CENG 382 - Analysis of Dynamic Systems
20221
Take Home Exam 3 Solutions

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1. (a) The linearization formula for 2D system is,

$$f(x) \approx Df(\tilde{x})(x-\tilde{x}) + f(\tilde{x})$$

where, $Df(x)$ is Jacobian matrix.
Let's calculate $Df(x)$ for this system,

$$Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 & 2x_2 \\ 6x_1 & -2 \end{bmatrix}$$

Now, we can apply linearization to find whether or not our fixed point is stable. Since, $f(\tilde{x}) = 0$ and $\tilde{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$f(x) \approx \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now, we need to find eigenvalues of $Df(\tilde{x})$ to state stability of this fixed point.

$$det(\lambda I - Df(\tilde{x})) = \begin{vmatrix} \lambda+1 & 0 \\ 0 & \lambda+2 \end{vmatrix} = (\lambda+1)(\lambda+2)$$

We can conclude that, our eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -2$. Since all eigenvalues have negative reel parts, the fixed point is stable.

(b) We need to show 3 conditions holds for the function V.

i. V is continuous and has continuous first partial derivatives.
ii. V(x) has a unique minimum at $\tilde{x} \in \Omega$
iii. The difference function $\dot{V}$ satisfies $\dot{V} \leq 0 \ \forall x \in \Omega$

Firstly, V is continuous since x_1 and x_2 are continuous. The partial derivatives of V are also continuous.
Secondly, since $x_1^2 \geq 0$ and $x_2^2 \geq 0$ always, the minimum of V is $\mathbf{0} = \overline{x}$
Finally, we need to calculate $\dot{V}$.

$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x_1}(t) + \frac{\partial V}{\partial x_2}\dot{x_2}(t)$$

$$\dot{V} = x_1(-x_1 + x_2^2) + \frac{x_2}{2}(-2x_2 + 3x_1^2)$$

If we reorder this equation, we can see the conditions more easily.

$$\dot{V} = x_1^2(\frac{3}{2}x_2 - 1) + x_2^2(x_1 - 1)$$

We can state that,

$$\dot{V} \leq 0 \quad \text{where,} \quad x_2 \leq \frac{2}{3} \quad \text{and} \quad x_1 \leq 1$$

Thus, the fixed point is stable. Since system converges to fixed point in these intervals.

2. Firstly, let's say that our Lyapunov function is,

$$V(x_1, x_2, x_3) = ax_1^2 + bx_2^2 + cx_3^2$$

where $a, b, c \in \mathbb{R}$ and $a, b, c \geq 0$.
We need to show 3 conditions holds for the function V.

i.) V is continuous.

ii.) V has a unique minimum at $\tilde{x} \in \Omega$

iii.) The difference function $\nabla V(x) = V(f(x)) - V(x)$ satisfies $\nabla V(x) \leq 0 \; \forall x \in \Omega$

Firstly, V is continuous since x_1 , x_2 and x_3 are continuous.
Secondly, since $x_1^2 \geq 0$, $x_2^2 \geq 0$ and $x_3^2 \geq 0$ always, the minimum of V is $\mathbf{0} = \overline{x}$
Finally, we need to calculate $\nabla V(x)$.

$$V(f(x)) = a(\frac{1}{2}x_1 + \frac{1}{2}x_2)^2 + b(\frac{1}{2}x_3)^2 + c(\frac{1}{2}x_1 - \frac{1}{2}x_2)^2$$

$$V(f(x)) = \frac{a+c}{4}x_1^2 + \frac{a+c}{4}x_2^2 + \frac{b}{4}x_3^2 + \frac{a-c}{2}x_1x_2$$

$$\nabla V(x) = V(f(x)) - V(x) = \frac{a+c-4a}{4}x_1^2 + \frac{a+c-4b}{4}x_2^2 + \frac{b-4c}{4}x_3^2 + \frac{a-c}{2}x_1x_2 \leq 0$$

In order to satisfy $V \leq 0$, we need to select a, b, c values by solving this equations.

i.) $c - 3a \leq 0$

ii.) $a + c - 4b \leq 0$

iii.) $b - 4c \leq 0$

iv.) $a - c = 0$

And,

$$\nabla V(x) = -\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - \frac{3}{4}x_3^2$$

Since x_1^2 , x_2^2 and x_3^2 are greater or equal to 0, we need to make their coefficients negative. Also, we need to get rid of from $\frac{a-c}{2}(x_1x_2)$ part since we cannot predict the sign of it.
If we give, $a = b = c = 1$, all of these equations hold. Therefore, our Lyapunov function is,

$$V(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

Since V exists, $\tilde{x} = \mathbf{0}$ is stable.

3. Poincare-Bendixson theorem, states that a two dimensional system $\dot{x} = f(x)$ has one of the following behaviors as $t \to \infty$ when f is continuous.

i.) Converge to a fixed point

ii.) Diverge to infinity

iii.) Approach a periodic orbit

First of all, we will find fixed point of the system and try linearization to find whether or not this fixed point is stable.

$$x_1 + x_2 - 4x_1(x_1^2 + x_2^2) = 0$$
$$-x_1 + x_2 - 4x_2(x_1^2 + x_2^2) = 0$$

We need to solve this two equations.

$$\frac{x_1 + x_2}{4x_1} = \frac{-x_1 + x_2}{4x_2}$$

$$x_1 x_2 + x_2^2 = -x_1^2 + x_1 x_2$$

Then we found that, $x_1^2 + x_2^2 = 0$. This indicates that $x_1 = x_2 = 0$. The only fixed point is $\overline{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Now, we will apply linearization,

$$Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 - 4(x_1^2 + x_2^2) + 8x_1^2 & 1 - 8x_1 x_2 \\ -1 - 8x_1 x_2 & 1 - 4(x_1^2 + x_2^2) + 8x_2^2 \end{bmatrix}$$

$$f(x) \approx Df(\tilde{x})(x - \tilde{x}) + f(\tilde{x}) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We need to look eigenvalues of $Df(\tilde{x})$

$$det(\lambda I - Df(\tilde{x})) = \begin{vmatrix} \lambda - 1 & -1 \\ 1 & \lambda - 1 \end{vmatrix} = \lambda^2 - 2\lambda + 2$$

Our eigenvalues are,

$$\lambda_1 = 1 + i$$
$$\lambda_2 = 1 - i$$

Since the real parts of the eigenvalues are greater than 0, it is unstable fixed point. Therefore, the system does not converge to a fixed point.

Now, we will look for Lyapunov function.

Firstly, V is continuous and the partial derivatives of V are continuous.
Secondly, minimum value of V is $\tilde{x}$.
Finally we need to look at $\dot{V}$

$$\dot{V} = x_1(x_1 + x_2 - 4x_1(x_1^2 + x_2^2)) + x_2(-x_1 + x_2 - 4x_2(x_1^2 + x_2^2))$$

When we reorder the terms we get,

$$\dot{V} = (x_1^2 + x_2^2)(1 - 4x_1^2 - 4x_2^2) \leq 0$$

Since $(x_1^2 + x_2^2) \geq 0$, we need to look at $(1 - 4x_1^2 - 4x_2^2) \leq 0$.
When,

$$\frac{1}{4} \leq x_1^2 + x_2^2$$

Lyapunov function exists and the fixed point is stable. x_1 and x_2 are bounded. Thus, we cannot say that the system diverges to infinity.
Since the system does not converge nor diverge, the system has a periodic limit cycle.

4. (a) In order to find fixed point of the system,

$$3 - x^2 = x \longrightarrow x^2 + x - 3 = 0$$

$$\Delta = b^2 - 4ac = 13$$

Fixed points of the system are,

$$\overline{x}_1 = \frac{-1 - \sqrt{13}}{2}$$

$$\overline{x}_2 = \frac{-1 + \sqrt{13}}{2}$$

(b)

$$f^2(x) = f(f(x)) = f(3 - x^2) = -x^4 + 6x^2 - 6$$

Now we need to solve $f^2(x) = x$ to find periodic points.

$$x^4 - 6x^2 + x + 6 = 0$$

Since the fixed points we found in the previous step are fixed points of $f^2(x)$ too. Therefore, in order to find the other fixed points, we can say that one component of this equation is x^2+x-3 since their prime period is 1.

$$x^4 - 6x^2 + x + 6 = (x + 1)(x - 2)(x^2 + x - 3) = 0$$

The other points are $x = -1$ and $x = 2$.

(c) In order to state the stability of these points we need to look at the derivatives of $f(x)$ and $f^2(x)$.

$$f'(x) = (3 - x^2)' = -2x$$

$$\frac{d}{dx}f^2(x) = \frac{d}{dx}f(f(x)) = f'(f(x))f'(x) = f'(3 - x^2)(-2x) = 12x - 4x^3$$

i.) For x = -1,

$$\frac{d}{dx}f^2(-1) = -12 + 4 = -8$$

It is unstable since, $|\frac{d}{dx}f^2(-1)| > 1$

ii.) For x = 2,

$$\frac{d}{dx}f^2(2) = 24 - 32 = -8$$

It is unstable since, $|\frac{d}{dx}f^2(-2)| > 1$

iii.) For x = $\frac{-1-\sqrt{13}}{2}$,

$$f'(\frac{-1 - \sqrt{13}}{2}) = 1 + \sqrt{13}$$

It is unstable since, $|f'(\frac{-1-\sqrt{13}}{2})| > 1$

iv.) For x = $\frac{-1+\sqrt{13}}{2}$,

$$f'(\frac{-1 + \sqrt{13}}{2}) = 1 - \sqrt{13}$$

It is unstable since, $|f'(\frac{-1+\sqrt{13}}{2})| > 1$