CENG 384 - Signals and Systems for Computer Engineers
Spring 2022
Homework 4

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August 5, 2022

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1. (a) Firstly,

$$y(t) = \int \left[\frac{dx(t)}{dt} - \int x(t)dt + x(t) - \int y(t)dt - 2y(t) \right] dt$$

Then, we take derivative of both sides,

$$\frac{dy(t)}{dt} = \frac{dx(t)}{dt} - \int x(t)dt + x(t) - \int y(t)dt - 2y(t)$$

Then, we take derivative again to get rid of integrals.

$$\frac{d^2y(t)}{dt^2} = \frac{d^2x(t)}{dt^2} - x(t) + \frac{dx(t)}{dt} - y(t) - 2\frac{dy(t)}{dt}$$

The differential equation becomes:

$$y(t) + 2y'(t) + y''(t) = -x(t) + x'(t) + x''(t)$$

(b)

$$x(t) = \delta(t) \leftrightarrow X(jw) = 1$$

And,

$$\sum_{k=0}^{N} a_k (jw)^k Y(jw) = \sum_{k=0}^{M} b_k (jw)^k X(jw)$$

And frequency response is,

$$H(jw) = \frac{Y(jw)}{X(jw)}$$

Now, we will use Fourier Transformation to the differential equation.

$$((jw)^2 + 2(jw) + 1)Y(jw) = ((jw)^2 + (jw) - 1)X(jw)$$

Since X(jw) = 1,

$$H(jw) = \frac{(jw)^2 + (jw) - 1}{(jw)^2 + 2(jw) + 1}$$

Which is,

$$H(jw) = 1 - \frac{1}{jw + 1} - \frac{1}{(jw + 1)^2}$$

(c) We need to apply Inverse Fourier Transformation.

$$h(t) \leftrightarrow H(jw)$$

We will use the Transform Table for Inverse Fourier Transformations.

$$1 \leftrightarrow \delta(t)$$

$$\frac{1}{jw + 1} \leftrightarrow e^{-t}u(t)$$

$$\frac{1}{(jw + 1)^2} \leftrightarrow te^{-t}u(t)$$

Therefore, the impulse response is:

$$h(t) = \delta(t) - e^{-t}u(t) - te^{-t}u(t)$$

(d) We will use Convolution Property here.

$$y(t) = h(t) * x(t) \leftrightarrow Y(jw) = H(jw)X(jw)$$

Now we need to Transform x(t). Since we found H(jw) before, we won't calculate it again. We found the Transformation in the Transformation Table.

$$x(t) = e^{-t}u(t) \leftrightarrow X(jw) = \frac{1}{jw+1}$$

Now we will calculate Y(jw).

$$Y(jw) = (1 - \frac{1}{jw+1} - \frac{1}{(jw+1)^2})(\frac{1}{jw+1})$$

$$Y(jw) = \frac{1}{jw+1} - \frac{1}{(jw+1)^2} - \frac{1}{(jw+1)^3}$$

Again we will use Transformation table to do Inverse Fourier Transformation.

$$y(t) = e^{-t}u(t) - te^{-t}u(t) - \frac{t^2}{2}e^{-t}u(t)$$

Which is equal,

$$y(t) = e^{-t}u(t)(1 - t - \frac{t^2}{2})$$

2. (a) To find impulse response of this system, we will put impulse function to x(t).

$$x(t) = \delta(t), \quad x(t+1) = \delta(t+1), \quad x(t-1) = \delta(t-1)$$

Then,

$$h'(t) = \delta(t+1) - \delta(t-1)$$

Now we take integral of both sides,

$$h(t) = \int_{-\infty}^{t+1} \delta(\tau)d\tau - \int_{-\infty}^{t-1} \delta(\tau)d\tau$$

Then the impulse response becomes,

$$h(t) = u(t+1) - u(t-1)$$

(b) In this part, we will use Time Shift Property of Fourier Transformation.

$$x(t - t_0) \leftrightarrow e^{-jwt_0}X(jw)$$

Thus,

$$x(t-1) \leftrightarrow e^{-jw}X(jw)$$

$$x(t+1) \leftrightarrow e^{jw}X(jw)$$

The equation becomes,

$$(jw)Y(jw) = (e^{jw} - e^{-jw})X(jw)$$

And the Frequency Response is:

$$H(jw) = \frac{Y(jw)}{x(jw)}$$

$$H(jw) = \frac{e^{jw} - e^{-jw}}{jw}$$

3. (a)

$$y[n] = x[n] * h_1[n] * h_2[n]$$

The overall impulse response is:

$$h[n] = h_1[n] * h_2[n]$$

Now we will use Fourier Transformation, and apply Convolution Property.

$$h[n] = h_1[n] * h_2[n] \leftrightarrow H(e^{jw}) = H_1(e^{jw})H_2(e^{jw})$$

Note that, $h_1[n] = h_2[n]$ and $H_1(e^{jw}) = H_2(e^{jw})$.

$$h_1[n] = (\frac{1}{2})^n u[n] \leftrightarrow H_1(e^{jw}) = \frac{1}{1 - \frac{1}{2}e^{-jw}}$$

The overall frequency response is

$$H(e^{jw}) = H_1(e^{jw})H_2(e^{jw}) = \frac{1}{(1 - \frac{1}{2}e^{-jw})^2}$$

(b) By Euler formula we can represent $x[n]$ as:

$$sin(\frac{\pi}{3} + \frac{\pi}{4}) = \frac{1}{2j}(e^{j\frac{\pi}{4}}e^{j\frac{\pi}{3}n} - e^{-j\frac{\pi}{4}}e^{-j\frac{\pi}{3}n})$$

By the Transformation Table, we have:

$$e^{j\frac{\pi}{3}n} \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \delta[w - \frac{\pi}{3} - 2\pi k]$$

$$e^{-j\frac{\pi}{3}n} \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \delta[w + \frac{\pi}{3} - 2\pi k]$$

So, Fourier Transform of the input:

$$X(e^{jw}) = \frac{\pi}{j} \sum_{k=-\infty}^{\infty} \left\{ e^{j\frac{\pi}{4}}\delta[w - \frac{\pi}{3} - 2\pi k] - e^{-j\frac{\pi}{4}}\delta[w + \frac{\pi}{3} - 2\pi k] \right\}$$

(c)

$$y[n] = x[n] * h[n]$$

Now we will use Fourier Transformation, and apply Convolution Property.

$$y[n] \leftrightarrow Y(e^{jw}) = X(e^{jw})H(e^{jw})$$

So Fourier Transform of the output:

$$Y(e^{jw}) = \frac{\pi}{j} \sum_{k=-\infty}^{\infty} \left\{ e^{j\frac{\pi}{4}}\delta[w - \frac{\pi}{3} - 2\pi k] - e^{-j\frac{\pi}{4}}\delta[w + \frac{\pi}{3} - 2\pi k] \right\} \left(\frac{1}{(1 - \frac{1}{2}e^{-jw})^2} \right)$$

4. (a) Note that,

$$h[n] \leftrightarrow H(e^{jw})$$

Let's say that,

$$h_1[n] = 2\delta[n], \qquad h_2[n] = 2^{-n}u[n]$$

Now, we will apply Fourier Transformation to $h_1[n]$ and $h_2[n]$

$$h_1[n] = 2\delta[n] \leftrightarrow H_1(e^{jw}) = 2$$

$$h_2[n] = 2^{-n}u[n] = (\frac{1}{2})^2 u[n] \leftrightarrow H_2(e^{jw}) = \frac{1}{1 - \frac{1}{2}e^{-jw}}$$

$$H(e^{jw}) = 2 + \frac{1}{1 - \frac{1}{2}e^{-jw}}$$

(b) In order to find difference equation, we will use Frequency response.

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = 2 + \frac{1}{1 - \frac{1}{2}e^{-jw}} = \frac{3 - e^{-jw}}{1 - \frac{1}{2}e^{-jw}}$$

Then,

$$(1 - \frac{1}{2}e^{-jw})Y(e^{jw}) = (3 - e^{-jw})X(e^{jw})$$

Then the difference equation becomes,

$$y[n] - \frac{1}{2}y[n-1] = 3x[n] - x[n-1]$$

(c) We can represent -1 in polar form as $e^{j\pi}$.
So, $(-1)^n = e^{j\pi n}$.
From the Transformation table, we have:

$$x[n] = e^{j\pi n} \leftrightarrow X(e^{jw}) = 2\pi \sum_{l=-\infty}^{\infty} \delta[w - \pi - 2\pi l]$$

$$y[n] = x[n] * h[n]$$

Now we will use Fourier Transformation, and apply Convolution Property.

$$y[n] \leftrightarrow Y(e^{jw}) = X(e^{jw})H(e^{jw})$$

Using $H(e^{jw})$ from part (a), $Y(e^{jw})$ is,

$$Y(e^{jw}) = 2\pi \sum_{l=-\infty}^{\infty} \delta[w - \pi - 2\pi l] \left(2 + \frac{1}{1 - \frac{1}{2}e^{-jw}} \right)$$