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Distribution type uncertainty due to sparse and imprecise data

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A R T I C L E I N F O

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A B S T R A C T

This paper proposes a likelihood-based methodology to quantify the distribution type uncertainty while fitting probability distributions to sparse and imprecise data. In probabilistic representation of uncertainty, it is common to assume a particular type of probability distribution (e.g. normal, lognormal, etc.) while fitting distributions to available data; once this type is chosen, the distribution parameters and the uncertainty in the distribution parameters are estimated. This paper analyzes the effect of the choice of the distribution type and quantifies the resulting uncertainty in the probabilistic characterization. Two approaches – Bayesian model averaging and Bayesian hypothesis testing – are investigated for the quantification of distribution type uncertainty. Two cases – competing distribution types and uncertainty regarding a single distribution type – are considered. Once the distribution type uncertainty in a particular random variable is quantified, the uncertainty in the distribution parameters is also quantified. Further, the three types of uncertainty – variability, distribution type uncertainty, and distribution parameter uncertainty – are propagated through a response function to calculate the effect of overall input distribution uncertainty on the response uncertainty.

1. Introduction

Uncertainty representation, quantification, and propagation are important steps in the performance prediction, risk, and reliability analysis of engineering systems. Researchers have pursued both probabilistic and non-probabilistic approaches for the treatment of uncertainty. Probabilistic approaches include frequentist [1,2] techniques, Bayesian methods [3,4], imprecise probabilities [5] and probability boxes [6]. Non-probabilistic methods include evidence theory [7–9], convex modeling of uncertainty [10], Zadeh's extension principle [11], fuzzy theory [12], etc. In these methods, several assumptions are made regarding the input uncertainty, and the system behavior model. Both these types of assumptions significantly affect output prediction, and hence decision making. This paper focuses on the assumptions involved in *input* uncertainty quantification, in the context of probabilistic approaches. Uncertainty in the system behavior model [13,14] has been studied by several researchers and is not the focus of this paper.

1.1. Probabilistic approach to sparse and imprecise data

Probabilistic methods operate by assigning probability distributions for the inputs based on available information, and then using an uncertainty propagation method such as Monte Carlo sampling, first-order reliability method (FORM), or

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second-order reliability method (SORM) to predict the output of the system under study. Even though these methods have achieved considerable attention in the literature [1], practical engineers still find it difficult to use these methods when only sparse and imprecise information about input variables is available. Consider a particular input quantity X that exhibits physical variability and is represented using a probability density function $f_X(x)$. Recent papers in the literature have pursued probabilistic methods to handle sparse and interval data [15–17]. Ferson et al. [15] and Zaman et al. [16] address interval data using a family of distributions; the former use a bounding p-box whereas the latter uses the Johnson family of distributions. Sankararaman and Mahadevan [17] use a likelihood-based method for constructing probability distributions using both sparse point and interval data. Two important issues are encountered in these methods: selection of the distribution type and estimation of the corresponding distribution parameters.

Conventionally, a distribution type (e.g. normal, lognormal, etc.) is assumed for X, and the parameters of this probability distribution (e.g. mean and standard deviation in the case of a normal distribution) are usually estimated based on techniques of statistical inference using observed data. When sufficient data is available, it may be reasonable to compute deterministic estimates of the parameters using frequentist techniques such as the method of moments [1], method of maximum likelihood [18], etc. It can be proved that, under some conditions, these deterministic estimates approach the true estimates as the data size approaches infinity [19]. When only finite data is available, there is uncertainty associated with the parameter estimates; the importance of this uncertainty in the distribution parameters increases especially when the data is sparse and/or imprecise. This uncertainty in the distribution parameters (which can be reduced with additional observational data) is one example of epistemic uncertainty. The topic of distribution parameter uncertainty has been studied by several researchers [20–22] and this has also been referred to as statistical uncertainty [1,2] or second-order uncertainty [23,24].

However, prior to the estimation of distribution parameters, a distribution type needs to be chosen for X; this choice of distribution type is another example of epistemic uncertainty because there might be a true underlying distribution type which may not be possible to identify due to the available sparse and imprecise data. Since this is representative of the form chosen to model the physical variability in X, the distribution type uncertainty can also be referred to as model form uncertainty. Note that this "model form" is not related to the form of the system behavior model through which the input uncertainty needs to be propagated.

The focus of this paper is to develop methods to quantify the uncertainty due to the distribution type assumption, or in other words, the uncertainty in the distribution type that arises due to sparse and imprecise data. Usually, when adequate data is available, the assumed distribution type can be verified by comparing against empirical distribution functions using statistical goodness-of-fit tests such as chi-square test [25] or Kolmogorov–Smirnov [26] test. Also, Anderson–Darling [27] and Cramer [28] tests are available for multi-variate distributions. In the presence of sparse and interval data, this approach may not be applicable because empirical distribution functions are not unique and are either bounded by a p-box [15] or represented using a family of distributions [16,17].

1.2. Distribution type uncertainty: preliminary concepts

This paper develops a likelihood-based methodology for the simultaneous quantification of the uncertainty in the distribution type and the distribution parameters of the quantity of interest. The proposed methodology is an extension of the authors' earlier work [17], where the distribution type was assumed to be known and therefore, the effect of distribution type uncertainty was not considered. In this paper, the methods of Bayesian model averaging [29,30], and Bayesian hypothesis testing [31,32] are investigated in order to include the effect of distribution type uncertainty in the likelihood-based methodology. The Bayesian model averaging approach is based on assigning weights to competing distribution types, and is applicable for comparing two or more models; the weights and the distribution parameters are estimated based on the available data. The Bayesian hypothesis testing approach computes the extent of support provided by data to the chosen distribution type and it can be used to assess competing models or to quantify the uncertainty regarding a given model. These Bayesian approaches have been previously used to validate system behavior models [14,33] which may be data-driven (regression or neural network) or physics-based (derived from first principles). The contribution of this paper is to use these methods in conjunction with the likelihood-based methodology in order to simultaneously quantify the uncertainty in the distribution type and distribution parameters.

Classical (frequentist) statistics addresses the uncertainty in the distribution parameters by estimating statistical confidence intervals for the distribution parameters [19]. In the context of classical statistics, the distribution parameter is deterministic but unknown, and hence it is not meaningful to talk about the probability distribution of the distribution parameter [4]. Hence, statistical confidence intervals should not be used for calculations such as uncertainty propagation, reliability analysis, etc. In contrast, the Bayesian approach treats probability distributions not as relative frequencies of occurrence but as states of belief or knowledge about the quantity of interest [3,4,34]. Hence, it allows the construction of probability distributions for the distribution parameters. This facilitates further probabilistic calculations, thereby enabling the application of the Bayesian model averaging and hypothesis testing methods.

Consider a variable X which is an input to a system behavior model. From the above discussion, there are three different types of uncertainty regarding this quantity: (1) physical variability (aleatory uncertainty); (2) distribution type (epistemic uncertainty); and (3) distribution parameters (epistemic uncertainty). Hence, the total uncertainty about X is the combination of all three types of uncertainty. When this input is propagated through a system model, all three types of uncertainty must be accounted for in a rigorous manner. If a Monte Carlo-based sampling technique is pursued, three loops of sampling are necessary, as shown in Fig. 1. If there is more than one possible distribution type, then the distribution type is treated as a discrete random variable and the distribution type is selected in the first (outermost) loop. In the second (inner) loop, a sample of the distribution parameter value is drawn. These two choices uniquely identify a probability density function for X. In the third (innermost) loop, samples of X are drawn.

Consider for the sake of illustration, a simple case where X could either be Lognormal or Weibull; thus, distribution type is a discrete variable with two possibilities. If it is lognormal, the distribution parameters are mean and standard deviation of $\ln(X)$. If it is Weibull, then the parameters are the scale parameter and shape parameter. If these distribution parameters are uncertain, then X can be represented using two families of distributions, Lognormal and Weibull, as shown in Fig. 2. The aim of this paper is to quantify (1) the extent to which the available data (sparse/imprecise) support the Lognormal vs. Weibull assumption and (2) the uncertainty in the parameters of these distributions.

Section 2 discusses fitting parametric distributions to sparse and imprecise data. Sections 3 and 4 present the Bayesian model averaging and hypothesis testing approaches to quantify the uncertainty in selecting a parametric distribution type. Section 5 discusses uncertainty propagation where the three types of uncertainty – physical variability, distribution type uncertainty, and distribution parameter uncertainty – in an input variable are propagated through a response function. Section 6 illustrates the proposed methods using two numerical examples – a mathematical example and an oscillator example – involving uncertainty propagation, considering both sparse and imprecise data.

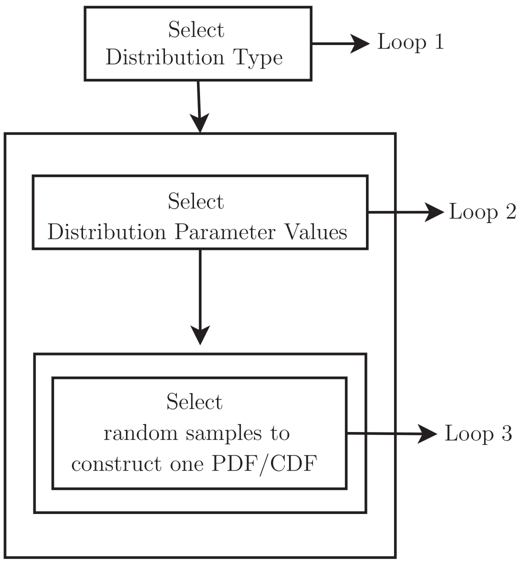

Fig. 1. Multiple loops of sampling.

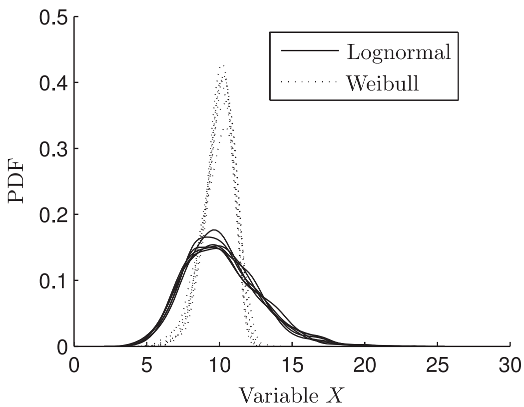

Fig. 2. Distribution type and distribution parameter uncertainty.

2. Fitting distributions to sparse and imprecise data

This section describes fitting probability distributions to sparse and imprecise data. In this section, the distribution type is assumed to be known, and therefore, parametric distributions are fitted to available data. This section provides only a summary of the approach; details can be found in Sankararaman and Mahadevan [17]. The case of unknown distribution type is considered in Sections 3–6.

Consider an input quantity X whose PDF type is known and the only unknown quantities are the distribution parameters. Let $f_X(x|\boldsymbol{P})$ denote the PDF. Note that this is a conditional distribution, i.e. conditioned on the realization of the distribution parameters $\boldsymbol{P}$, which need to be inferred from the evidence. If there is sufficient point-valued data, then there are several methods to estimate the distribution parameters $\boldsymbol{P}$. This section focuses on estimating the distribution parameters in the presence of sparse and imprecise data.

Assume that there are m point data x_i ($i=1$ to m) and n intervals $[a_i, b_i]$ ($i=1$ to n), available for an input X. If there are no point data, then $m=0$; similarly, if there are no interval data, then $n=0$. The principle of likelihood can be used to first estimate the parameters $\boldsymbol{P}$, and then to construct an unconditional probability distribution $f_X(x)$ with this information.

In the case of point data (x_i, $i=1$ to m), the likelihood function of $\boldsymbol{P}$ is [1,18]

$$L(\boldsymbol{P}) \propto \prod_{i=1}^{m} f_X(x = x_i|\boldsymbol{P}) \tag{1}$$

In the case of interval data (n intervals; $[a_i, b_i]$ ($i=1$ to n)), the likelihood of $\boldsymbol{P}$ can be constructed as [17,35]

$$L(\boldsymbol{P}) \propto \prod_{i=1}^{n} \int_{a_i}^{b_i} f_X(x|\boldsymbol{P})\, dx \tag{2}$$

Hence, the combined likelihood for both point data and interval data, assuming that the sources of these data are independent, can be constructed as [17,35]

$$L(\boldsymbol{P}) \propto \left[\prod_{i=1}^{m} f_X(x = x_i|\boldsymbol{P})\right]\left[\prod_{j=1}^{n} \int_{a_j}^{b_j} f_X(x|\boldsymbol{P})\, dx\right] \tag{3}$$

In addition to point and interval data, the procedure for the construction of the likelihood function can also easily accommodate noisy, incomplete and censored data.

2.1. Estimation of distribution parameters

The maximum likelihood estimate of $\boldsymbol{P}$ can be obtained by maximizing Eq. (3). Further, the uncertainty in the estimate of the distribution parameters can also be calculated using Bayes' theorem. Let $f_{\boldsymbol{P}}(\boldsymbol{P})$ denote the probability density of the parameters $\boldsymbol{P}$. Note that this may be a multivariate density function if there is more than one distribution parameter (for example, mean and standard deviation in the case of a normal distribution). By choosing a uniform prior distribution of $\boldsymbol{P}$ (over the domain of definition of $\boldsymbol{P}$), the posterior $f_{\boldsymbol{P}}(\boldsymbol{P})$ can be expressed in terms of the likelihood $L(\boldsymbol{P})$ calculated earlier in Eq. (3) as

$$f_{\boldsymbol{P}}(\boldsymbol{P}) = \frac{L(\boldsymbol{P})}{\int L(\boldsymbol{P})\, d\boldsymbol{P}} \tag{4}$$

The constant prior distribution gets canceled in the numerator and denominator.

The PDFs of the distribution parameters can be calculated using direct integration of Eq. (4). Alternatively, Markov Chain Monte Carlo sampling [36] methods such as Metropolis algorithm [37], Gibbs algorithm [38] or slice sampling [39] can be used to generate samples of the distribution parameters, without evaluating the constant of integration in Eq. (4). The method of slice sampling is used in this paper.

2.2. The predictive distribution

For a given realization of parameter $\boldsymbol{P}$, each model input X can be represented by a conditional PDF $f_X(x|\boldsymbol{P})$. If the distribution parameters ($\boldsymbol{P}$) are uncertain, as in this case, X is represented using a family of distributions, with each member of the family resulting from a particular realization of the distribution parameters.

The unconditional PDF for X, which includes both the variability in X and the uncertainty in the distribution parameters can be obtained as

$$f_X(x) = \int f_X(x|\boldsymbol{P}) f_{\boldsymbol{P}}(\boldsymbol{P})\, d\boldsymbol{P} \tag{5}$$

This unconditional PDF $f_X(x)$ has also been referred to as the predictive PDF [2,20] of X. Eq. (5) needs to be evaluated numerically and the resultant PDF is not parametric. Hence, the resultant PDF is not the same distribution type as the conditional $f_X(x|\boldsymbol{P})$. However, it includes both types of uncertainty—variability and distribution parameter uncertainty.

It is important to note that the integration in Eq. (5) does not involve the calculation of any performance function or a system behavior model (which is usually the more expensive calculation in practical problems of uncertainty propagation or reliability analysis). The integration in Eq. (5) only characterizes input uncertainty (combining both variability and distribution parameter uncertainty) before the "uncertainty propagation" stage, thereby leading to a single PDF for X. The advantage of this approach is that Eq. (5) allows the second and third loops of sampling in Fig. 1 to be collapsed into a single loop, thus enabling faster computation.

2.3. Extension to non-parametric distributions

The above described likelihood-based methodology requires the assumption of a distribution type for the quantity X. Sankararaman and Mahadevan [17] addressed this issue by developing a non-parametric approach that constructs the PDF using non-parametric interpolation methods (splines, Gaussian process) from sparse point and interval data. The non-parametric method does not assume any explicit distribution type or distribution parameters, and therefore represents all the three types of uncertainty – variability, distribution type uncertainty, and distribution parameter uncertainty – using a single probability distribution.

Since the present paper focuses on the quantification of distribution type uncertainty, non-parametric distributions will not be discussed in the rest of the paper. The following sections will extend the likelihood-based parametric approach to include distribution type uncertainty. The quantification of distribution type uncertainty will be integrated with the treatment of distribution parameter uncertainty, in order to simultaneously quantify both.

3. Bayesian model averaging approach

Consider a random variable X for which data is available. Let D denote the collection of all data, which comprises m point data x_i ($i=1$ to m) and n intervals $[a_i, b_i]$ ($i=1$ to n). The aim of the paper is to quantify the distribution type uncertainty in the probability distribution of X. The method of Bayesian model averaging is applicable when multiple competing distribution types are compared.

Consider N competing distribution types. The overall approach is to express the PDF $f_X(x)$ as a weighted sum of the densities of the competing distribution types. Let $f_X^k(x|\boldsymbol{\theta_k})$ denote the PDF of the kth competing distribution type ($k=1$ to N); in each PDF, $\boldsymbol{\theta_k}$ denotes the vector of distribution parameters, and $\boldsymbol{\theta} = \{\boldsymbol{\theta_k} : k=1 \text{ to } N\}$. The weights for each of the PDFs are given by $w=\{w_k\text{: } k=1 \text{ to } N\}$. Using Bayesian model averaging, the PDF of X can be expressed as

$$f_X(x|w,\boldsymbol{\theta}) = \sum_{k=1}^{N} w_k f_X^k(x|\boldsymbol{\theta_k}) \tag{6}$$

As $f_X(x|w,\boldsymbol{\theta})$ needs to be a valid PDF, the integral of this density function must be equal to unity, and hence,

$$\sum_{k=1}^{N} w_k = 1 \tag{7}$$

A likelihood-based estimation procedure similar to that in Section 2 is used here. The difference is that the combined likelihood of the weights and the distribution parameters, i.e. $L(\boldsymbol{\theta},w)$, is used now. Then the likelihood function is constructed as

$$L(\boldsymbol{\theta},w) \propto \left[\prod_{i=1}^{m} f_X(x=x_i|w,\boldsymbol{\theta})\right]\left[\prod_{j=1}^{n} \int_{a_j}^{b_j} f_X(x|w,\boldsymbol{\theta})\, dx\right] \tag{8}$$

This likelihood function can be maximized to obtain the maximum likelihood estimates of $\boldsymbol{\theta}$ and w. Further the uncertainty in the estimates can also be quantified using Bayesian inference, as in Eq. (4). Uniform priors bounded on [0,1] are chosen for $N-1$ weights and the Nth weight is calculated based on Eq. (7). Non-informative priors are chosen for the distribution parameters $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$. The prior distributions are multiplied by the likelihood function and then normalized to calculate the posterior distributions of $\boldsymbol{\theta}$ and w.

Two illustrations are presented below. The first example considers a large amount of data and two significantly different candidate distribution types. The second example considers a large amount of data and two candidate distribution types that are *not* significantly different from one another.

3.1. Illustration 1

Consider a case of 100 samples generated from an underlying normal distribution with mean and standard deviation equal to 100 units and 10 units respectively. Since the amount of data is large, it is easy to identify that the underlying distribution is, in fact, normal. However, this example is used only to demonstrate the Bayesian averaging method.

For the sake of illustration, assume that the two competing distribution types are normal ($N(\mu,\sigma)$) and uniform ($U(a,b)$), where μ and σ are the mean and standard deviation of the normal distribution, and a and b are the lower and upper bounds of the uniform distribution respectively. Since there are only two competing distributions, the weights are denoted as w (normal) and $1-w$ (uniform) respectively. The joint likelihood is evaluated for five quantities (w, μ, σ, a, and b), and the posterior distribution is estimated for each quantity using 10,000 samples from slice sampling [39]. The correctness of these posterior distributions can be easily verified since the samples were actually generated from a normal distribution N(100,10).

First, the PDF of the weight w is shown in Fig. 3. The estimated statistics/PDFs of distribution parameters are shown in Table 1 and Fig. 4.

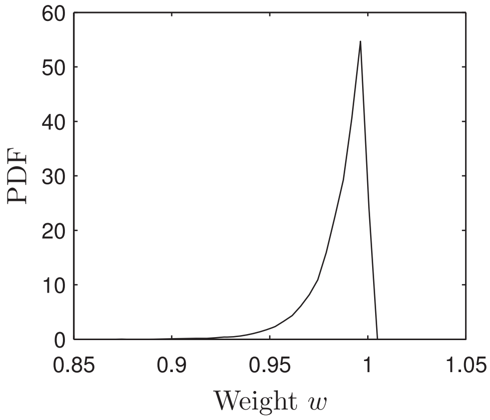

Fig. 3. PDF of weight w.

Table 1
Normal vs. uniform: results of Bayesian model averaging.

Quantity	Mean	Standard deviation	95% bounds
w	0.986	0.015	[0.949, 0.999]
μ	100.887	0.969	[99.078, 102.811]
σ	9.998	0.704	[8.752, 11.534]
a	18.193	9.584	[2.997, 43.800]
b	203.767	27.6065	[157.278, 239.324]

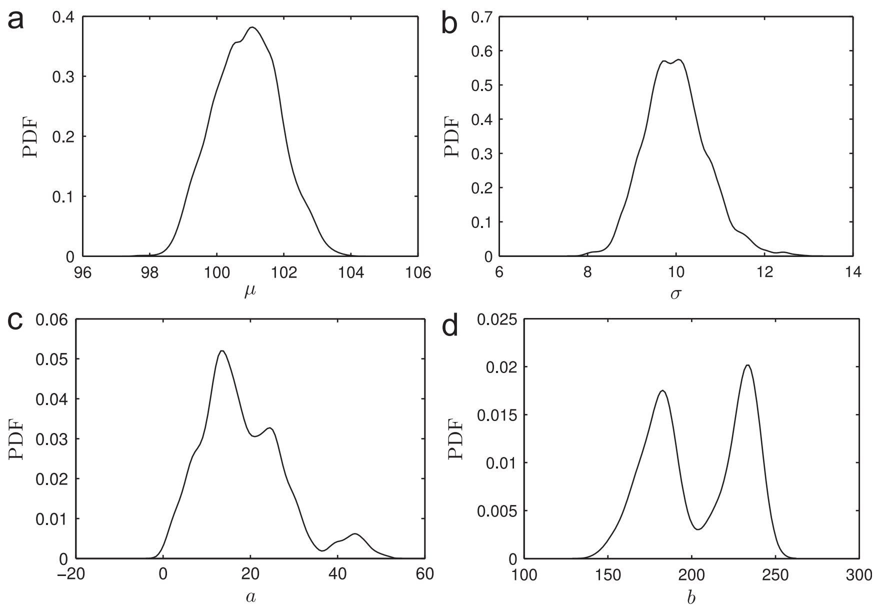

Fig. 4. PDFs of distribution parameters. (a) PDF of μ. (b) PDF of σ. (c) PDF of a. (d) PDF of b.

From Table 1, it can be clearly seen that the method isolates the data to come from a normal distribution. There is very high confidence in the result because the mean of w is very high; further, there is very little uncertainty about this conclusion. Also, the distribution parameters of the normal distribution are in good agreement with the actual values using which the data was simulated, and the uncertainty in the estimates of these distribution parameters is very small. Since the weight for the uniform distribution is very small, the distribution parameter estimates for the uniform distribution have high uncertainty.

The uncertainty in the estimate of the weight w is very low because of two reasons: (1) there is sufficient data to conclusively suggest a normal distribution and (2) the two competing distribution types, i.e. normal and uniform, are significantly different from each other.

It is obvious that, if there is only sparse data, then the uncertainty in the estimate of w will be high. However, even if there is sufficient data, it is hard to uniquely isolate one distribution type if the competing distribution types are not significantly different from one another, as shown next.

3.2. Illustration 2

Consider 100 samples generated from an exponential distribution with parameter $\mu = 1$, where μ is the mean. The PDF for this distribution is given by

$$f_X(x|\mu) = \frac{1}{\mu}\exp\left(-\frac{x}{\mu}\right) \tag{9}$$

For the sake of illustration, assume that the two competing distribution types are exponential and Rayleigh. While the former has one parameter (μ, the mean) as indicated in Eq. (9), the latter also has only one parameter (b, the mode of the distribution), and the Rayleigh PDF is given by

$$f_X(x|b) = \frac{x}{b^2}\exp\left(-\frac{x^2}{2b^2}\right) \tag{10}$$

Note that the exponential and Rayleigh distributions are not as significantly different from each other as the uniform and normal distributions. This is because both exponential and Rayleigh distributions can be viewed as special cases of the two-parameter Weibull distribution with shape parameters equal to one and two respectively. Since the Weibull distribution is commonly used to study time-dependent reliability, this example is of practical significance.

Similar to the previous example, the joint likelihood $L(w,\mu,b)$ is used to evaluate the posterior distributions of w, μ and b respectively. First, the PDF of the weight w is shown in Fig. 5 where w is the weight for the exponential distribution, and $1-w$ is the weight for the Rayleigh distribution.

The PDFs of the distribution parameter for each model-form (μ for exponential distribution and b for Rayleigh distribution) are shown in Fig. 6, and the numerical estimates are shown in Table 2.

The mean of w is about 0.75, which suggests a higher likelihood for the exponential distribution. However, there is significant uncertainty in w, leading to inconclusive distinction between the exponential and Rayleigh distributions. Also, the estimates of the distribution parameters suggest a higher likelihood for the exponential distribution, because μ in the exponential distribution has a much smaller uncertainty compared to b in Rayleigh distribution. That is, a "narrow" estimate of μ is sufficient to "explain" the available data whereas a "very wide" estimate of b is needed for the same. This is intuitive because the data actually originates from an exponential distribution. Also, the maximum likelihood estimate of μ is 1.0, which is exactly the same as the originally assumed value for μ used to generate the data.

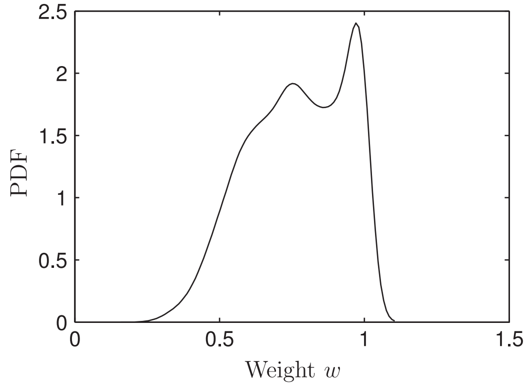

Fig. 5. PDF of weight w.

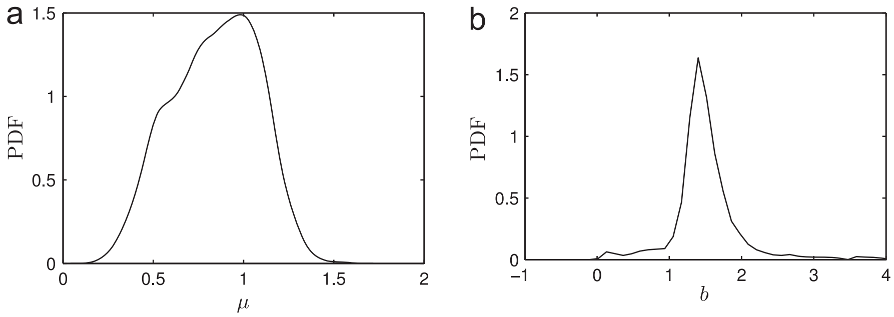

Fig. 6. PDFs of distribution parameters. (a) PDF of μ. (c) PDF of b.

Table 2
Exponential vs. Rayleigh: results of Bayesian model averaging.

Quantity	Mean	Standard deviation	95% bounds
w	0.746	0.158	[0.424, 0.988]
μ	0.840	0.239	[0.382, 1.255]
b	2.060	1.181	[0.561, 7.793]

3.3. Summary

This section investigated the use of the Bayesian model averaging approach to quantify the uncertainty in the distribution type and distribution parameters, thereby estimating both the confidence in a particular distribution type (through the mean value of w) and a measure of uncertainty in this confidence (through the standard deviation of w). One disadvantage of this approach is that it infers spurious interactions between competing distribution types while constructing the joint likelihood of weights and distribution parameters of all distribution types. As a result, this approach involves multi-dimensional integration; a significant amount of computational power may be required, if there are several competing distribution types. For example, if there were five competing distribution types, each with two distribution parameters, then the joint likelihood needs to be constructed for 14 quantities (four weights and 10 parameters), and a 14-dimensional integration is needed to quantify the distribution type uncertainty and estimate the distribution parameters.

The next section discusses the use of Bayesian hypothesis testing to quantify distribution type uncertainty; this approach provides a computationally efficient alternative and also directly computes the probability that the data supports a given distribution type.

4. Bayesian hypothesis testing approach

4.1. Two competing models: quantifying distribution type uncertainty

Consider two PDFs $f_X^1(x|\boldsymbol{\phi})$ and $f_X^2(x|\boldsymbol{\theta})$ that correspond to the two competing distribution types M_1 and M_2 respectively. In Bayesian hypothesis testing, M_1 and M_2 can be considered to be two competing hypotheses with prior probabilities $P(M_1)$ and $P(M_2)$ respectively. Using Bayes theorem, the posterior probabilities can be calculated based on the available data (D) as [14]

$$\frac{P(M_1|D)}{P(M_2|D)} = \frac{P(D|M_1)}{P(D|M_2)}\frac{P(M_1)}{P(M_2)} \tag{11}$$

The first term on the right hand side of Eq. (11) is referred to as the Bayes factor, denoted by B [40]:

$$B = \frac{P(D|M_1)}{P(D|M_2)} \tag{12}$$

The Bayes factor is the ratio of likelihoods of M_1 and M_2 and is a quantitative measure of extent of data support for model M_1 relative to the support for M_2. If $B > 1$, then the data D favors model M_1. Higher the Bayes factor, higher is the likelihood of the model M_1. In the absence of any prior preference between M_1 and M_2, assume equal prior probabilities, i.e. $P(M_1) = P(M_2) = 0.5$. Then, the posterior probabilities ($P(M_1|D)$ and $P(M_2|D)$) can be expressed in terms of the Bayes factor as

$$P(M_1|D) = \frac{B}{B+1}$$

$$P(M_2|D) = \frac{1}{B+1} \tag{13}$$

In order to implement this, the likelihood functions ($P(D|M_1)$ and $P(D|M_2)$) must be calculated. This is accomplished in two steps. In the first step, $P(D|M_1,\boldsymbol{\phi})$ is calculated using the data D available. Similar to the previous section, assume that m point data x_i ($i=1$ to m) and n intervals $[a_i, b_i]$ ($i=1$ to n) are available.

$$P(D|M_1,\boldsymbol{\phi}) \propto L(M_1,\boldsymbol{\phi}) = \prod_{i=1}^{m} f_X^1(x=x_i|\boldsymbol{\phi}) \prod_{j=1}^{n} \int_{a_j}^{b_j} f_X^1(x|\boldsymbol{\phi})\, dx \tag{14}$$

Similarly, $P(D|M_2,\boldsymbol{\theta})$ is also calculated. In the second step, these two quantities are used to calculate $P(D|M_1)$ and $P(D|M_2)$. Let $f_{\boldsymbol{\phi}}(\boldsymbol{\phi})$ denote the prior PDF of the distribution parameter $\boldsymbol{\phi}$. Using conditional probability, it follows that

$$L(M_1) \propto P(D|M_1) = \int P(D|M_1,\boldsymbol{\phi}) f_{\boldsymbol{\phi}}(\boldsymbol{\phi})\, d\boldsymbol{\phi} \tag{15}$$

If a uniform prior density is assigned for $\boldsymbol{\phi}$, then the above equation reduces to

$$L(M_1) \propto P(D|M_1) \propto \int P(D|M_1,\boldsymbol{\phi})\, d\boldsymbol{\phi} \tag{16}$$

Using Eq. (13), the posterior probability of model M_1, i.e. $P(M_1|D)$ can be calculated. Similar equations can be written for model M_2.

The evaluation of the above probabilities involves multi-dimensional integration; however, the number of dimensions is only equal to the number of distribution parameters for each individual distribution. In contrast, the Bayesian model averaging approach discussed earlier in Section 3 would require multi-dimensional integration with all weights and parameters together. Hence, Bayesian hypothesis testing is computationally more affordable in comparison with the Bayesian model averaging approach.

4.2. Extension to single and multiple distribution types

The case of two competing models was discussed above. This section extends the method to (1) addressing distribution type uncertainty in a single model and (2) quantifying the distribution type uncertainty for multiple models.

Consider the case when there is only one model M_1 and it is desired to calculate the distribution type uncertainty. This can be viewed as a hypothesis testing problem where the null hypothesis is that model M_1 is correct, and alternate hypothesis is that model M_2 is correct, where model M_2 is the opposite of model M_1. One possible approach is to choose model M_2 as a uniform distribution (non-informative). Hence, $f_X^2(x|\boldsymbol{\theta})$ is a uniform PDF; the PDFs of the lower and upper bounds are estimated based on the data and then "integrated out" to compute $P(M_1|D)$ and $P(M_2|D)$.

If there are more than two competing models, say n models, then the Bayes factor which was earlier defined as a ratio between two models can now be defined in terms of proportions as $P(D|M_1):P(D|M_2):P(D|M_3)\ldots P(D|M_n)$. Using equations analogous to those in the previous subsection, the probabilities $P(M_1|D)$, $P(M_2|D)$, $P(M_3|D)$ and so on until $P(M_n|D)$ can also be calculated.

4.3. Comparison with Bayesian model averaging: illustrations

First, consider the illustration in Section 3.1. Using the Bayes factor, the probabilities $P(M_1)$ and $P(M_2)$ are found to be 1 and 0 (up to 5th decimal place), thereby isolating the normal distribution with almost 100% confidence. This behavior is very similar to that in the Bayesian model averaging method, due to (1) sufficient data to uniquely identify the normal distribution; (2) significant difference between the two competing distribution types, normal and uniform.

Similar to the Bayesian model averaging (BMA) procedure, the PDFs of the distribution parameters using the Bayesian hypothesis testing (BHT) approach are also quantified, as shown in Fig. 7. Note that the results from both the approaches are shown for the sake of comparison. Since the normal distribution has been isolated with almost 100% confidence, the distribution parameters are shown only for normal distribution.

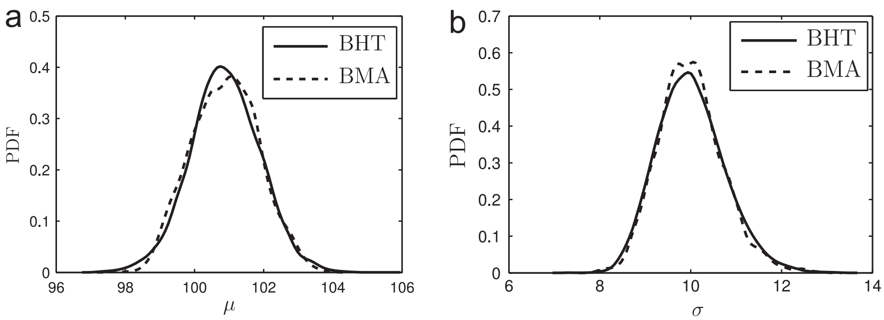

Fig. 7. PDFs of distribution parameters (Illustration 1 in Section 3.1). (a) PDF of μ. (b) PDF of σ.

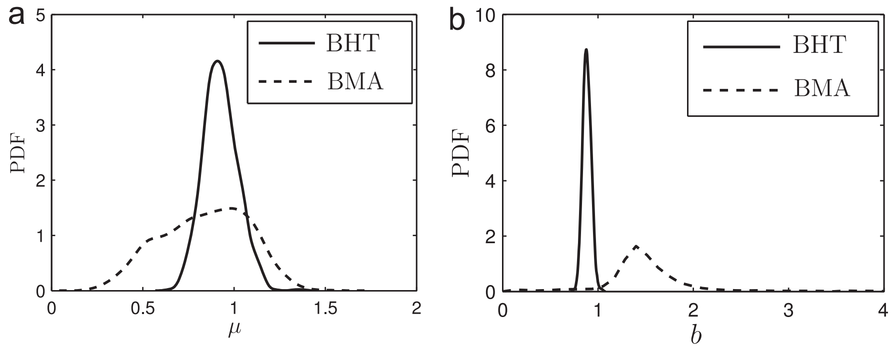

Fig. 8. PDFs of distribution parameters. (a) PDF of μ. (b) PDF of b.

Note that there is no significant difference between the PDFs of the distribution parameters estimated through the Bayesian hypothesis testing route or the model averaging route. This is expected because both the methods completely isolate the distribution type to normal distribution and hence, the PDFs of the distribution parameters are expected to be the same. The difference between the two methods is only in the quantification of the distribution type uncertainty and the computational effort.

Now, consider the illustration in Section 3.2, where the competing distributions – exponential(M_1) and Rayleigh (M_2) – are not significantly different from each other. The posterior probabilities $P(M_1|D)$ and $P(M_2|D)$ are estimated using Eq. (13) to be 1 and 0 respectively. Further, the PDFs of the distribution parameters, i.e. μ for the exponential distribution and b for the Rayleigh distribution are also quantified, as shown in Fig. 8.

There are two important observations. First, these results are considerably different from the Bayesian model averaging results. Second, the uncertainty (measured in terms of standard deviation) in the results from Bayesian hypothesis testing is much smaller than that from the model averaging approach.

This behavior is due to the conceptual differences between the two approaches. The Bayesian model averaging approach considers the joint likelihood of weights and parameters of all distribution types, thereby assuming interactions between all the parameters (where there is none). In contrast, the hypothesis testing approach only considers the joint likelihood of all parameters of a single distribution type and does not include interactions across multiple distribution types. As a result, the estimation of μ in the hypothesis testing approach is completely independent of b; on the contrary, these two parameters are estimated simultaneously in the model averaging approach. The results of Bayesian hypothesis testing have smaller uncertainty because fewer parameters are estimated with the same amount of data.

These differences were not seen in the first numerical example because the normal distribution and the uniform distribution are significantly different from each other and the data wholly supported the normal distribution; whereas the exponential and Rayleigh distributions are not.

4.4. Summary

The Bayesian hypothesis testing approach quantifies the distribution type uncertainty through the posterior probability ($P(M_1|D)$) which is deterministic in contrast with the model averaging approach which calculates a stochastic weight (w). It is clear that the Bayesian model averaging and Bayesian hypothesis testing methods are based on different assumptions; they are conceptually different and caution must be exercised while comparing the results of these methods. From the perspective of computational efficiency, it may be advantageous to use Bayesian hypothesis testing.

The Bayesian hypothesis testing method can also be used when the PDFs of the distribution parameters of two competing distribution types are readily available. For each realization of distribution parameter values, the Bayes factor is calculated, thereby leading to the PDF of the Bayes factor [33]. This approach is significantly different from the concern in this paper, where the probability that the model is correct and the PDF of the corresponding distribution parameters are estimated simultaneously using the available data, thereby leading to a single Bayes factor value which is easier for the purpose of decision making.

5. Uncertainty propagation through a system model

Consider the case where the quantity X is an input to a mathematical model ($Y=g(X)$), and all three types of uncertainty – physical variability, distribution type and distribution parameters – in X need to propagate through the system model to compute the uncertainty in the response Y. This section discusses the various issues in such uncertainty propagation and numerical implementation of the uncertainty propagation.

5.1. Propagation using Bayesian model averaging

In the Bayesian model averaging approach, the PDFs of w, $\boldsymbol{\phi}$, and $\boldsymbol{\theta}$ are all calculated simultaneously. In other words, the joint likelihood of these quantities is used to calculate the individual marginal PDFs. Based on Eq. (6), a given realization of w, $\boldsymbol{\phi}$, and $\boldsymbol{\theta}$ values lead to a particular PDF $f_X(x|w,\boldsymbol{\phi},\boldsymbol{\theta})$. Let $F_X(x|w,\boldsymbol{\phi},\boldsymbol{\theta})$ denote the corresponding CDF. For multiple values of w, $\boldsymbol{\phi}$, and $\boldsymbol{\theta}$. there exists a family of PDFs for X. Each PDF can be propagated through the above response function, and a family of PDFs for Y can be calculated.

However, in practice, it may not be possible to directly invert the CDF $F_X(x|w,\boldsymbol{\phi},\boldsymbol{\theta})$; thus a composite method is used [30]. For a set of sampled values of w, $\boldsymbol{\phi}$, and $\boldsymbol{\theta}$, this CDF can be inverted numerically. A uniform random number on [0, 1] is drawn. If this number is less than the sampled value of w, then a random sample of X is selected from the PDF $f_X^1(x|\boldsymbol{\phi})$; else a random sample of X is drawn from the PDF $f_X^2(x|\boldsymbol{\theta})$. Multiple such samples of X correspond to the PDF $f_X(x|w,\boldsymbol{\phi},\boldsymbol{\theta})$. This procedure is repeated for multiple samples of w, $\boldsymbol{\phi}$, and $\boldsymbol{\theta}$ to generate a family of distributions for X. Note that this procedure is different from the algorithm in Fig. 1; here, both the distribution type and distribution parameters are sampled at the same level, whereas in Fig. 1, they were sampled one within the other.

For the purpose of uncertainty propagation, the family of distributions approach may be computationally expensive because it needs two Monte Carlo loops, one within the other. Therefore, the family of distributions is replaced with a single, unconditional (predictive) PDF $f_X(x)$ by integrating over w, $\boldsymbol{\phi}$, and $\boldsymbol{\theta}$, as

$$f_X(x) = \int f_X(x|w,\boldsymbol{\phi},\boldsymbol{\theta}) f_w(w|D) f_{\boldsymbol{\phi}}(\boldsymbol{\phi}|D) f_{\boldsymbol{\theta}}(\boldsymbol{\theta}|D)\, dw\, d\boldsymbol{\phi}\, d\boldsymbol{\theta} \tag{17}$$

The above integral can be numerically evaluated using sampling. Generate *one* sample of X for *one* sample of w, $\boldsymbol{\phi}$, and $\boldsymbol{\theta}$. Repeat the procedure for multiple samples of w, $\boldsymbol{\phi}$, and $\boldsymbol{\theta}$, thereby leading to multiple samples of X. This sampling procedure may be referred to as single-loop sampling. The resultant unconditional, predictive PDF includes all three types of uncertainty – variability, distribution type and distribution parameter – and can be used for uncertainty propagation through the system behavior model $Y = g(X)$.

It may appear that the use of the unconditional, predictive PDF may lead to loss of information regarding the individual contributions of the three aforementioned types of uncertainty. It may be desirable to assess their individual contributions. Currently, this issue has been addressed only qualitatively through graphical visualization, as seen earlier in Fig. 2. Future research needs to address the rigorous quantification of the individual contributions of the three types of uncertainty.

5.2. Propagation using Bayesian hypothesis testing

While the Bayesian model averaging approach leads to a stochastic weight w, the Bayesian hypothesis testing approach leads to a deterministic posterior probability $P(M_1|D)$. Hence, obtaining the family of distributions is simpler than in the Bayesian model averaging approach because the uncertainty in the weight is not considered. However, the order of sampling is different. First, a uniform random number on [0, 1] is drawn. If this uniform random number is less than $P(M_1|D)$, then a random sample of $\boldsymbol{\phi}$ is selected and multiple samples of X are drawn from the PDF $f_X^1(x|\boldsymbol{\phi})$; this procedure is repeated for several samples of $\boldsymbol{\phi}$. If the uniform random number is greater than $P(M_1|D)$, then a random sample of $\boldsymbol{\theta}$ is selected and multiple samples of X are drawn from the PDF $f_X^2(x|\boldsymbol{\theta})$; this procedure is repeated for several samples of $\boldsymbol{\theta}$. This algorithm is, in fact, exactly the same as that in Fig. 1 and leads to a family of PDFs similar to that in Fig. 2.

As stated earlier, the family of PDFs approach is computationally expensive for the purpose of uncertainty propagation. Hence, a simultaneous sampling (single loop) approach, similar to that in Section 5.1 is used to construct a single, unconditional predictive PDF of X. Further, the unconditional, predictive PDF can be used for the purpose of uncertainty propagation through the system behavior model $Y = g(X)$.

6. Numerical example: sparse and imprecise data

This section discusses a numerical example, where sparse and imprecise data are available on an input quantity and it is desired to quantify the distribution type uncertainty in this quantity; further, the distribution parameter uncertainty is also quantified. The effects of all three types of uncertainty – variability, distribution type uncertainty, and distribution parameter uncertainty – on a response quantity are also computed.

This numerical example was developed (Challenge Problem A) in the Sandia Epistemic Uncertainty Workshop [41], where an uncertainty propagation problem with uncertain inputs was considered. There are two inputs a and b, and the model output is given by

$$y = (a+b)^a \tag{18}$$

Both a and b are described using sparse and imprecise data in this numerical example; assume that three intervals are available for a as ([0.5, 0.7], [0.3, 0.8], [0.1, 1.0]), and one point value (0.6) and three intervals ([0.4, 0.85], [0.2, 0.9], [0.0, 1.0]) are available for b. In the context of probabilistic representation, the interest here is to quantify the distribution type uncertainty in a and b respectively, and then propagate this distribution type uncertainty to quantify its effect on y.

Assume that the two competing distribution types are normal and uniform distributions for both a and b. The uncertainty in the distribution types are quantified using both the methods—Bayesian model averaging and Bayesian hypothesis testing.

6.1. Bayesian model averaging

For the quantity a, model M_1 is chosen as N(μ_a,σ_a), and model M_2 is chosen as U(L_a,U_a). Similarly, for the quantity b, model M_1 is chosen as N(μ_b,σ_b), and model M_2 is chosen as U(L_b,U_b). Let w_a and w_b denote the weights assigned to the normal distribution for the variables a and b respectively. Then, $1-w_a$ and $1-w_b$ denote the weights assigned to the uniform distribution for the variables a and b respectively. For the sake of illustration, the prior distribution for all the distribution parameters is chosen to be uniform on the interval [0, 1].

The PDFs of all the above quantities are estimated using the available data, and the mean values, the standard deviations, maximum likelihood estimates, and 95% bounds are shown in Table 3.

The results in Table 3 are difficult to interpret for a number of reasons, the primary reason being that all the estimates have very high degree of uncertainty (indicated by standard deviation). This happens because the method tries to estimate five parameters simultaneously using a small data set (three for a and four for b). As a result, the 95% bounds are too large to be useful. The PDFs of the weights w_a and w_b are almost uniform, suggesting that even the maximum likelihood estimates may not be useful. Also, consider the uniform distribution estimated for a; the estimates of the lower and higher bounds are so close but with very high standard deviations that it is difficult to derive any usefulness from such results.

Due to the large uncertainty in the input, further uncertainty propagation analysis is not useful. Instead, the Bayesian hypothesis testing approach is investigated next. Note that the hypothesis testing approach does not estimate more than two parameters simultaneously, and hence is expected to produce results that have less uncertainty.

6.2. Bayesian hypothesis testing

Using the Bayesian hypothesis testing approach proposed in Section 4, the probabilities $P(M_1|D)$ and $P(M_2|D)$ can be directly calculated for both a and b. Then, the PDFs of the distribution parameters (μ_a and σ_a for normal, and L_a and U_a for uniform) can also be calculated. The results of the distribution parameter estimation are shown in Table 4. Note that the estimation of the parameters of the normal distribution is totally independent of the estimation of the parameters of the uniform distribution, for both the variables a and b. However, this was not the case in the Bayesian model averaging approach.

Table 3
Bayesian model averaging: results.

Variable	Distribution type	Quantity	Mean	Standard deviation	95% bounds
a	Normal	w_a	0.36	0.23	[0.04, 0.76]
		μ_a	0.50	0.30	[0.03, 0.93]
		σ_a	0.20	0.25	[0.03, 0.75]
	Uniform	$1-w_a$	0.64	0.23	[0.01, 0.96]
		L_a	0.73	0.17	[0.42, 0.91]
		U_a	0.86	0.13	[0.61, 0.99]
b	Normal	w_b	0.58	0.29	[0.05, 0.97]
		μ_b	0.57	0.15	[0.23, 0.84]
		σ_b	0.35	0.53	[0.01, 1.34]
	Uniform	$1-w_b$	0.42	0.29	[0.03, 0.95]
		L_b	0.37	0.19	[0.04, 0.62]
		U_b	0.67	0.25	[0.29, 0.95]

Table 4
Bayesian hypothesis testing results.

Variable	$P(M\|D)$	Quantity	Mean	Standard deviation	95% bounds
a	Normal	μ_a	0.57	0.16	[0.16, 0.89]
	0.32	σ_a	0.23	0.20	[0.01, 0.80]
	Uniform	L_a	0.41	0.16	[0.05, 0.65]
	0.68	U_a	0.74	0.12	[0.54, 0.97]
b	Normal	μ_b	0.60	0.12	[0.30, 0.89]
	0.28	σ_b	0.17	0.15	[0.01, 0.64]
	Uniform	L_b	0.43	0.15	[0.08, 0.60]
	0.72	U_b	0.74	0.11	[0.60, 0.97]

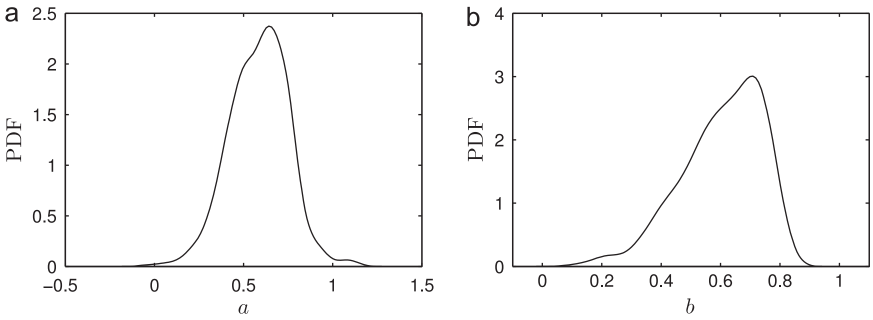

Fig. 9. PDFs of model inputs. (a) PDF of a. (b) PDF of b.

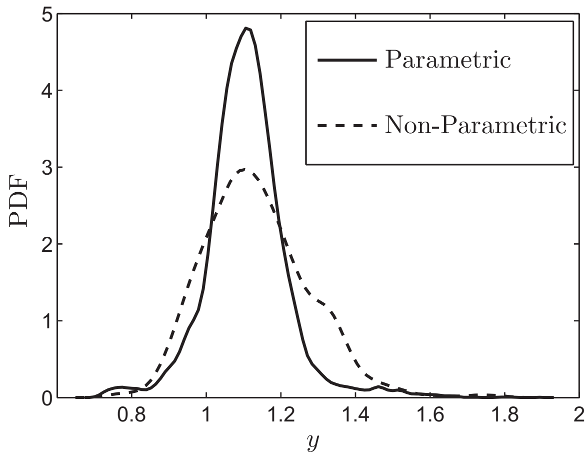

Fig. 10. PDF of model output y.

Once the uncertainty in the distribution type and the distribution parameters are estimated, then a and b are represented using a single, unconditional PDF each. The calculation of this unconditional PDF is using the simultaneous sampling approach explained earlier in Section 5.2. This single PDF accounts for physical variability, distribution type uncertainty and distribution parameter uncertainty, and hence renders the uncertainty propagation analysis efficient.

The unconditional PDF of a and the unconditional PDF of b is then propagated through Eq. (18) to calculate the PDF of Y. The PDFs of a and b are shown in Fig. 9.

Using simple Monte Carlo simulation, the PDF of y is then calculated and shown in Fig. 10. Since this PDF is calculated from multiple competing parametric PDFs, it is indicated as the parametric PDF. It accounts for all sources of uncertainty in the inputs—physical variability, distribution type uncertainty and distribution parameter uncertainty. Further, Fig. 10 also shows the result using the non-parametric approach (Section 2.3) by constructing non-parametric distributions for a and b.

6.3. Discussion

To begin with it is acknowledged that there is no unique correct or wrong answer to problems involving interval uncertainty [41,17]. Different researchers have pursued different approaches to tackle such problems and this paper presents one effective methodology for the analysis of interval uncertainty. It is seen that these different methods have led to comparable solutions, almost similar to one another. Kozine and Utkin [42], De Cooman and Troffaes [43], Ferson and Hajagos [6], and Zaman et al. [44] used imprecise coherent probabilities, natural extension rule in conjunction with imprecise probabilities, probability box (p-box), and Johnson family of distributions respectively in order to calculate the bounds on the output y. These approaches resulted in different, but comparable intervals, i.e. [0.9, 1.5], [1.0, 1.2], [0.8, 1.6], and [0.7, 2] respectively. The methodology proposed in this paper results in a probability distribution, where the distribution type uncertainty and distribution parameter uncertainty are individually quantified. It is also observed that the 90% bounds of this probability distribution ([0.9, 1.3]) is comparable with the solutions from the above methods.

Ferson et al. [41] mention four possible reasons for the observed discrepancies among the answers: (i) nesting (due to difference in approaches, one result may be nested in others), (ii) differences in truncation, i.e. whether or where the distributions were truncated to finite ranges, (iii) numerical approximation error and (iv) different representations of independence. While the solutions from different methodologies are similar, the proposed methodology has a few advantages. It is probabilistic, making it possible to use well-established uncertainty propagation methods such as

Monte Carlo simulation, FORM, SORM, etc. This can provide savings in computational effort, since FORM and SORM typically involve 10–20 evaluations of the system response function, and efficient sampling techniques (importance sampling, adaptive sampling, etc.) are available within Monte Carlo simulation. Further, different kinds of data can be combined and integrated into a single PDF, thereby making the uncertainty representation and propagation simple and straightforward.

7. Oscillator response prediction

This section presents the second challenge problem (Challenge Problem B) from the Sandia Epistemic Uncertainty Workshop [41]. A simple linear oscillator given by a mass–spring–damper system is acted on by a forcing function $Y\cos(\omega t)$, as shown in Fig. 11.

The equation of motion of the mass is given by

$$m\ddot{x}+c\dot{x}+kx=Y\cos(\omega t) \tag{19}$$

where m is the mass of the cart, k is the spring constant of the linear spring, and c is the damping coefficient of the linear damper. The amplitude and frequency of the force excitation are given by Y and ω respectively. The initial conditions for Eq. (19) are precisely known to be $x(t=0)=0$ and $\dot{x}(t=0)=0$. The analytical solution for the response, i.e. displacement x, can be computed continuously as a function of time [41].

The objective is to compute the uncertainty in the steady-state magnification factor (D) using data available on the parameters m, k, c, Y, and ω. These parameters are assumed to be independent of each other, and the steady-state magnification factor is computed as

$$D=\frac{k}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}} \tag{20}$$

The data available for the parameters are tabulated in Table 5.

The ultimate goal is to represent each quantity using a single PDF, which is computationally affordable for the purpose of uncertainty propagation through Eq. (20), in order to compute the uncertainty in D. As seen in Table 5, this example considers different types of data situations.

1. Precise known probability distribution for m: Since the probability distribution is well-defined, it can be directly included in uncertainty propagation.
2. Multiple interval data on each of the distribution parameters of the triangular distribution for k: Use the non-parametric approach [17] to compute the PDFs of k_{min}, k_{mode}, and k_{max}, and then calculate a single, unconditional PDF for k which can then be included in uncertainty propagation.

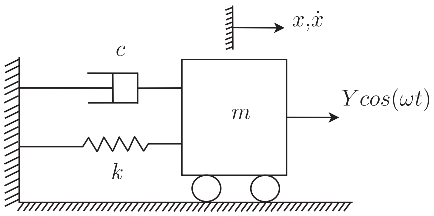

Fig. 11. Mass–spring–damper system acted on by a forcing function.

Table 5
ODE challenge problem: data.

Symbol	Quantity	Details	Numerical values
m	Mass	Triangular distribution, m_{min}, m_{mode}, m_{max}	$m_{min}=10$ $m_{mode}=11$ $m_{max}=12$
k	Spring	Triangular distribution, k_{min}, k_{mode}, k_{max}	$k_{min}=[90,100]$, $[80,110]$, $[60,120]$ $k_{mode}=[150,160]$, $[140,170]$, $[120,180]$ $k_{max}=[200,210]$, $[200,220]$, $[190,230]$
c	Damping	Multiple intervals	[5, 10], [15, 20], 25
ω	Frequency	Triangular distribution, ω_{min}, ω_{mode}, ω_{max}	$\omega_{min}=[2,2.3]$ $\omega_{mode}=[2.5,2.7]$ $\omega_{max}=[3,3.5]$

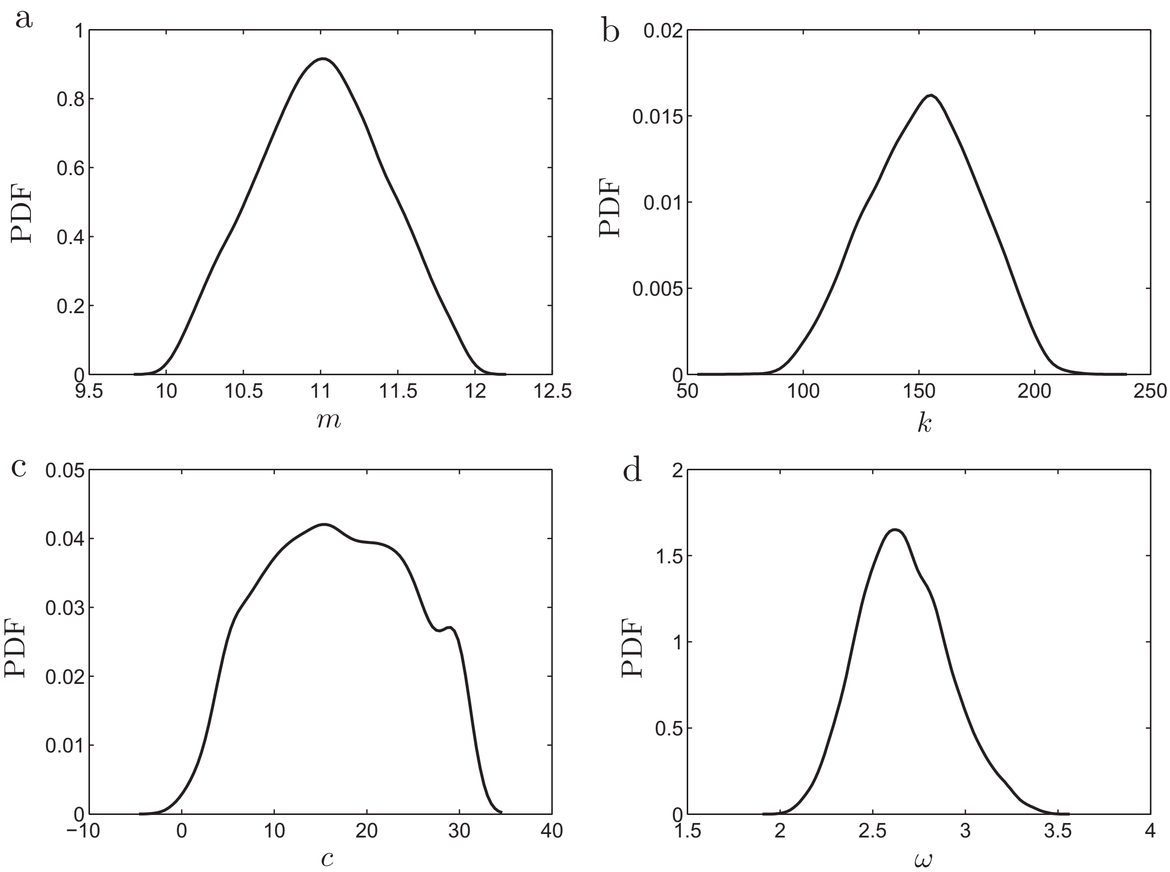

Fig. 12. PDFs of ODE parameters. (a) PDF of m. (b) PDF of k. (c) PDF of c. (d) PDF of ω.

3. Multiple interval data on c, leading to distribution type and parameter uncertainty: Use the Bayesian hypothesis testing approach using competing normal (M_1) and uniform (M_2) probability distributions. The Bayesian model averaging approach is not used since it leads to a large uncertainty in the estimates of the distribution parameters. Once the uncertainty in the distribution type ($P(M_1)=0.43$ and $P(M_2)=0.57$) and distribution parameters are estimated, then the method in Section 5.2 is used to construct a single PDF for c.
4. Single interval data on each of the distribution parameters of the triangular distribution for ω: Approximate each single interval with a uniform PDF, and compute a single unconditional PDF for ω, which can be used in uncertainty propagation.

The PDFs of parameters of the ordinary differential equation, i.e. stiffness k, mass m, damping c, and frequency ω are calculated as explained above and shown in Fig. 12. The uncertainty in the steady-state magnification factor, i.e. D is calculated by propagating the PDFs in Fig. 12 through Eq. (20) using Monte Carlo simulation, and the resultant PDF of D is shown in Fig. 13.

While the proposed methodology calculates a PDF for D ([1.35, 2.63] being the 90% bounds), Ferson and Hajagos [6] and Zaman et al. [44] calculated [1.17, 3.72] and [0.82, 1.89] respectively. As explained in the previous numerical example in Section 6, the proposed method not only provides comparable solution but also increases the computational efficiency by propagating a single, unconditional distribution for each quantity whereas the other methods [6,44] use double-loop Monte Carlo analysis for uncertainty propagation. Therefore, the proposed likelihood-based method provides a computationally efficient alternative for the treatment of distribution type and distribution parameter uncertainty.

8. Conclusion

This paper proposed a methodology to explicitly quantify the distribution type uncertainty while fitting parametric probability distributions to sparse and interval data. Two methods were developed to quantify the uncertainty due to the distribution type assumption. The first method was based on Bayesian model averaging which assigns weights to the competing distribution types, and simultaneously estimates the weights and the distribution parameters for all the competing distribution types. This method results in spurious interactions between competing distribution types, which

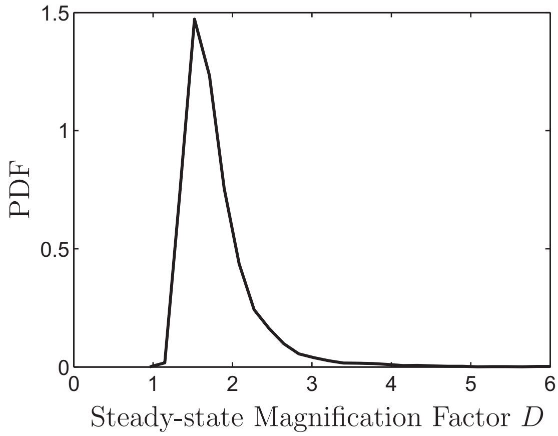

Fig. 13. PDF of *D*.

also results in multi-dimensional integration that considerably increases the computational effort. On the other hand, Bayesian hypothesis testing-based method directly computes the probability that the available data supports a particular distribution type. This method also uses multi-dimensional integration; however, the number of dimensions in each integration is limited to the number of distribution parameters in the individual distribution. Hence, the method provided a computationally efficient alternative to the Bayesian model averaging method.

In the context of uncertainty propagation, both methods are essentially computing a weighted average of the competing distribution types. However, in Bayesian model averaging, the weights are stochastic whereas in Bayesian hypothesis testing, the posterior probabilities are deterministic. As a result, Bayesian model averaging requires sampling the weights, in addition to the distribution parameters.

In the presence of sparse and imprecise data, the Bayesian averaging method was shown to yield results that have large uncertainty and may be difficult to interpret, whereas the hypothesis testing approach produced results that facilitates easier understanding and interpretation. In addition to the distribution type uncertainty, the distribution parameter uncertainty was also calculated.

The unconditional, predictive distribution is computationally efficient for uncertainty propagation; however, it combines the effect of physical variability, distribution type uncertainty, and distribution parameter uncertainty. In some applications, it may be desirable to separate the individual contributions of each source of uncertainty. Past research has addressed this issue only through graphical visualization. A quantitative approach needs to be developed in future work; such an analysis is useful in resource allocation for data collection, since physical variability is irreducible whereas epistemic uncertainty is reducible with additional data.

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