CENG 382 - Analysis of Dynamic Systems
20221
Take Home Exam 2 Solutions

Ünlü, Berke Can
e2381028@ceng.metu.edu.tr

December 10, 2022

---

1. (a) Let's define that , S = {0,1,2}, 0 is professional , 1 is skilled laborer, 2 is unskilled laborer. We can define the transition rules as follows,

   i. $p(0 \to 0) = 0.7$
   ii. $p(0 \to 1) = 0.2$
   iii. $p(0 \to 2) = 0.1$
   iv. $p(1 \to 0) = 0.2$
   v. $p(1 \to 1) = 0.6$
   vi. $p(1 \to 2) = 0.2$
   vii. $p(2 \to 0) = 0.1$
   viii. $p(2 \to 1) = 0.4$
   ix. $p(2 \to 2) = 0.5$

   Then we can construct state transition matrix as P.

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

   (b) We need to find $p(2 \xrightarrow{2} 0)$.

$$p(2 \xrightarrow{2} 0) = \sum_{k=0}^{2} P_{i,k}P_{k,j} = P_{2,0} * P_{0,0} + P_{2,1} * P_{1,0} + P_{2,2} * P_{2,0}$$

   We can insert the values from matrix P, and we got the result.

$$p(2 \xrightarrow{2} 0) = 0.1 * 0.7 + 0.4 * 0.2 + 0.5 * 0.1 = 0.20$$

   Also, $p(i \xrightarrow{m} j) = \sum_{k=0}^{m} P_{i,k}(P^{m-1})_{k,j}$. This indicates that, the (i,j) entry of P^m is the m-step transition probability of $p(i \xrightarrow{m} j)$.
   Where, P^2 is,

$$P^2 = \begin{bmatrix} 0.54 & 0.30 & 0.16 \\ 0.28 & 0.48 & 0.24 \\ 0.20 & 0.46 & 0.34 \end{bmatrix}$$

   As a result, we can say that,

$$p(2 \xrightarrow{2} 0) = (P^2)_{2,0} = 0.20$$

(c) We can find this probability by looking the P^2 matrix that was created in the previous part

$$p(0 \xrightarrow{2} 0) = (P^2)_{0,0} = 0.54$$

(d)

$$P^{100} = \begin{bmatrix} 0.3529 & 0.4118 & 0.2353 \\ 0.3529 & 0.4118 & 0.2353 \\ 0.3529 & 0.4118 & 0.2353 \end{bmatrix}$$

We need to look eigenvalues of P^T. We can compute eigenvalues from this equation:

$$det(\lambda I - A) = \begin{vmatrix} \lambda - 0.7 & -0.2 & -0.1 \\ -0.2 & \lambda - 0.6 & -0.2 \\ -0.1 & -0.4 & \lambda - 0.5 \end{vmatrix}$$

The eigenvalues of P^T are 1, 0.26 and 0.54. 1 must be an eigenvalue of P and P^T Since all the rows in P sum to 1, then we must have $P\mathbf{1} = \mathbf{1}$ where 1 is a column vector of all 1s.

1 is the eigenvalue of P and P^T with largest absolute value.

$$p(m) = p(0)P^m$$

We need to look eigenvectors/values of P^T since $p(m)$ are now row vectors not column vectors.

Eigenvector of $\lambda = 1$:

$$v = \begin{bmatrix} 1.5 & 1.75 & 1 \end{bmatrix}$$

If we rescale this vector by multiplying itself with $\frac{1}{1.5+1.75+1}$:

$$p^* = \frac{4}{17}\begin{bmatrix} 1.5 & 1.75 & 1 \end{bmatrix} = \begin{bmatrix} 0.3529 & 0.4118 & 0.2353 \end{bmatrix}$$

where p^* is the row vectors of P^{100}.

2. (a) Controllability matrix (n, nxm) is

$$M = \begin{bmatrix} B & AB & A^2B & .... & A^{N-1}B \end{bmatrix}$$

In our case, A is (3x3) matrix, B is (3x1) matrix. Therefore, M is (3x3) matrix. Note that, n = 3 and m = 1.
If a discrete-time system is controllable if and only if the M has rank n. The rank can be described as from among nm vectors, there are n vectors that are linearly independent, so the rank is n.
In our case, nm = 3 and n = 3. Therefore, every column of M should be linearly independent.

$$M = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$A^2B = A * AB = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

To find whether or not columns are linearly independent,

$$a\begin{bmatrix}1\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\-2\\0\end{bmatrix}+c\begin{bmatrix}0\\0\\-2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

This equation shows that $a=b=c=0$. These columns are linearly independent, the rank of M is 3. Therefore, we can say that the system is controllable.

(b) We can state that,

$$x(1)=Ax(0)+Bu(0)$$

$$x(2)=Ax(1)+Bu(1)=A^2x(0)+ABu(0)+Bu(1)$$

$$x(3)=\begin{bmatrix}4\\4\\4\end{bmatrix}=Ax(2)+Bu(2)=A^3x(0)+A^2Bu(0)+ABu(1)+Bu(2)$$

Where,

$$x(0)=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Our unknowns are, $u(0),u(1)$ and $u(2)$. Now let's find A^3. I will not compute A^2B, AB again since I did it in the previous part of this question.

$$A^3=\begin{bmatrix}0&0&1\\-2&0&-1\\0&1&0\end{bmatrix}*\begin{bmatrix}0&0&1\\-2&0&-1\\0&1&0\end{bmatrix}*\begin{bmatrix}0&0&1\\-2&0&-1\\0&1&0\end{bmatrix}=\begin{bmatrix}-2&0&-1\\2&-2&1\\0&-1&-2\end{bmatrix}$$

And,

$$A^3x(0)=\begin{bmatrix}-2&0&-1\\2&-2&1\\0&-1&-2\end{bmatrix}*\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}-3\\1\\-3\end{bmatrix}$$

Then, our equation becomes,

$$\begin{bmatrix}4\\4\\4\end{bmatrix}=\begin{bmatrix}-3\\1\\-3\end{bmatrix}+\begin{bmatrix}0\\0\\-2\end{bmatrix}u(0)+\begin{bmatrix}0\\-2\\0\end{bmatrix}u(1)+\begin{bmatrix}1\\0\\0\end{bmatrix}u(2)$$

$$\begin{bmatrix}7\\3\\7\end{bmatrix}=\begin{bmatrix}0\\0\\-2\end{bmatrix}u(0)+\begin{bmatrix}0\\-2\\0\end{bmatrix}u(1)+\begin{bmatrix}1\\0\\0\end{bmatrix}u(2)$$

Then, we can deduce that,

$$u(2)=7\qquad u(1)=-1.5\qquad u(0)=-3.5$$

3. (a) A discrete-time system x(k+1) = Ax(k), y(k) = Cx(k) is said to be observable if there is a finite index N such that knowing the output sequence y(0),y(1),....,y(N-1) is sufficient to determine the initial state x(0).
We can find these equations from x(k+1) = Ax(k) equation,

$$x_1(1)=x_3(0)$$

$$x_2(1)=-2x_1(0)-x_3(0)$$

$$x_3(1)=x_2(0)$$

$$x_1(2)=x_3(1)=x_2(0)$$

$$x_2(2) = -2x_1(1) - x_3(1) = -2x_3(0) - x_2(0)$$
$$x_3(2) = x_2(1) = -2x_1(0) - x_3(0)$$

We can find these equations from y(k) = Cx(k),

$$y(0) = -2x_2(0) - 4x_3(0)$$

$$y(1) = -2x_2(1) - 4x_3(1) = 4x_1(0) - 4x_2(0) + 2x_3(0)$$
$$y(2) = -2x_2(2) - 4x_3(2) = 8x_3(0) + 2x_2(0) + 8x_1(0)$$

After solving this system, we get the followings,

$$x_1(0) = \frac{5}{32}y(2) - \frac{1}{16}y(1) + \frac{9}{32}y(0)$$
$$x_2(0) = \frac{1}{8}y(2) - \frac{1}{4}y(1) + \frac{1}{8}y(0)$$
$$x_3(0) = -\frac{1}{16}y(2) + \frac{1}{8}y(1) - \frac{5}{16}y(0)$$

We found a finite output sequence to determine the initial state x(0). Therefore, the system is observable.

(b) Observability matrix (pn,n) is

$$M = \begin{bmatrix} C \\ CA \\ CA^2 \\ \dots \\ CA^{N-1} \end{bmatrix}$$

The discrete-time system x(k+1) = Ax(k) , y(k) = Cx(k) is observable if and only if M has rank n.

In our case, A is (3x3) matrix, C is (1x3) matrix. Therefore, M is (3x3) matrix. Note that, n = 3 and p = 1.
If a discrete-time system is controllable if and only if the M has rank n.

Now, let's construct Observability matrix M.

$$CA = \begin{bmatrix} 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 2 \end{bmatrix}$$

$$CA^2 = CA * A = \begin{bmatrix} 4 & -4 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 8 \end{bmatrix}$$

$$M = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -4 \\ 4 & -4 & 2 \\ 8 & 2 & 8 \end{bmatrix}$$

We can indicate that, if determinant of M is different than 0, it's columns are linearly independent.
$det(M) = -128$; therefore, rank of M is 3. The system is observable.
Also, the solution that I derived in previous question is correct.

4. In order to find fixed points of the system, we need to solve this equation,

$$x'(t) = 3x^2 - 3x^3 = 0$$
$$3x^2(1 - x) = 0$$

Therefore, the fixed points of the system are $x = 0$ and $x = 1$.

The formula of stability via linearization is,

$$f(x) \approx f(\tilde{x}) + f'(\tilde{x})(x - \tilde{x})$$

$f(\tilde{x}) = 0$ since $\tilde{x}$ is fixed point of the system.

Note that,

$$f'(x) = 6x - 9x^2$$

For $\tilde{x} = 1$,

$$f(x) \approx f(1) + f'(1)(x - 1)$$
$$f(x) \approx 0 + (-3)(x - 1) = -3x + 3$$

Coefficient of $x < 0$, therefore fixed point 1 is stable.

For $\tilde{x} = 0$,

$$f(x) \approx f(0) + f'(0)(x - 0)$$
$$f(x) \approx 0 + 0(x) = 0$$

Test fails.