MIDDLE EAST TECHNICAL UNIVERSITY
DEPARTMENT OF COMPUTER ENGINEERING

CENG 382

Fall 2022

Take Home Exam 1

Regulations:

- **Submission:** You need to submit a pdf file named "the1.pdf" to the odtuclass page of the course. You need to use the given template "the1.tex" to generate your pdf files. Otherwise you will receive zero.
- **Deadline:** 23:55, 14 November, 2022 (Monday).
- **Late Submission:** The solutions will be available after the deadline. Therefore, late submissions will not be allowed.

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1. (15 pts) For each of the below systems, classify them in terms of the following criteria and give a brief explanation/proof stating your reasoning.
   - linear vs. non-linear
   - time varying vs. time invariant
   - forced vs. unforced

   (a) $y(k+3) + 2y(k+1) - y(k) = 5k + 8$
   (b) $\ddot{y}(t) - (t+1)^2\dot{y}(t) - y(t) = 0$
   (c) $\ddot{y}(t) - 5\dot{y}(t) + 6y(t) = y^2(t) + 3$

2. (30 pts, 20 + 10 pts) Let $A = \begin{bmatrix} 2 & 2 \\ 5 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $x_0 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Consider the continuous time dynamical system $\dot{x}(t) = Ax(t) + b$.

   (a) Find an exact formula for $x(t)$.
   (b) Comment on the behavior of the system as $t \to \infty$.

3. (15 pts) Consider the following system.
$$\frac{dx(t)}{dt} = -7x(t) + 5$$
   Identify the fixed point of the system and its behavior as $t \to \infty$. Determine whether the fixed point is stable or not.

4. (10 pts) Consider the system represented by the following equation.
$$\frac{d^3x(t)}{dt^3} + t^3\frac{d^2x(t)}{dt^2} + (t+1)\frac{dx(t)}{dt} - x(t) = 2t + 1$$
   Represent this third order system as a system of first order equations by showing your steps.

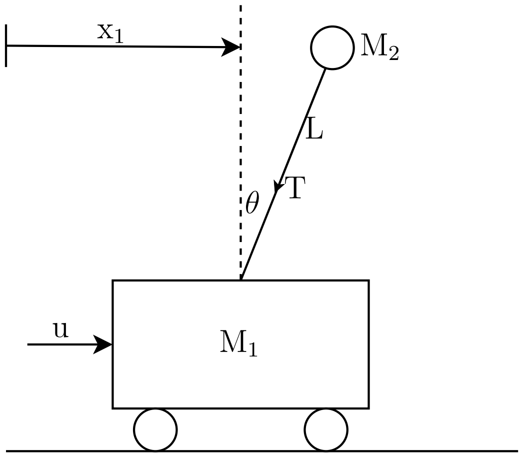
5. (30 pts, 10 pts each) Consider the system represented by the following equation.
$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k+2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

   (a) Find a fundamental set of solutions for the system for these two initial conditions $x^1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x^2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and write it in matrix form.
   (b) Using your answer in part (a), compute the state transition matrix, $\Phi(k, 0)$.
   (c) Find the fixed point of the system and comment on the system behavior as $k \to \infty$.

Ungraded Example Questions

1. Find the state transition matrix $\Phi(k,l)$ for the system $x(k+1) = \begin{bmatrix} \frac{k+2}{k+1} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} x(k)$. Comment on the behavior of the system as $k \to \infty$.

2. Consider the mechanical system below where a pendulum is attached to a moving cart with a rigid pole. L is the length of the pole, T is the tension on the pole created by the pendulum, M_1 and M_2 are the masses of the cart, and the pendulum, respectively.



   (a) (25 pts) Obtain the state equations for $\mathbf{x} = \begin{bmatrix} x_1 \\ \theta \\ \dot{x}_1 \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

   (b) (5 pts) Find an expression to calculate the equilibrium points using the definition of equilibrium points. (*Hint:* Think simple and give a brief explanation about your thought process.)

   Hints: In total, you will write down 3 equations. You can draw free body diagrams and make use of Newton's law ($F = ma$). First, apply Newton's law on the cart using the net force. Second, apply Newton's law on the pendulum in both horizontal and vertical directions. Notice that x_1, θ, u, T are all functions of t. Furthermore, at the final step, you can leave some variables as is (x_1, x_2, x_3, x_4) if there is not enough information to come to a distinct expression. (You do not need to draw the free body diagrams in your solutions, however, show each step of the derivation process clearly by writing down the equations.)

3. Let $A = \begin{bmatrix} \alpha & 1 \\ -2 & -3 \end{bmatrix}$ and $x_0 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Consider the discrete time dynamical system $x(k+1) = Ax(k)$.

   (a) Explain the behavior of the system as $k \to \infty$ conditioned on different values of α. For example, what is the condition on α such that the system converges to a fixed point as $k \to \infty$?

   (b) Find an exact formula for $x(k)$ when $\alpha = 0$ and verify your answer in part (a).