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where $h[\ ]$ = some function of its argument. Eq. (46) is an expression for the response surface as a function defined on the infinite dimensional space spanned by the random process $\alpha_x(\theta)$. The Karhunen-Loeve expansion of $\alpha_x(\theta)$ can be seen as an instrument for replacing this infinite dimensional space by a finite dimensional one. The response, then, becomes a function of the finite number of uncorrelated random variables appearing in the Karhunen-Loeve expansion. This can be formally written

$$u(\mathbf{x},\theta) = s[\{\xi_i\}] \quad \ldots\ldots (47)$$

The polynomial chaos spectral expansion then provides a convergent expansion of this continuous function in terms of a discrete number of coefficients. These coefficients are computed using the Galerkin scheme presented in the previous section. Once an explicit expression is obtained for the response surface $s[\ ]$, an expression for the failure surface $g[\{\xi_i\}]$ is immediately available by noting that

$$g[\{\xi_i\}] = s[\{\xi_i\}] - u_{\text{failure}}(x) \quad \ldots\ldots (48)$$

where $u_{\text{failure}}(x)$ denotes the capacity at failure, at point x, of the structural system. The probability of failure P_f can then be obtained by direct integration over the failure surface as

$$P_f = \int_{g[\{\xi_i\}]<0} g[\{\xi_i\}]\, p_{\xi_{i1} \ldots \xi_{in}} d\xi_{i1} \ldots d\xi_{in} \quad \ldots\ldots (49)$$

where $p_{\xi_{i1} \ldots \xi_{in}}$ = joint probability distribution function of the orthogonal random variables $\{\xi_{i1}, \ldots, \xi_{in}\}$. In general the shape of the failure surface is such that closed-form integration is not possible. A versatile numerical integration procedure can be based on simulation. This simulation can be readily implemented at a minimum computational cost. Specifically, realizations of the set $\{\xi_i\}_{i=1}^{M}$ are generated, and (48) is used to obtain the corresponding realizations of the failure surface. These realizations are then employed in a nonparametric estimation scheme (Becker and Chambers 1984) to obtain an approximation to the probabilistic content of the failure surface.

Applications

Random Flexural Beam

The first example involves a flexural beam, fixed at one end and subjected to a uniform transverse load along its length. The modulus of rigidity EI of the beam, involving its modulus of elasticity E and its mass moment of inertia I, is considered to be a Gaussian random process with known mean and covariance function. The covariance function used in this example is the exponentially decaying function given by the equation

$$C(x_1,x_2) = \sigma_{EI}^2\, e^{-(|x_1 - x_2|/b)} \quad \ldots\ldots (50)$$

where σ_{EI} = standard deviation of the random process; and b = correlation length of the random process. The eigenvalues and eigenvectors of this covariance kernel, required in the Karhunen-Loeve expansion of the associated random process, can be found in a number of references (Ghanem 1988). For this example, the constitutive matrix $\mathbf{D}^e(\mathbf{x},\theta)$ is reduced to a single term $EI^e(\mathbf{x},\theta)$ expressing the random modulus of rigidity of the beam. Interpolation of the displacements over each element is achieved using cubic

Hermite polynomials. The resulting interpolation matrix $\mathbf{H}_e(r)$ is given by the equation

$$\mathbf{H}^e(r) = [(1 - 3r^2 + 2r^3) l^e(x - 2r^2 + r^3)\ (3r^2 - 2r^3) l^e\ (-r^2 + r^3)] \ . \quad (51)$$

where l^e = length of element e. Further, r = a local coordinate system, and is given by the equation

$$r = \frac{x - x_1^e}{l^e} \quad (52)$$

where x_1^e = global coordinate of the first node of element e. The associated matrix $\mathbf{B}^e(r)$ is obtained by differentiating $\mathbf{H}^e(r)$ with respect to x, using the chain rule to provide for the coordinate transformation. Details of the implementation of the Karhunen-Loeve expansion for this problem are available in the literature (Ghanem 1988). Fig. 1 shows a schematic description of the problem. The beam is subdivided into 10 finite elements. Figs. 2 and 3 show the spectral content of the displacement and slope processes, respectively, for increasing orders of the polynomial chaos approximation. That is, Figs. 2 and 3 show the successive vectors $\mathbf{a}_i$ in (40). Recall from Table 1 that since two terms are used in the Karhunen-Loeve expansion for

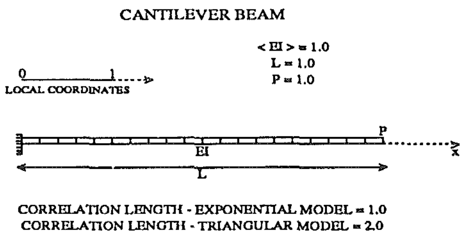

FIG. 1. Beam with Random Bending Rigidity under Uniform Load

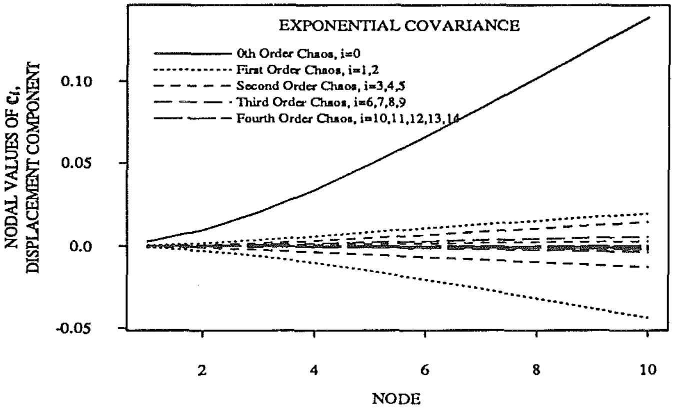

FIG. 2. Linear Interpolation of Nodal Values of Vector c_i of (40) for Beam Bending Problem

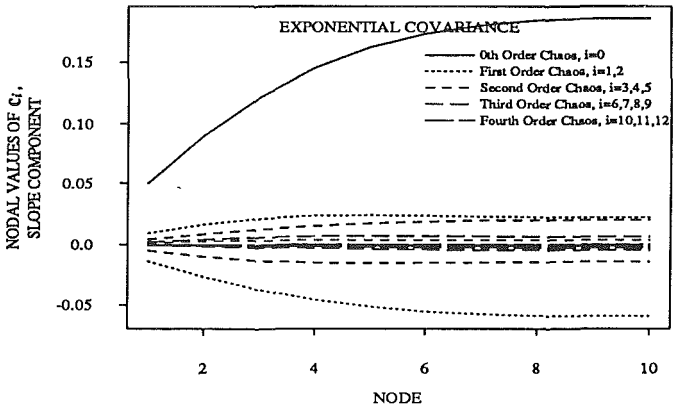

FIG. 3. Linear Interpolation of Nodal Values of Vector c_i of (40) for Beam Bending Problem

the material properties, $\mathbf{c}_0$ is the coefficient of the zeroth-order polynomial chaos; $\mathbf{c}_1$ and $\mathbf{c}_2$ are the coefficients for the first-order polynomial chaoses; $\mathbf{c}_3$, $\mathbf{c}_4$, and $\mathbf{c}_5$ correspond to the second-order polynomial chaoses; $\mathbf{c}_6$, $\mathbf{c}_7$, $\mathbf{c}_8$, and $\mathbf{c}_9$ are the coefficients of the third-order polynomial chaoses; and $\mathbf{c}_{10}$, $\mathbf{c}_{11}$, $\mathbf{c}_{12}$, $\mathbf{c}_{13}$, and $\mathbf{c}_{14}$ correspond to the fourth-order polynomial chaoses. Note the negligible contribution from higher-order terms. Due to the form of (42), the spectral content of the response is continually updated as the level of approximation is improved. That is, the numerical value of the components of the vectors $\mathbf{c}_i$, $i = 1, \ldots, P$ depends on the value of the parameter P, which represents the number of polynomial chaoses used in the expansion for the response. It is to be expected, however, that convergence of these vectors to their true value is reached with the addition of only few terms to the expansion. Indeed, Figs. 4 and 5 show the convergence of some of these spectral components as more terms are included in the approximation. Note the rapid convergence, indicating that a solution from a lower-level approximation can be efficiently employed to determine the solution for a higher-level approximation. In Figs. 2–5, values of the vectors $\mathbf{c}_i$ were computed only at the nodal points of the beam, and linear interpolation was used to complete the plots. Figs. 6–9 show results corresponding to the probability distribution of the displacement at the tip of the beam. The results are compared on the same plot with data obtained from a Monte Carlo simulation solution. The agreement is good, especially in the area around the tail of the distribution. It is this part of the probability distribution function that is of most relevance in a reliability analysis context.

Random Plane Stress

The second example involves a thin rectangular plate of dimensions L_x and L_y clamped at one side and subjected to a unit uniform in-plane tension applied at the opposite side, as shown in Fig. 10. The modulus of elasticity E of the plate is assumed to be a random process, with mean $\bar{E}$ and covariance function $C(x_1,y_1; x_2,y_2)$, reflecting the correlation of the process at two positions, (x_1,y_1) and (x_2,y_2), on the plate. The mean value $\bar{E}$ is assumed

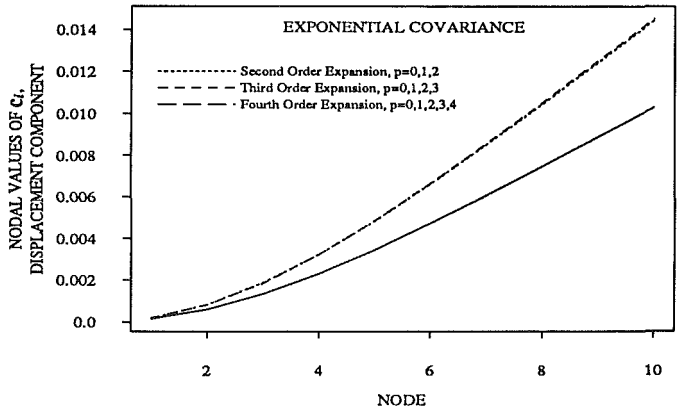

FIG. 4. Linear Interpolation of Nodal Values of Vector c_i of (40) for Beam Bending Problem

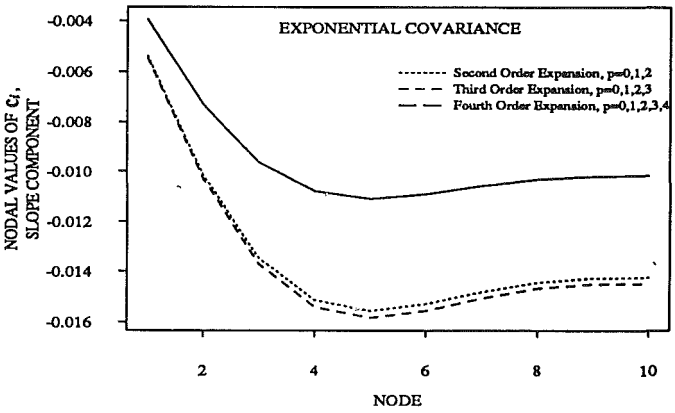

FIG. 5. Linear Interpolation of Nodal Values of Vector c_i of (40) for Beam Bending Problem

to be equal to 1; so are the sides of the plate. The covariance function is assumed to decay exponentially, and is given by the equation

$$C(x_1,y_1;\, x_2,y_2) = \sigma_E^2 e^{-(|x_1-x_2|/b_x)-(|y_1-y_2|/b_y)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53)$$

Here, b_x and b_y = correlation length of the process in the x- and y-directions, respectively; and σ_E = standard deviation of the modulus of elasticity. The deterministic finite-element analysis is implemented using bilinear interpolation functions. The resulting interpolation matrix over each finite element is given by the equation

$$\mathbf{H}^e(r_1,r_2) = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (54)$$

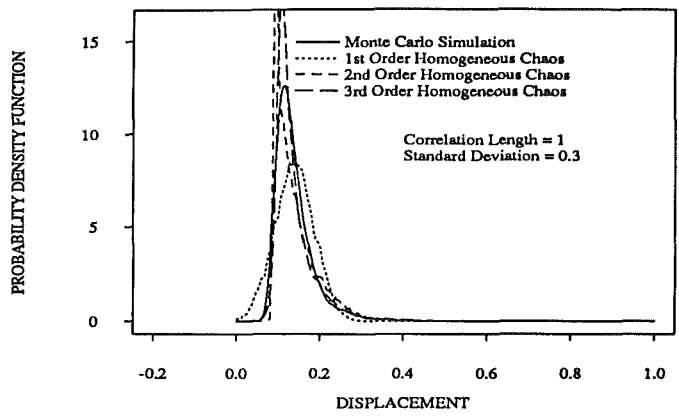

FIG. 6. Probability Density Function of Displacement at Tip of the Beam

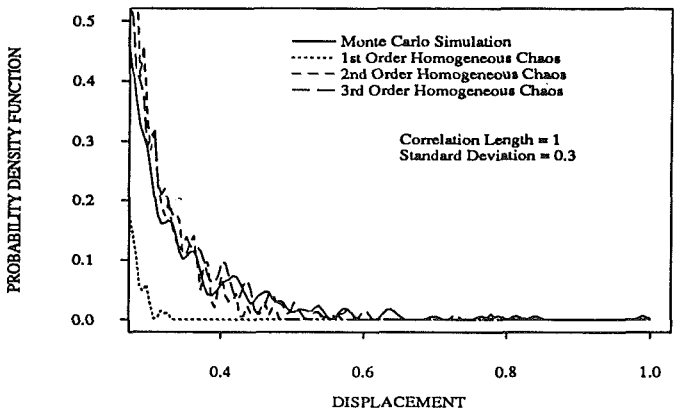

FIG. 7. Tail of Probability Density Function of Displacement at the Tip of Beam

where

$$N_1 = r_1(1 - r_2) \quad \ldots\ldots (55a)$$

$$N_2 = r_2(1 - r_1) \quad \ldots\ldots (55b)$$

$$N_3 = r_1 r_2 \quad \ldots\ldots (55c)$$

$$N_4 = (1 - r_1)(1 - r_2) \quad \ldots\ldots (55d)$$

and

$$r_1 = \frac{x - x_c}{l_x^e} \quad \ldots\ldots (56a)$$

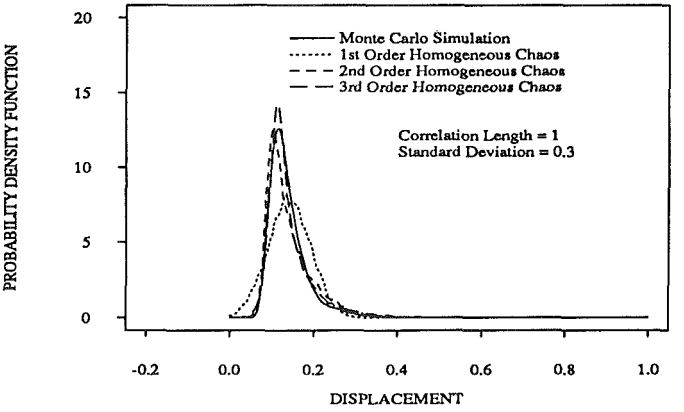

FIG. 8. Probability Density Function for Displacement at Tip of Beam

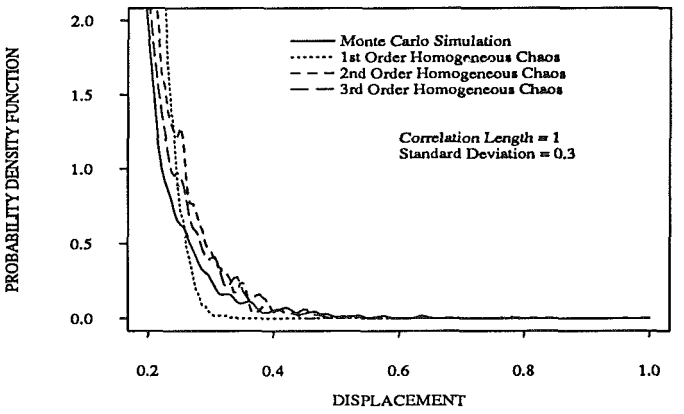

FIG. 9. Tail of Probability Density Function for Displacement at Tip of Beam

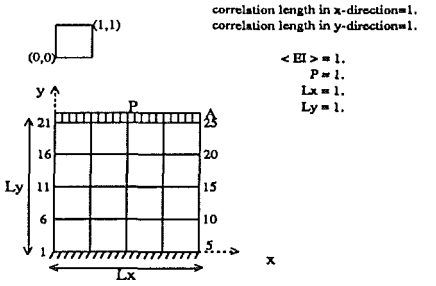

FIG. 10. Plate with Random Rigidity; Exponential Covariance Model

$$r_2 = \frac{y - y_c}{l_y^e} \quad \ldots\ldots (56b)$$

In (56), x_c and y_c = global coordinates of the lower-left node of the element; and l_x^e and l_y^e = dimensions of the rectangular finite elements in the x- and y-directions, respectively. The constitutive matrix relating the stresses to the strains over each element is given by the equation

$$\mathbf{D}^e = \frac{E^e(\mathbf{x},\theta)}{1 - \mu_e^2}\begin{bmatrix} 1 & \mu_e & 0 \\ \mu_e & 1 & 0 \\ 0 & 0 & (1 - \mu_e)/2 \end{bmatrix} = E^e(\mathbf{x},\theta)\mathbf{P}^e \quad \ldots\ldots (57)$$

where $E^e(\mathbf{x},\theta)$ and μ_e = modulus of elasticity and the Poisson ratio over each element, respectively; and $\mathbf{P}^e$ = a deterministic matrix defined by (57). The strain is related to the displacement through matrix $\mathbf{B}^e(r_1,r_2)$ given by the equation

$$\mathbf{B}^e(r_1,r_2) = \begin{bmatrix} \partial/\partial x_1 & 0 \\ 0 & \partial/\partial x_2 \\ \partial/\partial x_2 & \partial/\partial x \end{bmatrix} \mathbf{H}^e(r_1,r_2) \quad \ldots\ldots (58)$$

Note that in performing the foregoing differentiation, the chain rule must be used to account for the coordinate transformation. All the global matrices are assembled from elemental ones using standard procedures from the deterministic finite-element method (Akin 1982).

The next stage in the computations consists of solving the integral eigenvalue problem given by the equation

$$\lambda_n f_n(x_1,y_1) = \int_{-l_x/2}^{l_x/2}\int_{-l_y/2}^{l_y/2} C(x_1,y_1;\, x_2,y_2) f_n(x_2,y_2)\,dx_1 dy_1 \quad \ldots\ldots (59)$$

Substituting (53) for the covariance kernel, and assuming that

$$f_n(x_1,y_1) = f_i^{(x)}(x_1)\, f_j^{(y)}(y_1) \quad \ldots\ldots (60)$$

and

$$\lambda_n = \lambda_i^{(x)}\lambda_j^{(y)} \quad \ldots\ldots (61)$$

the solution of (59) reduces to the solution of the equation

$$\lambda_i^{(x)}\lambda_j^{(y)} f_i^{(x)}(x_1) f_j^{(y)}(y_1) = \int_{-l_x/2}^{l_x/2} e^{(-|x_1 - x_2|/b_x)} f_i^{(x)}(x_2)\,dx_2 \int_{-l_y/2}^{l_y/2} e^{(-|y_1 - y_2|/b_y)} f_j^{(y)}(y_2)\,dy_2 \quad \ldots\ldots (62)$$

Eq. (62) can be decoupled into two equations to be solved separately for $f_i(x)$ and $f_j(y)$. The solution of each of these integral equations is similar to the solution of the equation associated with the one-dimensional problem (Ghanem 1988). In the final expression for the eigenfunctions, it should be noted that to the same eigenvalue there corresponds two functions of the form given by (60), the second function being obtained from the first one by permuting the subscripts. Therefore, the complete normalized eigenfunctions are given by the equation

$$f_n(x,y) = \frac{1}{\sqrt{2}}\left[a_i^{(x)}(x) f_j^{(y)}(y) + a_j^{(x)}(x) f_i^{(y)}(y)\right] \quad \ldots\ldots (63)$$

In the expansion of the random process, the terms are ordered in descending order of the magnitude of the eigenvalues λ_n as given in (61).

Once the expansion for the system parameters is known, the solution procedure outlined in the previous section is implemented. Specifically, the matrices $\overline{\mathbf{K}}$ and $\mathbf{K}_m$ appearing in (13) are given explicitly as

$$\overline{\mathbf{K}} = \sum_e \overline{E}^e \int_{\Omega^e} [\mathbf{B}^{eT}(r_1,r_2)]\mathbf{P}^e[\mathbf{B}^e(r_1,r_2)]|\mathbf{J}^e|dr_1dr_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (64)$$

and

$$\mathbf{K}_m = \sum_e \int_{\Omega^e} a_m(r_1,r_2)[\mathbf{B}^{eT}(r_1,r_2)]\mathbf{P}^e[\mathbf{B}^e(r_1,r_2)]|\mathbf{J}^e|dr_1dr_2 \quad \ldots\ldots\ldots\ldots\ldots (65)$$

where $|\mathbf{J}^e|$ = determinant of the Jacobian of the transformation from the global to the local coordinates. The plate is subdivided into 16 elements ($NE = 16$), four along each direction. Two and four terms in the Karhunen-Loeve expansion are used ($M = 2$) and ($M = 4$). Polynomial chaoses of order up to and including three ($p = 3$) are used in the expansion given by (16). Fig. 11 shows the spectral content of the random variable corresponding to the longitudinal displacement at the corner A of the plate corresponding to $M = 4$. That is, Fig. 11 shows the component of the vector $\mathbf{c}_i$ (40) corresponding to node A on the plate for different values of the index i, ($i = 0, \ldots, 34$). Note the rapid decay, indicating negligible contribution from higher-order terms. Figs. 12–15 show successive approximations to the probability distribution function of this same random variable. Good agreement is observed with the Monte Carlo simulation results, especially around the tail area.

Conclusions

A new method was proposed for treating reliability problems involving structures with random media. The method is based on conceiving of the random aspect of the problem as being the realization of an additional dimension. Thus, random variables are viewed as functions with respect to this dimension. A basis is then identified in the space spanned by these

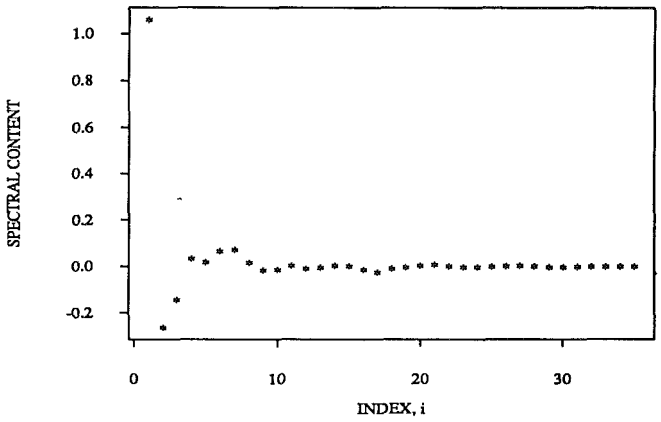

FIG. 11. Spectral Components of Longitudinal Displacement at Corner of Plate

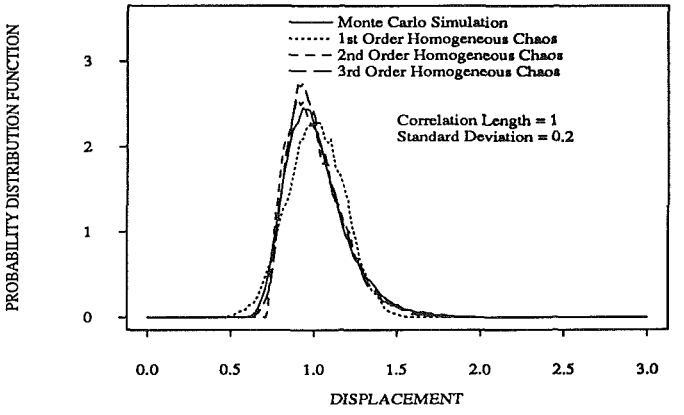

FIG. 12. Longitudinal Displacement at Free End of Rectangular Plate

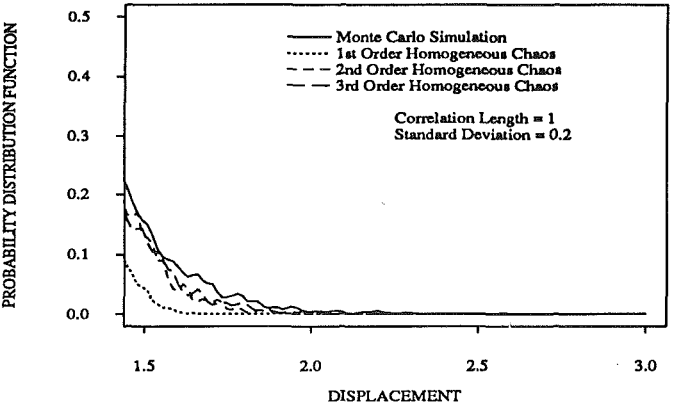

FIG. 13. Longitudinal Displacement at Free End of Rectangular Plate

functions, which consists of the polynomial chaoses. The random processes representing the material variability are expressed using their Karhunen-Loeve expansion. This expansion is incorporated into a spatial discretization in accordance with standard finite-element procedures and then coupled with a spectral expansion along the random dimension. The proposed formulation affords a theoretical rigor that is believed to be essential for critical applications of the stochastic finite-element method to reliability calculations. The spectral expansion permits an optimal representation of the random processes involved using a minimum number of random variables. As a by-product of the new method is an orthogonal expansion of the response surface. The coefficients in the expansion carry all the probabilistic information related to the response process. Once these coefficients have been computed, various statistics of interest, including probability distribution functions, can be readily evaluated. The proposed method was exemplified with two problems of interest. The results suggest that the first few terms in the spectral expansion of the response process are enough to adequately characterize the probabilistic content of the process. These conclusions are

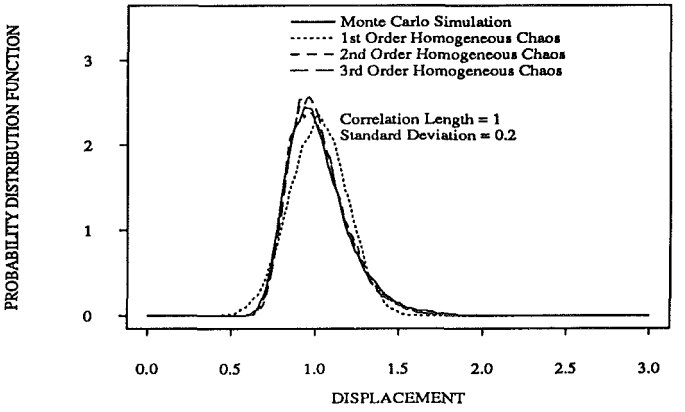

FIG. 14. Longitudinal Displacement at Free End of Rectangular Plate

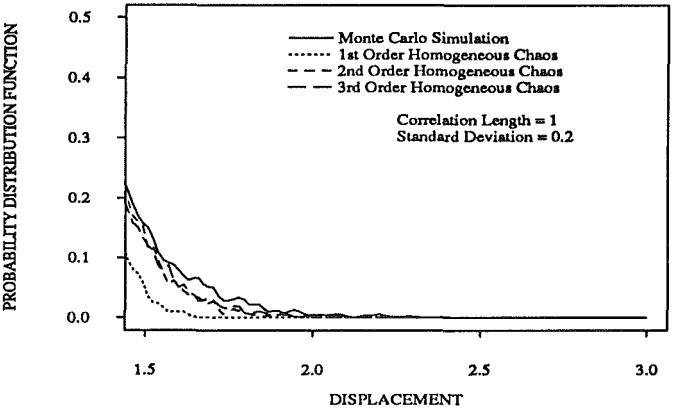

FIG. 15. Longitudinal Displacement at Free End of Rectangular Plate

reinforced by results comparing the probability distribution function of certain response variables with pertinent Monte Carlo simulation data.

Finally, it is emphasized that the representation of structural parameter uncertainty by (3), and of structural response uncertainty by (14), can obviously constitute the backbone of structural reliability methods combining these expeditious representations of uncertainty with well-established computational schemes such as those of FORM and SORM (Der-Kiureghian 1987). Specifically, once an explicit expression is available for the structural response surface, quantities such as the reliability index, and the coordinates of the design point can be readily evaluated using standard optimization algorithms. Further, these representations can be useful in treating reliability problems involving combined spatial and temporal uncertainty because they are applicable to both domains.

Acknowledgment

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