Simulation of stochastic processes by spectral representation

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The subject of this paper is the simulation of one-dimensional, uni-variate, stationary, Gaussian stochastic processes using the spectral representation method. Following this methodology, sample functions of the stochastic process can be generated with great computational efficiency using a cosine series formula. These sample functions accurately reflect the prescribed probabilistic characteristics of the stochastic process when the number N of the terms in the cosine series is large. The ensemble-averaged power spectral density or autocorrelation function approaches the corresponding target function as the sample size increases. In addition, the generated sample functions possess ergodic characteristics in the sense that the temporally-averaged mean value and the autocorrelation function are identical with the corresponding targets, when the averaging takes place over the fundamental period of the cosine series. The most important property of the simulated stochastic process is that it is asymptotically Gaussian as $N \to \infty$. Another attractive feature of the method is that the cosine series formula can be numerically computed efficiently using the Fast Fourier Transform technique. The main area of application of this method is the Monte Carlo solution of stochastic problems in engineering mechanics and structural engineering. Specifically, the method has been applied to problems involving random loading (random vibration theory) and random material and geometric properties (response variability due to system stochasticity).

CONTENTS

1. INTRODUCTION

In the last three decades or so, considerable progress has been made in applying stochastic process theory to the general area of engineering mechanics and structural engineering for the purpose of assuring the over-all structural safety at a higher level of reliability. The theory has been initially applied to problems involving random loading (random vibration theory) and during the last decade to problems involving random material and geometric properties (response variability due to system stochasticity). Typical problems in random vibration theory include among others the analysis of ship motions caused by ocean waves, the analysis of aircraft response to gust and maneuver loads, the response analysis of offshore structures to wave and wind forces, the study of vehicle vibrations caused by random roadway roughness and the response analysis of structures subjected to earthquake ground motion or atmospheric turbulence. The evaluation of response variability due to system stochasticity consists of performing the response analysis of structural systems with randomness in their material properties (e.g. elastic modulus) or geometry (e.g. dimensions of structural members), or both. The most widely-used method to solve random vibration problems is the frequency domain analysis, while perturbation techniques are the most frequently-used method to solve system stochasticity problems.

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While most researchers have concentrated on the use of the above-mentioned two methods of solution (frequency domain analysis and perturbation techniques), the first author of this paper has developed and advocated the use of Monte Carlo simulation techniques to solve both random vibration and system stochasticity problems (e.g. Shinozuka 1972, Shinozuka and Astill 1972, Shinozuka and Wen 1972, Astill, Nosseir and Shinozuka 1972, Wilkins, Wolff, Shinozuka and Cox 1975, Shinozuka and Lenoe 1976, Shinozuka 1977, Shinozuka, Yun and Vaicaitis 1977, Shinozuka, Fang, Kitutake and Matsui 1978, Shinozuka and Deodatis 1988a, Yamazaki, Shinozuka and Dasgupta 1988, Deodatis 1989, Deodatis, Shinozuka and Neal 1989, Billah and Shinozuka 1990, Billah and Shinozuka 1991). The major advantage of Monte Carlo simulation is that accurate solutions can be obtained for any problem whose deterministic solution (either analytical or numerical) is known. The only disadvantage of Monte Carlo simulation is that it is usually time-consuming. It is the authors' belief, however, that in the next decade, the further evolution of digital computers will enhance the usefulness of Monte Carlo simulation techniques in the area of engineering mechanics and structural engineering. Anyway, Monte Carlo simulation is the only method available to solve a large number of stochastic problems involving nonlinearity, system stochasticity, stochastic stability, parametric excitations, etc.

The most important part of the Monte Carlo simulation methodology is the generation of sample functions of the stochastic processes, fields or waves involved in the problem. The generated sample functions must accurately describe the probabilistic characteristics of the corresponding stochastic processes, fields or waves that may be either stationary or nonstationary, homogeneous or nonhomogeneous, one-dimensional or multi-dimensional, one-variate or multi-variate, Gaussian or non-Gaussian. The method that appears most amenable for generating such sample functions is the "spectral representation method." Although the concept of the method existed for some time (Rice 1954), it was Shinozuka (Shinozuka and Jan 1972, Shinozuka 1972) who first applied it for simulation purposes including multi-dimensional, multi-variate and nonstationary cases. Yang (1972, 1973) showed that the Fast Fourier Transform (FFT) technique can be used to dramatically improve the computational efficiency of the algorithm and proposed a formula to simulate random envelope processes. Shinozuka (1974) extended the application of the FFT technique to multi-dimensional cases. Recently, Deodatis and Shinozuka (1989) extended the spectral representation method to simulate stochastic waves, Yamazaki and Shinozuka (1988) proposed an iterative procedure to simulate non-Gaussian stochastic fields and Yamazaki and Shinozuka (1990) introduced statistical preconditioning to reduce the sample size. Finally, two review papers on the subject of simulation using the spectral representation method were written by Shinozuka (1987) and Shinozuka and Deodatis (1988b).

The present paper on simulation of 1D-1V stationary stochastic processes using the spectral representation method serves two purposes. The first is to serve as a detailed review of the method by gathering and compiling material that can only be found in several different papers. The second is to provide rigorous derivations and elaborations of certain important issues that have not been explicitly given in the past papers. These issues include, among others, asymptotic Gaussianness of the simulated stochastic process, nonergodic characteristics of an alternative spectral representation by Rice (1954), aliasing arising from the periodicity of the simulated stochastic process.

2. SPECTRAL REPRESENTATION OF STATIONARY STOCHASTIC PROCESSES

Let $f_0(t)$ be a 1D-1V stationary stochastic process with mean value equal to zero, autocorrelation function $R_{f_0f_0}(\tau)$ and two-sided power spectral density function $S_{f_0f_0}(\omega)$. Then, the following relations hold:

$$\mathcal{E}[f_0(t)] = 0 \tag{1}$$

$$\mathcal{E}[f_0(t+\tau)f_0(t)] = R_{f_0f_0}(\tau) \tag{2}$$

$$S_{f_0f_0}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{f_0f_0}(\tau)\mathrm{e}^{-i\omega\tau}d\tau \tag{3}$$

$$R_{f_0f_0}(\tau) = \int_{-\infty}^{\infty} S_{f_0f_0}(\omega)\mathrm{e}^{i\omega\tau}d\omega \tag{4}$$

where the last two equations constitute the well-known Wiener-Khintchine transform pair.

The following theorem is fundamental in the theory of 1D-1V stationary stochastic processes with mean value equal to zero (e.g. Yaglom 1962, Cramér and Leadbetter 1967).

To every real-valued 1D-1V stationary stochastic process $f_0(t)$ with mean value equal to zero and two-sided power spectral density function $S_{f_0f_0}(\omega)$, two mutually orthogonal real processes $u(\omega)$ and $v(\omega)$ with orthogonal increments $du(\omega)$ and $dv(\omega)$ can be assigned such that:

$$f_0(t) = \int_0^{\infty} [\cos(\omega t)\,du(\omega) + \sin(\omega t)\,dv(\omega)] \tag{5}$$

The processes $u(\omega)$ and $v(\omega)$ and the corresponding increments $du(\omega)$ and $dv(\omega)$ are defined for $\omega \geq 0$ and they satisfy the following requirements:

$$\mathcal{E}[u(\omega)] = \mathcal{E}[v(\omega)] = 0, \quad \text{for} \quad \omega \geq 0 \tag{6}$$

$$\mathcal{E}[u^2(\omega)] = \mathcal{E}[v^2(\omega)] = 2S_{F_0F_0}(\omega), \quad \text{for} \quad \omega \geq 0 \tag{7}$$

$$\mathcal{E}[u(\omega)v(\omega')] = 0, \quad \text{for} \quad \omega, \omega' \geq 0 \tag{8}$$

$$\mathcal{E}[du(\omega)] = \mathcal{E}[dv(\omega)] = 0, \quad \text{for} \quad \omega \geq 0 \tag{9}$$

$$\mathcal{E}[du^2(\omega)] = \mathcal{E}[dv^2(\omega)] = 2S_{f_0f_0}(\omega)d\omega, \quad \text{for} \quad \omega \geq 0 \tag{10}$$

$$\mathcal{E}[du(\omega)du(\omega')] = \mathcal{E}[dv(\omega)dv(\omega')] = 0, \quad \text{for} \quad \omega, \omega' \geq 0\ ;\ \omega \neq \omega' \tag{11}$$

$$\mathcal{E}[du(\omega)dv(\omega')] = 0, \quad \text{for} \quad \omega, \omega' \geq 0 \tag{12}$$

where it is assumed that $f_0(t)$ is associated with a differentiable power spectral distribution function $S_{F_0F_0}(\omega)$ whose derivative is the power spectral density function $S_{f_0f_0}(\omega)$:

$$\frac{dS_{F_0F_0}(\omega)}{d\omega} = S_{f_0f_0}(\omega), \quad \text{for} \quad \omega \geq 0 \tag{13}$$

The power spectral distribution function $S_{F_0F_0}(\omega)$ is finite in the limit as $\omega \to \infty$ which is equivalent to:

$$\int_0^{\infty} 2S_{f_0f_0}(\omega)d\omega < \infty \tag{14}$$

In Eq (9) through (12), $du(\omega)$ and $dv(\omega)$ are defined as:

$$du(\omega) = u(\omega + d\omega) - u(\omega) \tag{15}$$

$$dv(\omega) = v(\omega + d\omega) - v(\omega) \tag{16}$$

and the inequality $\omega \neq \omega'$ in Eq (11) implies that the frequency intervals $(\omega, \omega + d\omega)$ and $(\omega', \omega' + d\omega')$ are disjoint.

Equation (5) is now rewritten in the following form:

$$f_0(t) = \sum_{k=0}^{\infty} [\cos(\omega_k t)\,du(\omega_k) + \sin(\omega_k t)\,dv(\omega_k)] \tag{17}$$

where

$$\omega_k = k\,\Delta\omega \tag{18}$$

with sufficiently small but finite $\Delta\omega$, such that Eq (17) can be used for Eq (5).

If $du(\omega_k)$ and $dv(\omega_k)$ are defined as:

$$du(\omega_k) = X_k \tag{19}$$

$$dv(\omega_k) = Y_k \tag{20}$$

and if the X_ks and the Y_ks are independent random variables, with mean value equal to zero and standard deviation $(2S_{f_0f_0}(\omega_k)\Delta\omega)^{1/2}$, it is easy to show that all requirements imposed on $du(\omega_k)$ and $dv(\omega_k)$ [Eqs (9) to (12)] are satisfied. Then, substituting Eqs (19) and (20) into Eq (17), the following series representation is obtained:

$$f_0(t) = \sum_{k=0}^{\infty}[\cos(\omega_k t)\,X_k + \sin(\omega_k t)\,Y_k] \tag{21}$$

On the other hand, if $du(\omega_k)$ and $dv(\omega_k)$ are defined as (Shinozuka 1972):

$$du(\omega_k) = \sqrt{2}\,A_k \cos\Phi_k \tag{22}$$

$$dv(\omega_k) = -\sqrt{2}\,A_k \sin\Phi_k \tag{23}$$

in which

$$A_k = (2S_{f_0f_0}(\omega_k)\Delta\omega)^{1/2}, \qquad \omega_k = k\,\Delta\omega \tag{24}$$

and the Φ_ks are independent random phase angles uniformly distributed in the range $[0, 2\pi]$, it is a straightforward task to show that Eq (9) to (12) are again satisfied. Indeed, the following expressions can be written for Eq (9):

$$\mathcal{E}[du(\omega_k)] = \mathcal{E}[\sqrt{2}\,A_k\cos\Phi_k] = \sqrt{2}\,A_k\,\mathcal{E}[\cos\Phi_k]$$
$$= \sqrt{2}\,A_k\int_{-\infty}^{\infty}\cos\phi_k\,p_{\Phi_k}(\phi_k)d\phi_k \tag{25}$$

where $p_{\Phi_k}(\phi_k)$ is the probability density function of random phase angle Φ_k given by:

$$p_{\Phi_k}(\phi_k) = \begin{cases} \frac{1}{2\pi}, & 0 \le \phi_k \le 2\pi \\ 0, & \text{otherwise} \end{cases} \tag{26}$$

Then, Eq (25) can be written as:

$$\mathcal{E}[du(\omega_k)] = \sqrt{2}\,A_k\int_0^{2\pi}\cos\phi_k\,\frac{1}{2\pi}d\phi_k = 0 \tag{27}$$

and in exactly the same way it can be shown that $\mathcal{E}[dv(\omega_k)] = 0$.

The requirement described in Eq (10) can be expressed as:

$$\mathcal{E}[du^2(\omega_k)] = \mathcal{E}[2\,A_k^2\cos^2\Phi_k] = 2\,A_k^2\,\mathcal{E}[\cos^2\Phi_k]$$
$$= 2\,A_k^2\int_{-\infty}^{\infty}\cos^2\phi_k\,p_{\Phi_k}(\phi_k)d\phi_k$$
$$= 2\,A_k^2\int_0^{2\pi}\frac{1}{2}(1+\cos 2\phi_k)\frac{1}{2\pi}d\phi_k$$
$$= 2\,A_k^2\,\frac{1}{2} = 2S_{f_0f_0}(\omega_k)\Delta\omega \tag{28}$$

and in exactly the same way it can be shown that $\mathcal{E}[dv^2(\omega_k)] = 2S_{f_0f_0}(\omega_k)\Delta\omega$.

The condition described in Eq (11) can be written as:

$$\mathcal{E}[du(\omega_k)du(\omega_{k'})] = \mathcal{E}[2\,A_k\,A_{k'}\cos\Phi_k\cos\Phi_{k'}]$$
$$= 2\,A_k\,A_{k'}\,\mathcal{E}[\cos\Phi_k\cos\Phi_{k'}]$$
$$= 2\,A_k\,A_{k'}\,\mathcal{E}[\cos\Phi_k]\,\mathcal{E}[\cos\Phi_{k'}] \tag{29}$$

The last equality in Eq (29) is valid since Φ_k and $\Phi_{k'}$ are independent random phase angles for $k \neq k'$. Eventually, Eq (29) is written as:

$$\mathcal{E}[du(\omega_k)du(\omega_{k'})]$$
$$= 2\,A_k\,A_{k'}\int_0^{2\pi}\cos\phi_k\,\frac{1}{2\pi}d\phi_k\int_0^{2\pi}\cos\phi_{k'}\,\frac{1}{2\pi}d\phi_{k'}$$
$$= 0, \quad \text{for} \quad k \neq k' \tag{30}$$

and in exactly the same way it can be shown that $\mathcal{E}[dv(\omega_k)dv(\omega_{k'})] = 0$ for $k \neq k'$.

Finally, the requirement described in Eq (12) can be expressed as:

$$\mathcal{E}[du(\omega_k)dv(\omega_{k'})] = \mathcal{E}[-2\,A_k\,A_{k'}\cos\Phi_k\sin\Phi_{k'}]$$
$$= -2\,A_k\,A_{k'}\,\mathcal{E}[\cos\Phi_k\sin\Phi_{k'}] \tag{31}$$

When $k \neq k'$, the expected value appearing in the last term of Eq (31) can be written as follows, since Φ_k and $\Phi_{k'}$ are independent random phase angles:

$$\mathcal{E}[\cos\Phi_k\sin\Phi_{k'}] = \mathcal{E}[\cos\Phi_k]\,\mathcal{E}[\sin\Phi_{k'}] = 0, \quad \text{for} \quad k \neq k' \tag{32}$$

When $k = k'$, the term $\mathcal{E}[\cos\Phi_k\sin\Phi_{k'}]$ can be written as:

$$\mathcal{E}[\cos\Phi_k\sin\Phi_{k'}] = \mathcal{E}[\cos\Phi_k\sin\Phi_k] = \mathcal{E}\left[\frac{1}{2}\sin 2\Phi_k\right]$$
$$= \int_0^{2\pi}\sin 2\phi_k\,\frac{1}{2\pi}d\phi_k$$
$$= 0, \quad \text{for} \quad k = k' \tag{33}$$

Combining Eq (31) to (33), the following result is established:

$$\mathcal{E}[du(\omega_k)dv(\omega_{k'})] = 0, \quad \text{for} \quad k = k' \quad \text{as well as for} \quad k \neq k' \tag{34}$$

Therefore, all requirements imposed on $du(\omega_k)$ and $dv(\omega_k)$ [Eqs (9) to (12)] are satisfied by the expressions given in Eqs (22) and (23). Then, substituting Eqs (22) and (23) into Eq (17), the following series representation is obtained:

$$f_0(t) = \sum_{k=0}^{\infty}\Big[\cos(\omega_k t)\,\sqrt{2}\,(2S_{f_0f_0}(\omega_k)\Delta\omega)^{1/2}\cos\Phi_k$$
$$- \sin(\omega_k t)\,\sqrt{2}\,(2S_{f_0f_0}(\omega_k)\Delta\omega)^{1/2}\sin\Phi_k\Big]$$
$$= \sqrt{2}\sum_{k=0}^{\infty}(2S_{f_0f_0}(\omega_k)\Delta\omega)^{1/2}\cos(\omega_k t + \Phi_k) \tag{35}$$

It has been shown above that both series representation expressions displayed in Eqs (21) and (35) are consistent with the spectral representation theorem stated in Eq (5). In Section 4.1 it will be shown that when Eq (35) is used for simulation purposes, it produces ergodic sample functions, in the sense that the temporally-averaged mean value and autocorrelation function of each and every sample function are identical with the corresponding targets, when the averaging takes place over the fundamental period of the cosine series. On the other hand, in Section 4.3 it will be shown that sample functions produced by Eq (21) are not ergodic. This is the reason why Eq (35) will be used exclusively in the following.

The expressions given in Eq (21) (with X_k and Y_k being normally distributed) and (35) were mentioned in a celebrated paper by Rice (1954). As stated by Rice in that paper, the representation of Eq (21) was used by Einstein and Hopf (1910) for black-body radiation. Schottky (1918) also used it for representing the shot effect current, without taking the X_ks and Y_ks to be normally distributed.

3. SIMULATION OF STOCHASTIC PROCESSES

3.1 Simulation formula

Consider a 1D-1V stationary stochastic process $f_0(t)$ with mean value equal to zero, autocorrelation function $R_{f_0f_0}(\tau)$ and two-sided power spectral density function $S_{f_0f_0}(\omega)$. In the following, distinction will be made between the stochastic process $f_0(t)$ and its simulation $f(t)$.

From the infinite series representation displayed in Eq (35), it follows that the stochastic process $f_0(t)$ can be simulated by the following series as $N \to \infty$;

$$f(t) = \sqrt{2} \sum_{n=0}^{N-1} A_n \cos(\omega_n t + \Phi_n) \tag{36}$$

where

$$A_n = (2S_{f_0f_0}(\omega_n)\Delta\omega)^{1/2}, \qquad n = 0, 1, 2, \ldots, N-1 \tag{37}$$

$$\omega_n = n\Delta\omega, \qquad n = 0, 1, 2, \ldots, N-1 \tag{38}$$

$$\Delta\omega = \omega_u/N \tag{39}$$

and

$$A_0 = 0 \qquad \text{or} \qquad S_{f_0f_0}(\omega_0 = 0) = 0 \tag{40}$$

In Eq (39), ω_u represents an upper cut-off frequency beyond which the power spectral density function $S_{f_0f_0}(\omega)$ may be assumed to be zero for either mathematical or physical reasons. As such, ω_u is a fixed value and hence $\Delta\omega \to 0$ as $N \to \infty$ so that $N\Delta\omega = \omega_u$. The following criterion is usually used to estimate the value of ω_u:

$$\int_0^{\omega_u} S_{f_0f_0}(\omega)d\omega = (1-\epsilon)\int_0^{\infty} S_{f_0f_0}(\omega)d\omega \tag{41}$$

where $\epsilon << 1$ (e.g. $\epsilon = 0.01, 0.001$).

Note that if a linear random vibration problem is considered, then

$$\int_0^{\omega_u} |H(\omega)|^2 S_{f_0f_0}(\omega)d\omega = (1-\epsilon)\int_0^{\infty} |H(\omega)|^2 S_{f_0f_0}(\omega)d\omega \tag{42}$$

is used as criterion, where $|H(\omega)|$ is the frequency response function of the system involved in the problem. Then, the response can be directly simulated using $|H(\omega)|^2 S_{f_0f_0}(\omega)$ as power spectral density function in Eq (36).

The $\Phi_0, \Phi_1, \Phi_2, \ldots, \Phi_{N-1}$ appearing in Eq (36) are independent random phase angles distributed uniformly over the interval $[0, 2\pi]$.

Under the condition of Eq (40), it is easy to show that the simulated stochastic process $f(t)$ given by Eq (36) is periodic with period T_0:

$$T_0 = 2\pi/\Delta\omega \tag{43}$$

Equation (43) indicates that the smaller $\Delta\omega$, or equivalently the larger N under a specified upper cut-off frequency value ω_u, the longer the period of the simulated stochastic process.

Another very important point is that the simulated stochastic process $f(t)$ is asymptotically Gaussian as $N \to \infty$ because of the multi-variate central limit theorem. The Gaussianness of stochastic process $f(t)$ as $N \to \infty$ is examined in Section 3.3. The rate of convergence of random variable $f(t = t_s)$ to Gaussianness as a function of N is examined in Section 3.4, where t_s is any specific time instant.

A sample function $f^{(i)}(t)$ of the simulated stochastic process $f(t)$ can be obtained by replacing the sequence of random phase angles $\Phi_0, \Phi_1, \Phi_2, \ldots, \Phi_{N-1}$ with their respective i-th realizations $\phi_0^{(i)}, \phi_1^{(i)}, \phi_2^{(i)}, \ldots, \phi_{N-1}^{(i)}$:

$$f^{(i)}(t) = \sqrt{2} \sum_{n=0}^{N-1} A_n \cos(\omega_n t + \phi_n^{(i)}) \tag{44}$$

The condition set in Eq (40) is necessary (and must be forced if $S_{f_0f_0}(0) \neq 0$) to guarantee that the temporal average and the temporal autocorrelation function of any sample function $f^{(i)}(t)$ are identical to the corresponding targets, $\mathcal{E}[f_0(t)] = 0$ and $R_{f_0f_0}(\tau)$, respectively (see Section 4.1).

At this point it should be noted that when generating sample functions of the simulated stochastic process according to Eq (44), the time step Δt separating the generated values of $f^{(i)}(t)$ in the time domain has to obey the condition:

$$\Delta t \leq 2\pi/2\omega_u \tag{45}$$

The condition set on Δt in Eq (45) is necessary in order to avoid aliasing according to the sampling theorem (e.g. Bracewell 1986).

Another interesting point is that the generated values of $f^{(i)}(t)$ according to Eq (44) are bounded as follows:

$$f^{(i)}(t) \leq \sqrt{2} \sum_{n=0}^{N-1} A_n = \sqrt{2} \sum_{n=0}^{N-1} (2S_{f_0f_0}(\omega_n)\Delta\omega)^{1/2} \tag{46}$$

In Section 6.1 it will be demonstrated that for a specific form of the power spectral density function, the above bound is large enough for all practical applications, even for relatively small values of N. It is obvious that this bound can be easily calculated for any form of the power spectral density function to be used.

It will be shown now that the ensemble expected value $\mathcal{E}[f(t)]$ and the ensemble autocorrelation function $R_{ff}(\tau)$ of the simulated stochastic process $f(t)$ are identical to the corresponding targets, $\mathcal{E}[f_0(t)] = 0$ and $R_{f_0f_0}(\tau)$, respectively.

(i) Show that: $\mathcal{E}[f(t)] = \mathcal{E}[f_0(t)] = 0.$

Proof:

Utilizing the property that the operations of mathematical expectation and summation are commutative, the ensemble expected value of the simulated stochastic process $f(t)$ can be written as:

$$\mathcal{E}[f(t)] = \mathcal{E}[\sqrt{2} \sum_{n=0}^{N-1} A_n \cos(\omega_n t + \Phi_n)]$$

$$= \sqrt{2} \sum_{n=0}^{N-1} A_n \, \mathcal{E}[\cos(\omega_n t + \Phi_n)] \tag{47}$$

in which A_n was defined in Eq (37). By virtue of Eq (26), the expected value $\mathcal{E}[\cos(\omega_n t + \Phi_n)]$ can be shown to be equal to zero in the following manner:

$$\mathcal{E}[\cos(\omega_n t + \Phi_n)]$$

$$= \int_{-\infty}^{\infty} p_{\Phi_n}(\phi_n) \cos(\omega_n t + \phi_n) d\phi_n$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cos(\omega_n t + \phi_n) d\phi_n = 0 \tag{48}$$

Combining now Eqs (47) and (48), it is easy to show that:

$$\mathcal{E}[f(t)] = 0 = \mathcal{E}[f_0(t)] \tag{49}$$

(ii) Show that: $R_{ff}(\tau) = R_{f_0 f_0}(\tau)$

Proof:

Utilizing again the property that the operations of mathematical expectation and summation are commutative, the ensemble autocorrelation function of the simulated stochastic process $f(t)$ can be written as:

$$\begin{aligned} R_{ff}(\tau) &= \mathcal{E}[f(t+\tau)f(t)] \\ &= \mathcal{E}\left\{\sqrt{2}\sum_{n=0}^{N-1} A_n \cos[\omega_n(t+\tau)+\Phi_n] \right. \\ &\quad \left. \times\sqrt{2}\sum_{m=0}^{N-1} A_m \cos(\omega_m t+\Phi_m)\right\} \\ &= 2\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} A_n A_m \\ &\quad \times \mathcal{E}\left\{\cos[\omega_n(t+\tau)+\Phi_n]\cos(\omega_m t+\Phi_m)\right\} \end{aligned} \tag{50}$$

Since random variables Φ_n $(n = 0, 1, 2, \ldots, N-1)$ are independent, the expected value shown in Eq (50) can be written as follows for $n \neq m$:

$$\begin{aligned} &\mathcal{E}\left\{\cos[\omega_n(t+\tau)+\Phi_n]\cos(\omega_m t+\Phi_m)\right\} \\ &= \mathcal{E}\left\{\cos[\omega_n(t+\tau)+\Phi_n]\right\}\,\mathcal{E}\left\{\cos(\omega_m t+\Phi_m)\right\} \\ &= 0\cdot 0 = 0, \quad \text{for} \quad n \neq m \end{aligned} \tag{51}$$

Because of Eq (51), only the terms with $n = m$ remain in Eq (50). Hence, the ensemble autocorrelation function of the simulated stochastic process $f(t)$ can be written as:

$$\begin{aligned} R_{ff}(\tau) &= 2\sum_{n=0}^{N-1} A_n^2\,\mathcal{E}\left\{\cos[\omega_n(t+\tau)+\Phi_n]\cos(\omega_n t+\Phi_n)\right\} \\ &= 2\sum_{n=0}^{N-1} A_n^2\,\frac{1}{2}\,\mathcal{E}\left\{\cos(2\omega_n t+\omega_n\tau+2\Phi_n)+\cos(\omega_n\tau)\right\} \\ &= \sum_{n=0}^{N-1} A_n^2\left[\mathcal{E}\left\{\cos(2\omega_n t+\omega_n\tau+2\Phi_n)\right\}+\mathcal{E}\left\{\cos(\omega_n\tau)\right\}\right] \\ &= \sum_{n=0}^{N-1} A_n^2\left[0+\cos(\omega_n\tau)\right] = \sum_{n=0}^{N-1} A_n^2\cos(\omega_n\tau) \end{aligned} \tag{52}$$

In deriving Eq (52), the following trigonometric identity was used:

$$\cos A\cos B = \frac{1}{2}\left\{\cos(A+B)+\cos(A-B)\right\} \tag{53}$$

Substituting Eq (37) into Eq (52) and taking the limit as $\Delta\omega \to 0$ and $N \to \infty$, while keeping in mind that $\omega_u = N\,\Delta\omega$ is constant and that $S_{f_0 f_0}(\omega) = 0$ for $|\omega| \geq \omega_u$, leads to:

$$R_{ff}(\tau) = 2\int_0^\infty S_{f_0 f_0}(\omega)\cos(\omega\tau)d\omega = \int_{-\infty}^{\infty} S_{f_0 f_0}(\omega)\,e^{i\omega\tau}d\omega \tag{54}$$

In deriving Eq (54), the fact that the two-sided power spectral density function is a real and even function of the frequency ω was used.

Comparing finally Eqs (4) and (54) and considering that the stochastic process was simulated over the time interval $[0, T]$ according to Eq (36), it is evident that:

$$R_{ff}(\tau) = R_{f_0 f_0}(\tau), \quad \text{for} \quad 0 \leq \tau \leq T \tag{55}$$

Obviously, the value of T appearing in Eq (55) can be less or equal to the period T_0 defined in Eq (43). If, in addition, $T = T_0$ and always for the same limiting conditions as those considered in deriving Eq (55):

$$S_{ff}(\omega) = S_{f_0 f_0}(\omega), \quad \text{for} \quad 0 \leq \omega \leq \omega_u \tag{56}$$

3.2 Rate of convergence of ensemble autocorrelation function to target autocorrelation function

It has just been shown that the ensemble autocorrelation function $R_{ff}(\tau)$ is equal to the target autocorrelation function $R_{f_0 f_0}(\tau)$. However, this equality holds in the limit as $N \to \infty$, as can be readily seen in the process of obtaining Eq (54) from Eq (52). Therefore, in order to examine the rate of convergence of $R_{ff}(\tau)$ to $R_{f_0 f_0}(\tau)$ as a function of variable N appearing in Eq (52), it is necessary to estimate the following error:

$$E = \int_0^{\omega_u} 2S_{f_0 f_0}(\omega)\cos\omega\tau\,d\omega - \sum_{n=0}^{N-1} 2S_{f_0 f_0}(\omega_n)\cos\omega_n\tau\,\Delta\omega \tag{57}$$

where

$$\omega_n = n\Delta\omega, \qquad \Delta\omega = \omega_u/N \tag{58}$$

According to Conte and de Boor (1980), this error is equal to:

$$E = \frac{\omega_u^2}{2N}\left[\frac{d}{d\omega}\left(S_{f_0 f_0}(\omega)\cos\omega\tau\right)\right]_{\omega^*} \tag{59}$$

where the symbol $[\ ]_{\omega^*}$ denotes evaluation of the derivative inside the brackets at $\omega = \omega^*$ with $0 \leq \omega^* \leq \omega_u$. Equation (59) requires that the power spectral density function $S_{f_0 f_0}(\omega)$ has a continuous first derivative in the interval $[0, \omega_u]$. The above discussion shows that the rate of convergence of the ensemble autocorrelation function $R_{ff}(\tau)$ to the target autocorrelation function $R_{f_0 f_0}(\tau)$ is proportional to $1/N$.

At this point it is interesting to note that if ω_n is defined as $\omega_n = n\Delta\omega + \Delta\omega/2$ $(n = 0, 1, 2, \ldots, N-1)$ instead of the definition given in Eq (38) and again in Eq (58), then according to Conte and de Boor (1980) the error defined in Eq (57) becomes:

$$E = \frac{\omega_u^3}{24N^2}\left[\frac{d^2}{d\omega^2}\left(S_{f_0 f_0}(\omega)\cos\omega\tau\right)\right]_{\omega^*} \tag{60}$$

Equation (60) requires that the power spectral density function $S_{f_0 f_0}(\omega)$ has a continuous second derivative in the interval $[0, \omega_u]$. Therefore, if the expression $\omega_n = n\Delta\omega + \Delta\omega/2$ is used instead of the expression $\omega_n = n\Delta\omega$ in the simulation formula shown in Eq (36), the rate of convergence of $R_{ff}(\tau)$ to $R_{f_0 f_0}(\tau)$ becomes proportional to $1/N^2$. It should be noted that if the expression $\omega_n = n\Delta\omega + \Delta\omega/2$ is used, then the simulated stochastic process $f(t)$ according to Eq (36) is periodic with period $T_0 = 4\pi/\Delta\omega$, but the Fast Fourier Transform technique (see Section 5) cannot be applied to Eq (36). Therefore, the expression $\omega_n = n\Delta\omega + \Delta\omega/2$ can only be used in the case of simulation by summation of cosines.

3.3 Gaussianness of simulated stochastic process

According to Laning and Battin (1956), the bi-variate central limit theorem states the following:

Let $\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_N$ be N independent random bi-variate vectors defined as:

$$\mathbf{Z}_n = (X_n, Y_n), \quad n = 1, 2, \ldots, N \tag{61}$$

The first and second moments of $\mathbf{Z}_n$ are given by:

$$\mathcal{E}[\mathbf{Z}_n] = (\mathcal{E}[X_n], \mathcal{E}[Y_n]) \tag{62}$$

$$\sigma^2_{X_n} = \mathcal{E}[X_n^2] - \mathcal{E}[X_n]^2, \quad \sigma^2_{Y_n} = \mathcal{E}[Y_n^2] - \mathcal{E}[Y_n]^2 \tag{63}$$

$$\mathrm{Cov}[X_n, Y_n] = \mathcal{E}[X_n Y_n] - \mathcal{E}[X_n]\mathcal{E}[Y_n] \tag{64}$$

The vector sum $\mathbf{Z}$ is defined as:

$$\mathbf{Z} = \sum_{n=1}^{N} \mathbf{Z}_n = \left(\sum_{n=1}^{N} X_n, \sum_{n=1}^{N} Y_n \right) = (X, Y) \tag{65}$$

The first and second moments of $\mathbf{Z}$ are calculated as:

$$\mathcal{E}[\mathbf{Z}] = \left(\sum_{n=1}^{N} \mathcal{E}[X_n], \sum_{n=1}^{N} \mathcal{E}[Y_n] \right) \tag{66}$$

$$\sigma^2_X = \sum_{n=1}^{N} \sigma^2_{X_n}, \quad \sigma^2_Y = \sum_{n=1}^{N} \sigma^2_{Y_n} \tag{67}$$

$$\mathrm{Cov}[X, Y] = \sum_{n=1}^{N} \mathrm{Cov}[X_n, Y_n] \tag{68}$$

The bi-variate central limit theorem states that, under certain restrictions, the random vector $\mathbf{Z} = (X, Y)$ approaches a bi-variate Gaussian variable with first and second moments shown in Eqs (66) through (68), in the limit as $N \to \infty$.

According to Laning and Battin (1956), one sufficient condition for the truth of the bi-variate central limit theorem is the following:

$$\lim_{N\to\infty} \frac{1}{\sigma_X^3} \sum_{n=1}^{N} \mathcal{E}[|X_n - \mathcal{E}[X_n]|^3] = 0 \tag{69}$$

$$\lim_{N\to\infty} \frac{1}{\sigma_Y^3} \sum_{n=1}^{N} \mathcal{E}[|Y_n - \mathcal{E}[Y_n]|^3] = 0 \tag{70}$$

The generalization of this theorem to m-variate cases follows almost immediately and the m-variate central limit theorem can also be stated in an analogous form.

The bi-variate central limit theorem presented above will be used now to show that the simulated stochastic process according to Eq (36) is Gaussian in the limit as $N \to \infty$. Specifically, it will be shown that vector $\mathbf{Z}$ defined as:

$$\mathbf{Z} = (X, Y) = (f(t_X), f(t_Y)) \tag{71}$$

is asymptotically normal in the limit as $N \to \infty$, where $f(t_X)$ and $f(t_Y)$ are the values of the simulated stochastic process [Eq (36)] at two distinct time instants t_X and t_Y.

Following the notation of Eq (61), X_n and Y_n are defined as:

$$X_n = \sqrt{2} A_n \cos(\omega_n t_X + \phi_n) \tag{72}$$

$$Y_n = \sqrt{2} A_n \cos(\omega_n t_Y + \phi_n) \tag{73}$$

and $\mathbf{Z}_n$ is given by

$$\mathbf{Z}_n = (X_n, Y_n), \quad n = 0, 1, 2, \ldots, N-1 \tag{74}$$

Vectors $\mathbf{Z}_n$ $(n = 0, 1, \ldots, N-1)$ are indeed independent since random phase angles ϕ_n $(n = 0, 1, \ldots, N-1)$ are independent. The first and second moments of $\mathbf{Z}_n$ are easily calculated as:

$$\mathcal{E}[X_n] = \mathcal{E}[Y_n] = 0 \tag{75}$$

$$\sigma^2_{X_n} = \sigma^2_{Y_n} = A_n^2 \tag{76}$$

$$\mathrm{Cov}[X_n, Y_n] = A_n^2 \cos[\omega_n (t_Y - t_X)] \tag{77}$$

Then, vector $\mathbf{Z}$ is defined as

$$\mathbf{Z} = (X, Y) = \left(\sum_{n=0}^{N-1} X_n, \sum_{n=0}^{N-1} Y_n \right) \tag{78}$$

and its first and second moments are computed as

$$\mathcal{E}[X] = \mathcal{E}[Y] = 0 \tag{79}$$

$$\sigma^2_X = \sigma^2_Y = \sum_{n=0}^{N-1} A_n^2 \tag{80}$$

$$\mathrm{Cov}[X, Y] = \sum_{n=0}^{N-1} A_n^2 \cos[\omega_n (t_Y - t_X)] \tag{81}$$

In order to show that vector $\mathbf{Z}$ defined in Eq (71) is asymptotically normal as $N \to \infty$, it is sufficient to prove that:

$$\lim_{N\to\infty} \frac{1}{\sigma_X^3} \sum_{n=0}^{N-1} \mathcal{E}[|X_n|^3] = 0 \tag{82}$$

The expected value $\mathcal{E}[|X_n|^3]$ is calculated as:

$$\begin{aligned} \mathcal{E}[|X_n|^3] &= \mathcal{E}[|\sqrt{2} A_n \cos(\omega_n t_X + \phi_n)|^3] \\ &= 2\sqrt{2} A_n^3 \ \mathcal{E}[|\cos(\omega_n t_X + \phi_n)|^3] \\ &= 2\sqrt{2} A_n^3 \int_0^{2\pi} |\cos(\omega_n t_X + \phi_n)|^3 \frac{1}{2\pi} d\phi_n \\ &= 2\sqrt{2} A_n^3 \, 4/3\pi = 8\sqrt{2} A_n^3 / 3\pi \end{aligned} \tag{83}$$

In deriving Eq (83), the fact that A_n [Eq (37)] is a non-negative quantity has been used.

Combining now Eqs (80) and (83) and considering that σ_X is finite, the following result is obtained:

$$\lim_{N\to\infty} \frac{1}{\sigma_X^3} \sum_{n=0}^{N-1} \mathcal{E}[|X_n|^3] = \frac{8\sqrt{2}}{3\pi\sigma_X^3} \lim_{N\to\infty} \sum_{n=0}^{N-1} A_n^3 \tag{84}$$

Using Eqs (37) and (39), the limit appearing in Eq (84) can be written as:

$$\begin{aligned} \lim_{N\to\infty} \sum_{n=0}^{N-1} A_n^3 &= \lim_{N\to\infty} \sum_{n=0}^{N-1} [2 S_{f_0 f_0}(n\Delta\omega)\Delta\omega]^{3/2} \\ &= 2\sqrt{2} \lim_{N\to\infty} \left(\frac{\omega_u}{N} \right)^{3/2} \sum_{n=0}^{N-1} [S_{f_0 f_0}(n\Delta\omega)]^{3/2} \leq \\ &\leq 2\sqrt{2} (\omega_u)^{3/2} \lim_{N\to\infty} \left(\frac{1}{N} \right)^{3/2} N (S_{\max})^{3/2} \\ &= 2\sqrt{2} (\omega_u S_{\max})^{3/2} \lim_{N\to\infty} \frac{1}{\sqrt{N}} = 0 \end{aligned} \tag{85}$$

where

$$S_{\max} = \text{maximum}[S_{f_0 f_0}(n\Delta\omega)], \qquad n = 0, 1, \ldots, N-1 \tag{86}$$

and $S_{\max}$ is assumed to be finite.

Considering then that A_n is a non-negative quantity, the validity of Eq (82) is proven immediately. As a consequence, vector $\mathbf{Z}$ defined in Eq (71) is asymptotically normal in the limit as $N \to \infty$. The generalization to m-variate cases (m = arbitrary positive integer) follows immediately, proving the Gaussianness of the simulated stochastic process.

3.4 Rate of convergence of simulated stochastic process to Gaussianness

The following question arises now: how fast is $f(t)$ converging to Gaussianness as a function of N? In order to find an answer, the method developed by Yang (1973) will be used.

The characteristic function $M_f(\theta)$ of a stochastic process $f(t)$ is defined as (e.g. Lin 1967):

$$M_f(\theta) = \mathcal{E}\left[\mathrm{e}^{i\theta f(t)}\right] = \int_{-\infty}^{\infty} p(f)\, \mathrm{e}^{i\theta f}\, df \tag{87}$$

where $p(f)$ is the first-order probability density function of stochastic process $f(t)$. Therefore, the density function $p(f)$ can be obtained as the inverse Fourier transform of the characteristic function $M_f(\theta)$:

$$p(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M_f(\theta)\, \mathrm{e}^{-i\theta f}\, d\theta \tag{88}$$

Substituting now Eq (36) into Eq (87), the following expression is obtained for the characteristic function of the simulated stochastic process $f(t)$:

$$\begin{aligned} M_f(\theta) &= \mathcal{E}\left\{\exp\left[i\theta\sqrt{2}\sum_{n=0}^{N-1} A_n \cos(\omega_n t + \Phi_n)\right]\right\} \\ &= \int_0^{2\pi} N-fold \int_0^{2\pi} \exp\left[i\theta\sqrt{2}\sum_{n=0}^{N-1} A_n \cos(\omega_n t + \phi_n)\right] \\ &\quad \times \left(\frac{1}{2\pi}\right)^N d\phi_0 d\phi_1 \cdots d\phi_{N-1} \end{aligned} \tag{89}$$

In writing Eq (89), the fact that $\Phi_0, \Phi_1, \ldots, \Phi_{N-1}$ are independent random phase angles distributed uniformly over the interval $[0, 2\pi]$ was utilized. Note that the expression for A_n was given in Eq (37). Equation (89) can then be written as:

$$\begin{aligned} M_f(\theta) &= \prod_{n=0}^{N-1} \left(\frac{1}{2\pi}\right) \int_0^{2\pi} \exp\left[i\theta\sqrt{2}A_n \cos(\omega_n t + \phi_n)\right]\, d\phi_n \\ &= \prod_{n=0}^{N-1} J_0(\theta\sqrt{2}A_n) \end{aligned} \tag{90}$$

where J_0 is the zero-order Bessel function of the first kind. The integral representation of J_0 used in the derivation of Eq (90) is taken from Korn and Korn (1968).

Substituting now Eq (90) into Eq (88), the following expression is obtained for the first-order probability density function $p(f)$ of stochastic process $f(t)$:

$$p(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{n=0}^{N-1} J_0(\theta\sqrt{2}A_n)\, \mathrm{e}^{-i\theta f}\, d\theta \tag{91}$$

The density function $p(f)$ of the simulated stochastic process $f(t)$ can be calculated by performing numerically the inverse Fourier transform indicated in Eq (91). This calculation can be performed efficiently by using the Fast Fourier Transform (FFT) technique. Then, the difference between the density function $p(f)$ calculated from Eq (91) for a specific value of N and the Gaussian density function can be examined.

Yang (1973) is using the following discretization:

$$A_n^2 = 2S_{f_0 f_0}(\omega_n)\Delta\omega_n = \sigma^2/N \tag{92}$$

where σ is the standard deviation of stochastic process $f(t)$. It is reminded that:

$$\int_0^{\infty} 2S_{f_0 f_0}(\omega) d\omega = \sigma^2 \tag{93}$$

The discretization described in Eq (92) is therefore one into intervals of equal area. Substituting Eq (92) into Eq (91) it follows that:

$$p(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} J_0^N\left(\theta\sqrt{2}\sigma/\sqrt{N}\right)\, \mathrm{e}^{-i\theta f}\, d\theta \tag{94}$$

Finally, considering without loss of generality that the standard deviation σ of the simulated stochastic process $f(t)$ is normalized to unity, Eq (94) can be written as:

$$p(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} J_0^N\left(\theta\sqrt{2/N}\right)\, \mathrm{e}^{-i\theta f}\, d\theta \tag{95}$$

It can be observed from Eq (95) that the discretization of Eq (92) allows the investigation of the convergence of $p(f)$ to the standardized Gaussian density function as a function of N, in a very general fashion: $p(f)$ calculated from Eq (95) is independent of the particular form of the power spectral density function $S_{f_0 f_0}(\omega)$. Of course, if the discretization of Eq (37) is used instead of that of Eq (92), then Eq (91) must be used to investigate the convergence of the density function, instead of Eq (95). It is believed (Yang 1973), however, that the results obtained by using Eq (91) would not have any significant difference from those obtained by using Eq (95). The exceedence probability $\bar{\Phi}(x)$ is defined now as:

$$\bar{\Phi}(x) = \int_x^{\infty} p(f)\, df \tag{96}$$

Numerical computations for $\bar{\Phi}(x)$ are performed using Eq (95) for different values for x and N. It is noted that the inverse Fourier transform shown in Eq (95) is performed numerically using the FFT technique. The results are displayed in Table 1 along with the exact values of the standardized Gaussian exceedence probability for the sake of comparison.

Table 1 gives a clear picture of the rate of convergence of the simulated stochastic process $f(t)$ to Gaussianness, as a function of N. The following conclusion can be drawn by studying Table 1: the rate of convergence to Gaussianness decreases as x increases. This indicates the rather obvious fact that the tail portion of the density function $p(f)$ converges to Gaussianness much slower than the central portion. However, if N is as large as 1000, the error is as small as 3% even at 5σ range.

TABLE 1. First-order exceedence probability of simulated stochastic process $\bar{\Phi}(x) = \int_x^{\infty} p(f)\, df$.

N \ x	1σ (10^0)	2σ (10^{-1})	3σ (10^{-2})	4σ (10^{-4})	5σ (10^{-6})
50	0.1593	0.2262	0.1250	0.2345	0.1250
100	0.1590	0.2268	0.1300	0.2745	0.1956
200	0.1588	0.2272	0.1325	0.2954	0.2385
400	0.1587	0.2274	0.1338	0.3060	0.2619
600	0.1587	0.2274	0.1342	0.3096	0.2701
800	0.1587	0.2274	0.1344	0.3114	0.2742
1000	0.1587	0.2275	0.1345	0.3125	0.2767
10000	0.1587	0.2275	0.1350	0.3164	0.2858
Gaussian	0.1587	0.2275	0.1350	0.3167	0.2867

4. ERGODICITY OF SIMULATED STOCHASTIC PROCESSES

4.1 Ergodicity

A very important property of the simulated stochastic process is the following: each and every sample function given by Eq (44) is ergodic in the mean value and in correlation.

It will be shown now that the temporal average and the temporal autocorrelation function of any sample function $f^{(i)}(t)$ are identical to the corresponding targets, $\mathcal{E}[f_0(t)] = 0$ and $R_{f_0 f_0}(\tau)$, respectively. In addition, it will be shown that these two identities are valid only when the length of the sample function $f^{(i)}(t)$ is either equal to the period T_0 or when it approaches infinity.

(i) Show that: $< f^{(i)}(t) >_T = \mathcal{E}[f_0(t)] = 0$ either when $T = T_0$ or as $T \to \infty$

Proof:

The temporal average $< f^{(i)}(t) >_T$ of sample function $f^{(i)}(t)$ over a time interval equal to T can be written as

$$\begin{aligned} < f^{(i)}(t) >_T &= \frac{1}{T} \int_0^T f^{(i)}(t)dt \\ &= \frac{\sqrt{2}}{T} \int_0^T \sum_{n=0}^{N-1} A_n \cos(\omega_n t + \phi_n^{(i)})dt \\ &= \frac{\sqrt{2}}{T} \sum_{n=0}^{N-1} A_n \int_0^T \cos(\omega_n t + \phi_n^{(i)})dt \end{aligned} \quad (97)$$

where A_n was defined in Eq (37) and ω_n in Eq (38). The integrand $\cos(\omega_n t + \phi_n^{(i)})$ is a periodic function of t with period equal to T_0/n $(n = 1, 2, \ldots, N-1)$. Taking also into account that for $n = 0$ it was assumed that $A_0 = 0$, it is obvious that

$$< f^{(i)}(t) >_T = 0 \quad \text{when} \quad T = T_0 \quad (98)$$

with T_0 given by Eq (43).

Considering now the following result of temporal integration:

$$\begin{aligned} &\lim_{T\to\infty} \frac{\sqrt{2}}{T} \sum_{n=0}^{N-1} A_n \int_0^T \cos(\omega_n t + \phi_n^{(i)})dt \\ &= \sum_{n=0}^{N-1} A_n \lim_{T\to\infty} \frac{\sqrt{2}}{T\omega_n} [\sin(\omega_n t + \phi_n^{(i)})]_0^T = 0 \end{aligned} \quad (99)$$

it is straightforward to write

$$< f^{(i)}(t) >_T = 0 \quad \text{when} \quad T \to \infty \quad (100)$$

Hence, it has been shown in Eqs (98) and (100) that:

$$< f^{(i)}(t) >_T = 0 = \mathcal{E}[f_0(t)] \text{ when either } T = T_0 \text{ or } T \to \infty \quad (101)$$

The meaning of Eq (101) is that the temporal average of any sample function $f^{(i)}(t)$ is identical to the target mean value when the length of the sample function is either equal to the period T_0 or when it approaches infinity.

(ii) Show that: $R_{ff}^{(i)}(\tau) = R_{f_0 f_0}(\tau)$ either when $T = T_0$ or as $T \to \infty$

Proof:

The temporal autocorrelation function $R_{ff}^{(i)}(\tau)$ of sample function $f^{(i)}(t)$ over a time interval equal to T can be written as:

$$\begin{aligned} &R_{ff}{}^{(i)}(\tau) \\ &= < f^{(i)}(t+\tau) f^{(i)}(t) >_T \\ &= \frac{1}{T} \int_0^T f^{(i)}(t+\tau) f^{(i)}(t) dt \\ &= \frac{2}{T} \int_0^T \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} A_n A_m \\ &\quad \times \cos[\omega_n(t+\tau) + \phi_n^{(i)}] \cos(\omega_m t + \phi_m^{(i)}) dt \\ &= \frac{1}{T} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} A_n A_m \\ &\quad \times \int_0^T \{\cos[(\omega_n + \omega_m)t + \omega_n \tau + \phi_n^{(i)} + \phi_m^{(i)}] \\ &\quad + \cos[(\omega_n - \omega_m)t + \omega_n \tau + \phi_n^{(i)} - \phi_m^{(i)}]\} dt \end{aligned} \quad (102)$$

The integral appearing in Eq (102) can be computed by considering the following two cases.

For $n \neq m$, the integral shown in Eq (102) becomes:

$$\begin{aligned} &\int_0^T \cos[(\omega_n + \omega_m)t + \omega_n \tau + \phi_n^{(i)} + \phi_m^{(i)}]dt \\ &\quad + \int_0^T \cos[(\omega_n - \omega_m)t + \omega_n \tau + \phi_n^{(i)} - \phi_m^{(i)}]dt \\ &= 0 + 0 = 0, \quad \text{when} \quad T = T_0 \end{aligned} \quad (103)$$

since integrands $\cos[(\omega_n + \omega_m)t + \omega_n \tau + \phi_n^{(i)} + \phi_m^{(i)}]$ and $\cos[(\omega_n - \omega_m)t + \omega_n \tau + \phi_n^{(i)} - \phi_m^{(i)}]$ are periodic functions of t with periods equal to $T_0/(n+m)$ and $T_0/|n-m|$ $(n = 0, 1, 2, \ldots, N-1;\ m = 0, 1, 2, \ldots, N-1;\ n \neq m)$, respectively.

For $n = m$, the integral shown in Eq (102) reduces to

$$\begin{aligned} &\int_0^T \cos(2\omega_n t + \omega_n \tau + 2\phi_n^{(i)})dt + \int_0^T \cos(\omega_n \tau)dt \\ &= 0 + T\cos(\omega_n \tau) = T\cos(\omega_n \tau) \quad \text{when} \quad T = T_0 \end{aligned} \quad (104)$$

In deriving Eq (104), the fact was used that the integrand $\cos(2\omega_n t + \omega_n \tau + 2\phi_n^{(i)})$ is a periodic function of t with period equal to $T_0/2n$ $(n = 1, 2, \ldots, N-1)$ and that for $n = 0$, A_0 is assumed to be zero; $A_0 = 0$.

Combining now Eqs (102), (103) and (104), the temporal autocorrelation function $R_{ff}^{(i)}(\tau)$ of sample function $f^{(i)}(t)$ becomes

$$\begin{aligned} R_{ff}^{(i)}(\tau) = < f^{(i)}(t+\tau) f^{(i)}(t) >_T &= \frac{1}{T} \sum_{n=0}^{N-1} A_n^2 \, T \cos(\omega_n \tau) \\ &= \sum_{n=0}^{N-1} A_n^2 \cos(\omega_n \tau), \quad \text{for} \quad T = T_0 \end{aligned} \quad (105)$$

Comparing eventually Eqs (52) and (105) and taking advantage of Eqs (54) and (55), the following conclusion can be drawn:

$$\begin{aligned} R_{ff}^{(i)}(\tau) = < f^{(i)}(t+\tau) f^{(i)}(t) >_T = R_{f_0 f_0}(\tau), \\ \text{for} \quad 0 \leq \tau \leq T, \quad \text{only when} \quad T = T_0 \end{aligned} \quad (106)$$

It is noted that Eq (106) is valid in the limit as $\Delta\omega \to 0$ and $N \to \infty$, while keeping in mind that $\omega_u = N\Delta\omega$ is constant and that $S_{f_0 f_0}(\omega) = 0$ for $|\omega| \geq \omega_u$. Consequently, under the same limiting conditions:

$$S_{ff}^{(i)}(\omega) = S_{f_0 f_0}(\omega), \quad \text{for} \quad 0 \leq \omega \leq \omega_u \quad \text{only when} \quad T = T_0 \quad (107)$$

Finally, using the following results of temporal integration in Eq (102):
For $n \neq m$:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \cos[(\omega_n + \omega_m)t + \omega_n\tau + \phi_n^{(i)} + \phi_m^{(i)}]dt + \lim_{T\to\infty} \frac{1}{T} \int_0^T \cos[(\omega_n - \omega_m)t + \omega_n\tau + \phi_n^{(i)} - \phi_m^{(i)}]dt = 0 + 0 = 0 \tag{108}$$

For $n = m$:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \cos(2\omega_n t + \omega_n\tau + 2\phi_n^{(i)})dt + \lim_{T\to\infty} \frac{1}{T} \int_0^T \cos(\omega_n\tau)dt = 0 + \cos(\omega_n\tau) \lim_{T\to\infty} \frac{1}{T} T = \cos(\omega_n\tau) \tag{109}$$

it is easy to show that under the same limiting conditions as those of Eq (106):

$$R_{ff}^{(i)}(\tau) = < f^{(i)}(t+\tau)f^{(i)}(t) >_T = R_{f_0f_0}(\tau) \quad \text{as} \quad T \to \infty \tag{110}$$

and consequently:

$$S_{ff}^{(i)}(\omega) = S_{f_0f_0}(\omega), \quad \text{for} \quad 0 \leq \omega \leq \omega_u \quad \text{as} \quad T \to \infty \tag{111}$$

The meaning of Eqs (106) and (110) is that the temporal autocorrelation function of any sample function $f^{(i)}(t)$ is identical to the target autocorrelation function when the length of the sample function is either equal to the period T_0 or when it approaches infinity.

4.2 Rate of convergence of temporal autocorrelation function to target autocorrelation function

In Section 4.1 it has been shown that the temporal autocorrelation function $R_{ff}^{(i)}(\tau)$ is equal to the target autocorrelation function $R_{f_0f_0}(\tau)$. However, this equality holds in the limit as $\Delta\omega \to 0$ and $N \to \infty$, exactly in the same way as described in the process of obtaining Eq (54) from Eq (52). It is therefore obvious that the rate of convergence of the temporal autocorrelation function $R_{ff}^{(i)}(\tau)$ to the target autocorrelation function $R_{f_0f_0}(\tau)$ is the same with the rate of convergence of the ensemble autocorrelation function $R_{ff}(\tau)$ to the target autocorrelation function $R_{f_0f_0}(\tau)$, which was examined in detail in Section 3.2. Consequently, the reader is referred to the results obtained in Section 3.2.

4.3 Non-ergodic characteristics of series expression in Eq (21)

In Section 2 it has been stated that although both series expressions displayed in Eqs (21) and (35) are consistent with the spectral representation theorem stated in Eq (5), Eq (35) is used exclusively in this work because it produces ergodic sample functions. The fact that Eq (21) produces non-ergodic sample functions will be demonstrated in the following.

From the infinite series representation displayed in Eq (21), it follows that the stochastic process $f_0(t)$ can be simulated by the following series as $N \to \infty$

$$f(t) = \sum_{n=0}^{N-1} [\cos(\omega_n t)\, X_n + \sin(\omega_n t)\, Y_n] \tag{112}$$

where ω_n is given by Eqs (38) and (39). The X_ns and the Y_ns are independent random variables, with mean value equal to zero and standard deviation equal to $A_n = (2S_{f_0f_0}(\omega_n)\Delta\omega)^{1/2}$. Considering that Eq (40) is again valid, it is easy to show that the simulated stochastic process $f(t)$ given by Eq (112) is periodic with period $T_0 = 2\pi/\Delta\omega$. In addition, $f(t)$ is obviously Gaussian if the X_ns and Y_ns are normally distributed. But even if the X_ns and Y_ns are not normally distributed, it can be shown that $f(t)$ is still asymptotically Gaussian, under certain conditions, using the m-variate central limit theorem, as was done in Section 3.3.

At this point it is interesting to note that the series representation shown in Eq (112) may also be represented (Anderson 1971) as:

$$f(t) = \sum_{n=0}^{N-1} Q_n \cos(\omega_n t - \theta_n) \tag{113}$$

where:

$$Q_n^2 = X_n^2 + Y_n^2 \tag{114}$$

$$\tan\theta_n = Y_n/X_n \tag{115}$$

If X_n and Y_n are normally distributed, Q_n^2 is a chi-square variable with 2 degrees of freedom and θ_n is uniformly distributed between 0 and 2π and is independent of Q_n^2.

Going back to Eq (112), a sample function $f^{(i)}(t)$ of the simulated stochastic process $f(t)$ can be obtained by replacing the two sequences of random variables $X_0, X_1, \ldots, X_{N-1}$ and $Y_0, Y_1, \ldots, Y_{N-1}$ with their respective i-th realizations $x_0^{(i)}, x_1^{(i)}, \ldots, x_{N-1}^{(i)}$ and $y_0^{(i)}, y_1^{(i)}, \ldots, y_{N-1}^{(i)}$:

$$f^{(i)}(t) = \sum_{n=0}^{N-1} [\cos(\omega_n t)\, x_n^{(i)} + \sin(\omega_n t)\, y_n^{(i)}] \tag{116}$$

Provided that $x_0^{(i)}$ and $y_0^{(i)}$ are both set equal to zero, it is easy to show that each and every sample function $f^{(i)}(t)$ given by Eq (116) is ergodic in mean value. However, it will be shown now that $f^{(i)}(t)$ is not ergodic in correlation.

The temporal autocorrelation function $R_{ff}^{(i)}(\tau)$ of sample function $f^{(i)}(t)$ over a time interval equal to T can be written as:

$$\begin{aligned} R_{ff}^{(i)}(\tau) =< f^{(i)}(t+\tau)f^{(i)}(t) >_T &= \frac{1}{T}\int_0^T f^{(i)}(t+\tau)f^{(i)}(t)dt \\ &= \frac{1}{T}\int_0^T \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \Big[\cos[\omega_n(t+\tau)]x_n^{(i)} + \sin[\omega_n(t+\tau)]y_n^{(i)}\Big] \\ &\quad \times \Big[\cos(\omega_m t)x_m^{(i)} + \sin(\omega_m t)y_m^{(i)}\Big]dt \\ &= \frac{1}{T}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} x_n^{(i)}x_m^{(i)} \int_0^T \cos[\omega_n(t+\tau)]\cos(\omega_m t)dt \\ &\quad + \frac{1}{T}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} x_n^{(i)}y_m^{(i)} \int_0^T \cos[\omega_n(t+\tau)]\sin(\omega_m t)dt \\ &\quad + \frac{1}{T}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} y_n^{(i)}x_m^{(i)} \int_0^T \sin[\omega_n(t+\tau)]\cos(\omega_m t)dt \\ &\quad + \frac{1}{T}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} y_n^{(i)}y_m^{(i)} \int_0^T \sin[\omega_n(t+\tau)]\sin(\omega_m t)dt \\ &= \frac{1}{T}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} x_n^{(i)}x_m^{(i)} \frac{1}{2}\int_0^T [\cos(\omega_n t + \omega_n\tau + \omega_m t) \end{aligned}$$

$$+\cos(\omega_n t+\omega_n\tau-\omega_m t)]dt$$
$$+\frac{1}{T}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}x_n^{(i)}y_m^{(i)}\,\frac{1}{2}\int_0^T[\sin(-\omega_n t-\omega_n\tau+\omega_m t)$$
$$+\sin(\omega_n t+\omega_n\tau+\omega_m t)]dt$$
$$+\frac{1}{T}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}y_n^{(i)}x_m^{(i)}\,\frac{1}{2}\int_0^T[\sin(\omega_n t+\omega_n\tau-\omega_m t)$$
$$+\sin(\omega_n t+\omega_n\tau+\omega_m t)]dt$$
$$+\frac{1}{T}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}y_n^{(i)}y_m^{(i)}\,\frac{1}{2}\int_0^T[\cos(\omega_n t+\omega_n\tau-\omega_m t)$$
$$-\cos(\omega_n t+\omega_n\tau+\omega_m t)]dt \tag{117}$$

Using now the same procedure as the one used in Section 4.1 to obtain Eqs (106) and (110), it can be shown that:

$$R_{ff}^{(i)}(\tau)=<f^{(i)}(t+\tau)f^{(i)}(t)>_T$$
$$=\sum_{n=0}^{N-1}\frac{1}{2}\left(x_n^{(i)^2}+y_n^{(i)^2}\right)\cos(\omega_n\tau)$$
$$\text{for either}\quad T=T_0\quad\text{or}\quad T\to\infty \tag{118}$$

Comparing eventually Eqs (52) and (118) and taking advantage of Eq (54), it is obvious that:

$$R_{ff}^{(i)}(\tau)\neq R_{f_0f_0}(\tau),\quad\text{for either}\quad T=T_0\quad\text{or}\quad T\to\infty \tag{119}$$

Therefore, each and every sample function given by Eq (116) is not ergodic in correlation.

5. USE OF FAST FOURIER TRANSFORM (FFT) TECHNIQUE

The cost of digitally generating sample functions of the simulated stochastic process can be drastically reduced by using the FFT technique (e.g. Brigham 1988). In order to take advantage of the FFT technique, Eq (44) is rewritten in the following form:

$$f^{(i)}(p\,\Delta t)=\text{Re}\left\{\sum_{n=0}^{M-1}B_n\exp[i(n\Delta\omega)(p\Delta t)]\right\},$$
$$p=0,1,\ldots,M-1 \tag{120}$$

where Re indicates the real part and B_n stands for

$$B_n=\sqrt{2}\,A_n\,e^{i\phi_n^{(i)}},\qquad n=0,1,\ldots,M-1 \tag{121}$$

with A_n being the expression defined in Eq (37) as

$$A_n=(2S_{f_0f_0}(n\Delta\omega)\Delta\omega)^{1/2},\qquad n=0,1,\ldots,M-1 \tag{122}$$

and

$$\Delta\omega=\omega_u/N \tag{123}$$

with ω_u being the upper cut-off frequency beyond which the power spectral density function $S_{f_0f_0}(\omega)$ may be assumed to be zero for either mathematical or physical reasons.

Note again that sample function $f^{(i)}(p\,\Delta t)$ shown in Eq (120) is periodic with period T_0:

$$T_0=2\pi/\Delta\omega \tag{124}$$

Hence, Δt and $\Delta\omega$ are related in the following way:

$$M\,\Delta t=T_0=2\pi/\Delta\omega \tag{125a}$$

$$\Delta t=2\pi/(M\,\Delta\omega) \tag{125b}$$

$$\Delta t\,\Delta\omega=2\pi/M \tag{125c}$$

With the aid of Eq (125c), Eq (120) can be written as:

$$f^{(i)}(p\,\Delta t)=\text{Re}\left\{\sum_{n=0}^{M-1}B_n\,e^{inp\,2\pi/M}\right\},\quad p=0,1,\ldots,M-1 \tag{126}$$

It has been established in Eq (45) that in order to avoid aliasing (assuming that $S(\omega)=0$ for $\omega>\omega_u$), the time step Δt has to obey the condition:

$$\Delta t\le 2\pi/2\omega_u \tag{127}$$

Combining now Eqs (123), (125b) and (127), the following condition is established between N and M:

$$M\ge 2N \tag{128}$$

Finally, the $\phi_n^{(i)}$ $(n=0,1,\ldots,M-1)$ appearing in Eq (126) are the i-th realizations of the independent random phase angles Φ_n $(n=0,1,\ldots,M-1)$ which are all distributed uniformly over the interval $[0,2\pi]$.

Instead of using Eq (44) involving straightforward summation of cosines for digitally generating sample functions of the simulated stochastic process, the FFT technique can be readily used on Eq (126) resulting in a drastic reduction of computer cost. In order to take full advantage of the FFT technique, M must be an integer power of two:

$$M=2^\mu \tag{129}$$

where μ is a positive integer. The condition set in Eq (129) can be sometimes rather restrictive. FFT algorithms were therefore developed removing this condition. Unfortunately, such algorithms are not quite as fast as the algorithms using the condition of Eq (129). For a description of FFT algorithms the reader is referred, for example, to Brigham (1988).

At this point it is reminded that the following conditions must be imposed when using Eq (126):

$$B_0=0 \tag{130}$$

and:

$$B_n=0,\quad\text{for}\quad N\le n\le M-1 \tag{131}$$

The condition set in Eq (130) is equivalent to the condition set in Eq (40), while Eq (131) represents the condition that the power spectral density function $S_{f_0f_0}(\omega)$ is assumed to be zero for values of the frequency larger that ω_u.

Lately, Winterstein (1990) indicated that the Fast Hartley Transform may be used to increase the computational efficiency of the simulation.

6. NUMERICAL EXAMPLES

6.1 Stochastic process description

Consider a 1D-1V stationary stochastic process $f_0(t)$ with mean value equal to zero, autocorrelation function $R_{f_0f_0}(\tau)$ given by:

$$R_{f_0f_0}(\tau)=\sigma^2\,\frac{b^4(b^2-3\tau^2)}{(b^2+\tau^2)^3},\qquad -\infty<\tau<\infty \tag{132}$$

and corresponding two-sided power spectral density function $S_{f_0f_0}(\omega)$ given by:

$$S_{f_0f_0}(\omega)=\tfrac{1}{4}\,\sigma^2\,b^3\,\omega^2\,e^{-b|\omega|},\qquad -\infty<\omega<\infty \tag{133}$$

It should be mentioned that the expressions for $R_{f_0 f_0}(\tau)$ and $S_{f_0 f_0}(\omega)$ shown in Eqs (132) and (133), respectively, satisfy the Wiener-Khintchine theorem [see Eqs (3) and (4)]. Parameter σ appearing in Eqs (132) and (133) is the standard deviation of stochastic process $f_0(t)$ since:

$$\int_{-\infty}^{\infty} S_{f_0 f_0}(\omega) d\omega = R_{f_0 f_0}(0) = \sigma^2 \tag{134}$$

In deriving Eq (134), the expressions for $S_{f_0 f_0}(\omega)$ and $R_{f_0 f_0}(\tau)$ shown in Eqs (133) and (132), respectively, were used.

Parameter b appearing in Eqs (132) and (133) is proportional to the correlation distance of the stochastic process as indicated in Shinozuka and Deodatis (1988a).

In the following, σ and b are set both equal to unity:

$$\sigma = 1 \qquad \text{and} \qquad b = 1 \text{ sec} \tag{135}$$

In order to perform the generation of sample functions of stochastic process $f_0(t)$, an upper cut-off frequency ω_u as defined in Eq (41) must be specified. The value of:

$$\omega_u = 4\pi \simeq 12.6 \text{ rad/sec} \tag{136}$$

is established by using the criterion described in Eq (41) with $\epsilon = 3.23\,10^{-4}$. If parameter N appearing in Eq (39) is chosen to be:

$$N = 128 \tag{137}$$

then, $\Delta\omega$ and T_0 are calculated using Eqs (39) and (43), respectively, as:

$$\Delta\omega = 4\pi/128 \simeq 0.0982 \text{ rad/sec} \tag{138}$$

$$T_0 = 64 \text{ sec} \tag{139}$$

Finally, the condition set on the time step Δt by Eq (45) is satisfied when:

$$\Delta t \leq 0.25 \text{ sec} \tag{140}$$

Before proceeding with the generation of sample functions of stochastic process $f_0(t)$, the upper bound for $f^{(i)}(t)$ given by Eq (46) is calculated using $N = 128$ and the expression for $S_{f_0 f_0}(\omega)$ shown in Eq (133) as:

$$f^{(i)}(t) \leq 12.5 \tag{141}$$

Considering that the standard deviation of stochastic process $f_0(t)$ is equal to $\sigma = 1$ [see Eq (135)], it is obvious that the upper bound shown in Eq (141) is large enough for all practical applications. Note that it is equal to more than twelve standard deviations!

In the following, realizations of random phase angles distributed uniformly over the interval $[0, 2\pi]$ will be needed for the generation of sample functions of stochastic process $f_0(t)$. These realizations are obtained using a uniform $[0, 1]$ random number generator and then multiplying the uniform $[0, 1]$ deviates by 2π.

6.2 Simulation by summation of cosines and zooming-in

One sample function of stochastic process $f_0(t)$ is generated with the aid of Eq (44), using a time step $\Delta t = 0.25$ sec. The generation is performed at 256 equidistant points over a length equal to one period, since $256 \times 0.25 = 64$ sec $= T_0$. This sample function is plotted in Fig 1a. The temporal autocorrelation function $R_{ff}^{(i)}(\tau)$ of this generated sample function is calculated and plotted against the target autocorrelation function $R_{f_0 f_0}(\tau)$ in Fig 1b. As can be seen in Fig 1b, $R_{ff}^{(i)}(\tau)$ coincides with $R_{f_0 f_0}(\tau)$ thus confirming the conclusion drawn in Section 4.1 that the temporal autocorrelation function of any sample function $f^{(i)}(t)$ is identical to the target autocorrelation function when the length of the sample function is equal to the period T_0.

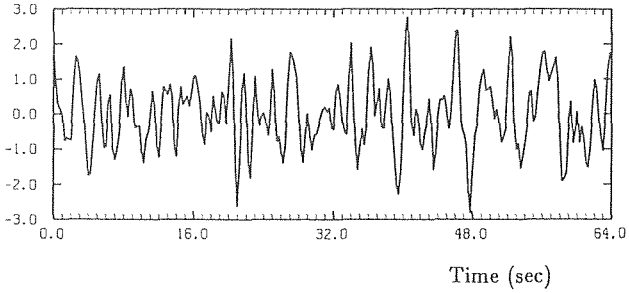

FIG 1a. Generated sample function using Eq (44) at 256 points with $\Delta t = 0.25$ sec.

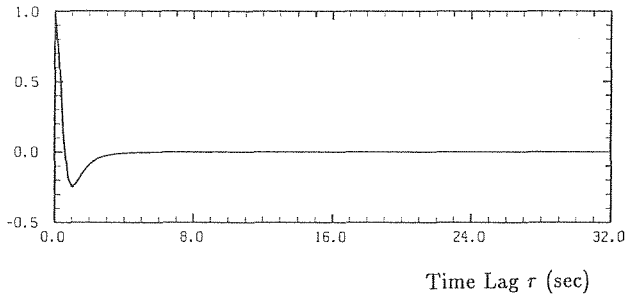

FIG 1b. Temporal autocorrelation function of generated sample function shown in Fig 1a (dotted line) and corresponding target autocorrelation function (continuous line). Note that dotted line and continuous line completely coincide.

Another sample function is generated using a time step $\Delta t = 0.50$ sec. The generation is performed at 128 equidistant points, again over a length equal to one period, since $128 \times 0.50 = 64$ sec $= T_0$. Note that the time step $\Delta t = 0.50$ sec is violating the condition set in Eq (140). This sample function is plotted in Fig 2a. Its temporal autocorrelation function $R_{ff}^{(i)}(\tau)$ is calculated and plotted against the target autocorrelation function $R_{f_0 f_0}(\tau)$ in Fig 2b. As can be seen in Fig 2b, $R_{ff}^{(i)}(\tau)$ differs from $R_{f_0 f_0}(\tau)$, although the length of the sample function is equal to the period T_0, because the condition on Δt set by Eq (140) was violated.

Another sample function is then generated using a time step $\Delta t = 0.25$ sec satisfying the condition set by Eq (140). The generation is now performed at 200 equidistant points, over a length less than one period, since $200 \times 0.25 = 50$ sec $< T_0$. This sample function is plotted in Fig 3a. Its temporal autocorrelation function $R_{ff}^{(i)}(\tau)$ is calculated and plotted against the target autocorrelation function $R_{f_0 f_0}(\tau)$ in Fig 3b. As can be seen in Fig 3b, $R_{ff}^{(i)}(\tau)$ differs considerably from $R_{f_0 f_0}(\tau)$ because the length of the sample function is less than the period T_0. Consider now that 1000

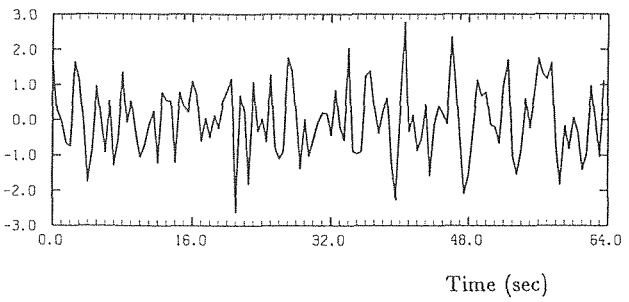

FIG 2a. Generated sample function using Eq (44) at 128 points with $\Delta t = 0.50$ sec.

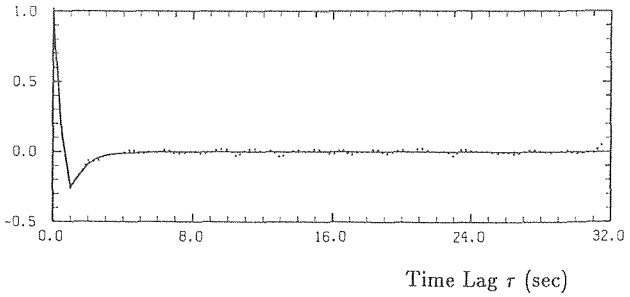

FIG 2b. Temporal autocorrelation function of generated sample function shown in Fig 2a (dotted line) and corresponding target autocorrelation function (continuous line).

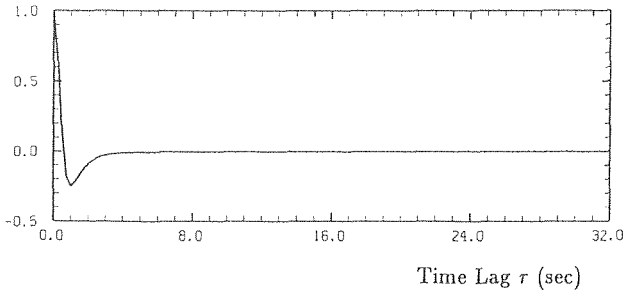

FIG 3c. Ensemble autocorrelation function of 1,000 generated sample functions (each at 200 points with $\Delta t = 0.25$ sec - dotted line) and corresponding target autocorrelation function (continuous line). Note that dotted line and continuous line practically coincide.

sample functions are generated, each one at 200 equidistant points and using the same time step $\Delta t = 0.25$ sec. Therefore, each one of these 1000 sample functions is generated over a length equal to $200 \times 0.25 = 50$ sec. The ensemble autocorrelation function $R_{ff}(\tau)$ of these 1000 sample functions is calculated and plotted against the target autocorrelation function $R_{f_0 f_0}(\tau)$ in Fig 3c. Figure 3c shows that $R_{ff}(\tau)$ practically coincides with $R_{f_0 f_0}(\tau)$ thus confirming the conclusion drawn in Section 3.1 that the ensemble autocorrelation function is identical to the target autocorrelation function, even when the length of the sample functions is less than the period T_0.

One of the most important characteristics of simulating by summation of cosines using Eq (44) is the possibility of zooming-in. The simulation formula in Eq (44) allows the simulation to be performed in any narrow time window using a time step as small as desired. It is obvious that there is no restriction in the number of points to be generated and the time step separating them. Note that the time step can even be different between different points.

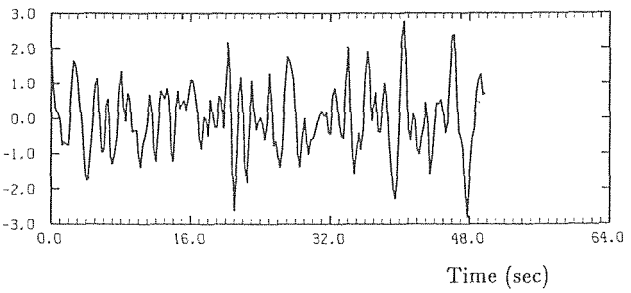

FIG 3a. Generated sample function using Eq (44) at 200 points with $\Delta t = 0.25$ sec

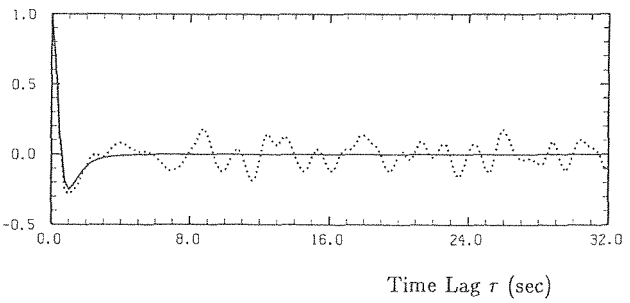

FIG 3b. Temporal autocorrelation function of generated sample function shown in Fig 3a (dotted line) and corresponding target autocorrelation function (continuous line).

6.3 Simulation by FFT and zooming-in

One sample function of stochastic process $f_0(t)$ is generated with the aid of Eq (126), using the following value for M:

$$M = 1024 = 2^{10} \tag{142}$$

Note that the values of ω_u and N are still given by Eqs (136) and (137), respectively; while the value of M shown in Eq (142) satisfies the condition shown in Eq (128). As a consequence the value of the time step Δt is calculated using Eq (125b) as:

$$\Delta t = 0.0625 \text{ sec} \tag{143}$$

From Eq (126) it is obvious that the generation is performed at $M = 1024$ equidistant points over a length equal to one period, since $1024 \times 0.0625 = 64$ sec $= T_0$. This sample function is plotted in Fig 4.

At this point, the following interesting remarks must be made, concerning the generation of sample functions of stochastic process $f_0(t)$ using the FFT technique:

(i) The length of every sample function is always equal to one period. This fact is described in Eq (125a).

(ii) The time step Δt in the time domain is calculated using Eq (125b). This calculated value of Δt will automatically satisfy the condition set in Eq (127), as long as the condition set in Eq (128) is satisfied.

(iii) The value of Δt can be decreased by increasing the value of M in accordance with Eq (125b). For example, suppose that M points are generated with a time step Δt defined by Eq (125b). If the value of M is then doubled, the value of Δt will be cut in half. This is the way to zoom-in when using the FFT technique for simulation. A minor drawback is that the zooming-in is performed over the whole period T_0 and not over a narrow time window. However, zooming-in over the whole period T_0 when using the FFT technique might require less computing time than zooming-in

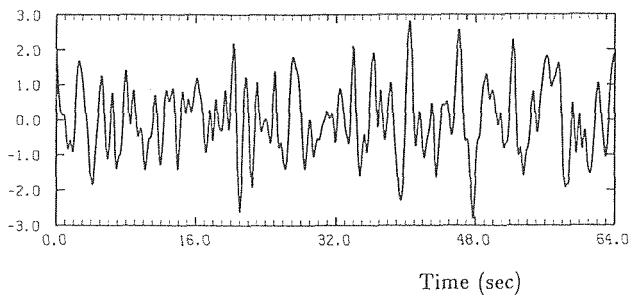

FIG 4. Generated sample function using Eq (126) at 1024 points with $\Delta t = 0.0625$ sec

over a narrow time window when using summation of cosines (see Section 6.4).

(iv) The temporal autocorrelation function $R_{jj}^{(i)}(\tau)$ of every sample function is identical to the target autocorrelation function $R_{f_0 f_0}(\tau)$, since the length of every sample function is equal to the period T_0 (provided that the condition set in Eq (128) is satisfied).

6.4 Comparison between simulation by summation of cosines and simulation by FFT

The major advantage of simulating using the FFT technique [Eq (126)], when compared to simulating by summation of cosines [Eq (44)], is the dramatic reduction in computing time. According to Brigham (1988), if it is assumed that computing time is proportional to the number of multiplications performed, then the approximate ratio of summation of cosines to FFT computing time is given by:

$$2M/\mu \tag{144}$$

where μ was defined in Eq (129). Note that Eq (144) yields for $M = 1024 = 2^{10}$ a computational reduction of more than 200 to 1!

Two minor disadvantages of simulating using the FFT technique are the following:

(i) The length of generated sample functions is always equal to one period.

(ii) In order to reduce the value of the time step Δt in the time domain, the value of M has to be increased [see Eq (125)].

However, both of these disadvantages are far outweighed by the dramatic reduction in computing time.

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