CENG 384 - Signals and Systems for Computer Engineers
Spring 2022
Homework 2

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1. (a) According to the block diagram we have,

$$y(t) = \int \left[x(t) - 2\frac{dx(t)}{t} - 2\int y(t)dt + 3y(t) \right] dt$$

We need to take derivative once,

$$\frac{dy(t)}{dt} = \frac{d}{dt}\int \left[x(t) - 2\frac{dx(t)}{dt} - 2\int y(t)dt + 3y(t) \right]$$

$$\frac{dy(t)}{dt} = x(t) - 2\frac{dx(t)}{dt} - 2\int y(t)dt + 3y(t)$$

We need to take derivative once more,

$$\frac{d^2y(t)}{dt^2} = \frac{dx(t)}{dt} - 2\frac{d^2x(t)}{dt^2} - 2y(t) + 3\frac{dy(t)}{dt}$$

So, we get,

$$y''(t) = x'(t) - 2x''(t) - 2y(t) + 3y'(t)$$

The differential equation which represents the system given by block diagram:

$$y''(t) - 3y'(t) + 2y(t) = -2x''(t) + x'(t)$$

(b) Since this is a linear constant coefficient differential equation,

$$y(t) = y_h(t) + y_p(t)$$

where $y_h(t)$ is the homogeneous solution and $y_p(t)$ is particular solution. We have also initially rest conditions, which is $y(0) = y'(0) = y''(0) = 0$.

- For particular solution, $y_p(t) = Kx(t)$,

$$Kx''(t) - 3Kx'(t) + 2Kx(t) = -2x''(t) + x'(t)$$

$$K(e^{-t} + 4e^{-2t}) - 3K(-e^{t} - 2e^{-2t}) + 2K(e^{-t} + e{-}2t) = -2(e^{-t}) + (-e^{-t} - 2e^{-2t})$$

$$6Ke^{-t} + 12Ke^{-2t} = -3e^{-t} - 10e^{-2t}$$

By initially rest conditions, at $t = 0$, we have $K = \frac{-13}{18}$.
Then, particular solution, $y_p(t) = \frac{-13}{18}(e^{-t} + e^{-2t})$ for $t > 0$.

- For homogeneous solution, $y_h(t) = Ce^{\alpha t}$, we have:

$$\alpha^2 Ce^{\alpha t} - 3\alpha Ce^{\alpha t} + 2Ce^{\alpha t} = 0$$

$$Ce^{\alpha t}\left(\alpha^2 - 3\alpha + 2\right) = 0$$

From the equation above, we have $\alpha_1 = 2$ and $\alpha_2 = 1$. Therefore,

$$y_h(t) = C_1e^{2t} + C_2e^{t}$$

Finally, we have the solution $y(t) = y_h(t) + y_p(t)$ as:

$$y(t) = \frac{-13}{18}(e^{-t} + e^{-2t}) + C_1e^{2t} + C_2e^{t}$$

To find constants C_1 and C_2, we use initially rest conditions,

$$y(0) = C_1 + C_2 - \frac{26}{18} = 0$$

$$y'(0) = C_1 + 2C_2 + \frac{13}{18} + \frac{26}{18} = 0$$

Then, we have $C_1 = \frac{-65}{18}$ and $C_2 = \frac{91}{18}$. At the end, the solution $y(t)$ is:

$$y(t) = -\frac{65}{18}e^{2t} + \frac{91}{18}e^{t} - \frac{13}{18}(e^{-t} + e^{-2t})$$

2. (a)

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$x[k] = 0$ except for $k = 1$ and $k = -2$. We write the equation again for these k values,

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[1]h[n-1] + x[-2]h[n+2]$$

And, we have $x[1] = 1$ and $x[-2] = 3$.

$$y[n] = x[n] * h[n] = h[n-1] + 3h[n+2]$$

So, we have the solution as,

$$y[n] = x[n] * h[n] = 2\delta[n+1] - \delta[n] + 6\delta[n+4] - 3\delta[n+3]$$

The graph is below.

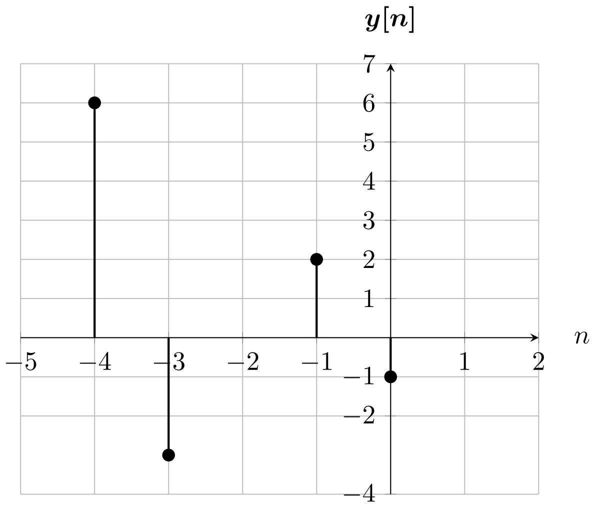

Figure 1: n vs. $y[n]$.

(b)

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Since $x[k] = 1 for - 1 \leq k < 2$, we write the equation as,

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1]$$

So,

$$y[n] = x[n] * h[n] = h[n+1] + h[n] + h[n-1]$$

Since $h[n] = u[n-4] - u[n-6]$, we have,

$$y[n] = x[n] * h[n] = u[n-3] - u[n-5] + u[n-4] - u[n-6] + u[n-5] - u[n-7]$$

We know that $u[n] - u[n-1] = \delta$ and $u[n-1] - u[n-2] = \delta[n-1]$. In the first equation, if we replace $u[n-1]$ with $\delta[n-1] + u[n-2]$ by second equation, we get $u[n] - u[n-2] = \delta[n] + \delta[n-1]$. Using this fact, we have,

$$u[n-3] - u[n-5] = \delta[n-3] + \delta[n-4]$$

$$u[n-4] - u[n-6] = \delta[n-4] + \delta[n-5]$$
$$u[n-5] - u[n-7] = \delta[n-5] + \delta[n-6]$$

Summing above equations side by side, we get the solution as,

$$y[n] = x[n]h[n] = \delta[n-3] + 2\delta[n-4] + 2\delta[n-5] + \delta[n-6]$$

The graph is below.

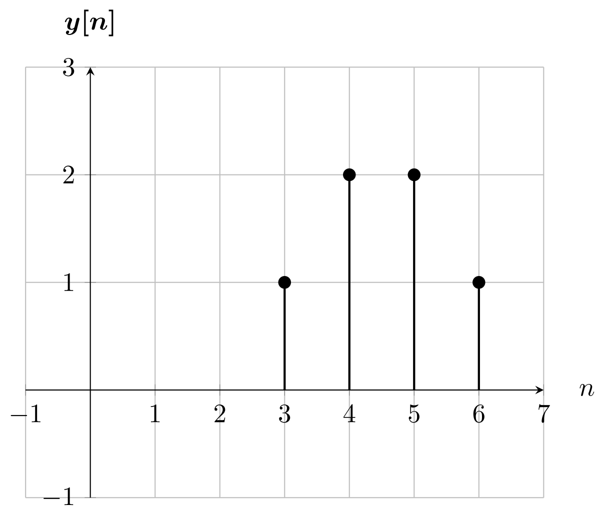

Figure 2: n vs. $y[n]$.

3. (a)

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-\frac{1}{2}(t-\tau)}u(t-\tau)d\tau$$

Since,

$$u(\tau) = \begin{cases} 1 & \tau \geq 0 \\ 0 & otherwise \end{cases}$$
$$u(t-\tau) = \begin{cases} 1 & t \geq \tau \\ 0 & otherwise \end{cases}$$

We have,

$$y(t) = \int_0^t e^{-\tau}e^{-\frac{1}{2}(t-\tau)}d\tau = e^{-\frac{1}{2}t}\int_0^t e^{-\tau}e^{\frac{\tau}{2}}d\tau = e^{\frac{1}{2}t}\int_0^t e^{-\frac{\tau}{2}}d\tau$$
$$y(t) = e^{-\frac{1}{2}t}\left[-2e^{-\frac{\tau}{2}}\Big|_0^t\right] = e^{-\frac{1}{2}t} + 2\left(-2e^{-\frac{1}{2}t}\right)$$

At the end, we have,

$$y(t) = \left(-2e^{-t} + 2e^{-\frac{1}{2}t}\right)u(t)$$

(b)

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} (u(\tau) - u(\tau-4))e^{-3(t-\tau)}u(t-\tau)d\tau$$
$$y(t) = \int_{-\infty}^{\infty} u(\tau)e^{-3(t-\tau)}u(t-\tau)d\tau - \int_{-\infty}^{\infty} u(\tau-4)e^{-3(t-\tau)}u(t-\tau)d\tau$$

- For $0 \leq t < 4$, since $u(t-4) = 0$ for $t < 4$, we have:

$$y(t) = \int_{-\infty}^{\infty} u(\tau)e^{-3(t-\tau)}u(t-\tau)d\tau = \int_0^t e^{-3(t-\tau)}d\tau = e^{-3t}\int_0^t e^{3\tau}d\tau = e^{-3t}\left[\frac{1}{3}e^{3\tau}\Big|_0^t\right]$$
$$y(t) = e^{-3t}\left[\frac{1}{3}e^{3t} - \frac{1}{3}\right] = \frac{1}{3} - \frac{e^{-3t}}{3} = \frac{1}{3}(1 - e^{-3t})u(t)$$

- For $t \geq 4$, we have,

$$y(t) = \int_{-\infty}^{\infty} u(\tau)e^{-3(t-\tau)}u(t-\tau)d\tau - \int_{-\infty}^{\infty} u(\tau-4)e^{-3(t-\tau)}u(t-\tau)d\tau = \int_0^t e^{-3(t-\tau)}d\tau - \int_4^t e^{-3(t-\tau)}d\tau$$

We have the first operand from above, $\int_0^t e^{-3(t-\tau)}d\tau = \frac{1}{3}(1-e^{-3t})$. We need to find second operand.

$$\int_4^t e^{-3(t-\tau)}d\tau = e^{-3t}\left[\frac{1}{3}e^{3\tau}\Big|_4^t\right] = \frac{e^{-3t}}{3}(e^{3t}-e^{12}) = \frac{1}{3}(1-e^{12-3t})$$

So,

$$y(t) = \frac{1}{3}(1-e^{-3t}) - \frac{1}{3}(1-e^{12-3t}) = \frac{1}{3}\left[e^{12-3t}-e^{-3t}\right] = \frac{e^{-3t}}{3}(e^{12}-1)u(t)$$

4. (a)

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x(\tau-3)d\tau = \int_{-\infty}^{t} x(\tau-3)e^{-(t-\tau)}u(t-\tau)d\tau$$

Now, we will variable change of variables method.

$$\tau - 3 = v$$

$$d\tau = dv$$

Now, we will apply this change into our integral.

$$\int_{-\infty}^{t-3} x(v)e^{-(t-v-3)}u(t-v-3)dv$$

Now, we found x(v), and the rest of the components are belong to h(t-v).

$$h(t-v) = e^{-(t-v-3)}u(t-v-3)$$

$$h(t) = e^{-(t-3)}u(t-3)$$

Note that,

$$h(t) = \begin{cases} 0 & t < 3 \\ e^{-(t-3)} & t \geq 3 \end{cases}$$

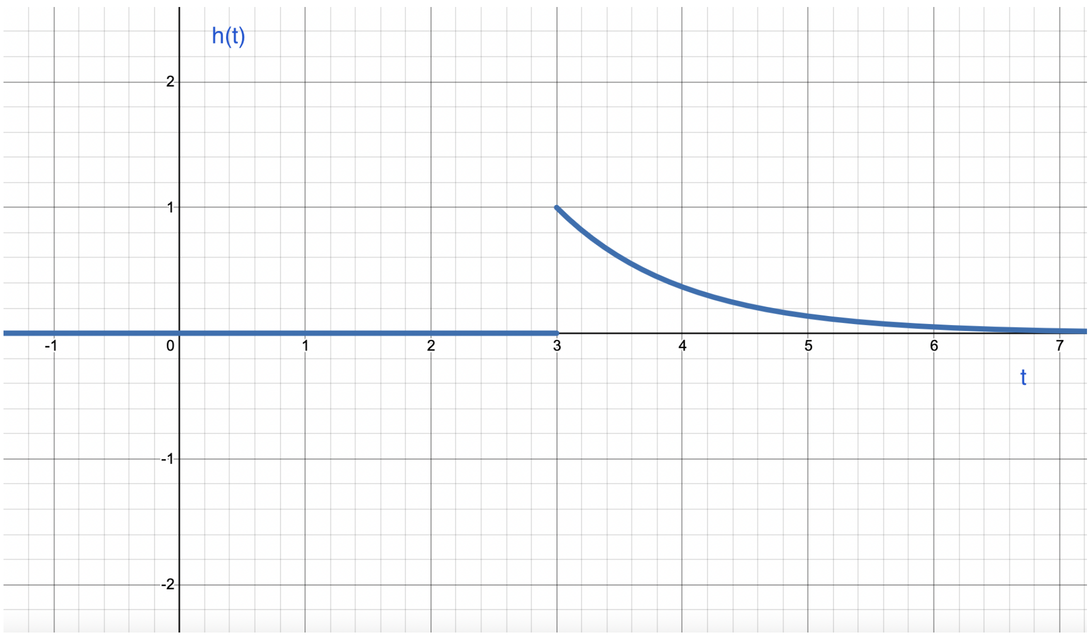

Figure 3: t vs h(t)

(b)

$$h(t-\tau) = e^{-(t-\tau-3)}u(t-\tau-3)$$
$$x(\tau) = u(\tau+2) - u(\tau-1)$$

Now, we will put these functions into the convolution integral that we found above.

$$\int_{-\infty}^{t-3} (u(\tau+2) - u(\tau-1))e^{-(t-\tau-3)}u(t-\tau-3)d\tau$$

We will change the variable of this integral.

$$v = \tau + 3$$
$$dv = d\tau$$

The integral becomes,

$$\int_{-\infty}^{t} (u(v-1) - u(v-4))e^{-(t-v)}u(t-v)dv$$

Since we know $h(t) = 0$ when $t < 3$, the solution of this integral for $t < 3$ is 0.
Also, since $u(\tau-4)$ is 0 when t ¡ 4, we will seperate our solution to two parts.

First part: For $3 \leq t \leq 4$

$$\int_{3}^{t} u(v-1)e^{-(t-v)}u(t-v)dv$$
$$\int_{3}^{t} e^{-(t-v)}dv$$
$$e^{-t}(e^{t} - e^{3}) = 1 - e^{-t+3}$$

Second part: For $4 < t$

$$\int_{3}^{t} u(v-1)e^{-(t-v)}u(t-v)dv - \int_{4}^{t} u(v-4)e^{-(t-v)}u(t-v)dv$$
$$\int_{3}^{t} e^{-(t-v)}dv - \int_{4}^{t} e^{-(t-v)}dv$$
$$[e^{-t}(e^{t} - e^{3})] - [e^{-t}(e^{t} - e^{4})] = e^{-t+4} - e^{-t+3}$$

Finally, our response is:

$$y(t) = \begin{cases} 0 & t < 3 \\ 1 - e^{-t+3} & 3 \leq t \leq 4 \\ e^{-t+4} - e^{-t+3} & 4 < t \end{cases}$$

5. (a) We know the inverse of $h_1[n]$, and we can find $h_1[n]$ by using the following formula:

$$h_1[n] * h_1^{-1}[n] = \delta[n] = u[n] - u[n-1]$$
$$h_1[n] = \delta[n] - \frac{1}{2}\delta[n-1]$$

Now we need to find $h_1[n] * h_1[n]$.

$$h_1[n] * h_1[n] = \sum_{k=-\infty}^{\infty} h_1[k]h_1[n-k]$$

$$h_1[n] * h_1[n] = \sum_{k=-\infty}^{\infty} (\delta[k] - \frac{-1}{2}\delta[k-1])(\delta[n-k] - \frac{-1}{2}\delta[n-k-1])$$

$$h_1[n] * h_1[n] = \sum_{k=-\infty}^{\infty} \delta[k]\delta[n-k] - \sum_{k=-\infty}^{\infty} \frac{1}{2}\delta[k]\delta[n-k-1] - \sum_{k=-\infty}^{\infty} \frac{1}{2}\delta[k-1]\delta[n-k] + \sum_{k=-\infty}^{\infty} \frac{1}{4}\delta[k-1]\delta[n-k-1]$$

$$h_1[k] = \begin{cases} \delta[0] & for \quad k = 0 \\ \frac{-1}{2}\delta[0] & for \quad k = 1 \\ 0 & otherwise \end{cases}$$

Therefore, our equation becomes:

$$h_1[n] * h_1[n] = \delta[n] - \frac{1}{2}\delta[n-1] - \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2]$$

And finally,

$$h_1[n] * h_1[n] = \delta[n] - \delta[n-1] + \frac{1}{4}\delta[n-2]$$

(b)

$$h[n] = h_0[n] * (h_1[n] * h_1[n])$$

$$h[n] * h_1^{-1}[n] * h_1^{-1}[n] = h_0[n] * h_1[n] * h_1[n] * h_1^{-1}[n] * h_1^{-1}[n]$$

Since we know that, $h_1^{-1}[n] * h_1[n] = \delta[n]$, and $h[n]\ ast\delta[n] = h[n]$, our equation becomes:

$$h[n] * h_1^{-1}[n] * h_1^{-1}[n] = h_0[n]$$

Now, in order to find $h_0[n]$, we need to calculate $h_1^{-1}[n] * h_1^{-1}[n]$ first.

$$h_1^{-1}[n] * h_1^{-1}[n] = \sum_{-\infty}^{\infty} h_1^{-1}[k]h_1^{-1}[n-k]$$

$$h_1^{-1}[n] * h_1^{-1}[n] = \sum_{k=-\infty}^{\infty} (\frac{1}{2})^k u[k] (\frac{1}{2})^{(n-k)} u[n-k] = \sum_{k=-\infty}^{\infty} (\frac{1}{2})^n u[k]u[n-k]$$

$$h_1^{-1}[n] * h_1^{-1}[n] = (\frac{1}{2})^n \sum_{k=-\infty}^{\infty} u[k]u[n-k]$$

Since, $u(k) = 1$ for $k \geq 0$ and $u(n-k) = 1$ for $k \leq n$,

$$h_1^{-1}[n] * h_1^{-1}[n] = (\frac{1}{2})^n \sum_{k=0}^{n} u[k]u[n-k] = (\frac{1}{2})^n \sum_{k=0}^{n} 1$$

And finally,

$$h_1^{-1}[n] * h_1^{-1}[n] = (\frac{1}{2})^n (n+1)u[n]$$

Now, we need to convolute $h[n]$ and $h_1^{-1}[n] * h_1^{-1}[n]$.

$$h[n] = 4\delta[n] + \delta[n-1] - 3\delta[n-3] + \delta[n-4]$$

$$h[n] * h_1^{-1}[n] * h_1^{-1}[n] = \sum_{k=-\infty}^{\infty} (4\delta[k] + \delta[k-1] - 3\delta[k-3] + \delta[k-4]) \frac{1}{2}^{(n-k)} (n-k+1)u[n-k]$$

Since h[n] is constructed with impulse functions, there are 4 k values that makes h[k] $\neq 0$. These are $k = 0$, $k = 1$, $k = 3$ and $k = 4$.

So, the convolution becomes:

$$h_0[n] = 4(\frac{1}{2})^n\ (n+1)\ u[n] + (\frac{1}{2})^{(n-1)}\ n\ u[n-1] + -3(\frac{1}{2})^{(n-3)}\ (n-2)\ u[n-3] + (\frac{1}{2})^{(n-4)}\ (n-3)\ u[n-4]$$

(c)

$$y[n] = x[n] * h_0[n]$$

$$y[n] = sum_{k=-\infty}^{\infty} x[k]h_0[n-k]$$

There are two steps since $x[n] = \delta[n] + \delta[n-2]$.

So, we can seperate $y[n]$ as, $y[n] = y_1[n] + y_2[n]$

$$y_1[n] = \delta[n] * h_0[n]$$

$$y_2[n] = \delta[n-2] * h_0[n]$$

We know that,

$$x[n] * \delta[n] = x[n]$$

$$x[n] * \delta[n-n_0] = x[n-n_0]$$

We can find both convolutions respectively.

$$y_1[n] = 4(\frac{1}{2})^n\ (n+1)\ u[n] + (\frac{1}{2})^{(n-1)}\ n\ u[n-1] + -3(\frac{1}{2})^{(n-3)}\ (n-2)\ u[n-3] + (\frac{1}{2})^{(n-4)}\ (n-3)\ u[n-4]$$

$$y_2[n] = 4(\frac{1}{2})^{(n-2)}\ (n-1)\ u[n-2] + (\frac{1}{2})^{(n-3)}\ (n-2)\ u[n-3] + -3(\frac{1}{2})^{(n-5)}\ (n-4)\ u[n-5] + (\frac{1}{2})^{(n-6)}\ (n-5)\ u[n-6]$$

$$y[n] = y_1[n] + y_2[n]$$