3.12 If random variables X and Y are independent and exponentially distributed, and

$$U = X + Y, \quad V = \frac{X}{X + Y},$$

determine the joint probability density function of U and V. Are U and V independent?

3.13 Let $Z = A\cos(\omega t + X)$. If A and X are independent random variables and X is uniformly distributed in the interval $(-\pi, \pi)$, show that

$$p(z) = \int_{-\infty}^{-|z|} \frac{p(a)}{\pi(a^2 - z^2)^{1/2}} da + \int_{|z|}^{\infty} \frac{p(a)}{\pi(a^2 - z^2)^{1/2}} da.$$

3.14 If random variables X and Y are independent with

$$p(x) = \frac{x}{a^2}\exp\left(-\frac{x^2}{2a^2}\right), \quad p(y) = \frac{y}{b^2}\exp\left(-\frac{x^2}{2b^2}\right),$$

and $Z = X/Y$, show that

$$p(z) = \begin{cases} \dfrac{2a^2}{b^2}\dfrac{z}{(z^2 + a^2/b^2)}, & z \geq 0 \\ 0, & \text{otherwise,} \end{cases}$$

$$P[X \leq hY] = \frac{h^2}{h^2 + a^2/b^2}, \quad h > 0.$$

3.15 X_1 and X_2 are independent and uniformly distributed random variables in the range $[0, 1]$. If

$$Y_1 = (-2\ln X_1)^{1/2}\cos 2\pi X_2, \quad Y_2 = (-2\ln X_1)^{1/2}\sin 2\pi X_2,$$

Show that Y_1 and Y_2 are $N(0, 1)$.

3.16 If X: $N(\mu, \sigma)$, show that

$$E[X^n] = E[X^{n-1}] + (n - 1)\sigma^2 E[X^{n-2}].$$

4

Theory of Random Processes

4.1 Introduction

The theory of random processes has evolved as a generalization of the concept of random variables. In the preceding chapter we saw that a random variable can be used to model random events for which the outcomes of repeated trials are a real or complex number. In many problems the outcome of a trial is not a number but a function of one or more parameters, such as time or space or both. In such cases, the outcome of each trial is called a *realization* or a *sample function* and the collection of all possible sample functions is called the *ensemble* of the *random process*. For a fixed value of the parameters, a random process is a random variable, and therefore it may be looked upon as a parametered family of random variables. Figure 4.1 shows the realizations of a random process $X(t)$, where t is the parameter—say, time. The superscript $j = 1, 2, \ldots$ in $x^j(t)$ represents the jth realization of the random process $X(t)$. At fixed times $t = t_1$ and t_2, $X(t_1)$ and $X(t_2)$ are random variables and $x^j(t_1)$ and $x^j(t_2)$ their jth realizations.

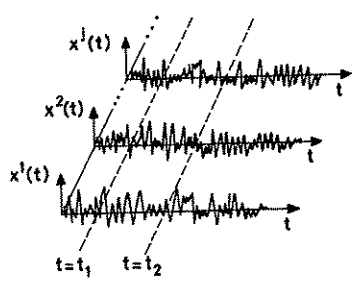

Figure 4.1 Ensemble of a random process.

The random processes have found increasing application as models of a large class of natural phenomena. Some examples of the phenomena modeled as random processes are

1. particles in suspension undergoing brownian motion as a function of time,
2. ground motion at a point during earthquakes as a function of time,
3. unevenness of a road surface as a function of distance along the center line, and
4. the pressure field due to jet noise as a function of both the time and space coordinates.

In the literature, random processes are also called *random functions*, *stochastic processes*, and *time series* if the indexing parameter is time. (Crandall and Mark, 1963, have given a lucid description of the nature of random processes and random vibration.)

4.2 Definition and Probability Description

4.2.1 Definition

A random process $X(\omega, t)$ is defined as a set function of two arguments $\omega \in \Omega$ and $t \in T$, where Ω is the sample space of the family of random variables $X(\cdot, t)$ and T is the indexing set of the parameter t.

A random process $X(\omega, t)$ may be described in two different ways. It may be treated as a family of random variables $\{X(\cdot, t): t \in T\}$ or as a set of functions $\{X(\omega, \cdot): \omega \in \Omega\}$ on T. If the indexing set T is finite, a random process is a random vector; it is a random sequence if T is countably infinite. A random process is said to be *discrete* or *continuous* depending on whether it is a family of discrete or continuous random variables. Thus a random process may be classified into one of the following four categories:

1. continuously parametered continuous random processes;
2. continuously parametered discrete random processes;
3. discretely parametered continuous random processes; and
4. discretely parametered discrete random processes.

In the sequel the argument ω is dropped and the random process $X(\omega, t)$ is denoted $X(t)$. Many random processes are characterized by more than one indexing parameter. Such processes are called *multiparametered random processes*. For example, $X(t, \bar{s})$, $t \in T$, $\bar{s} \in \bar{S}$, may describe the random process model of the pressure field due to jet noise that varies randomly with time t and space coordinates $\bar{s}$ over the domains T and $\bar{S}$, respectively.

4.2.2 Probability Description

Since a random process $X(t)$ reduces to a set of random variables at fixed instants of time $t = t_1, t_2, \ldots, t_n \ldots$, its probability structure may be defined by a hierarchy of joint probability density functions:

$$\begin{aligned} &p(x_1, t_1), \\ &p(x_1, t_1; x_2, t_2), \\ &\vdots \\ &p(x_1, t_1; x_2, t_2; \ldots; x_n, t_n), \\ &\vdots \end{aligned} \tag{4.1}$$

where $x_i = x(t_i)$. In (4.1), the definition of the probability density function of a random variable [equation (3.5)], is extended to include the effect of the parameter t, so that

$p(x_1, t_1)dx_1$ = probability that $X(t)$ lies in the interval $(x_1, x_1 + dx_1)$ at time t_1,

$p(x_1, t_1; x_2, t_2)$ = joint probability that $X(t)$ lies in the interval $(x_1, x_1 + dx_1)$ at time t_1, and in the interval $(x_2, x_2 + dx_2)$ at time t_1,

$\vdots$

Each probability density function must satisfy the following conditions:

$$\begin{aligned} &p(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) \geq 0 \text{ and is symmetric in} \\ &x_1, t_1; x_2, t_2; \ldots; x_n, t_n; \\ &p(x_1, t_1; x_2, t_2; \ldots; x_m, t_m) = \\ &\underbrace{\int \cdots \int}_{n-m} p(x_1, t_1; x_2, t_2; \ldots; x_n, t_n)\, dx_{m+1} \cdots dx_n, \quad m < n. \end{aligned} \tag{4.2}$$

Alternatively, the probability structure may be defined by a hierarchy of joint characteristic functions

$$\begin{aligned} &M(\theta_1, t_1) = E[\exp(i\theta_1 X(t_1)], \\ &M(\theta_1, t_1; \theta_2, t_2) = E[\exp(i\theta_1 X(t_1) + i\theta_2 X(t_2))], \\ &\vdots \\ &M(\theta_1, t_1; \theta_2, t_2; \ldots; \theta_n, t_n) = E\left[\exp\left(\sum_{j=1}^{n} i\theta_j X(t_j)\right)\right]. \end{aligned} \tag{4.3}$$

The concept of characteristic function can be generalized to that of a *characteristic functional* if the parameter t is treated as continuous:

$$M[\theta(t)] = E\left[\exp\left\{i\int_T \theta(t)X(t)\,dt\right\}\right], \tag{4.4}$$

where the function $\theta(t)$ belongs to a class of functions for which the integral $\int_T \theta(t)X(t)\,dt$ is meaningful.

The joint probability description of two or more random processes can be expressed in terms of joint probability density functions of random processes sampled at specific values of indexing parameters. For example, for random processes $X(t)$ and $Y(s)$

$$\begin{aligned} &p(x_1, t_1);\quad p(y_1, s_1),\\ &p(x_1, t_1; y_1, s_1),\\ &p(x_1, t_1; x_2, t_2; y_1, s_1),\\ &\vdots \end{aligned} \tag{4.5}$$

Stationary Random Processes

A random process is said to be *stationary* if its probability structure is invariant under arbitrary translations of the indexing parameter. Thus $X(t)$ is stationary for all $t_1, t_2, \ldots, t_n$ and an arbitrary constant a if for all n

$$p(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = p(x_1, t_1 + a; x_2, t_2 + a; \ldots; x_n, t_n + a). \tag{4.6}$$

If the above property holds for a particular n, the random process is called *stationary of order n*. It can be shown that if a process is stationary of order n, then it is also stationary of all orders less than n. In particular, let $a = -t_1$. Equations (4.1) reduce to

$$\begin{aligned} &p(x_1, t_1) = p(x_1, 0) \quad \text{(independent of time)},\\ &p(x_1, t_1; x_2, t_2) = p(x_1, 0; x_2, t_2 - t_1) \quad \text{(function of } t_2 - t_1),\\ &\vdots\\ &p(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = p(x_1, 0; x_2, t_2 - t_1; \ldots; x_n, t_n - t_1). \end{aligned} \tag{4.7}$$

The definition of stationary random processes imposes the mathematical requirements that the realizations of these processes must extend from $-\infty$ to $+\infty$. Physically, stationarity implies a measure of temporal uniformity in the characteristics of the factors contributing to the randomness. The concept of stationarity is analogous to steady-state property of deterministic functions of time. In practice, no random process can be truly stationary. However, long segments of random process realizations

exhibiting uniform characteristics can be treated as stationary. Some examples of stationary random processes are the gust component of wind velocity at a point under steady weather conditions and base excitation experienced by a vehicle moving over a prepared surface (such as road or rail) with constant velocity. The random processes that are not stationary are called *nonstationary*. If the indexing parameter of a random process is other than time and its probability structure is independent of a shift in parametric origin, it is customary to call the random process *homogeneous*.

4.3 Expected Values: Moments

The expected values of the functions of random processes can be obtained by a simple generalization of the expectation operator defined in the previous chapter for random variables. Thus

$$E[f(X(t))] = \int_{-\infty}^{\infty} [f(x)]p(x, t)\,dx, \tag{4.8}$$

$$E[f(X(t), Y(s))] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [f(x, y)]p(x, t; y, s)\,dx\,dy. \tag{4.9}$$

4.3.1 Moments

Let $f(X(t)) = X^n(t)$, and

$$m_n(t) = E[X^n(t)] = \int_{-\infty}^{\infty} x^n p(x, t)\,dx; \tag{4.10}$$

$m_n(t)$ is called the *nth moment of the random process*. For $n = 1$ and 2,

$$m_1(t) = E[X(t)] = \int_{-\infty}^{\infty} xp(x, t)\,dx, \tag{4.11}$$

$$m_2(t) = E[X^2(t)] = \int_{-\infty}^{\infty} x^2 p(x, t)\,dx; \tag{4.12}$$

$m_1(t)$ and $m_2(t)$ are called the *mean* and *mean square of the random process* $X(t)$. The square root of $m_2(t)$ is called the *root mean square (rms)* of $X(t)$. We shall denote the mean $m_1(t)$ by $\mu(t)$. Let

$$f(X(t_1), X(t_2)) = X(t_1)X(t_2)$$

and

$$\phi(t_1, t_2) = E(X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 p(x_1, t_1; x_2, t_2)dx_1 dx_2; \tag{4.13}$$

$\phi(t_1, t_2)$ is called the *autocorrelation* function of $X(t)$.

4.3.2 Central Moments

Consider

$$f(X(t_1), X(t_2)) = (X(t_1) - \mu(t_1))(X(t_2) - \mu(t_2));$$

in this case

$$K(t_1, t_2) = E[(X(t_1) - \mu(t_1))(X(t_2) - \mu(t_2))] = \phi(t_1, t_2) - \mu(t_1)\mu(t_2). \tag{4.14}$$

Set $t_1 = t_2 = t$ in equation (4.14):

$$K(t, t) = E[X(t) - \mu(t)]^2] = \sigma^2(t). \tag{4.15}$$

$K(t_1, t_2)$ is called the *autocovariance function* and $\sigma(t)$ the *standard deviation.*

Consider two random processes $X(t)$ and $Y(t)$; then

$$\begin{aligned} K_{XY}(t_1, t_2) &= E[(X(t_1) - \mu_X(t_1))(Y(t_2) - \mu_Y(t_2))] \\ &= \phi_{XY}(t_1, t_2) - \mu_X(t_1)\mu_Y(t_2), \end{aligned} \tag{4.16}$$

where subscripts $X, Y, \ldots$ are used when required to distinguish the random processes being referred to. $K_{XY}(t_1, t_2)$ and $\phi_{XY}(t_1, t_2)$ are called the *cross-covariance* and *cross-correlation functions,* respectively. The *normalized covariance,* or the *correlation coefficient,* is given by

$$\rho_{XY}(t_1, t_2) = \frac{K_{XY}(t_1, t_2)}{\sigma_X(t_1)\sigma_Y(t_2)}. \tag{4.17}$$

In the definitions of the moment functions, *auto* refers to the same random process, while *cross* indicates different random processes.

The moment functions of random processes have the following properties:

1. From the definition given by (4.13) and (4.14), it is clear that the correlation and covariance functions are symmetric in t_1 and t_2:

$$\begin{aligned} &\phi(t_1, t_2) = \phi(t_2, t_1), \\ &K(t_1, t_2) = K(t_2, t_1), \\ &K_{XY}(t_1, t_2) = K_{YX}(t_2, t_1) \neq K_{XY}(t_2, t_1). \end{aligned} \tag{4.18}$$

2. $\phi(t_1, t_2)$ is a nonnegative definite function:

$$\sum_{j=1}^{n} \sum_{i=1}^{n} \phi(t_i, t_j)h(t_i)h^*(t_j) \geq 0, \tag{4.19}$$

where $h(t)$ is an arbitrary complex function and $h^*(t)$ is its complex conjugate.

3. Consider two random processes $X(t)$ and $Y(t)$. It is clear that

$$E[|\{X(t_1) - \mu_X(t_1)\} \pm \{Y(t_2) - \mu_Y(t_2)\}|^2] \geq 0.$$

Expand the expression within the bracket and take the expectation of the sum:

$$K_{XX}(t_1, t_1) + K_{YY}(t_2, t_2) \pm 2K_{XY}(t_1, t_2) \geq 0,$$

or

$$K_{XX}(t_1, t_1) + K_{YY}(t_2, t_2) \geq \pm 2K_{XY}(t_1, t_2). \tag{4.20}$$

4. The addition of a deterministic function to a random function does not change the covariance function. Consider

$$Y(t) = X(t) + \eta(t), \tag{4.21}$$

where $X(t)$ is a random process with mean $\mu_X(t)$ and covariance function $K_{XX}(t_1, t_2)$ and $\eta(t)$ is deterministic function. Then

$$\mu_Y(t) = \mu_X(t) + \eta(t) \tag{4.22}$$

and

$$\begin{aligned} K_{YY}(t_1, t_2) &= E[\{Y(t_1) - \mu_Y(t_1)\}\{Y(t_2) - \mu_Y(t_2)\}] \\ &= E[\{X(t_1) - \mu_X(t_1)\}\{X(t_2) - \mu_X(t_2)\}] = K_{XX}(t_1, t_2). \end{aligned} \tag{4.23}$$

In particular, if $\eta(t) = -\mu_X(t)$,

$$Y(t) = X(t) - \mu_X(t),$$

which has a zero mean and possesses the same covariance function as $X(t)$. Thus while considering the covariance property of a random process, there is no loss of generality in assuming that it has a zero mean. It is convenient to make this assumption to simplify the algebra. In view of this property, we shall assume the mean to be zero, unless specified otherwise, and drop the distinction between correlation and covariance functions.

4.3.3 Stationary Random Processes

From the definition of the moment functions and the properties of stationary random processes given by (4.7), it is clear that

$$\mu(t) = \mu = \text{constant}, \tag{4.24}$$

$$\sigma(t) = \sigma = \text{constant}, \tag{4.25}$$

$$\phi(t_1, t_2) = R(t_2 - t_1) = R(\tau), \tag{4.26}$$

$$K(t_1, t_2) = \Gamma(t_2 - t_1) = \Gamma(\tau), \tag{4.27}$$

$$K_{XY}(t_1, t_2) = \Gamma_{XY}(t_2 - t_1) = \Gamma_{XY}(\tau), \tag{4.28}$$

where $t_1 = t$ and $t_2 = t + \tau$. The temporal uniformity in the characteristics of a stationary random process is obvious from the fact that its mean and standard deviation are independent of time. It is possible that a random process may satisfy (4.24)–(4.27) but may not satisfy (4.7) beyond a certain value of n. Such processes are called *stationary in the wide sense.*

For stationary random processes, the properties of the ~~covariance~~ correlation functions given by (4.18)–(4.20) simplify to

1. $R(\tau)$, an even function of τ:

$$R(t_2 - t_1) = R(t_1 - t_2),$$

or

$$R(\tau) = R(-\tau). \tag{4.29}$$

2. $R(\tau)$, a nonnegative definite function:

$$\sum_{j=1}^{n} \sum_{i=1}^{n} R(t_j - t_i)h(t_i)h^*(t_j) \geq 0, \tag{4.30}$$

where $h(t)$ is an arbitrary complex function.

3. $R_{XX}(0) + R_{YY}(0) \geq 2R_{XY}(\tau)$: (4.31)

Set $X(t) = Y(t)$ in equation (4.31):

$$|R(\tau)| \leq R(0); \tag{4.32}$$

that is, $R(\tau)$ is maximum at $\tau = 0$, so that

$$\frac{d}{d\tau}R(\tau)\bigg|_{\tau=0} = 0$$

and

$$\frac{d^2R(\tau)}{d\tau^2}\bigg|_{\tau=0} \leq 0. \tag{4.33}$$

Moreover,

4. If a random process $X(t)$ does not contain periodic components, then

$$\lim_{\tau\to\infty} R(\tau) = 0. \tag{4.34}$$

A typical plot of autocorrelation function of a stationary random process is shown in figure 4.2.

4.3.4 Ergodic Random Processes

The expected values of the functions of random processes are obtained by taking averages across the ensemble and are called *ensemble averages.* For

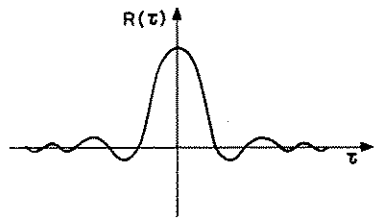

Figure 4.2 Autocorrelation function of a stationary random process.

a given sample function $x(t)$ of a stationary random process, we can define averages over time, called the *temporal averages*:

$$\langle X(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,dt, \tag{4.35}$$

$$\langle X(t)X(t+\tau)\rangle = \lim_{T\to\infty} \frac{1}{T-\tau}\int_0^{T-\tau} x(t)x(t+\tau)\,dt. \tag{4.36}$$

A subclass of stationary random processes is said to be *ergodic in the mean* if

$$\langle X(t)\rangle = E[X(t)] = \mu, \tag{4.37}$$

and *ergodic in correlation* if

$$\langle X(t)X(t+\tau)\rangle = E[X(t)X(t+\tau)] = R(\tau). \tag{4.38}$$

Thus the ergodic property holds if ensemble averages and temporal averages are equal. It may be noted that the ergodic property makes it possible to obtain the moment functions of a stationary random process from a single long record. Physically, ergodicity implies that a sufficiently long record of a stationary random process contains all the statistical information about the random phenomenon. In practical applications, often only one or two records are available, so that ergodicity is commonly assumed; when more records become available, the ergodicity assumption can be verified.

4.3.5 Complex Random Processes

We have defined a real random process as a parametered family of random variables that takes on real values. In the solution of many problems, the analysis can be simplified by assuming that the random process can take on complex values (for a detailed treatment, see Yaglom, 1952). Let

$$X(t) = U(t) + iV(t) \tag{4.39}$$

where $U(t)$ and $V(t)$ are real random processes. The moment functions of a complex random process are defined as

$$\mu_X(t) = E[X(t)] = E[U(t)] + iE[V(t)], \tag{4.40}$$

$$\begin{aligned} K_{XX}(t_1, t_2) &= E[\{X^*(t_1) - \mu_X^*(t_1)\}\{X(t_2) - \mu_X(t_2)\}] \\ &= [K_{UU}(t_1, t_2) + K_{VV}(t_1, t_2)] \\ &\quad + i[K_{VU}(t_1, t_2) - K_{VU}(t_2, t_1)]. \end{aligned} \tag{4.41}$$

Set $t_1 = t_2 = t$:

$$K_{XX}(t, t) = \sigma^2(t) = E[|X(t) - \mu_X(t)|^2], \tag{4.42}$$

which is real. From equation (4.41) it is clear that

$$K_{XX}(t_1, t_2) = K_{XX}^*(t_2, t_1). \tag{4.43}$$

$X(t)$ is stationary if $U(t)$ and $V(t)$ are jointly stationary and

$$R_{XX}(\tau) = R_{XX}^*(-\tau). \tag{4.44}$$

Example 4.1 Consider a complex random process defined by

$$X(t) = Ae^{i\omega t} = A\cos\omega t + iA\sin\omega t,$$

where A is a random variable with zero mean and ω is a deterministic parameter. Then

$$\mu_X(t) = 0,$$

$$\begin{aligned} K_{XX}(t_1, t_2) &= E[Ae^{-i\omega t_1}Ae^{i\omega t_2}] \\ &= E[A^2][\cos\omega(t_2 - t_1) + i\sin\omega(t_2 - t_1)]. \end{aligned}$$

Thus $X(t)$ is at least stationary in the wide sense.

4.3.6 Random Processes with Independent Increments

Consider a random process $X(t), t \in T$: $[0, \infty)$. $X(t)$ is said to be a *random process with independent increments* if for all choices of indices $t_0 < t_1 < t_2 < \cdots < t_n$, the n random variables

$$Z(t_i, t_{i-1}) = \{X(t_i) - X(t_{i-1})\}, \quad i = 1, 2, \ldots, n,$$

are independent. It is easy to show that the characteristic function of the random process $X(t)$ can be expressed as

$$\begin{aligned} M_X(\theta_1, t_1; \theta_2, t_2; \ldots; \theta_n, t_n) &= M_X(\theta_1 + \theta_2 + \cdots + \theta_n; t_1) \\ &\quad \cdot \prod_{k=2}^{n} M_Z(\theta_k + \cdots + \theta_n; t_k, t_{k-1}). \end{aligned} \tag{4.45}$$

Thus the joint distribution of any n random variables $X(t_i)$, $i = 1, 2, \ldots, n$, can be expressed in terms of the distributions of $X(t)$ and $\{X(s) - X(t)\}$ for all nonnegative t and s. The random process $X(t)$ is said to have *stationary independent increments* if $\{X(t_i + a) - X(t_{i-1} + a)\}$ has the same distribution as $\{X(t_i) - X(t_{i-1})\}$ for all choices of i and every $a > 0$, besides being independent.

Example 4.2 Consider a random process $X(t)$ with following properties:

1. $\{X(t), t \geq 0\}$ has stationary independent increments;
2. $X(t)$ is normally distributed, with $E[X(t)] = 0$ for every $t > 0$; and
3. $X(0) = 0$.

Since $X(0) = 0$ and $X(t)$ has independent increments, the distribution of $X(t)$ is completely determined by the distribution of the increment $Z(s, t) = \{X(s) - X(t)\}$ for any $s > t$. It is clear that $E[Z(s, t)] = E[X(s) - X(t)] = 0$. Further, since $Z(s, t)$ is normal,

$$M_Z(\theta; s, t) = \exp\left[-\frac{1}{2}\theta^2 \operatorname{Var}(Z(s, t))\right]. \tag{4.46}$$

Since $X(t)$ has stationary independent increments, it can be shown, [equation (4.66)] that

$$\operatorname{Var}(Z(s, t)) = \sigma^2|s - t|, \quad s \geq t \geq 0, \tag{4.47}$$

where σ^2 is a positive constant.

The random process $X(t)$ is called *Wiener process*. Its distribution is determined by properties (1)–(3) up to a parameter σ^2. Wiener processes (Wiener, 1923) were introduced to model the brownian motion, and have since found application in a large class of problems in quantum mechanics and other areas.

4.3.7 Generalized Chebyshev Inequality

Let $X(t), t \in T$: $[a, b]$, be a random process. It can be shown that

$$P\left[\left|\underset{a\leq t\leq b}{X(t)} - \mu_X(t)\right| \geq \alpha\right] \leq \frac{1}{2\alpha^2}[\sigma_X^2(a) + \sigma_X^2(b)] + \frac{1}{\alpha^2}\int_a^b \sigma_X(t)\sigma_{\dot{X}}(t)\,dt, \tag{4.48}$$

where $\alpha > 0$ and $\dot{X} = dX/dt$. Equation (4.48) is called the *generalized Chebyshev inequality*.

Consider a random process $Y(t)$ with $\mu_Y(t) = 0$. Then

$$\sup_{a\leq t\leq b} |Y(t)|$$

is a random variable, where sup denotes the supremum, the least upper bound. From the Chebyshev inequality [equation (3.54)],

$$P[\sup_{a\le t\le b}|Y(t)|\ge\alpha]\le\frac{1}{\alpha^2}E[\sup_{a\le t\le b}Y^2(t)]. \tag{4.49}$$

Since

$$Y^2(t)=Y^2(a)+\int_a^t\frac{d}{du}(Y^2(u))\,du=Y^2(b)-\int_t^b\frac{d}{du}(Y^2(u))\,du,$$

for any $t\in T$, it follows that

$$Y^2(t)\le\frac{1}{2}[Y^2(a)+Y^2(b)]+\int_a^b|\dot{Y}Y|\,du.$$

In particular,

$$\sup_{a\le t\le b}[Y^2(t)]\le\frac{1}{2}[Y^2(a)+Y^2(b)]+\int_a^b|Y\dot{Y}|\,du.$$

Take expectation of both sides:

$$E[\sup_{a\le t\le b}(Y^2(t))]\le\frac{1}{2}[\sigma_Y^2(a)+\sigma_Y^2(b)]+\int_a^b E[|Y\dot{Y}|]\,du. \tag{4.50}$$

From the Schwarz inequality,

$$E[|Y\dot{Y}|]\le\{E[Y^2]E[\dot{Y}^2]\}^{1/2}. \tag{4.51}$$

Substitute (4.50) and (4.51) in (4.49):

$$P[\sup_{a\le t\le b}|Y(t)|\ge\alpha]\le\frac{1}{2\alpha^2}[\sigma_Y^2(a)+\sigma_Y^2(b)]+\frac{1}{\alpha^2}\int_a^b\sigma_Y(t)\sigma_{\dot{Y}}(t)\,dt. \tag{4.52}$$

Equation (4.52) reduces to (4.48) if $Y(t)=X(t)-\mu_X(t)$. The generalized Chebyshev inequality provides an upper bound estimate of the event $\{|X(t)-\mu_X(t)|\ge\alpha: a\le t\le b\}$ in terms of the mean $\mu_X(t)$ and standard deviations of the random processes $X(t)$ and $\dot{X}(t)$. It may be noted that the upper bound estimate does not stipulate any specific distribution.

4.4 Operations on Random Processes

The probabilistic models of a large class of physical systems involve the calculus of random processes, primarily the operations of differentiation and integration. Since a random process is defined in a probabilistic sense, the calculus of the random processes must also be developed within a probabilistic framework. The development of calculus is based on the concept of *convergence*. For random processes, convergence can be defined in several ways. We shall confine our discussion to the notion of *mean square convergence* and state the necessary and sufficient conditions for the continuity, differentiability, and integrability of random processes

without proof. For a derivation of these conditions the reader is referred to the texts on probability and stochastic processes, such as Parzen (1960), Loève (1963), and Lin (1967).

4.4.1 Limit

A random sequence X_n is said to *converge in the mean square to a random variable* X if

$$\lim_{n\to\infty}E[|X_n-X|^2]=0. \tag{4.53}$$

We shall abbreviate this to

$$\text{l.i.m.}_{n\to\infty}X_n=X, \tag{4.54}$$

where l.i.m. denotes *limit in the mean*. For (4.53) to hold it is necessary that $E[X_n^2]<\infty$; in such a case, X is called a *second-order random variable*.

A random process $X(t)$ satisfying the condition $E[X^2(t)]<\infty$ is called a *second-order random process*. Such a process converges in mean square to a random variable X as $t\to t_0$ if and only if the limit as $t, s\to t_0$ of the autocorrelation function $\phi(t,s)$ exists and is finite, no matter how t and s approach t_0. Further, it can be shown that the operations of l.i.m. and expectation commute; that is,

$$\lim_{t\to t_0}E[X(t)]=E[\text{l.i.m.}_{t\to t_0}X(t)]=E[X]. \tag{4.55}$$

4.4.2 Continuity

A random process $X(t)$ is said to be *continuous in the mean square* if

$$\text{l.i.m.}_{h\to 0}X(t+h)=X(t). \tag{4.56}$$

The necessary and sufficient conditions for (4.56) to hold are

$$\phi(t,s)<\infty \text{ and is continuous on } t=s. \tag{4.57}$$

4.4.3 Differentiation

The *mean square derivative* of the random process $X(t)$ is defined as

$$\frac{d}{dt}X(t)=\dot{X}(t)=\text{l.i.m.}_{h\to 0}\frac{X(t+h)-X(t)}{h}. \tag{4.58}$$

The necessary and sufficient conditions for $X(t)$ to be *differentiable in the mean square* are

$$\frac{\partial^2\phi(t,s)}{\partial t\,\partial s}<\infty \quad \text{and} \quad \text{is continuous on } t=s. \tag{4.59}$$

If $X(t)$ is stationary, $\phi(t, s) = R(s - t) = R(\tau)$ and (4.59) reduces to

$$\frac{d^2R(\tau)}{d\tau^2} < \infty \quad \text{and} \quad \text{is continuous on } \tau = 0. \tag{4.60}$$

It can be shown that if (4.59) holds, then

$$\frac{d}{dt}E[X(t)] = E\left[\frac{dX(t)}{dt}\right] = E[\dot{X}(t)]; \tag{4.61}$$

that is, the operations of expectation and differentiation commute. It follows that

$$\begin{aligned}
&\frac{\partial}{\partial t}\phi_{XY}(t, s) = \frac{\partial}{\partial t}E[X(t)Y(s)] = E[\dot{X}(t)Y(s)] = \phi_{\dot{X}Y}(t, s), \\
&\frac{\partial^2}{\partial s\,\partial t}\phi_{XY}(t, s) = \frac{\partial^2}{\partial s\,\partial t}E[X(t)Y(s)] = E[\dot{X}(t)\dot{Y}(s)] = \phi_{\dot{X}\dot{Y}}(t, s).
\end{aligned} \tag{4.62}$$

In particular, if $X(t)$ is weakly stationary, then

$$\begin{aligned}
&\frac{\partial}{\partial t}R_{XX}(t - s) = R_{\dot{X}X}(t - s), \quad \text{or} \quad \frac{d}{d\tau}R_{XX}(\tau) = R_{\dot{X}X}(\tau), \\
&\frac{\partial^2}{\partial s\,\partial t}R_{XX}(t - s) = -R_{\dot{X}\dot{X}}(t - s), \quad \text{or} \quad \frac{d^2}{d\tau^2}R_{XX}(\tau) = -R_{\dot{X}\dot{X}}(\tau), \\
&\frac{\partial^2}{\partial t^2}R_{XX}(t - s) = R_{\dot{X}\dot{X}}(t - s), \quad \text{or} \quad \frac{d^2}{d\tau^2}R_{XX}(\tau) = R_{\dot{X}\dot{X}}(\tau).
\end{aligned} \tag{4.63}$$

Thus, the mean and correlation functions of a differentiable random process and its derivatives can be obtained directly from (4.61) and (4.62) without any knowledge of the distribution function of the parent random process. Further, since $[\partial R_{XX}(\tau)/\partial\tau]_{\tau=0} = 0$, (4.33) and (4.63) become

$$\dot{R}_{XX}(0) = R_{\dot{X}X}(0) = E[\dot{X}(t)X(t)] = 0. \tag{4.64}$$

Two random processes $X(t)$ and $Y(t)$ are said to be *orthogonal* if $E[X(t)Y(t)] = 0$. Thus a stationary random process and its derivative process are orthogonal.

Instances of random processes that are continuous but not differentiable occur often in applications. Such cases arise, for example, where a random process is obtained as a result of the summation of an infinitely large number of infinitely small mutually independent increments. Consider a random process $X(t)$ formed by the addition of independent, identically distributed random increments Δ_i with zero mean and standard deviation σ. Let the instants of the increments of the random process be uniformly distributed in time and have an arrival rate ν. Let ν be so large that even

for small values of time interval Δt, the number of increments $n = \nu\,\Delta t$ can be assumed to be an integer. Then the increment of the random process $X(t)$ in the interval Δt can be expressed as the sum

$$\Delta X(t) = \sum_{i=1}^{n} \Delta_i. \tag{4.65}$$

The variance of $\Delta X(t)$ is given by

$$E[|\Delta X(t)|^2] = n\sigma^2 = \nu\sigma^2\,\Delta t. \tag{4.66}$$

Thus the square of the increment of the process is proportional to Δt and not $(\Delta t)^2$, as would be the case for a differentiable process. The above example explains both the mathematical basis and the physical conditions for the generation of nondifferentiable random processes.

Example 4.3

1. Consider a stationary random process $X(t)$ with autocorrelation function

$$R(\tau) = \sigma^2 e^{-\alpha|\tau|}, \quad \alpha > 0.$$

Figure 4.3 shows a plot of $R(\tau)$ and its derivative with respect to τ. It is seen that $R(\tau)$ is continuous and bounded at $\tau = 0$, and therefore $X(t)$ is continuous. However, the derivative of $R(\tau)$ is discontinuous at $\tau = 0$; therefore (4.60) is not satisfied, and $X(t)$ is not differentiable.

2. Consider a stationary random process $X_1(t)$ with autocorrelation function

$$R_1(\tau) = \sigma^2 e^{-\alpha_1\tau^2}, \quad \alpha_1 > 0.$$

Again $X_1(t)$ is continuous since $R_1(\tau)$ is bounded and continuous at $\tau = 0$. Further, $R_1(\tau)$ is continuously differentiable for any value of τ, and therefore $X_1(t)$ is a differentiable random process.

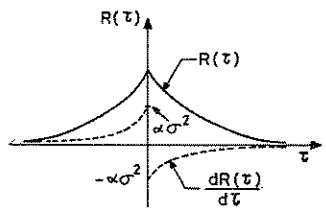

Figure 4.3 Autocorrelation function $R(\tau) = \sigma^2 e^{-\alpha|\tau|}$ and its derivative with respect to τ.

4.4.4 Integration

Let $X(t)$ be a random process and $h(t, \tau)$ be a bounded deterministic function. Then $Z(\tau)$, defined by

$$Z(\tau) = \int_a^b h(t, \tau)X(t)\,dt, \tag{4.67}$$

is a transformed random process with τ as the indexing parameter. It can be shown that $Z(\tau)$ exists in the mean as a limit of the Riemann sum if and only if

$$\left| \int_a^b \int_a^b \phi(t_1, t_2)h(t_1, \tau_1)h^*(t_2, \tau_2)\,dt_1\,dt_2 \right| < \infty \quad \text{for all } \tau_1 \text{ and } \tau_2. \tag{4.68}$$

It can also be shown that if the condition given by (4.68) is satisfied, then

$$E[Z(\tau)] = \int_a^b h(t, \tau)E[X(t)]\,dt, \tag{4.69}$$

$$E[Z(\tau_1)Z(\tau_2)] = \int_a^b \int_a^b h(t_1, \tau_1)h^*(t_2, \tau_2)E[X(t_1)X(t_2)]\,dt_1\,dt_2, \tag{4.70}$$

where $h^*(t, \tau)$ is the complex conjugate of $h(t, \tau)$. Thus integration and expectation operators commute.

4.4.5 General Linear Operations

The differentiation and integration operations are specific examples of linear operations. Let the random process $Y(t)$ be generated from $X(t)$ by the operation

$$Y(t) = L[X(t)], \tag{4.71}$$

where L is a mathematical operator.

The operator L is said to be a *linear homogeneous operator* if it satisfies the following two conditions:

$$L[Cg(t)] = CL[g(t)]; \tag{4.72a}$$

$$L[g_1(t) + g_2(t)] = L[g_1(t)] + L[g_2(t)], \tag{4.72b}$$

where C is a constant and $g(t)$, $g_1(t)$, and $g_2(t)$ are arbitrary functions. The sum of a linear homogeneous operator and a specified function is called a *linear nonhomogeneous operator*. Thus if L is a linear homogeneous operator and $g(t)$ is some function, an operator L_1 defined by

$$L_1[X(t)] = L[X(t)] + g(t) \tag{4.73}$$

is a *linear nonhomogeneous operator*, which does not satisfy the conditions expressed by (4.72). Operators that do not satisfy (4.72) or (4.73) are called *nonlinear operators*.

Let L be linear homogeneous operator. If such an operator commutes with the expectation operator, it follows that

$$E[Y(t)] = E[L[X(t)]] = L[E[X(t)]] \tag{4.74}$$

and

$$K_{YY}(t_1, t_2) = L_{t_1}^* L_{t_2} K_{XX}(t_1, t_2), \tag{4.75}$$

where subscripts t_1 and t_2 indicate that in the first case the operator acts with respect to the variable t_1 and in the second case with respect to t_2. These properties also hold for nonhomogeneous linear operators and are independent of the distribution function of $X(t)$. The nonlinear operators do not posses such simple properties.

4.5 Spectral Properties of Random Processes

Harmonic analysis provides a very powerful tool for the analytical and experimental treatment of a large class of physical problems. To carry out the harmonic analysis, it is necessary to determine the spectral properties of the functions involved. If the functions are periodic, their spectral properties may be determined through Fourier series representation in terms of a discrete set of frequencies that are integral multiples of the fundamental frequency. A nonperiodic function, if absolutely integrable, admits a Fourier integral representation in terms of a continuous band of frequencies. In both cases, the Parseval theorem relates the mean square value of a function to the square of its spectral ordinates and thus provides an insight into the spectral distribution of the energy associated with the function. The application of Fourier analysis has led to the development of the frequency domain approach for solving a large class of problems. This approach is particularly suitable for vibration problems in which frequency is a very important parameter.

We have defined a random process as an ensemble of random functions. The Fourier analysis of each sample function, if admissible, provides the spectral characteristics of the function. However, our primary interest is in the spectral properties of the moment functions—in particular, the correlation functions. We shall see later that the frequency domain approach provides a very powerful analytical and experimental tool for the analysis and design of systems operating in a random vibration environment.

Consider a random process $X(t)$. Without any loss of generality, let the process be assumed to have a zero mean. In the first instance, let the process be expressed as a sum of harmonic functions with a discrete set of frequencies ω_j and random amplitudes A_j and B_j:

$$X(t) = \sum_{j=1}^{n} (A_j \cos \omega_j t + B_j \sin \omega_j t), \tag{4.76}$$

where A_j and B_j are random variables with zero mean. It is convenient to express the above representation in the complex form:

$$X(t) = \sum_{j=-n}^{n} S_j e^{i\omega_j t}, \tag{4.77}$$

where ω_{-j} denotes $-\omega_j$ and the S_j are complex random variables. The covariance function of $X(t)$ is given by

$$\begin{aligned} K(t_1, t_2) &= E\left[\left(\sum_{j=-n}^{n} S_j^* e^{-i\omega_j t_1}\right)\left(\sum_{k=-n}^{n} S_k e^{i\omega_k t_2}\right)\right] \\ &= \sum_{j=-n}^{n} \sum_{k=-n}^{n} E[S_j^* S_k] e^{i(\omega_k t_2 - \omega_j t_1)} = \sum_{j=-n}^{n} \sum_{k=-n}^{n} \hat{\Phi}_{jk} e^{i(\omega_k t_2 - \omega_j t_1)}, \end{aligned} \tag{4.78}$$

where $\hat{\Phi}_{jk} = E[S_j^* S_k]$ is called the *generalized spectral function.*

Consider the special case

$$E[S_j^* S_k] = \Phi_j \delta_{jk}, \tag{4.79}$$

where δ_{jk} is the Kronecker delta:

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases} \tag{4.80}$$

Substitute (4.79) in (4.78) and use (4.80):

$$K(t_1, t_2) = \sum_{j=-n}^{n} \Phi_j e^{i\omega_j (t_2 - t_1)} = R(t_2 - t_1), \tag{4.81}$$

which shows that if condition (4.79) is satisfied, then $X(t)$ is stationary. Set $t_1 = t_2 = t$ in (4.81):

$$\sigma^2 = E[X^2(t)] = R(0) = \sum_{j=-n}^{n} \Phi_j = \sum_{j=-n}^{n} E[|S_j|^2]. \tag{4.82}$$

In many physical problems, $X^2(t)$ may represent an energylike quantity. For example, if $X(t)$ is the displacement of a linear spring, $X^2(t)$ is proportional to the strain energy stored in the spring. If $X(t)$ represents the velocity of a rigid body, $X^2(t)$ is proportional to the kinetic energy of the body. In electrical systems, if $X(t)$ represents the current or the voltage, $X^2(t)$ is proportional to the power. Thus σ^2 represents average energy and Φ_j the part of the average energy contributed by the frequency ω_j. The Φ_j are called *energy spectra* of the random process $X(t)$. It is clear that the Φ_j are real and positive and that $\Phi_j = \Phi_{-j}$.

The spectral representation of the random process can be generalized by considering the Fourier integral representation. Consider a random

process $\hat{X}(\omega)$ defined by

$$\hat{X}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t) e^{i\omega t} dt. \tag{4.83}$$

Comparing (4.83) with (4.67) shows that $\hat{X}(\omega)$ is a transformed random process with ω as the indexing parameter. $\hat{X}(\omega)$ exists in the mean if and only if

$$|E[\hat{X}(\omega_1)\hat{X}^*(\omega_2)]| = \left| \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(t_1, t_2) e^{-i(\omega_2 t_2 - \omega_1 t_1)} dt_1 \, dt_2 \right| < \infty \tag{4.84}$$

for every ω_1 and ω_2.

If condition (4.84) is satisfied, $X(t)$ can be expressed in terms of $\hat{X}(\omega)$ by the Fourier inversion theorem and

$$X(t) = \int_{-\infty}^{\infty} \hat{X}(\omega) e^{-i\omega t} d\omega. \tag{4.85}$$

Thus $\hat{X}(\omega) d\omega$ represents the spectral ordinate associated with the frequency ω. $E[X(\omega_1)X^*(\omega_2)]$ is called the *generalized power spectral density* and is denoted $\hat{\Phi}(\omega_1, \omega_2)$. It is seen from (4.84) that $\hat{\Phi}(\omega_1, \omega_2)$ is the 2-dimensional Fourier transform of $K(t_1, t_2)$. By the inversion theorem, or by applying (4.70) to (4.85),

$$K(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\Phi}(\omega_1, \omega_2) e^{i(\omega_2 t_2 - \omega_1 t_1)} d\omega_1 \, d\omega_2. \tag{4.86}$$

Equations (4.84) and (4.86) are valid if $K(t_1, t_2)$ is piecewise continuous, bounded, and absolutely integrable over the domain of t_1 and t_2.

The transformed random process $\hat{X}(\omega)$, which is the Fourier transform of $X(t)$, exists if and only if condition (4.84) is satisfied. This condition is not satisfied by the stationary random processes, and therefore such processes do not admit Fourier representation in the form of (4.83) and (4.85). To show this, let us consider a stationary random process $X(t)$. By definition a stationary random process has an infinite duration. Consider a finite segment of the random process of duration on $2T$ and denote it $X_T(t)$. Let

$$X_T(t) = \begin{cases} X(t), & -T \leq t \leq T \\ 0, & \text{otherwise.} \end{cases} \tag{4.87}$$

Clearly $\lim_{T \to \infty} X_T(t) \to X(t)$. The Fourier transform of $X_T(t)$ can be expressed as

$$\hat{X}(\omega, T) = \frac{1}{2\pi} \int_{-T}^{T} X(t) e^{i\omega t} dt; \tag{4.88}$$

$\hat{X}(\omega, T)$ will exist in the mean if and only if

$$E[\hat{X}(\omega_1, T)\hat{X}^*(\omega_2, T)] = \frac{1}{(2\pi)^2}\int_{-T}^{T}\int_{-T}^{T} R(t_2 - t_1)e^{-i(\omega_2 t_2 - \omega_1 t_1)}\,dt_1\,dt_2 \tag{4.89}$$

is bounded for all ω_1 and ω_2. By setting $t_2 - t_1 = \tau$ and $\omega_1 = \omega_2 = \omega$ in (4.89) it can be shown (Lin, 1967) that

$$E[|X(\omega, T)|^2] = \frac{1}{(2\pi)^2}\int_{-2T}^{2T} (2T - |\tau|)R(\tau)e^{-i\omega\tau}\,d\tau. \tag{4.90}$$

It is clear that the right-hand side of (4.90) becomes unbounded as $T \to \infty$. Hence the Fourier transform of $X(t)$ does not exist.

Multiply both sides of (4.85) by π/T and take the limit as $T \to \infty$:

$$\lim_{T\to\infty}\frac{\pi}{T}E[|\hat{X}(\omega, T)|^2] = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-2T}^{2T} R(\tau)e^{-i\omega\tau}\,d\tau - \lim_{T\to\infty}\frac{1}{2\pi}\int_{-2T}^{2T}\frac{|\tau|}{T}e^{-i\omega\tau}\,d\tau. \tag{4.91}$$

It can be shown that

$$\lim_{T\to\infty}\frac{1}{2\pi}\int_{-2T}^{2T}\frac{|\tau|}{T}e^{-i\omega\tau}\,d\tau = 0, \tag{4.92}$$

and therefore

$$\lim_{T\to\infty}\frac{\pi}{T}E[|\hat{X}(\omega, T)|^2] = \Phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}\,d\tau. \tag{4.93}$$

The integral in (4.93) exits if $R(\tau)$ is piecewise continuous, bounded, and absolutely integrable. These conditions are satisfied by most random processes.

It is seen that $\Phi(\omega)$ is the Fourier transform of the autocorrelation function $R(\tau)$. Hence by the inversion theorem

$$R(\tau) = \int_{-\infty}^{\infty}\Phi(\omega)e^{i\omega\tau}\,d\omega \tag{4.94}$$

and

$$\sigma^2 = R(0) = E[X^2(t)] = \int_{-\infty}^{\infty}\Phi(\omega)\,d\omega. \tag{4.95}$$

As discussed earlier, σ^2 represents the average energy, and therefore $\Phi(\omega)$ gives the spectral distribution of average energy of a stationary random process. It is called the *power spectral density* (*psd*). The psd of a random process may be obtained either by first computing the autocorrelation

function and then taking its Fourier transform or by taking the ensemble average of the Fourier transform of a random process of sufficiently long duration in accordance with (4.93). With the development of the fast Fourier transform technique (FFT; Cooley, Lewis, and Welch, 1969), the latter approach has become computationally more efficient.

From (4.93) and the fact that

$$R(\tau) \quad \text{is an even function of } \tau, \tag{4.96}$$

it is clear that $\Phi(\omega)$ is real and

$$\Phi(\omega) = \frac{1}{\pi}\int_0^{\infty} R(\tau)\cos\omega\tau\,d\tau, \tag{4.97}$$

$$R(\tau) = \int_0^{\infty}\Phi(\omega)\cos\omega\tau\,d\omega. \tag{4.98}$$

Equations (4.93), (4.94), (4.97), and (4.98) are the well-known *Wiener-Khinchine relations* between the psd and the autocorrelation function of a stationary random process.

The psd of random process generated by differentiating a stationary random process can be obtained very simply from the Wiener-Khinchine relations using the differential properties of the autocorrelation function given by (4.63). Consider the random process

$$\dot{X}(t) = \frac{dX(t)}{dt}.$$

From (4.63) and (4.94),

$$R_{\dot{X}X}(\tau) = -\frac{d^2}{d\tau^2}R_{XX}(\tau) = \int_{-\infty}^{\infty}\omega^2\Phi_{XX}(\omega)e^{i\omega\tau}\,d\omega. \tag{4.99}$$

Since $\dot{X}(t)$ is a stationary random process,

$$R_{\dot{X}\dot{X}}(\tau) = \int_{-\infty}^{\infty}\Phi_{\dot{X}\dot{X}}(\omega)e^{i\omega\tau}\,d\omega. \tag{4.100}$$

Comparing (4.99) and (4.100) shows that

$$\Phi_{\dot{X}\dot{X}}(\omega) = \omega^2\Phi_{XX}(\omega). \tag{4.101}$$

The above relation holds if and only if the derivative process exists—that is, condition (4.60) is satisfied. It is clear that (4.60) is equivalent to

$$\int_{-\infty}^{\infty}\omega^2\Phi_{XX}(\omega)\,d\omega < \infty. \tag{4.102}$$

The psd of higher-order derivatives, if they exist, can be derived by repeated application of the above results. Thus if

$$Y(t) = \frac{d^n}{dt^n}X(t), \tag{4.103}$$

then

$$\Phi_{YY}(\omega) = (\omega)^{2n}\Phi_{XX}(\omega), \tag{4.104}$$

provided

$$\int_{-\infty}^{\infty} (\omega)^{2n}\Phi_{XX}(\omega)\,d\omega < \infty. \tag{4.105}$$

Example 4.4

1. Consider a stationary random process $X(t)$ with autocorrelation function $R(\tau) = \sigma^2 e^{-\alpha|\tau|}$. From (4.93), the psd of $X(t)$ is given by

$$\Phi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\sigma^2 e^{-\alpha|\tau|}\,d\tau = \frac{\sigma^2}{2\pi}\left[\int_{-\infty}^{0} e^{(\alpha - i\omega)\tau}\,d\tau + \int_{0}^{\infty} e^{-(\alpha + i\omega)\tau}\,d\tau\right]$$

$$= \frac{\sigma^2}{2\pi}\left[\frac{1}{\alpha - i\omega} + \frac{1}{\alpha + i\omega}\right] = \frac{\sigma^2}{\pi}\frac{\alpha}{\omega^2 + \alpha^2}.$$

A plot of $\Phi(\omega)$ for three values of α is shown in figure 4.4. It is seen that as α increases, $\Phi(\omega)$ becomes flatter and approaches a constant value, $\Phi_0 = \sigma^2/\pi\alpha$, over a wide range of frequencies. For large values of α, the random process thus approaches the idealized white noise random process, which has a constant psd at all frequencies.

The variance of $X(t)$ is given by

$$\mathrm{Var}[X(t)] = \int_{-\infty}^{\infty} \Phi(\omega)\,d\omega = \frac{\sigma^2\alpha}{\pi}\int_{-\infty}^{\infty}\frac{1}{\omega^2 + \alpha^2}d\omega.$$

The integrand has poles at $\omega = \pm i\alpha$ and tends to zero as $1/\omega^2$ for $\omega \to \infty$. Hence from Cauchy residue theorem (appendix A),

$$\mathrm{Var}[X(t)] = \frac{\sigma^2\alpha}{\pi}2\pi i\frac{1}{2\alpha i} = \sigma^2.$$

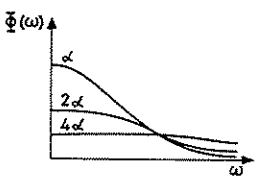

Figure 4.4 Power spectral density $\Phi(\omega) = (\sigma^2/\pi)\,(\alpha/\omega^2 + \alpha^2)$.

From (4.101), the psd of the derivative process $\dot{X}(t)$ is given by

$$\Phi_{\dot{X}\dot{X}}(\omega) = \omega^2\Phi_{XX}(\omega) = \frac{\sigma^2}{\pi}\frac{\alpha\omega^2}{\omega^2 + \alpha^2}.$$

We have seen in example 4.3 that the random process $X(t)$ is not differentiable. The same conclusion can be drawn from (4.102) because the integral becomes unbounded.

2. Consider a stationary random process $X_1(t)$ with autocorrelation function

$$R_{X_1X_1}(\tau) = \sigma e^{-\alpha_1\tau^2}, \quad \alpha_1 > 0.$$

It can be shown that the psd of $X_1(t)$ is given by

$$\Phi_{X_1X_1}(\omega) = \frac{\sigma^2}{2\sqrt{\pi\alpha_1}}e^{-\omega^2/4\alpha_1^2}.$$

Clearly, the variance of $X_1(t) = \sigma^2$. The psd of the derivative process $\dot{X}_1(t)$ is given by

$$\Phi_{\dot{X}_1\dot{X}_1}(\omega) = \frac{\omega^2\sigma^2}{2\sqrt{\pi\alpha_1}}e^{-\omega^2/4\alpha_1^2}.$$

Since the psd of the derivative process decreases exponentially as $\omega \to \infty$, (4.105) is satisfied for all n, and therefore $X_1(t)$ is differentiable.

4.5.1 Evolutionary Random Processes

We have seen that a stationary random process does not possess a Fourier transform, and therefore does not admit the Riemann integral representation [equation (4.85)]. However, by starting with a discrete frequency representation as in (4.77) and generalizing it to a continuous spectrum of frequencies through a limiting process, it can be shown that both stationary and nonstationary random process can be expressed by the Fourier-Stieltjes integral (appendix A)

$$X(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\,dS(\omega). \tag{4.106}$$

Comparing (4.106) with (4.85) shows that the Stieltjes integral differs from the Riemann integral in that $\hat{X}(\omega)\,d\omega$ is replaced by the increment of some function $S(\omega)$. The Stieltjes integral reduces to the Riemann integral if the function $S(\omega)$ is differentiable. The functions $S(\omega)$ associated with stationary random processes do not satisfy this property.

The covariance function of a random process defined by (4.106) can be expressed as

$$K(t_1, t_2) = E\left[\int_{-\infty}^{\infty} e^{-i\omega_1 t_1}\, dS(\omega_1) \times \int_{-\infty}^{\infty} e^{i\omega_2 t_2}\, dS^*(\omega_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\omega_2 t_2 - \omega_1 t_1)} E[dS(\omega_1)\, dS^*(\omega_2)]. \tag{4.107}$$

By comparing (4.107) with (4.86), one can relate the generalized power spectral density to the function $S(\omega)$ by the relation

$$\hat{\Phi}(\omega_1, \omega_2)\, d\omega_1\, d\omega_2 = E[dS(\omega_1)\, dS^*(\omega_2)]. \tag{4.108}$$

If $X(t)$ is stationary, by writing $t_2 = t_1 + \tau$ (4.107) can be expressed as

$$R(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\{\omega_2(t_1+\tau) - \omega_1 t_1\}} E[dS(\omega_1)\, dS^*(\omega_2)]. \tag{4.109}$$

Comparing (4.109) with (4.94) shows that they are equivalent if

$$E[dS(\omega_1)\, dS^*(\omega_2)] = \Phi(\omega_1)\delta(\omega_2 - \omega_1)\, d\omega_1\, d\omega_2 \tag{4.110}$$

where $\delta(\omega_2 - \omega_1)$ is the Dirac delta function. Hence a stationary random process with psd $\Phi(\omega)$ admits a Stieltjes integral representation of the form (4.106), provided (4.110) is satisfied.

The Stieltjes integral representation of stationary random processes can be used to define a special class of nonstationary random processes having *evolutionary spectral density*. Consider a random process $Y(t)$ defined by

$$Y(t) = \int_{-\infty}^{\infty} A(t, \omega) e^{-i\omega t}\, dS(\omega), \tag{4.111}$$

where $A(t, \omega)$ is a deterministic function of ω and t, called the *modulating function*, and $S(\omega)$ is associated with $X(t)$ through (4.106) and (4.110). We shall assume that $Y(t)$ is zero for $t < 0$, and therefore $A(t, \omega) = 0$ for $t < 0$. Without any loss of generality $A(t, \omega)$ can be normalized so that

$$\sup_{t,\omega} |A(t, \omega)| = 1.$$

Under this assumption, if $A(t, \omega) = 1$, then $Y(t) = X(t)$. The autocorrelation function of $Y(t)$ is given by

$$K_{YY}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(t_1, \omega_1) A^*(t_2, \omega_2) e^{i(\omega_2 t_2 - \omega_1 t_1)} \cdot E[dS(\omega_1)\, dS^*(\omega_2)].$$

Substitute for $E[dS(\omega_1)\, dS^*(\omega_2)]$ from (4.110) and use the property of the Dirac delta function:

$$K_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} A(t_1, \omega) A^*(t_2, \omega) e^{i\omega(t_2 - t_1)} \Phi_{XX}(\omega)\, d\omega. \tag{4.112}$$

In particular, the mean square value of $Y(t)$ is obtained by setting $t_1 = t_2 = t$ in (4.112), so that

$$\sigma_Y^2(t) = \int_{-\infty}^{\infty} |A(t, \omega)|^2 \Phi_{XX}(\omega)\, d\omega \tag{4.113}$$

$$= \int_{-\infty}^{\infty} \Phi_{YY}(t, \omega)\, d\omega, \tag{4.114}$$

where

$$\Phi_{YY}(t, \omega) = |A(t, \omega)|^2 \Phi_{XX}(\omega); \tag{4.115}$$

$\Phi_{YY}(t, \omega)$ represents the spectral distribution of average energy as a function of time and is therefore called the *evolutionary spectral density* (Priestley, 1965). The concept of evolutionary spectral density provides an elegant spectral theory for a class of nonstationary processes within the framework of classical concepts such as energy and frequency. It depicts how the spectral properties of a random process evolve as a function of time. A closely related concept, called the *instantaneous power spectra*, was proposed by Page (1952) and has been used by Liu (1972); for a comprehensive treatment of random processes with evolutionary psd, see Priestley (1981a, b).

The random processes with evolutionary spectral density have been used to model a large class of physical problems. In many applications, the random process is assumed to be a *uniformly modulated process*; that is, it can be factored as

$$Y(t) = \psi(t) X(t) \tag{4.116}$$

where $\psi(t)$ is a deterministic envelope function and $X(t)$ is a stationary random process. For such processes (4.109)–(4.115) hold with $A(\omega, t)$ replaced by $\psi(t)$. Nigam (1982) has shown that the distribution of the first derivative of the phase of the Fourier transform of a uniformly modulated white noise process can be expressed in terms of the first three moments of the intensity function $I(t) = \psi^2(t)$.

4.5.2 Spectral Classification of Stationary Random Processes

A stationary random process is classified as a *broadband* or *narrowband* random process depending on the nature of its psd. A random process is said to be a broadband random process if its psd has significant values over a wide range of frequencies. A typical sample function (a), the auto-

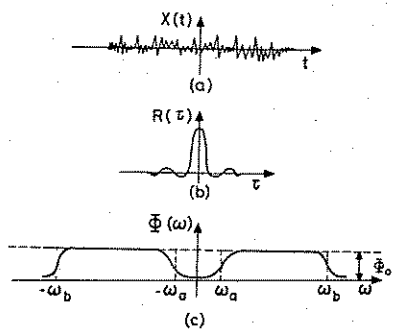

Figure 4.5 Broadband random process: (a) sample function, (b) autocorrelation function, and (c) power spectral density. The dashed line denotes white noise idealization.

correlation function (b), and the psd (c) of a broadband random process are shown in figure 4.5.

A common idealization of a broadband random process is to assume $\Phi(\omega) = \Phi_0$ for all frequencies. Such a process is called *white noise*. The white noise idealization of a broadband process is shown by dashed line in figure 4.5. Set $\Phi(\omega) = \Phi_0$ in (4.94) and use (A.19):

$$R(\tau) = \Phi_0 \int_{-\infty}^{\infty} e^{i\omega\tau} d\omega = 2\pi\Phi_0 \delta(\tau). \tag{4.117}$$

Thus the autocorrelation function of white noise is a delta function, implying no correlation for $\tau \neq 0$, which is not realistic for a continuous process. Also, white noise implies infinite variance and is therefore physically unrealizable. In spite of these defects, white noise idealization provides a meaningful mathematical model of broadband random processes under certain conditions: The *correlation time* of a stationary random process is defined as

$$t_c = \int_0^{\infty} \tau R(\tau)\, d\tau \Big/ \int_0^{\infty} R(\tau)\, d\tau;$$

if $t_c \ll t$, where t is the time scale of interest, correlation can be neglected, which implies white noise idealization.

Another idealization of broadband random processes, *band-limited white noise*, is defined by

$$\Phi(\omega) = \begin{cases} \Phi_0, & \omega_a < |\omega| < \omega_b \\ 0, & \text{otherwise,} \end{cases} \tag{4.118}$$

or

$$R(\tau) = 2\Phi_0 \frac{\sin \omega_b \tau - \sin \omega_a \tau}{\tau}. \tag{4.119}$$

The band-limited white noise idealization of the broadband random process is shown by dashed line in figure 4.5. Band-limited white noise can be physically realized quite closely and is commonly specified for qualification testing. Some of the examples of commonly occurring broadband processes are (1) pressure fluctuation on the surface of a flight vehicle due to boundary layer turbulence and jet noise, (2) atmospheric turbulence, and (3) ground acceleration during earthquakes.

A random process is said to be a *narrowband process* if its psd has significant values over a narrow frequency band around a central frequency. A typical sample function (a), the autocorrelation function (b), and the psd (c) of a narrowband random process are shown in figure 4.6. A narrowband random process resembles a harmonic function with randomly varying amplitudes. Generally such processes arise as the response of strongly resonant systems to random excitation.

4.5.3 Cross-Spectral Properties of Random Processes

So far we have discussed the spectral properties of a single random process. Often it becomes necessary to consider the joint behavior of two or more random processes. In the time domain, the joint properties of random processes are described by cross-covariance and cross-correlation functions defined by (4.16). The joint spectral properties can be described analogously by a cross-spectral density function. Consider two random processes $X(t)$ and $Y(t)$. The generalized cross psd is defined as

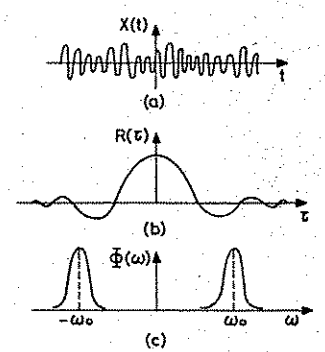

Figure 4.6 Narrowband random process: (a) sample function, (b) autocorrelation function, and (c) power spectral density.

$$\hat{\Phi}_{XY}(\omega_1, \omega_2) = E[\hat{X}(\omega_1)\hat{Y}^*(\omega_2)]$$
$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{XY}(t_1, t_2)e^{-i(\omega_2 t_2 - \omega_1 t_1)}\,dt_1\,dt_2. \tag{4.120}$$

By the Fourier inversion theorem

$$K_{XY}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{XY}(\omega_1, \omega_2)e^{i(\omega_2 t_2 - \omega_1 t_1)}\,d\omega_1\,d\omega_2. \tag{4.121}$$

If $X(t)$ and $Y(t)$ are stationary, the cross psd is defined by

$$\Phi_{XY}(\omega) = \lim_{T\to\infty}\frac{\pi}{T}E[\hat{X}(\omega, T)\hat{Y}^*(\omega, T)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XY}(\tau)e^{-i\omega\tau}\,d\tau, \tag{4.122}$$

and by the Fourier inversion theorem

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} \Phi_{XY}(\omega)e^{i\omega\tau}\,d\omega. \tag{4.123}$$

The cross psd is generally complex and has following properties:

$$\Phi_{XY}(-\omega) = \Phi_{XY}^*(\omega) = \Phi_{YX}(\omega). \tag{4.124}$$

It can be expressed as

$$\Phi_{XY}(\omega) = C_{XY}(\omega) - iQ_{XY}(\omega), \tag{4.125}$$

where $C_{XY}(\omega)$ is called the *cospectrum* and $Q_{XY}(\omega)$ is called the *quad-spectrum*. From (4.124) it is clear that $C_{XY}(\omega)$ is a real-valued even function of ω and that $Q_{XY}(\omega)$ is a real-valued odd function of ω. It can be shown that

$$|\Phi_{XY}(\omega)|^2 \le \Phi_{XX}(\omega)\Phi_{YY}(\omega); \tag{4.126}$$

hence $\Phi_{XY}(\omega)$ exists if $\Phi_{XX}(\omega)$ and $\Phi_{YY}(\omega)$ exist. The normalized cross psd function is called the *coherence function* and is defined as

$$\gamma_{XY}^2(\omega) = \frac{|\Phi_{XY}(\omega)|^2}{\Phi_{XX}(\omega)\Phi_{XX}(\omega)}. \tag{4.127}$$

It follows from (4.126) that

$$0 \le \gamma_{XY}^2(\omega) \le 1. \tag{4.128}$$

If $X(t)$ and $Y(t)$ are treated as the elements of a 2-dimensional vector, the psd matrix of the vector can be defined as

$$\mathbf{\Phi}(\omega) = \begin{bmatrix}\Phi_{XX}(\omega) & \Phi_{XY}(\omega)\\ \Phi_{YX}(\omega) & \Phi_{YY}(\omega)\end{bmatrix}. \tag{4.129}$$

In view of (4.124), $\mathbf{\Phi}(\omega)$ is a hermitian matrix.

The cross-spectral properties of two random processes can be readily extended to a system of n random processes. Treating n stationary random processes as elements of an n-dimensional vector, the $n \times n$ psd matrix can be defined in the same way as (4.129).

4.5.4 Spectral Moments of Stationary Random Processes

The power spectral density function $\Phi(\omega)$ defined by (4.93) is called the *two-sided psd* since its argument ω takes on values from $-\infty$ to ∞. From (4.96) it is seen that $\Phi(\omega)$ is an even function of ω, and therefore (4.95) can be expressed as

$$\sigma^2 = \int_{-\infty}^{\infty}\Phi(\omega)\,d\omega = \int_0^{\infty} 2\Phi(\omega)\,d\omega = \int_0^{\infty} G(\omega)\,d\omega, \tag{4.130}$$

where $G(\omega) = 2\Phi(\omega)$. $G(\omega)$ is called the *one-sided psd* since its argument ω takes on only positive values.

Several important properties of a stationary random process can be expressed in terms of the first few moments of the one-sided psd $G(\omega)$ (Vanmarcke, 1972). Let

$$\lambda_n = \int_0^{\infty}\omega^n G(\omega)\,d\omega, \quad n = 0, 1, 2, \ldots, \tag{4.131}$$

and

$$\gamma_n = \left(\frac{\lambda_n}{\lambda_0}\right)^{1/n}, \quad n = 1, 2; \tag{4.132}$$

λ_n is called the *nth spectral moment* of the random process. Clearly $\lambda_0 = \sigma^2$ is the variance of the random process; γ_1 and γ_2, defined by (4.132), are the normalized moments and have the dimension of circular frequency. The first normalized moment, γ_1, called the *central frequency*, may be interpreted as the abscissa of the "centroid" of the "spectral mass" $G(\omega)$, as shown in figure 4.7. The second normalized moment, γ_2, is analogous to the "radius of gyration" of $G(\omega)$ about the origin.

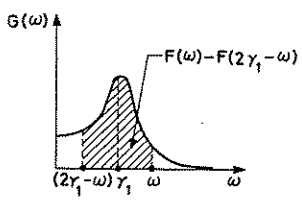

Figure 4.7 Spectral moments of a stationary random process.

A parameter q, called the *spectral parameter*, may be defined in terms of the spectral moments:

$$q = \left(1 - \frac{\lambda_1^2}{\lambda_0 \lambda_2}\right)^{1/2} = \frac{(\gamma_2^2 - \gamma_1^2)^{1/2}}{\gamma_2} = \frac{\gamma_s}{\gamma_2}, \tag{4.133}$$

where γ_s is analogous to the "radius of gyration" of $G(\omega)$ about its central frequency γ_1. Thus parameter q depends only on the dispersion of $G(\omega)$ about its central frequency. From the Schwarz inequality, $0 \le \dfrac{\lambda_1^2}{(\lambda_0 \lambda_2)} \le 1$, and therefore $0 \le q \le 1$.

It is of interest to note that the normalized spectral density function $\lambda_0^{-1} G(\omega)$ is similar to a probability density function in some respects. Both are nonnegative and have a unit area. Hence γ_1, γ_2, and γ_s may be interpreted as the "mean," the "mean square," and the "standard deviation" of the function $\lambda_0^{-1} G(\omega)$. The *partial mean square* of a stationary random process may be defined as

$$F(\omega) = \lambda_0^{-1} \int_0^{\omega} G(u)\, du; \tag{4.134}$$

$F(\omega)$ increases monotonically from zero (at $\omega = 0$) to one (at $\omega \to \infty$) and is similar to the probability distribution function. By an argument similar to the one involving the Chebyshev inequality, it can be shown (figure 4.7) that

$$F(\omega) - F(2\gamma_1 - \omega) \ge 1 - \left(\frac{\omega - \gamma_1}{\gamma_s}\right)^{-2} \ge 1 - q^2 \left(\frac{\omega - \gamma_2}{\gamma_2}\right)^{-2},$$

$$\gamma_1 \le \omega \le 2\gamma_1,$$

and

$$F(\omega) \ge 1 - q^2 \left(\frac{\omega - \gamma_2}{\gamma_2}\right)^{-2}, \quad \omega \ge 2\gamma_1 \tag{4.135}$$

since $F(2\gamma_1 - \omega) = 0$ for $\omega \ge 2\gamma_1$.

Example 4.5 Consider a stationary random process with an ideal band-limited one-sided psd

$$G(\omega) = \begin{cases} G_0, & \omega_a \le \omega \le \omega_b \\ 0, & \text{otherwise.} \end{cases}$$

The first three spectral moments of the random process are given by

$$\lambda_0 = G_0(\omega_b - \omega_a), \quad \lambda_1 = \frac{G_0}{2}(\omega_b^2 - \omega_a^2), \quad \lambda_2 = \frac{G_0}{3}(\omega_b^3 - \omega_a^3).$$

The spectral parameter q, can be expressed as

$$q = \left[1 - \frac{3\omega_m^2(\omega_b - \omega_a)}{\omega_b^3 - \omega_a^3}\right]^{1/2}, \tag{4.136}$$

where $\omega_m = (\omega_a + \omega_b)/2$ is the midband frequency. By setting

$$Q = \frac{\omega_m}{\omega_b - \omega_a}$$

it can be shown that

$$q = \frac{1}{Q}\left[\frac{1}{12 + Q^{-2}}\right]^{1/2}.$$

For a narrowband random process, $(\omega_b - \omega_a) \ll \omega_m$; that is, $Q \gg 1$ and

$$q \simeq 1/(\sqrt{12}Q). \tag{4.137}$$

4.6 Multiparameter Random Processes

So far we have discussed the properties of single parameter random processes. A multiparameter random process is a function of two or more indexing parameters. Let $\bar{u}$ be a vector with components $u_1, u_2, \ldots, u_n$ representing n indexing parameters of a random process $X(\bar{u})$. The random process $X(\bar{u})$ is said to be *homogeneous in the wide sense* if

$$E[X(\bar{u})] = \mu \tag{4.138}$$

and

$$K(\bar{u}^1, \bar{u}^2) = E[\{X(\bar{u}^1) - \mu\}\{X(\bar{u}^2) - \mu\}] = \Gamma(\bar{u}^2 - \bar{u}^1) = \Gamma(\bar{v}), \tag{4.139}$$

where $\bar{v} = \bar{u}^2 - \bar{u}^1$. Without any loss of generality, let $\mu = 0$. Then $\Gamma(\bar{v}) = R(\bar{v})$ and

$$\Phi(\bar{\omega}) = \frac{1}{(2\pi)^n} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{} e^{-i(\bar{\omega}\cdot\bar{v})} R(\bar{v})\, dv_1\, dv_2 \cdots dv_n. \tag{4.140}$$

Take the inverse Fourier transform:

$$R(\bar{v}) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{} e^{i(\bar{\omega}\cdot\bar{v})} \Phi(\bar{\omega})\, d\omega_1\, d\omega_2 \cdots d\omega_n. \tag{4.141}$$

Thus the psd of a multiparameter homogeneous random process is a function of n frequencies $\omega_1, \omega_2, \ldots, \omega_n$ and forms a transform pair with the autocorrelation function $R(\bar{v})$.

A special case of multiparameter homogeneous random process is an

isotropic random process. An isotropic random process depends only on the distance $\bar{v}$ between two points of the n-dimensional vector space formed by $\bar{u}$ and does not depend on the direction of the vector $\bar{v}$. By introducing n-dimensional spherical coordinates with direction $\bar{\omega}$ as the polar axis and carrying out the integration over $n-1$ angular coordinates, it can be shown (Sveshnikov, 1966) that

$$\Phi(\omega) = \frac{2^{(n-2)/2}k_n}{(2\pi)^n}\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)\int_0^\infty J_{(n-2)/2}(\omega v)\frac{R(v)}{(\omega v)^{(n-2)/2}}v^{n/2}\,dv \tag{4.142}$$

and

$$R(v) = 2^{(n-2)/2}k_n\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)\int_0^\infty J_{(n-2)/2}(\omega v)\frac{\Phi(\omega)}{(\omega v)^{(n-2)/2}}d\omega, \tag{4.143}$$

where k_n is the $(n-2)$-fold integral of the factor of the jacobian of transformation and $J_{(n-2)/2}(\omega v)$ is the Bessel function of order $(n-2)/2$. Thus the psd and the correlation functions of an isotropic multiparameter random process are functions of a single variable, as in the case of a single-parameter random process. However, they do not form a Fourier transform pair, as in the case of a single-parameter random process. The results of linear operations on multiparameter random processes are similar to the results for a single-parameter random process.

Exercises

4.1 A random process $X(t)$ is given by

$$X(t) = A\cos\omega t + B\sin\omega t,$$

where ω is a constant and A and B are independent, identically distributed normal random variables with zero means and standard deviations σ. Show that

$$E[X(t)] = 0,$$

$$K(t_1, t_2) = \sigma^2\cos\omega(t_2 - t_1) = \sigma^2\cos\omega\tau,$$

$$p(x, t) = \frac{1}{\sqrt{2\pi}\sigma}e^{-x^2/2\sigma^2},$$

and

$$p(x_1, t_1, x_2, t_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\cos^2\omega\tau}}\exp\left[-\frac{x_1^2 - 2x_1x_2\cos\omega\tau + x_2^2}{2\sigma^2(1-\cos^2\omega\tau)}\right].$$

4.2 A random process $X(t)$ is given by $X(t) = e^{-At}$, where A is a random variable with probability density function $p(a)$. Find $E[X(t)]$, $E[X(t_1)X(t_2)]$, and $p(x, t)$.

4.3 A random process $X(t)$ is defined as

$$X(t) = \int_0^\infty [U(\omega)\cos\omega t + V(\omega)\sin\omega t]\,d\omega,$$

where $U(\omega)\,d\omega$ and $V(\omega)\,d\omega$ are random variables with zero means and variances equal to $G(\omega)\,d\omega$. Also

$$E[U(\omega_1)V(\omega_2)] = 0 \quad \text{for all } \omega_1, \omega_2 > 0,$$

$$E[U(\omega_1)U(\omega_2)] = E[V(\omega_1)V(\omega_2)] = 0 \quad \text{for } \omega_1 \neq \omega_2.$$

Show that $X(t)$ is stationary with

$$E[X(t)] = 0, \quad R(\tau) = \int_0^\infty G(\omega)\cos\omega\tau\,d\omega.$$

Further, if

$$\tilde{X}(t) = \int_0^\infty [U(\omega)\cos\omega t - V(\omega)\sin\omega t]\,d\omega,$$

show that $E[X(t)\tilde{X}(t)] = 0$.

4.4 Determine in each case whether the random processes with the autocorrelation functions given below are differentiable in the mean:

a. $R(\tau) = \sigma^2 e^{-\alpha|\tau|}\cos\beta\tau$, $\alpha > 0$;

b. $R(\tau) = \sigma^2 e^{-\alpha|\tau|}(1 + \alpha|\tau|)$, $\alpha > 0$;

c. $R(\tau) = \sigma^2 e^{-\alpha\tau^2}\cos\beta\tau$, $\alpha > 0$;

d. $R(\tau) = a^2/(a^2 + \tau^2)$.

Determine the autocorrelation function of the derivative processes for the cases where the parent processes are differentiable.

4.5 Consider a random process $X(t)$ defined by $Y(t) = \int_0^t X(s)\,ds$. If $X(s)$ is a real stationary random process with zero mean, show that

$$K_{YY}(t_1, t_2) = \int_0^{t_2}(t_2 - \tau)R_{XX}(\tau)\,d\tau + \int_0^{t_1}(t_1 - \tau)R_{XX}(\tau) - \int_0^{t_2-t_1}(t_2 - t_1 - \tau)R_{XX}(\tau)\,d\tau$$

and

$$\mathrm{Var}[Y(t)] = 2\int_0^t (t - \tau)R_{XX}(\tau)\,d\tau.$$

4.6 Determine the expression for the psd of each random process defined in exercise 4.4. Check differentiability using (4.102).

4.7 Determine the psd of the random process $X(t)$ given by

a. $X(t) = ae^{iYt}$, where Y is a random variable with probability density function $p(y)$, and

b. $X(t) = a\cos(Yt + B)$, where a is real, B is a uniformly distributed random variable with range $(-\pi, \pi)$, and Y is the same as in (a).

4.8 A random process $Z(t)$ is given by $Z(t) = X(t)Y(t)$, where $X(t)$ and $Y(t)$ are independent, stationary random processes. Determine the expression for the psd of $Z(t)$ in terms of the psd of $X(t)$ and $Y(t)$.

4.9 A random process $Z(t)$ is given by $Z(t) = X(t) + X(t - T)$, where $X(t)$ is a stationary random process with psd

$$\Phi_{XX}(\omega) = \frac{C}{\omega^2 + \alpha^2}$$

and T is a constant. Determine the psd of $Z(t)$ in terms of the psd of $X(t)$.

4.10 The one-sided psd of a stationary narrowband random process is given by

$$G(\omega) = \frac{G_0}{(\omega_0^2 - \omega^2)^2 + 4\zeta^2\omega^2\omega_0^2}, \quad 0 \le \omega \le \infty.$$

Show that

$$\lambda_0 = \frac{\pi G_0}{4\zeta\omega_0^3},$$

$$\lambda_1 = \frac{\omega_0^2}{[\omega_0^2(1 - \zeta^2)]^{1/2}} \frac{\pi G_0}{4\zeta\omega_0^3}\left[1 - \frac{1}{\pi}\tan^{-1}\left(\frac{2\zeta(1 - \zeta^2)^{1/2}}{(1 - 2\zeta^2)^{1/2}}\right)\right],$$

$$\lambda_2 = \frac{\pi G_0}{4\zeta\omega_0},$$

and

$$q = \frac{2}{\sqrt{\pi}}\zeta^{1/2} \quad \text{for } \zeta \ll 1.$$

Approximate the actual psd by a band-limited psd centered at ω_0 with

$$G'(\omega) = \begin{cases} \dfrac{G_0}{\omega_0^4(2\zeta)^2}, & \omega_0\left(1 - \dfrac{\pi\zeta}{2}\right) \le \omega \le \omega_0\left(1 + \dfrac{\pi\zeta}{2}\right) \\ 0, & \text{otherwise} \end{cases}$$

and show that

$$\frac{q}{q'} = \frac{4\sqrt{3}}{\pi^{3/2}}\zeta^{-1/2} \simeq 12.5 \quad \text{for } \zeta = 0.01.$$

From inequality (4.135) determine the upper bound on the partial mean square value contributed by the bandwidth $2\zeta\omega_0$ centered at ω_0 for $\zeta \ll 1$.

4.11 Consider a uniformly modulated random process

$$Y(t) = \begin{cases} \psi(t)X(t), & 0 \le t \le T \\ 0, & \text{otherwise,} \end{cases}$$

where $X(t)$ is white noise with $R_{XX}(t_2 - t_1) = 2\pi\Phi_0\delta(t_2 - t_1)$. Show that $\hat{Y}(\omega)$, the Fourier transform of $Y(t)$, is a complex, homogeneous random process with ω as the indexing parameter. Further, show that the autocovariance and the psd-like functions of $\hat{Y}(\omega)$ are given by

$$R_{\hat{Y}\hat{Y}}(\omega_2 - \omega_1) = \frac{\Phi_0}{2\pi}\int_0^T I(t)\exp[-i(\omega_2 - \omega_1)t]\,dt$$

and

$$\Phi_{\hat{Y}\hat{Y}}(\Omega) = \begin{cases} \dfrac{\Phi_0}{2\pi}I(\Omega), & 0 \le \Omega \le T \\ 0, & \text{otherwise,} \end{cases}$$

where $I(t) = \psi^2(t)$ is the intensity function associated with $Y(t)$ and Ω is the frequencylike parameter associated with the indexing parameter ω.