CENG 384 - Signals and Systems for Computer Engineers
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Homework 3

Ünlü, Berke Can
e2381028@ceng.metu.edu.tr

Altıntaş, Buğra Burak
e2380079@ceng.metu.edu.tr

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1. (a) By Euler's formula, we have,

$$sin(\frac{\pi}{5}t) = \frac{1}{2j}\left(e^{j\frac{\pi}{5}t} - e^{-j\frac{\pi}{5}t}\right)$$

$$cos(\frac{\pi}{4}t) = \frac{1}{2}\left(e^{j\frac{\pi}{4}t} + e^{-j\frac{\pi}{4}t}\right)$$

So,

$$x(t) = \frac{1}{2j}\left(e^{j\frac{\pi}{5}t} - e^{-j\frac{\pi}{5}t}\right) + \frac{1}{2}\left(e^{j\frac{\pi}{4}t} + e^{-j\frac{\pi}{4}t}\right)$$

To find spectral coefficients of $x(t)$, we need to use synthesis equation here. Since greatest common divisor of $\frac{\pi}{5}$ and $\frac{\pi}{4}$ is $\frac{\pi}{20}$, $\omega_0 = \frac{\pi}{20}$.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

For $\omega_0 = \frac{\pi}{20}$, we have the $x(t)$ in the form,

$$x(t) = \frac{1}{2j}\left(e^{j4\omega_0 t} - e^{-j4\omega_0 t}\right) + \frac{1}{2}\left(e^{j5\omega_0 t} + e^{-j5\omega_0 t}\right)$$

From this equation, we have nonzero coefficients a_4, a_{-4}, a_5, a_{-5}.

$$a_4 = \frac{1}{2j} \qquad a_{-4} = -\frac{1}{2j} \qquad a_5 = a_{-5} = \frac{1}{2}$$

(b) Since n is integer, $sin4\pi n = 0$ and $cos(2\pi n) = 1$ for all values of n.
So we have,

$$x[n] = \frac{3}{2} + e^{j\pi n}$$

By using synthesis equation,

$$x[n] = \frac{3}{2} + e^{j\pi n} = \sum_{k=<N>} a_k e^{jk\omega_0 n}, \quad \omega_0 = \pi$$

We have nonzero coefficients a_0 and a_1.

$$a_0 = \frac{3}{2} \qquad a_1 = 1$$

2. We have,

$$a_1 = 2j \qquad a_{-1} = -2j \qquad a_2 = a_{-2} = 2 \qquad a_3 = 2j \qquad a_{-3} = -2j$$

We can write the synthesis equation as,

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=-3}^{3} a_k e^{jk\omega_0 n} \quad \omega_0 = \frac{2\pi}{7}$$

$$x[n] = \sum_{k=-3}^{3} a_k e^{jk\omega_0 n} = a_{-1}e^{-j\omega_0 n} + a_1 e^{j\omega_0 n} + a_{-2}e^{-2j\omega_0 n} + a_2 e^{2j\omega_0 n} + a_{-3}e^{-3j\omega_0 n} + a_3 e^{3j\omega_0 n}$$

So,

$$x[n] = -2je^{-j\omega_0 n} + 2je^{j\omega_0 n} + 2e^{-2j\omega_0 n} + 2e^{2j\omega_0 n} + -2je^{-3j\omega_0 n} + 2je^{3j\omega_0 n}$$

By Euler's formula, we have the $x[n]$ in the form,

$$x[n] = -4sin(\omega_0 n) + 4cos(2\omega_0 n) - 4sin(\omega_0 n)$$

Since $4cos(2\omega_0 n) = 4sin(2\omega_0 n - \frac{\pi}{2})$, we have,

$$x[n] = -4sin(\omega_0 n) + 4sin(2\omega_0 n - \frac{\pi}{2}) - 4sin(\omega_0 n), \qquad \omega_0 = \frac{2\pi}{7}$$

$$x[n] = -4sin\left(\frac{2\pi}{7}n\right) - 4sin\left(\frac{4\pi}{7}n - \frac{\pi}{2}\right) - 4sin\left(\frac{6\pi}{7}n\right)$$

3. (a) By Euler's formula,

$$sin\left(\frac{\pi}{8}t\right) = \frac{1}{2j}(e^{j\frac{\pi}{8}t} - e^{-j\frac{\pi}{8}t}) = \frac{1}{2j}e^{j\frac{\pi}{8}t} - \frac{1}{2j}e^{-j\frac{\pi}{8}t}$$

Then, using synthesis equation,

$$x(t) = sin\left(\frac{\pi}{8}t\right) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \frac{1}{2j}e^{j\frac{\pi}{8}t} - \frac{1}{2j}e^{-j\frac{\pi}{8}t}$$

So, $a_1 = \frac{1}{2j}$ and $a_{-1} = -\frac{1}{2j}$

(b) By Euler's formula,

$$cos\left(\frac{\pi}{8}t\right) = \frac{1}{2}(e^{j\frac{\pi}{8}t} + e^{-j\frac{\pi}{8}t}) = \frac{1}{2}e^{j\frac{\pi}{8}t} + \frac{1}{2}e^{-j\frac{\pi}{8}t}$$

Then, using synthesis equation,

$$x(t) = cos\left(\frac{\pi}{8}t\right) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \frac{1}{2}e^{j\frac{\pi}{8}t} + \frac{1}{2}e^{-j\frac{\pi}{8}t}$$

So, $a_1 = a_{-1} = \frac{1}{2}$

(c) Let Fourier series coefficients for $x(t)$ be a_k, Fourier series coefficients for $y(t)$ be b_k and Fourier series coefficients for $z(t)$ be c_k.
By multiplication property,

$$c_k = a_k * b_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

From part a, we have a_k. So,

$$c_k = a_{-1}b_{k+1} + a_1 b_{k-1}$$

By inspecting b_k from part b, we can say that c_k may have nonzero values only for $k = -2$, $k = 0$ and $k = 2$.

$$c_{-2} = a_{-1}b_{-1} + a_1 b_{-3} = -\frac{1}{2j}\frac{1}{2} + 0 = -\frac{1}{4j}$$

$$c_0 = a_{-1}b_1 + a_1 b_{-1} = -\frac{1}{2j}\frac{1}{2} + \frac{1}{2j}\frac{1}{2} = 0$$

$$c_2 = a_{-1}b_3 + a_1 b_1 = 0 + \frac{1}{2j}\frac{1}{2} = \frac{1}{4j}$$

So, Fourier series coefficients for $z(t)$ are $c_{-2} = -\frac{1}{4j}$ and $c_2 = \frac{1}{4j}$

4. We can write x(t) in terms of sin functions, since it is real and odd periodic function.

$$\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$$

Also, $a_0, a_1, a_2, a_{-1}, a_{-2}$ are not zero.

Let's say that, one component of the x(t) is:

$$-6sin(\pi t)$$

By the Euler Formula,

$$-6sin(\pi t) = \frac{-6}{2j}(e^{j2\frac{\pi}{2}t} - e^{-j2\frac{\pi}{2}t}) = 3j(e^{j2\frac{\pi}{2}t} - e^{-j2\frac{\pi}{2}t})$$

This supports $a_2 = 3j$. Also, it can ease our job since we found a_{-2}.

Then, the equation becomes,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = a_0 + a_1 e^{j\frac{\pi}{2}t} + a_{-1}e^{-j\frac{\pi}{2}t} + 3je^{j2\frac{\pi}{2}t} - 3je^{-j2\frac{\pi}{2}t}$$

Again, since sin is odd and real periodic function, we will try to write a_1 and a_{-1} in terms of sin function. This means that, $a_1 = a_{-1}$.
Let's say that,

$$\alpha = 2ja_1$$

$$\alpha sin(\frac{\pi}{2}t) = \frac{\alpha}{2j}(e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t}) = a_1(e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t})$$

Now, x(t) becomes:

$$x(t) = a_0 + \alpha sin(\frac{\pi}{2}t) - 6sin(\pi t)$$

For the last requirements in the list can be solved by Parseval's Equality.
Parseval's Equality states that:

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{\forall k} |a_k|^2$$

$$\frac{1}{4}\int_0^4 |x(t)|^2 dt = \sum_{\forall k} |a_k|^2 = |a_0|^2 + |a_1|^2 + |a_2|^2 + |a_{-1}|^2 + |a_{-2}|^2 = 18$$

- $|a_2|^2 = 9$
- $|a_{-2}|^2 = 9$
- $|a_1|^2 = \frac{\alpha^2}{4}$
- $|a_{-1}|^2 = \frac{\alpha^2}{4}$

Now our equation becomes,

$$|a_0|^2 + \frac{\alpha^2}{2} + 18 = 18$$

$$|a_0|^2 + \frac{\alpha^2}{2} = 0$$

Since x(t) is real and odd periodic function, $a_0 = a_1 = a_{-1} = 0$. In order to solve the equation above, we need complex numbers, however both α and a_0 must be real.

$$x(t) = -6sin(\pi t))$$

5. (a) We will use analysis formula to find the coefficients.

$$a_k = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-jk\omega_0 n}$$

The period is 5, and $\omega_0 = \frac{2\pi}{9}$

$$a_k = \frac{1}{9}\sum_{n=0}^{4} x[n]e^{-jk\omega_0 n}$$

$$a_k = \frac{1}{9}(x[0] + x[1]e^{-jk\frac{2\pi}{9}} + x[2]e^{-jk\frac{4\pi}{9}} + x[3]e^{-jk\frac{6\pi}{9}} + x[4]e^{-jk\frac{8\pi}{9}})$$

$$a_k = \frac{1}{9}(1 + e^{-jk\frac{2\pi}{9}} + e^{-jk\frac{4\pi}{9}} + e^{-jk\frac{6\pi}{9}} + e^{-jk\frac{8\pi}{9}})$$

$$a_k = \begin{cases} \frac{5}{9} & k = 0, \pm N, \pm 2N... \\ \frac{1}{9}(1 + e^{-jk\frac{2\pi}{9}} + e^{-jk\frac{4\pi}{9}} + e^{-jk\frac{6\pi}{9}} + e^{-jk\frac{8\pi}{9}}) & k \neq 0, \pm N, \pm 2N... \end{cases}$$

(b) We will use analysis formula to find the coefficients.

$$b_k = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-jk\omega_0 n}$$

The period is 5, and $\omega_0 = \frac{2\pi}{9}$

$$b_k = \frac{1}{9}\sum_{n=0}^{3} y[n]e^{-jk\omega_0 n}$$

$$b_k = \frac{1}{9}(y[0] + y[1]e^{-jk\frac{2\pi}{9}} + y[2]e^{-jk\frac{4\pi}{9}} + y[3]e^{-jk\frac{6\pi}{9}})$$

$$b_k = \frac{1}{9}(1 + e^{-jk\frac{2\pi}{9}} + e^{-jk\frac{4\pi}{9}} + e^{-jk\frac{6\pi}{9}})$$

$$b_k = \begin{cases} \frac{4}{9} & k = 0, \pm N, \pm 2N... \\ \\ \frac{1}{9}(1 + e^{-jk\frac{2\pi}{9}} + e^{-jk\frac{4\pi}{9}} + e^{-jk\frac{6\pi}{9}}) & k \neq 0, \pm N, \pm 2N... \end{cases}$$

(c) The relationship between the spectral coefficients of the input and output pairs of a discrete time LTI system is:

$$b_k = a_k H(e^{jk\omega_0})$$

Therefore, the frequency response is:

$$H(e^{jk\omega_0}) = \begin{cases} \frac{4}{5} & k = 0, \pm N, \pm 2N... \\ \\ \frac{\frac{1}{9}(1+e^{-jk\frac{2\pi}{9}}+e^{-jk\frac{4\pi}{9}}+e^{-jk\frac{6\pi}{9}})}{\frac{1}{9}(1+e^{-jk\frac{2\pi}{9}}+e^{-jk\frac{4\pi}{9}}+e^{-jk\frac{6\pi}{9}}+e^{-jk\frac{8\pi}{9}})} & k \neq 0, \pm N, \pm 2N... \end{cases}$$