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Reliability analysis under epistemic uncertainty

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ABSTRACT

This paper proposes a probabilistic framework to include both aleatory and epistemic uncertainty within model-based reliability estimation of engineering systems for individual limit states. Epistemic uncertainty is considered due to both data and model sources. Sparse point and/or interval data regarding the input random variables leads to uncertainty regarding their distribution types, distribution parameters, and correlations; this statistical uncertainty is included in the reliability analysis through a combination of likelihood-based representation, Bayesian hypothesis testing, and Bayesian model averaging techniques. Model errors, which include numerical solution errors and model form errors, are quantified through Gaussian process models and included in the reliability analysis. The probability integral transform is used to develop an auxiliary variable approach that facilitates a single-level representation of both aleatory and epistemic uncertainty. This strategy results in an efficient single-loop implementation of Monte Carlo simulation (MCS) and FORM/SORM techniques for reliability estimation under both aleatory and epistemic uncertainty. Two engineering examples are used to demonstrate the proposed methodology.

1. Introduction

Reliability analysis is concerned with the assessment of system performance in the presence of uncertainty, which has generally been classified into two types: aleatory (natural variability) and epistemic (lack of knowledge). The reliability estimate is affected by both types of uncertainty; however, extensive previous literature in model-based reliability analysis has predominantly considered the former type and not the latter. While several alternative frameworks have been explored to represent uncertainty, this paper considers the probabilistic framework. In this context, the probability of failure is represented as

$$P_f = \int_{g(\boldsymbol{X})\le 0} f_{\boldsymbol{X}}(\boldsymbol{x})\, d\boldsymbol{x} \tag{1}$$

where P_f is the probability of failure, $\boldsymbol{X}$ is the vector of input random variables, $f_{\boldsymbol{X}}(\boldsymbol{x})$ is the joint probability distribution function (PDF) of $\boldsymbol{X}$, $g(\boldsymbol{X})$ is the performance (limit state) function, and $g(\boldsymbol{X})\le 0$ represents the failure domain. Different types of Monte Carlo simulation methods, as well as analytical integration techniques such as first-order and second-order reliability methods (FORM, SORM), have been developed [1] to evaluate Eq. (1).

The evaluation of the multi-dimensional integral in Eq. (1) can be difficult; therefore, First Order Reliability Methods (FORM) approximate the limit state function, which could be non-linear, with a first-order (linear) approximation while the Second Order Reliability Methods (SORM) estimate the failure probability by employing a second-order approximation to the limit state. Refer to [1] for more details.

Due to insufficient information, uncertainty may arise about the exact values of deterministic variables or the distribution characteristics of random variables in Eq. (1). This is referred to as statistical uncertainty. Several theories, both probabilistic [2–4] and non-probabilistic [5], have been used to represent this type of epistemic uncertainty. Some of the approaches include interval analysis [5], convex models [6], fuzzy sets and possibility theory [7], evidence theory [8], Bayesian probability theory [3] and imprecise probabilities [9].

This paper uses a Bayesian probabilistic approach to model epistemic uncertainty about the input random variables. A random variable may be represented using a parametric (e.g., normal) or a non-parametric distribution. A parametric distribution is associated with a distribution type and distribution parameters. If the distribution type of an input variable X is known but the distribution parameters are uncertain, then X can be represented by a distributional p-box. If the distribution type is also uncertain, then X may be represented by a free p-box [10].

Further, it may be difficult to obtain joint data on all the variables in the system due to limited resources. In such cases, the correlations between variables are also uncertain. In some cases, qualitative information that some variables are positively or

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negatively correlated might be available. It is desirable to include such information in reliability estimation; however, methods to include correlation uncertainty in reliability estimation are not yet fully explored.

A simple implementation of reliability analysis in the presence of only distribution parameter uncertainty may be through a nested double-loop MCS or a nested MCS-FORM/SORM approach where the distribution parameters are sampled in the outer loop, and for each realization of the parameters, failure probability is calculated in the inner loop using MCS or FORM/SORM. The result of double-loop analysis may be described through an average of all estimates of failure probability or by representing the failure probability itself as a distribution [11]. The hierarchical formulation of uncertainty sources according to the Rosenblatt transformation [12] offers a more convenient approach compared to double-loop sampling. Recently, Sankararaman and Mahadevan [13] proposed a new method of single-loop sampling using the concept of an auxiliary variable, based on the probability integral transform [1], in which samples of a variable are obtained through simultaneous sampling of parameters and CDF values. This approach is used in this paper for faster computation.

The statistical uncertainty discussed above constitutes one category of epistemic uncertainty; another category is model uncertainty [14]. Models are built to explain the real world phenomena and frequently involve assumptions, simplifications and generalizations. Models may be based on first principles (physics-based) or derived from data (data-driven). Model uncertainty represents the inability of these models to accurately represent the true physical behavior of the system. Uncertainty due to a model may be due to three sources: (1) model parameters, due to limited data; (2) numerical solution errors that arise from the methodology adopted in solving the model equations; and (3) model form errors, which arise due to assumptions and simplifications made in the development of models. Model calibration is used to estimate the model parameters using input-output data. Model verification can be used to quantify numerical solution errors (e.g., finite element discretization error, surrogate model error, round-off error, etc.). Model form errors can be estimated by comparing the model predictions against physical observations (e.g., model validation tests). Whenever physical observations are used for either model calibration or model validation, measurement uncertainties also arise, and these contribute to the uncertainty in the model prediction. Discretization error arises when the solution of the continuum domain is computed using numerical techniques (e.g., finite element methods) which involve discretization of the continuum domain. Surrogate models are often used in uncertainty quantification, reliability analysis and design optimization when high fidelity physics models are computationally expensive. The estimation of surrogate model error involves comparing the output of the original model with the surrogate model.

The next stage after quantification of different types of epistemic uncertainty is their inclusion in a framework for reliability estimation. This paper proposes a probabilistic framework to include both forms of epistemic uncertainty and aleatory uncertainty in reliability analysis. The main issue is that reliability analysis techniques such as MCS, FORM, etc. are wrapped around deterministic physics models, i.e., for a fixed input value, the model output is deterministic. In the presence of model uncertainty, the model output is not deterministic even for a fixed input. When variability and input statistical uncertainty are added, the model output is in the form of multiple probability distributions. Further, the various uncertainty sources and errors do not combine in a simple manner; they occur at different stages of the analysis, and their combination could be nonlinear, nested or iterative. Reliability analysis in the presence of multiple sources and types of uncertainty is thus not straightforward; this paper seeks to overcome this challenge. Current FORM-based reliability analysis methods have included either parameter uncertainty [15] or model errors [16] but distribution type uncertainty, uncertain correlations or combination of several uncertainty sources have not been considered. Similarly, Monte Carlo-based methods have not considered the various epistemic uncertainty sources in reliability analysis.

The overall contribution of the paper is a comprehensive and systematic framework for quantifying and aggregating the contributions of different types of epistemic uncertainty (statistical and model uncertainties) in a manner suitable for *reliability analysis* using FORM and Monte Carlo sampling. The key contributions of this paper can be summarized as follows – (1) quantification of different types of statistical uncertainty (distribution parameter and distribution type uncertainty, and uncertainty about correlations) and model uncertainty (model form and numerical solution errors) within a probabilistic framework; (2) development of a novel FORM-based approach to include different types of epistemic uncertainty (data, model) along with aleatory uncertainty within reliability analysis by utilizing the concepts of auxiliary variable and theorem of total probability; and (3) development of a single-loop Monte Carlo sampling approach for the inclusion of both aleatory and epistemic uncertainty in reliability estimation.

The rest of the paper is organized as follows. In Section 2, procedures to quantify various types of epistemic uncertainty are presented. Section 3 develops the proposed methodologies (using FORM and Monte Carlo sampling) for reliability estimation in the presence of aleatory and epistemic uncertainty. In Section 4, a structural reliability example and a fluid–structure interaction problem (airplane wing) are used to demonstrate the application of the proposed methods. Concluding remarks are provided in Section 5.

2. Representation of epistemic uncertainty

In this section, procedures for the representation of epistemic uncertainty due to data and model sources are discussed in order to facilitate reliability analysis through MCS and FORM techniques.

2.1. Distribution parameter uncertainty

In the presence of sparse point data on X, two approaches may be used to construct the probability distributions of distribution parameters Θ (using a Bayesian perspective). The first approach is to use resampling methods such as Jack-knife and Bootstrap [17] to generate multiple values of Θ that are used to construct their distributions; the second approach is to use a likelihood-based representation of the available data to construct distributions of Θ using Bayes' theorem [18]. The likelihood-based approach can be extended to accommodate interval data and to construct parametric as well as non-parametric distributions [19]; this approach is adopted in this paper.

Let a dataset D for a variable X consist of n point data $p_i(i = 1 \text{ to } n)$ and m interval data $[a_j, b_j](j = 1 \text{ to } m)$. The likelihood function for the distribution parameters $\boldsymbol{\Theta}$ can be constructed as

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f_X(x=p_i \mid \boldsymbol{\theta}) \prod_{j=1}^{m} \left[F_X(x=b_j\boldsymbol{\theta}) - F_X(x=a_j \mid \boldsymbol{\theta}) \right] \tag{2}$$

where $f_X(x)$ and $F_X(x)$ represent the PDF and CDF of variable X respectively. After constructing the likelihood function, the distributions of the distribution parameters are obtained using Bayes' theorem as

$$f''_{\Theta}(\theta) = \frac{L(\theta)f'_{\Theta}(\theta)}{\int_{\Omega^{\Theta}} L(\theta)f'_{\Theta}(\theta)d\theta} \tag{3}$$

where $f'_{\Theta}(\theta)$ and $f''_{\Theta}(\theta)$ refer to the joint prior and posterior distributions of the distribution parameters. Ω^{Θ} refers to the domain of the Θ.

Sometimes, point data and/or interval data may be directly available on the distribution parameters. One scenario where point and interval data are available is when domain experts provide their opinions about the possible values for the model parameters. In this case, some experts might provide point data and some other experts might provide intervals. In such a case, a non-parametric distribution can be directly fit to represent the model parameters using available data. Since, data is available from multiple sources, it can be assumed independent. A detailed procedure for the construction of a non-parametric distribution is described in [19]. Only important steps are presented here for the sake of completeness.

Let the data available on Θconsist of r point data, $\theta_p^i (i = 1 to\ r)$ and s interval data $\left[\theta_a^j, \theta_b^j\right](j = 1 to\ s)$. From the available data, the range of Θ is obtained by observing the maximum and minimum values. The domain is then discretized into Q points, given by $\theta_1, \theta_2, \ldots \theta_Q$ and the PDF values at these discretized points are represented by $\alpha_1, \alpha_2, \ldots \alpha_Q$. Using $(\theta_k, \alpha_k)(k = 1 to\ Q)$, the probability density function can be constructed through an interpolation technique (e.g., linear, spline-based or Gaussian-process interpolation) over these Qpoints. The likelihood function, in this case, can be defined as

$$L(\alpha) = \prod_{i=1}^{r} f_{\Theta}(\theta = \theta_p^i \mid \alpha) \prod_{j=1}^{s} \left[F_{\Theta}\left(\theta = \theta_b^i \alpha\right) - F_{\Theta}(\theta = \theta_a^i \mid \alpha) \right] \tag{4}$$

These PDF values, $\alpha_k (k = 1 to\ Q)$, can be obtained by maximizing the above likelihood function, subject to the following constraints.

(1) $\alpha_k \geq 0$
(2) $f_{\Theta}(\theta) \geq 0$ for all values of θ
(3) $\int_{\Omega^{\Theta}} f_{\Theta}(\theta) = 1$

Since the α's represent the PDF values, they should be greater than or equal to zero, at any value of θ; this is specified in the first two constraints and finally, the area under the PDF is equal to 1, specified in the third constraint. The likelihood approach helps to construct either parametric or non-parametric distributions of a variable or its distribution parameters (Eqs. (2, 4)) and either of these options can be used depending on the available data and analysis requirements.

2.2. Distribution type uncertainty

In the presence of sparse data, when the distribution type is not known precisely, a set of competing distribution types can be obtained through a combination of prior knowledge and results from statistical goodness-of-fit tests. This uncertainty in the distribution type can be represented in a Bayesian fashion. Initially, when no data or prior knowledge is available, all the distribution types can be assumed equally probable and therefore, equal weights can be assigned to them (prior weights). When any new data is available, these weights can be updated (posterior weights) using Eq. (5). What this means is that distribution type is unknown, and this epistemic uncertainty is represented by different possible categories with prior weights that are updated with new data. In other words, the epistemic uncertainty regarding distribution type is represented by a categorical variable within the proposed Bayesian framework. Given a dataset D, the ratio of the posterior probabilities of two distribution types $f_X(x|\theta_c)$ and $f_X(x|\theta_d)$, is calculated as

$$\frac{Pr\ (f_X(x \mid \theta_c) \mid D)}{Pr\ (f_X(x \mid \theta_d) \mid D)} = \frac{Pr\ (D \mid f_X(x \mid \theta_c))}{Pr\ (D \mid f_X(x \mid \theta_d))} \frac{Pr\ (f_X(x\theta_c))}{Pr\ (f_X(x\theta_d))} \tag{5}$$

where $Pr\ (f_X(x|\theta_c))$ and $Pr\ (f_X(x|\theta_c))$ refer to the prior probabilities of the two distribution types and $\frac{Pr\ (D|f_X(x|\theta_c))}{Pr\ (D|f_X(x|\theta_d))}$ refers to the ratio of likelihoods (also referred to as Bayes factor B). The Bayes factor is a quantitative measure of extent of data support for $f_X(x|\theta_c)$ relative to support for $f_X(x|\theta_d)$. If $B > 1$, the dataset D supports $f_X(x\theta_c)$ over $f_X(x|\theta_d)$. With the list of possible distribution types, one could either construct a composite distribution, i.e., a weighted average of all the possible distribution types using Bayesian model averaging (BMA) [19,20], or select a single distribution type that best explains the observations. The weights for averaging or selection can be computed using Bayesian hypothesis testing (BHT) [19,21] by comparing the likelihoods of possible distribution types, as shown in Eq. (5). When BMA is used, a composite distribution can be constructed as

$$f_X(x \mid \theta) = \sum_{k=1}^{N} w_k f_X(x \mid \theta_{\boldsymbol{k}}) \tag{6}$$

where $f_X(x|\theta)$ represents the composite distribution, $f_X(x|\theta_{\boldsymbol{k}})(k = 1 to\ N)$ refers to each of the N possible distribution types with $w_k, \theta_{\boldsymbol{k}}$ representing their weights, distribution parameters and $\theta = \{\theta_{\boldsymbol{k}}: k = 1 to\ N\}$. Note that the weights are proportional to the posterior probabilities calculated using Eq. (5).

2.3. Uncertainty about correlations

As stated in Section 1, adequate data may not be always available to characterize the correlations between the input variables. Similar to the distribution parameters, parametric or non-parametric distributions may be used to represent the uncertainty regarding correlation coefficients. In the case of a parametric approach, bounded distributions such as a beta distribution may be used because correlation coefficients lie between −1 and 1. In a non-parametric approach, the procedure in Section 2.1 may be used to construct a non-parametric PDF to represent the uncertainty regarding the correlation coefficients. The latter procedure is illustrated in this paper.

2.4. Model uncertainty

Section 1 introduced different types of model errors. Formulations for their quantification are presented in this subsection. The model error ϵ_m is composed of solution approximation errors and model form error. Methods for the quantification of two prominent solution approximation errors (discretization error and surrogate model error) are discussed below, followed by the quantification of model form error.

Let g_{true} represent the response of the true system, g_{model} represent the prediction of the mathematical model, and ϵ_{mf} represent the model form error in the mathematical model; then the three variables are related as

$$g_{true}(\boldsymbol{X}) = g_{model}(\boldsymbol{X}) + \epsilon_{mf}(\boldsymbol{X}) \tag{7}$$

2.4.1. Discretization error

Often the mathematical model may be evaluated using a numerical technique such as the finite element method. Let g_h represent the prediction of the numerical technique. The accuracy of the solution depends on the mesh size h used in the analysis. Since evaluation at mesh size h~0 is computationally unaffordable in many cases, the finite element model is evaluated at coarse mesh sizes at the same input value and a model is built to estimate the output at mesh size h~0. Traditionally, Richardson's extrapolation [22] has been used to quantify discretization error. Other methods include discretization error transport equations [23] and residual/recovery methods in finite elements [24]. Recently, Rangavajhala et al. [25] proposed construction of a Gaussian Process (GP) model as an enhancement to Richardson's extrapolation, to alleviate the need for monotonic and asymptotic convergence. Predictions are carried out at several values of h and the corresponding numerical model outputs (g_h) are obtained. A GP model is constructed with this training data, and used to estimate the "corrected" output g_{cor} at mesh size h~0. Formulae used in the GP model are given below.

2.4.2. Surrogate model error

In many applications, a surrogate model g_{sm} is constructed to replace the expensive physics computational model and used in activities such as calibration and optimization that require repeated runs of the computational model. Surrogate models in the reliability analysis literature include polynomial chaos models, Gaussian process or Kriging models, radial basis function models, support vector machines and neural networks. A Gaussian Process (GP) surrogate model is illustrated in this paper. The training data for the GP model is the set of input values and the corresponding outputs from the expensive computational model. Note that instead of using the raw output g_h from the computational model for training, we use g_{cor} mentioned above.

$$(\boldsymbol{g_{sm}} \mid \boldsymbol{X_T}, \boldsymbol{g_{sm_T}}, \boldsymbol{X}, \boldsymbol{\Theta_{g_{sm}}}) \sim \boldsymbol{N}(\boldsymbol{\mu}, \boldsymbol{\sigma}) \tag{8}$$

$$\boldsymbol{\mu} = \boldsymbol{\mu}_0 + \boldsymbol{K_{PT}}\left(\boldsymbol{K_{TT}} + \boldsymbol{\sigma_n^2 I}\right)^{-1}(\boldsymbol{\epsilon m_{X_T}} - \boldsymbol{\mu_{X_T}}) \tag{9}$$

$$\boldsymbol{\sigma} = \boldsymbol{K_{PP}} - \boldsymbol{K_{PT}}\left(\boldsymbol{K_{TT}} + \boldsymbol{\sigma_n^2 I}\right)\boldsymbol{K_{TP}} \tag{10}$$

In Eq. (8), $\boldsymbol{g_{sm}}$ represents the surrogate model prediction at $\boldsymbol{X}$; $\boldsymbol{X_T}$ and $\boldsymbol{g_{sm_T}}$ correspond to the training inputs and the surrogate model predictions at $\boldsymbol{X_T}$ for building the GP model. $\boldsymbol{\Theta_{g_{sm}}}$ refers to the parameters of the GP (e.g., length-scale parameters in a squared-exponential correlation function). In Eq. (9), $\boldsymbol{\mu_{X_T}}, \boldsymbol{\mu}_0$ represent the mean values of the GP at the training inputs $\boldsymbol{X_T}$ and the prediction input $\boldsymbol{X}$. In Eq. (10), $\boldsymbol{K_{TT}}$ is the covariance matrix between the training inputs, $\boldsymbol{K_{TP}}$ is the vector representing the covariance between the prediction input ($\boldsymbol{X}$) and the training inputs ($\boldsymbol{X_T}$). $\boldsymbol{K_{PT}}$ represents the transpose of $\boldsymbol{K_{TP}}$, and $\boldsymbol{K_{PP}}$ represents the covariance matrix between the test inputs. $\boldsymbol{\sigma_n^2}$ represents the variance in the model predictions, and $\boldsymbol{I}$is the identity matrix. To account for the error associated with the surrogate model prediction g_{sm}, another set of evaluations (S_{set}) can be made at different input values and compared against g_{cor} computed at these new input values. The surrogate model error at the input values, $\boldsymbol{X} \in S_{set}$ is then calculated as

$$\epsilon_{se}(\boldsymbol{X}) = g_{cor}(\boldsymbol{X}) - g_{sm}(\boldsymbol{X}) \tag{11}$$

Note that both g_{cor} and g_{sm} are stochastic because of the use of the GP model, therefore ϵ_{se} is also stochastic. Using the set of input values $\boldsymbol{X} \in S_{set}$ and their corresponding surrogate model errors, another model, $g_{se}(\boldsymbol{X})$ is constructed to estimate the surrogate model error at any value of $\boldsymbol{X}$. Thus, we have an estimate of the mathematical model output based on the surrogate model, and the associated error. Eq. (7) becomes

$$g_{true}(\boldsymbol{X}) = g_{sm}(\boldsymbol{X}) + g_{se}(\boldsymbol{X}) + \epsilon_{mf}(\boldsymbol{X}) \tag{12}$$

2.4.3. Model form error

The next step is the quantification of model form error ϵ_{mf}, which may or may not be a function of input $\boldsymbol{X}$. The prediction accuracy in the estimation of the model form error depends on the number of experimental tests that can be undertaken. However, in complex engineering systems, only a limited number of experimental tests can be carried out due to high computational expense. For a given number of experimental tests (dependent on the cost constraints), we can compute the model form errors corresponding to the input values of these tests, by comparing model predictions against physical observations, and then build a GP model to estimate ϵ_{mf} at any value of $\boldsymbol{X}$. To obtain training data for building the GP model, a set of physical experiments (M_{set}) are conducted. If $g_{obs}(\boldsymbol{X})$ and $\epsilon_{obs}(\boldsymbol{X})$ denote the experimental observations and the corresponding observation errors respectively, then

$$g_{obs}(\boldsymbol{X}) = g_{true}(\boldsymbol{X}) + \epsilon_{obs}(\boldsymbol{X}) \tag{13}$$

Combining Eqs. (12) and (13),

$$g_{obs}(\boldsymbol{X}) - \epsilon_{obs}(\boldsymbol{X}) = g_{sm}(\boldsymbol{X}) + g_{se}(\boldsymbol{X}) + \epsilon_{mf}(\boldsymbol{X}) \tag{14}$$

Thus, the model form error at the input $\boldsymbol{X} \in M_{set}$ can be obtained as

$$\epsilon_{mf}(\boldsymbol{X}) = g_{obs}(\boldsymbol{X}) - \epsilon_{obs} - g_{sm}(\boldsymbol{X}) - g_{se}(\boldsymbol{X}) \tag{15}$$

After obtaining the model form errors at a set of input values, a surrogate model $g_{mf}(\boldsymbol{X})$ may be constructed to quantify the model form error to use for predictions at other unobserved values of $\boldsymbol{X}$. An additive uncertainty model has often been used to represent model discrepancy (such as in the Kennedy and O'Hagan calibration framework). However, multiplicative models can also be used. Thus, the overall prediction (g_{pred}) after accommodating the various errors can be written as

$$g_{pred}(\boldsymbol{X}) = g_{sm}(\boldsymbol{X}) + g_{se}(\boldsymbol{X}) + g_{mf}(\boldsymbol{X}) \tag{16}$$

One advantage of fitting GP models is that the overall model error, $\epsilon_m(\boldsymbol{X}) = g_{se}(\boldsymbol{X}) + g_{mf}(\boldsymbol{X})$ also follows a GP, due to the summation.

3. Including epistemic uncertainty in reliability analysis

After quantifying various types of epistemic uncertainty in Section 2, the next challenge is to include these estimates in reliability analysis. Two commonly used approaches, MCS and FORM, are considered in this section. The proposed approaches can also be adapted to other reliability analysis methods.

3.1. Using Monte Carlo simulation to handle epistemic uncertainty

3.1.1. Distribution parameter uncertainty alone

If a random variable X has fixed distribution parameters Θ, then X can be represented by a single PDF. Samples of X are obtained using the inverse CDF method, i.e., $x = F_X^{-1}(u|\Theta)$, where u is a uniform random number in [0, 1]. In the case where distribution parameters Θ are uncertain, each realization of the distribution parameters leads to a different PDF for X; thus X may be represented by a family of PDFs. Conventional methods for sampling in the presence of distribution parameter uncertainty, which include nested double-loop MCS and nested MCS-FORM approach were mentioned in Section 1. This is computationally expensive; therefore a single-loop approach is presented below.

3.1.1.1. Single loop sampling using an auxiliary variable. An auxiliary variable u may be defined to represent the contribution of aleatory uncertainty to the overall uncertainty [13] using the probability integral transform as

$$u = F_X(x\Theta = \theta) = \int_{-\infty}^{x} f_X(w \mid \Theta = \theta)dw \tag{17}$$

where u is the auxiliary variable, θ is a realization of the distribution parameter Θ, w is a dummy variable used for integration, and $F_X(x|\Theta=\theta)$ denotes the CDF value of variable X for a realization of Θ. The variability in X is represented by this auxiliary variable u, which follows a uniform distribution between 0 and 1 (since the CDF values range from 0 to 1). Also, it can be observed that the auxiliary variable is statistically independent of Θ [13]. The statistical independence between u and Θ helps us to simultaneously generate samples of u and Θ, thus being able to use a single loop instead of a nested double-loop for sampling. After generating samples of u and Θ, samples of X are obtained using $x = F_X^{-1}(u|\Theta=\theta)$. This approach helps to generate one sample value of X for each realization of u and Θ. Several samples of X are generated and propagated through the limit state for reliability estimation.

3.1.2. Distribution type uncertainty and parameter uncertainty

When the distribution type is unknown, a composite distribution may be assumed for the random variable, with the weights of candidate distributions derived using likelihood comparisons to each other. A new discrete variable d_X, is introduced to represent the distribution type uncertainty in a random variable X. The values taken by d_X are the distribution types X can take. The probability masses (weights) of d_X values are obtained by comparing the likelihoods of observing the available data with each of the distribution types, as mentioned in Section 2.2. A simplistic approach to sampling X in the presence of distribution type uncertainty and parameter uncertainty is by a nested three-loop process, where distribution type is sampled in the outermost loop, distribution parameters in the middle loop, and CDF values in the innermost loop. However, using the auxiliary variable representation, the nested three-loop sampling can be collapsed into a single-loop sampling procedure, represented as $x = F^{-1}(u|\Theta=\theta, d_X=d_X^*)$. For each random realization of auxiliary variable (u), distribution type (d_X), and distribution parameters (Θ) drawn from their corresponding distributions, one sample of the input variable X can be obtained using the inverse CDF.

3.1.3. Sampling with uncertain correlations, distribution parameter uncertainty, and distribution type uncertainty

The procedure for correlated sampling under fixed parameters, correlation coefficients and known distribution type is well known [26]; this procedure is first extended for sampling under distribution parameter and distribution type uncertainty, and then to include uncertain correlations.

Correlated input variables can be first transformed from the original space to a correlated standard normal space, from which they are transformed to an uncorrelated reduced normal space using orthogonal decomposition. Samples are generated in the uncorrelated space and are transformed to the correlated standard normal space, and to the original space using the inverse CDF, conditioned on the distribution parameters [1]. In the presence of parameter and distribution type uncertainty, this transformation cannot be carried out uniquely because the input variable cannot be represented using a single PDF but a family of PDFs. To overcome this problem, marginal unconditional PDFs can be constructed first, defined as

$$f_X(x) = \int f_X(x\theta) f_{\Theta}(\theta) d\theta \tag{18}$$

Note that in the presence of distribution type uncertainty, the conditional distribution $f_{X|\Theta}(x\theta)$ refers to the composite distribution, shown in Eq. (6). The transformation from correlated standard normal space to the original space in the presence of distribution parameter and distribution type uncertainty can be carried out in two steps – (1) constructing an unconditional CDF corresponding to the unconditional PDF; (2) using this unconditional CDF for the inverse transformation. Samples of each X can be generated using the auxiliary variable for several realizations of distribution parameters and distribution types, as shown in Sections 3.1.1 and 3.1.2. Using the samples of X, an empirical cumulative distribution function (eCDF) [27] is constructed, which represents the unconditional CDF. This eCDF can be used for inverse CDF computations from standard normal space to the original space.

The samples generated above are for fixed correlation coefficients. In the presence of uncertain correlation coefficients, samples of each correlation coefficient can be drawn from its corresponding non-parametric distribution (Section 2.3). One sample of each correlation coefficient can be used to create a correlation matrix and a correlated sample can be obtained using the procedure described above. Several samples of each correlation coefficient can be drawn, thereby generating several correlated samples of the input variables. These samples can then be propagated through the system model for reliability estimation.

3.1.4. Inclusion of model errors

In Section 2.4, the overall model error $\epsilon_m(\boldsymbol{X})$ is represented by a Gaussian process, thus the model error follows a normal distribution at every point $\boldsymbol{X}$, with parameters (mean and standard deviation) dependent on the value of $\boldsymbol{X}$. For uncertainty propagation in the presence of parameter and distribution type uncertainty, samples of $\boldsymbol{u}$ and $\boldsymbol{\Theta}$ are obtained and samples of $\boldsymbol{X}$ are obtained accordingly (Sections 3.1.1 and 3.1.2). Once $\boldsymbol{X}$ is sampled, the parameters of the normal distribution of model error at $\boldsymbol{X} = \boldsymbol{x}$ can be determined as

$$\left(\boldsymbol{\epsilon_{m_X}} \mid \boldsymbol{X_T}, \boldsymbol{\epsilon_{m_{X_T}}}, \boldsymbol{X}, \boldsymbol{\Theta_{\epsilon_m}}\right) \sim \boldsymbol{N}(\boldsymbol{\mu}, \boldsymbol{\sigma}) \tag{19}$$

In Eq. (19), $\boldsymbol{\epsilon_{m_X}}$ represents the model error at $\boldsymbol{X}$; $\boldsymbol{X_T}$ and $\boldsymbol{\epsilon_{m_{X_T}}}$ correspond to the training inputs and the model errors at $\boldsymbol{X_T}$ for building the GP model. $\boldsymbol{\Theta_{\epsilon_m}}$ refers to the parameters of the GP. A sample of model error is drawn from this normal distribution.

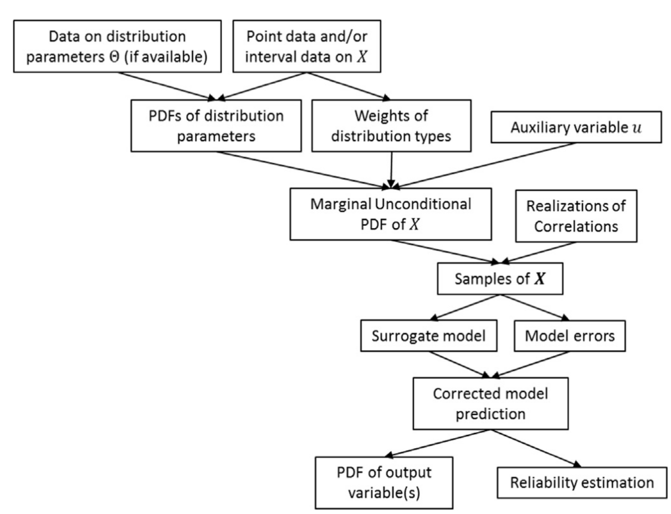

Fig. 1. Flowchart for reliability analysis using single-loop Monte Carlo simulation.

Samples of $\boldsymbol{X}$ are then propagated through the physics surrogate model and then model error is added for reliability analysis. Thus, parameter uncertainty, distribution type uncertainty, uncertain correlations and model errors can all be integrated in the framework of MCS-based reliability analysis. Fig. 1 shows the flowchart for reliability analysis using single-loop MCS.

3.2. Using FORM to handle epistemic uncertainty

3.2.1. Inclusion of distribution parameter uncertainty within FORM

An essential step in FORM analysis is to transform the random variable values at each iteration into an equivalent normal space. In the presence of uncertain distribution parameters, it is not possible to obtain a single PDF and CDF value for the equivalent normal transformation ; this problem can be resolved by using the auxiliary variable approach, developed in Section 3.1.1. Therefore, X can be replaced by u and Θin the limit state equation, and FORM analysis is carried out with $u$and Θinstead of X. Thus, each X variable is replaced by $\dim(\Theta)+1$ new variables in the FORM analysis. In the limit state equation, if N, M represent the number of variables with uncertain distribution parameters, and variables with known distribution parameters, then FORM analysis is carried out on $\left[\sum_{i=1}^{N}\left(\dim(\Theta_i)+1\right)\right]+M$ variables, i.e.,

$$g=g\left(\boldsymbol{X_N},\boldsymbol{X_M}\right)=g\left(\boldsymbol{u_N},\boldsymbol{\Theta_N},\boldsymbol{X_M}\right) \tag{20}$$

Eq. (20) corresponds to the limit state equation defined in terms of original variables, $\boldsymbol{X}$, which are decomposed into variables with parameter uncertainty, denoted by $\boldsymbol{X_N}$and variables without parameter uncertainty, denoted by $\boldsymbol{X_M}$. $\boldsymbol{X_N}$ are further represented in terms of auxiliary variables $(\boldsymbol{u_N})$ and distribution parameters $(\boldsymbol{\Theta_N})$. In FORM, the gradient of the limit state function is numerically calculated because the transformation of each variable in $\boldsymbol{X_N}$ to its auxiliary variable and distribution parameters is its inverse CDF which in general cannot be expressed analytically (e.g., Gaussian CDF). In each iteration, the value of each variable in $\boldsymbol{X_N}$ is calculated as $X_N=F^{-1}(u_N\mid\boldsymbol{\theta_N})$, in order to compute $g(\boldsymbol{X_N},\boldsymbol{X_M})$. With the increase in the number of epistemic variables, the number of input variables to FORM increases. As the reviewer pointed out, this could possibly result in the optimization problem being ill-posed. One way to overcome this problem is to reduce the input space through dimension reduction techniques. Variance-based global sensitivity analysis is one possible approach for dimension reduction where the sensitivity of each variable (original input variable, auxiliary variable, distribution parameter) is estimated and if the sensitivity is less than a pre-defined threshold, it can be assumed deterministic at its mean value [28].

For example, let $g=g(X_1,X_2,X_3)$ be a limit state function, consisting of three Gaussian random variables (X_1,X_2,X_3). Let $\Theta_{\boldsymbol{i}}=\{\mu_i,\sigma_i\}(i=1{,}2{,}3)$ where $\Theta_{\boldsymbol{i}}$ represents the set of parameters of each X_i. Assume that the parameters of X_3 are deterministic but those of X_1,X_2 are uncertain, i.e., $X_N=\{X_1,X_2\}$ and $X_M=\{X_3\}$. Also, $\dim(\Theta_i)=2$ for $i=1{,}2$. Therefore, FORM analysis is carried out on 7variables $(u_1,\mu_1,\sigma_1,u_2,\mu_2,\sigma_2,X_3)$. The values of X_1 and X_2 at each iteration are calculated as $X_1=F^{-1}(u_1\mid\mu_1,\sigma_1)$ and $X_2=F^{-1}(u_2\mid\mu_2,\sigma_2)$.

3.2.2. Inclusion of distribution type uncertainty and parameter uncertainty within FORM

The variables d_X that are used to describe distribution type uncertainty are discrete variables. FORM cannot be applied to discrete variables, but the limitation can be overcome by using the theorem of total probability. The probability of failure can be computed as [29],

$$P_f=\sum_{k=1}^{n}P\left(\boldsymbol{X_{discrete,k}}\right)P(g\left(\boldsymbol{X_{cont}}\left(\boldsymbol{X_{discrete,k}}\right)\le 0\right) \tag{21}$$

where $P(\boldsymbol{X_{discrete,k}})$ refers to the probability that the discrete variables (i.e., distribution type variables) assume a particular value k from the set of possible values. $P(g(\boldsymbol{X_{cont}}\boldsymbol{X_{discrete,k}})\le 0)$ refers to the failure probability conditioned on a realization of the discrete variables.

If there are n variables in the limit state equation and each variable X_i has m_i possible distributions, then a total of $\prod_{i=1}^{n}m_i$ possible combinations exist. FORM needs to be carried out for all these combinations and the overall probability of failure is obtained according to Eq. (21). For illustration, consider a performance function $g(X)$ with three variables X_1,X_2,X_3. Let each of the three variables have three possible distribution choices – normal, lognormal or Type 1 extreme value distribution. Thus, there are 27 possible combinations for the distribution types between X_1,X_2,X_3. Assume that the probability of each of these variables following a particular distribution is known from prior knowledge or from Bayesian hypothesis testing. For each of the 27 cases, the probability of each combination can be calculated easily if all three distribution type variables are independent (i.e., the distribution type of one variable does not affect the distribution type of another variable). For each of the 27 cases, the probability of failure is obtained separately, and the overall failure probability is evaluated using Eq. (21).

3.2.3. Inclusion of uncertain correlations within FORM

In Sections 3.2.1 and 3.2.2, the MPP search within FORM is carried out in the equivalent normal space corresponding to the auxiliary variables $\boldsymbol{u}$ and distribution parameters $\boldsymbol{\Theta}$. A key difficulty in this approach is in transferring the information on correlation coefficients between the $\boldsymbol{X}$ variables to the space of auxiliary variables $(\boldsymbol{u})$ and distribution parameters $(\boldsymbol{\Theta})$.

This difficulty is overcome by using the concept of unconditional PDF, developed earlier in Section 3.1.3. Unconditional PDFs are constructed for individual random variables that have distribution type and/or distribution parameter uncertainty; note that the resulting unconditional PDF is numerical and non-parametric. FORM analysis can now be carried out with these unconditional distributions of the $\boldsymbol{X}$ variables. The implementation of FORM in the presence of fixed correlations is well known. This procedure can be extended to the case where the correlation coefficients among the $\boldsymbol{X}$ variables are uncertain.

The uncertainty regarding the correlation coefficients may be represented by constructing non-parametric distributions for each of the correlation coefficients based on available information (Section 2.3). The domain of each of the correlation coefficients is uniformly discretized and the corresponding PDF values are calculated using the non-parametric PDF. Using the discretized values of the correlation coefficients, several realizations of the correlation matrices can be obtained. FORM is carried out using each of the realizations of the correlation matrices and failure probabilities are calculated corresponding to each of the correlation matrices. For each correlation matrix, a measure of weight can be derived which is proportional to the product of PDF values of individual correlation coefficients. The overall failure probability can be obtained by taking a weighted average of all the individual failure probabilities.

3.2.4. Inclusion of model errors within FORM

The overall model error (ϵ_m), as stated in Section 2.4, is modeled using a Gaussian process, i.e., it follows a normal distribution with parameters (μ,σ) dependent on the value of $\boldsymbol{X}$, i.e., for each value of $\boldsymbol{X}$ we get a different normal distribution for model error. Similar to Section 3.1.1, the model error $\epsilon_m(\boldsymbol{X})$ can be represented using an auxiliary variable (u_{ϵ_m}) and distribution parameters $(\mu(\boldsymbol{X}),\sigma(\boldsymbol{X}))$ using the relation, $\epsilon_m=F^{-1}(u_{\epsilon_m}\mid\mu(\boldsymbol{X}),\sigma(\boldsymbol{X}))$. Note that F in the above relation is a normal CDF since model error is represented by a normal distribution at each X due to the use of the GP model. Since the distribution parameters μ and σ are functions of X, the auxiliary variable (u_{ϵ_m}) is the only additional variable that needs to be included in the FORM framework to account for model errors. Thus, the total number of variables in FORM analysis is equal to $\left[\sum_{i=1}^{N}\left(\dim\left(\Theta_i\right)+1\right)\right]+M+1$. The terms in the above expression are defined in Section 3.2.1. The '1' at the end of the expression is due to the inclusion of the auxiliary variable for model error. Thus, distribution parameter uncertainty, distribution type uncertainty and model errors are incorporated in the FORM framework for reliability analysis.

The benefits of using an auxiliary variable approach within FORM-based reliability analysis are given below. The use of an auxiliary variable approach enables us to perform a single-level FORM analysis considering both the aleatory uncertainty and distribution parameter uncertainty, as opposed to a traditional double-loop approach where multiple FORM analyses were carried out for several realizations of distribution parameters. In the presence of uncertain correlations between the inputs, the correlation information cannot be transferred from the input space to the space of auxiliary variable and distribution parameters. Therefore, unconditional distributions are constructed for the inputs using the auxiliary variable approach, which are then used to perform multiple FORM analyses with several realizations of the correlation matrices. Thus, the auxiliary variable approach enables us to perform FORM analysis under distribution parameter uncertainty and uncertain correlations.

In comparison with MCS, there exist a few differences in the implementation of reliability analysis using FORM given below.

(1) The major difference is in handling the distribution type uncertainty. Since FORM cannot be carried out for discrete variable (distribution type uncertainty), a double-loop approach is used. For several realizations of the distribution type uncertainty in the outer-loop, FORM analysis is carried out in the inner-loop considering parameter uncertainty and model errors. The results from multiple FORM analyses are then aggregated using the total probability theorem to estimate the failure probability. In contrast, a single-level implementation can be used for MCS-based reliability analysis in the presence of distribution parameter uncertainty, distribution type uncertainty and model error.
(2) When handling uncertain correlations, the first step in both the methods is the construction of unconditional PDFs for inputs with distribution parameter and distribution type uncertainty. Similar to handling distribution type uncertainty, uncertain correlations in FORM are handled in a double-loop approach with several realizations of correlation matrices in the outer loop and FORM analysis is carried out, conditioned on the correlation matrices, in the inner loop. The results from multiple FORM analyses are then aggregated to obtain a failure probability estimate. However, MCS can easily use a single-level implementation where for every realization of the correlation matrix, one sample of the inputs is obtained. For several realizations of correlation matrices, corresponding samples of inputs are obtained and propagated through the limit state to estimate the failure probability estimate.

3.3. Comparison of computational effort between FORM and MCS

The introduction of auxiliary variables to account for different epistemic uncertainty sources introduces additional variables in the reliability analysis. Therefore, it is of interest to evaluate the increase in computational effort due to this strategy. The effect on both FORM and MCS methods is discussed in this section. While the computational effort in FORM is affected by the number of variables, the computational effort in MCS is only affected by the magnitude of the failure probability. The various factors affecting computational effort are discussed below.

Consider a limit state function with n input variables associated with distribution parameter and distribution type uncertainty, m variables with precisely known distribution type and parameters, and a model error term. For illustration, let there be k possible distribution types for each input variable with distribution type uncertainty. First, computational expense is calculated without considering correlations; correlations are considered later.

The number of variables in each FORM analysis is equal to $\left[\sum_{i=1}^{n}\left(\dim\left(\Theta_i\right)+1\right)\right]+m+1$ and therefore, we require $\left[\sum_{i=1}^{n}\left(\dim\left(\Theta_i\right)+1\right)\right]+m+1$ gradients. If the computational model is a black box (e.g., commercial finite element software) and thus a finite difference (forward) approach is used for gradient computation, the number of function evaluations in each iteration is equal to $\left[\sum_{i=1}^{n}\left(\dim\left(\Theta_i\right)+1\right)\right]+m+2$. That is, we require one function evaluation at the current iteration point, and $\left[\sum_{i=1}^{n}\left(\dim\left(\Theta_i\right)+1\right)\right]+m+1$ additional evaluations for gradients of each of the inputs in FORM analysis. If each FORM analysis takes p iterations on average to converge, then the total number of function evaluations is equal to $\left(\left[\sum_{i=1}^{n}\left(\dim\left(\Theta_i\right)+1\right)\right]+m+2\right)\times p\times k^n$ where $\dim\left(\Theta_i\right)$ represent the number of distribution parameters for each of the epistemic variables and k^n represents the number of combinations of distribution types. Thus, it can be seen that the number of function evaluations increases rapidly with the number of uncertain distribution types. In the presence of a large number of combinations, some pruning techniques such as branch-and-bound [30] can be used to reduce the number of combinations.

Next, we consider uncertain correlations between the $n+m$ variables. Consider the case when all the variables are correlated; therefore, there exist $\frac{(n+m)\times(n+m-1)}{2}$ correlation coefficients. In the worst-case scenario, assume that all the correlation coefficients are uncertain and non-parametric distributions are constructed for each of the correlation coefficients. As mentioned in Section 3.2.3, in the presence of uncertain correlations, FORM is carried out using the unconditional distributions of the original input variables since it is not possible to transfer the correlation information from X space to the space of (u,Θ). Let the PDF of each correlation coefficient be discretized into l values. Therefore, the total number of FORM analyses is equal to $l^{\frac{(n+m)\times(n+m-1)}{2}}$, where $l^{\frac{(n+m)\times(n+m-1)}{2}}$ refers to the number of combinations of the correlation matrices. If each FORM analysis requires p iterations, then the number of gradient evaluations is equal to $(n+m+1)$ (including the model error), and therefore, the number of function evaluations include $(n+m+2)$ in each iteration. Thus, the total number of function evaluations is equal to $(n+m+2)\times p\times l^{\frac{(n+m)\times(n+m-1)}{2}}$. Similar to the previous case, the number of function evaluations increase rapidly with the number of uncertain correlation coefficients.

In the case of Monte Carlo sampling, the number of samples (N) (i.e., function evaluations) required can be calculated using Eq. (22) [1]

$$\varepsilon\% = 200\%\times\sqrt{\frac{(1-p_f)}{N\times p_f}} \tag{22}$$

where ε represents the percentage coefficient of variation (COV) in the failure probability estimate (p_f). Eq. (22) shows that the number of samples required in MCS only depends on the magnitude of the failure probability and not on the number of variables. An increase in the number of variables does increase the sampling effort a little, but this is usually very small compared to the effort in function evaluations.

For illustration, let the failure probability be 0.0005. The number of Monte Carlo samples required for a desired COV of 10% in the failure probability estimate is equal to 800,000. In FORM, let $p = 10$, $n = 5$, $m = 3$, $k = 2$ and let each epistemic variable be associated with two distribution parameters. The total number of function evaluations in FORM is equal to 6400, which is small compared to MCS. However, if uncertain correlations are also considered, then the number of function evaluations (for$l = 4$) is of the order of 10^{18}, which is practically impossible. In cases with a large number of uncertain correlations, either MCS or FORM with strong pruning techniques should be used. From this example, it is seen that FORM or MCS could be more efficient in different situations, depending on the number of epistemic quantities and the magnitude of failure probability.

In summary, Sections 2 and 3 developed (1) methods to quantify different types of statistical uncertainty (distribution parameters, distribution type, uncertainty about correlations) and model uncertainty (model error and numerical solution errors) in a probabilistic framework; (2) a novel approach to include epistemic uncertainty, within FORM- based reliability analysis using the concepts of auxiliary variable and theorem of total probability; and (3) a single-loop Monte Carlo sampling approach, instead of an expensive traditional nested-loop sampling approaches, for the inclusion of both aleatory and epistemic uncertainty sources in reliability estimation.

4. Numerical examples

4.1. Example 1a: Distribution parameter uncertainty, distribution type uncertainty, and model error

Consider the limit state function

$$g(X_1, X_2, X_3) = X_1X_2 - X_3 + \epsilon_m \tag{23}$$

where $X_i (i = 1,2,3)$ are independent random variables with X_1, X_2 representing the resistance variables and X_3representing a load variable on a structural system. (For example, in a cantilever beam subjected to a moment at the free end as shown in Fig. 2, X_1, X_2, X_3 represent the material strength, section modulus and bending moment respectively).

Several candidate distribution types are considered for X_1, X_2 and X_3 based on available data D_1, D_2, and D_3 respectively. The candidate distribution types and their probabilities are shown in Table 1.

In Table 2, the assumed statistics of mean and standard deviation of each variable are provided. (No physical units are used). The overall model error (ϵ_m) is assumed to follow a normal distribution with zero mean and a standard deviation of 50.

Both Monte Carlo simulations and FORM are used to solve this problem. For each combination of distribution types of input variables, failure probabilities are calculated using FORM by substituting each X_i in the performance function with an auxiliary variable and its distribution parameters. Since all three input variables are associated with parameter uncertainty, FORM is

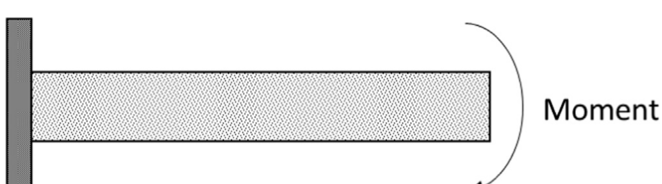

Fig. 2. Example 1: Cantilever beam subjected to a moment at the free end.

Table 1
Example 1: Probabilities of distribution types of variables X_1, X_2, X_3.

Distribution type	X_1	X_2	X_3
Lognormal	0.6	0	0
Normal distribution	0.4	1	0.1
Type 1 Extreme Value Distribution	0	0	0.9

Table 2
Example 1: Statistics of mean and standard deviation of X_1, X_2, X_3.

Variable	Parameter	Distribution type	Mean	Standard deviation
X_1	μ_1	Normal	38	4
	σ_1	Lognormal	3.8	0.35
X_2	μ_2	Normal	54	4.5
	σ_2	Lognormal	2.7	1
X_3	μ_3	Normal	1300	50
	σ_3	Lognormal	100	30

Table 3
Example 1: Failure probability estimates for different input distributions.

Distribution of X_1	Distribution of X_2	Distribution of X_3	Probability that X_1, X_2, X_3 follow the designated distributions	Probability of failure using FORM
Normal	Normal	Normal	0.04	0.0272
Normal	Normal	Type 1 EVD	0.36	0.0264
Lognormal	Normal	Normal	0.06	0.0239
Lognormal	Normal	Type 1 EVD	0.54	0.0231

carried out with 10 variables (including the model error). Considering the distribution type uncertainty, 4 FORM analyses have been carried out. The resulting failure probabilities are shown in Table 3.

The overall failure probability, using FORM results, is calculated using the theorem of total probability as $P(g \le 0) = \sum_i [P(g \le 0C_i)*P(C_i)] = 0.0245$. Here, C_i represents each combination of distribution types of X_1, X_2, X_3. In FORM analysis, the step size for gradient computation is assumed to be equal to 0.1% of the order of magnitude of the input variable, i.e., if the input is of the order n (e.g.,2×10^n), then the step size is assumed to be 10^{n-3}. In this work, we have not performed any experimental study for optimal step size selection in the numerical examples; we agree that in realistic problems one may need to experiment with step size in order to identify the optimum step size. Each FORM analysis, on an average, required 9 iterations which corresponds to 99 function evaluation. Considering all distribution type combinations, 396 function evaluations are performed (Section 3.3).

The above problem is also solved with MCS using the approach in Section 3.1. To sample the distribution type of X_1, a uniform number in [0, 1] is generated. If the random number lies in [0, 0.6], X_1 is assumed to follow a lognormal distribution, else, a normal distribution. A similar procedure is followed for sampling the distribution type of X_3. Since all X_i are independent, their samples can be generated separately. Thus, sampling of $X_i(i = 1,2,3)$ is performed. 20,000 samples are generated and substituted in the limit state equation, to obtain the failure probability as 0.022. The number of samples was chosen in a way such that the percentage coefficient of variation in the failure probability estimate, computed using Eq. (22) [1], is within 10%.

To illustrate the importance of including epistemic uncertainty in reliability estimation; the same problem is solved without considering statistical uncertainty and model error. X_1, X_2, X_3 are assumed to follow lognormal, normal and Type 1 extreme value distribution since these distribution types have the highest probabilities from Table 1. The distribution parameters are assumed to be deterministic, taking the mean values in Table 2. The failure probability, using FORM is calculated to be 0.0009, which is much smaller than the probability computed when statistical uncertainty and model error were included earlier. This is expected, as the amount of uncertainty in the system is higher, the estimate of failure probability is also higher. Using MCS, the failure probability is obtained to be 0.001, when epistemic uncertainty is not considered.

4.2. Example 1a. Discussion

In FORM, since X_1 and X_3 have distribution type uncertainty with each variable taking two possible distribution types, four FORM analyses are carried out, one for each combination. The failure probabilities computed using FORM (0.0245) and MCS (0.022) approaches are in good agreement with each other. In addition, the comparison of results with and without epistemic uncertainty shows the importance of including epistemic uncertainty in reliability estimation. Example 1a shows the results with and without considering epistemic uncertainty in reliability analysis. From the results, it can be seen that there is a significant deviation in the failure probability estimates. Using FORM, the failure probability estimates when epistemic uncertainty is considered and not considered is equal to 0.0245 and 0.0009 respectively. If the threshold failure probability is assumed at 0.001, then the structure would have been deemed *safe* when epistemic uncertainty is not considered. Due to the inclusion of epistemic uncertainty, the estimated failure probability is greater than the threshold; therefore, the structure is *unsafe*.

If the limit state function has several variables, and if each variable has several possible distribution types, then the number of combinations of distribution types might be quite large. In the presence of limited computational resources, some pruning techniques are needed to reduce the number of combinations. A probability threshold may be defined, such that if the probability of a combination is less than the threshold, then that combination is discarded. The threshold may be defined in such a way that the sum of probabilities of combinations that need to be evaluated are greater than 90% or 95% depending on the computational resources and accuracy requirements.

FORM uses gradient-based approaches, which could sometimes result in local optima, and not the global optimum; this largely depends on the starting point. This issue arises even when only aleatory uncertainty sources are considered. Often, this issue is addressed by carrying out the optimization with multiple starting points to identify the global optimum; in the case of FORM, the global optimum is the point of maximum failure probability.

4.3. Example 1b. Additional uncertainty regarding correlation coefficients

In this example, correlation uncertainty between variables is considered along with distribution parameter uncertainty, distribution type uncertainty in the input random variables and model error in the limit state function used in Example 1a. The statistics for the distribution parameters, distribution type and model error used in Example 1a are also used in this example. Suppose that variables X_1 and X_2 are known to be correlated from prior knowledge, but the correlation coefficient is uncertain, in the range [0.3, 0.7] given by an expert. Both FORM and Monte Carlo simulations are used to solve this problem. The correlation coefficient between X_1 and X_2 is assumed to be a uniform distribution between 0.3 and 0.7. If the data on correlation coefficients were a combination of point and interval data, a non-parametric distribution can be constructed (Section 2.3). The first step of analysis in both FORM and MCS approaches in dealing with uncertain correlations (see Sections 3.1.3, 3.2.3) is the construction of marginal unconditional PDF for each X_i. The procedure for the construction of unconditional PDF is provided in Section 3.1.3.

In MCS, for each sample of correlation coefficient drawn, a correlated sample of X_1 and X_2 is obtained using the procedure explained in Section 3.1.3. Since X_3 is independent of X_1 and X_2, samples of X_3 are obtained separately. The generated samples of $\boldsymbol{X}$ are then propagated through the limit state function for reliability analysis. 12,000 samples are generated and substituted in the limit state equation, to obtain the failure probability as 0.0368. The number of samples for MCS was chosen in a way such that the percentage error in the failure probability estimate is within 10%, calculated using Eq. (22).

In FORM, several samples of correlation coefficient ρ are drawn and FORM is carried out using each sample of the correlation coefficient and the marginal unconditional PDFs. For each value of the correlation coefficient, a failure probability is obtained and the overall failure probability is computed using weighted averaging with weights being proportional to the PDF values corresponding to the sampled value of the correlation coefficient, as stated in Section 3.2.3. Since the correlation coefficient ρ is represented using a uniform distribution in this example, the PDF values at each of the ρ values are equal and therefore, the overall failure probability in this case, is simply the average of all the individual failure probabilities. The overall failure probability for this example using FORM is obtained as 0.0352.

This example demonstrates the application of the proposed methodologies for reliability analysis in the presence of both aleatory uncertainty and different types of epistemic uncertainty (both data and model sources). For higher dimensional problems and in the presence of large number of uncertain correlation coefficients, the number of realizations of correlation matrices (Section 3.2.3) increases, and performing FORM for each of the correlation matrices leads to high computational effort. In such cases, MCS may even tend to be more efficient than FORM.

4.4. Example 2. Fluid–structure interaction

In this example, the proposed reliability methods are demonstrated with aero-elastic analysis of an airplane wing. A cantilever wing with NACA 0012 airfoil [31] is used. The solid wing is made of an isotropic material whose Young's Modulus E is not known precisely, but a collection of sparse point data and interval data are available. The Poisson ratio of the material is assumed to be precisely known as 0.2. A free stream density of 0.4135 kg/m^3 and a free stream pressure of 2.65 x 10^4 N/m^2 are assumed. The angle of attack, altitude and Mach number are assumed to be epistemic, and sparse data are assumed to be available. Also, geometric variables such as length of the wing, ratio of chord lengths and the back-sweep angle are all assumed to be random variables with known distributions. The data available on all the above mentioned variables are given in Table 4.

The output variable of interest is maximum stress in the wing and the limit state function is defined as

$$g(s) = s_{limit} - s \tag{24}$$

where s_{limit} represents the failure stress, $s_{limit}=1000KPa$ (deterministic). The coupled fluid–structure interaction analysis is carried out in ANSYS. An incompressible turbulent flow model is used for

Table 4
Example 2: Data available on input variables.

Variable	Point data	Interval data
Young's Modulus E (x 10^{11})	1.451,1.458	[1.433,1.457] [1.422,1.466] [1.451,1.455]
Angle of attack (α)	No point data	[4.68,5.75] [4.59,5.57] [4.72,5.66]
Mach number (m)	0.21	[0.188,0.233] [0.173,0.215] [0.162,0.222]
Altitude (A)	10,000	[9500,10700] [9850,11000] [9110,10200]
Length (L)	Normal distribution with $\mu = 3$ and $\sigma = 0.3$	
Ratio of chords	Lognormal distribution with $\mu = 1.45$ and $\sigma = 0.2$	
Back sweep angle(β)	Normal distribution with $\mu = 0.42$ and $\sigma = 0.03$	

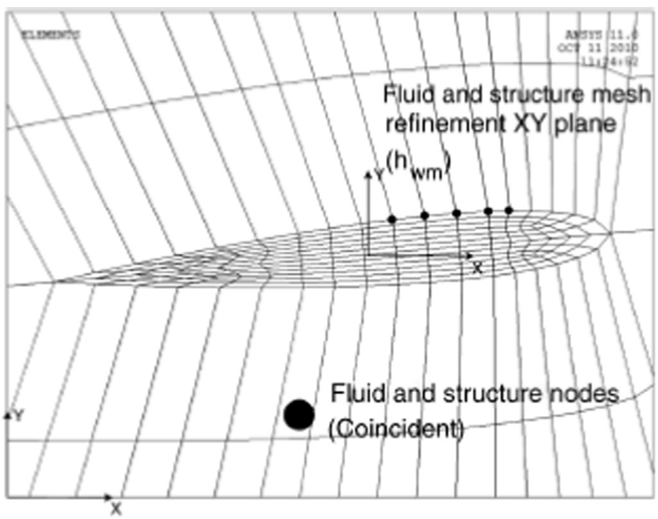

Fig. 3. Example 2: Fluid and structural meshes.

computational fluid dynamics (CFD) analysis and a non-linear solver is used for structural finite element analysis (FEA). Simulations are performed at different combinations of input variable values to calculate the stress output values, and at each input value, several mesh sizes are used for the construction of a GP model, in order to predict the corresponding stress values at an infinitesimally small mesh size, Fig. 3 shows the airfoil structural mesh along with the surrounding fluid mesh. The set of inputs and the corresponding stress values (corrected for discretization error) are used to build a GP surrogate model. Then, another set of simulations at different input values are carried out to construct a GP model of the surrogate model error. A synthetic dataset with added experimental error was used to train a third GP model for model form error. Since reliability estimation is the main focus of this paper, the details regarding the construction of the GP models have been omitted. Refer to [32] for details.

The set of inputs and the corresponding stress values (corrected for discretization error) are used to build a GP surrogate model. Then, another set of simulations at different input values are carried out to construct a GP model of the surrogate model error. A synthetic dataset with added experimental error was used to train a third GP model for model form error. Since reliability estimation is the main focus of this paper, the details regarding the construction of the GP models have been omitted. Refer to [32] for details.

The next step after uncertainty quantification is uncertainty propagation for reliability estimation. Since data for some inputs are available in the form of point and interval data, non-parametric distributions using spline-based interpolations are constructed for the corresponding variables using procedures described in Section 2.1. Fig. 4 represents the posterior PDFs of the altitude, angle of attack, Mach number and Young's modulus. Both FORM and Monte Carlo simulations are used for reliability estimation.

FORM analysis is carried out using 8 variables (7 input variables shown in Table 4 and an auxiliary variable for the Gaussian Process model). Here, the surrogate model and the model errors are both represented using Gaussian processes. Since summation of Gaussian processes is also a Gaussian process, one auxiliary variable is sufficient in FORM analysis, rather than using a separate auxiliary variable for surrogate model and model errors. The failure probability is calculated as 0.00041. Since this example does not consider distribution type uncertainty or uncertain correlations, one FORM analysis is sufficient for reliability analysis. The FORM analysis has taken 72 function evaluations to estimate the failure probability.

In Monte Carlo approach, samples of all the inputs are generated using the inverse CDF and are propagated through the model, after accounting for all the errors, to calculate the maximum stress in the wing. From these stress values, the number of samples when the stress value exceeds the failure stress is obtained and the failure probability is calculated. The number of samples is chosen such that the percentage error in the failure probability estimate is within 10%. 1,000,000 samples are used and the failure probability is calculated as 0.00039. In Example 1, reliability estimation is demonstrated using FORM and MCS when the input variables with epistemic uncertainty are represented using parametric distributions (with uncertain distribution types, distribution parameters and correlations) and model error as a random variable. In Example 2, reliability estimation is demonstrated when non-parametric distributions are used to represent the input random variables with epistemic uncertainty, and model error is estimated using Gaussian process models. Thus, reliability estimation is successfully accomplished through different options to integrate natural variability, statistical uncertainty and model uncertainty, using either FORM or Monte Carlo methods.

5. Conclusion

This paper proposed a systematic probabilistic framework to include both aleatory uncertainty and epistemic uncertainty in reliability analysis. The First-Order Reliability Method (FORM), which has largely dealt with aleatory uncertainty in the literature, is extended to accommodate epistemic uncertainty due to uncertain distribution parameters, distribution type, uncertain correlations, and model errors using auxiliary variables, based on probability integral transform and the theorem of total probability. Gaussian Process (GP) models are used to quantify different types of model errors as well as the overall model error term. Traditional sampling methods in the presence of statistical and model uncertainty would involve nested three-loop sampling. The proposed Monte Carlo methodology results in a single loop sampling approach due to the use of auxiliary variables. In FORM, each limit state variable with uncertain distribution parameters is substituted by $\dim(\Theta)+1$ variables where $\dim(\Theta)$ represents the number of distribution parameters. However, the number of variables in FORM increases rapidly when considering limit states with many variables that have uncertain distribution types and distribution parameters. As a result, FORM may be difficult to use in high-dimensional problems. Uncertainty about correlations is included in the reliability analysis using non-parametric distributions that are fit to available data on correlation coefficients. For several realizations of the correlation matrices, correlated samples are drawn and propagated through the system model using single-loop MCS. In FORM, the uncertainty about correlations is handled

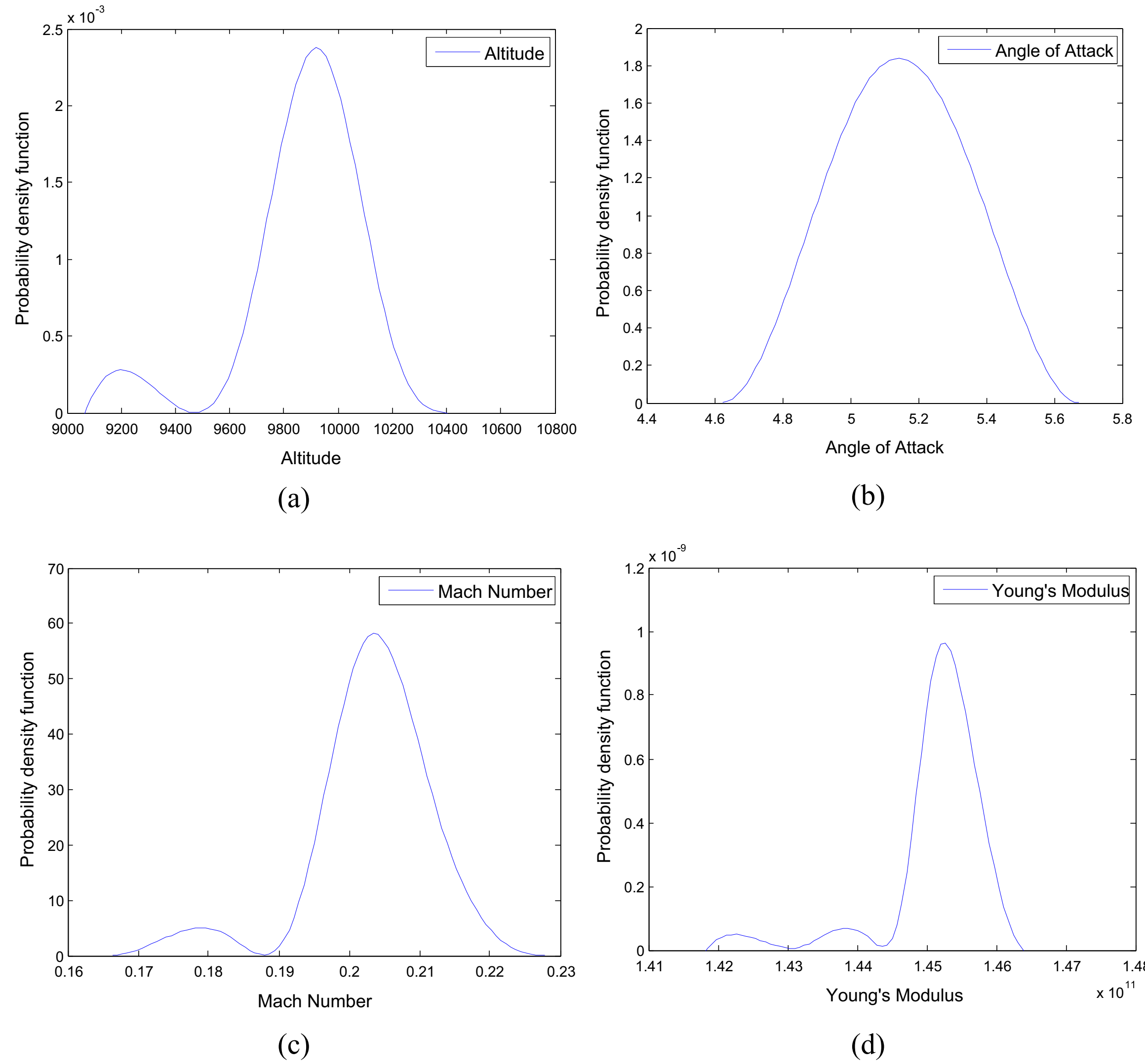

Fig. 4. Example 2: PDF of (a) Altitude (b) Angle of Attack (c) Mach number (d) Young's modulus.

by performing FORM using unconditional distributions at several realizations of correlation matrices, and weighted averaging is used to obtain the overall reliability estimate.

In this study, reliability calculations are carried out for a single limit state under several forms of epistemic uncertainty. Future work needs to address system reliability estimation with multiple limit states under epistemic uncertainty. The proposed framework can be extended to include additional types of epistemic uncertainty such as model parameter uncertainty, data processing errors, truncation errors, and round off errors. Depending on the specific application problem, different sources of error and uncertainty may have different levels of importance, and sensitivity analysis can be used to identify the dominant sources and include them in the reliability analysis. The auxiliary variable approach described in Section 2 can also be used to quantify the relative contributions of various aleatory and epistemic uncertainty sources to the overall output uncertainty, based on global sensitivity analysis [13,33].

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