CENG 384 - Signals and Systems for Computer Engineers
Spring 2022
Homework 1

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1. (a) i)
First of all, $\overline{z}$ is conjugate of z.
So, $\overline{z} = x - jy$
Let us solve the equality and find the x and y.

$$2x + 2yj - 9 = 4j - x + jy$$

Then, we put the unknowns to left side, and others to right side.

$$3x + jy = 9 + 4j$$

Therefore, we got the followings.

$$x = 3 \qquad y = 4$$

Thus,

$$z = 3 + 4j$$

In order to find $|z|$, we need to find r.
Polar form of a complex number, (it comes from Euler's Identity)

$$z = re^{j\Theta}$$

Where,

$$|z| = r = \sqrt{a^2 + b^2}, \ \ a = Re\{z\}, \ \ b = Im\{z\}$$

Therefore,

$$|z| = r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Eventually we get this for $|z|^2$,

$$|z|^2 = 5 * 5 = 25$$

ii)
In the previous part of this question, we found r of the polar form of z.
Now, we need to find the angle between the $x - axis$ and the complex number z.

$$\Theta = arctan(\frac{Im\{z\}}{Re\{z\}})$$

Now we put the numbers and found the angle,

$$\Theta = arctan(\frac{4}{3}) = 0.927295218 \ \ radians = 53.13 \ \ degrees$$

Ultimately, we got the polar form of z.

$$z = re^{j\Theta} = 5e^{j53.13^\circ}$$

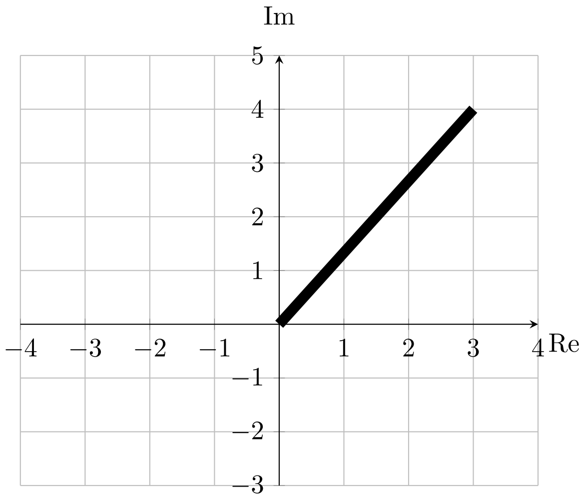

Figure 1: Plot of z

The degree between x-axis and z is 53.13° and the size of the line is $r = 5$.

(b) Firstly, note that $-j = j^3$; therefore, we can rewrite the equation as,

$$z^3 = -27j = 27j^3 = 3^3j^3 = (3j)^3$$

Then, take cuberoot of both sides,

$$\sqrt[3]{z^3} = \sqrt[3]{(3j)^3}$$

Finally, we got,

$$z = 3j$$

Polar form of z is $re^{j\Theta}$. In order to find r and Θ, we will use these formulas:

$$|z| = r = \sqrt{(Re\{z\})^2 + (Im\{z\})^2}$$

$$\Theta = arctan(\frac{Im\{z\}}{Re\{z\}})$$

Note that, $Re\{z\} = 0$ and $Im\{z\} = 3$.
If we put these number into equations above, we got the followings:

$$|z| = r = \sqrt{0^2 + 3^2} = 3$$

$$\Theta = \frac{\pi}{2}$$

Finally, we got the polar form of z.

$$z = 3e^{j\frac{\pi}{2}}$$

(c) Firstly, multiply both sides with $(\sqrt{3} - j)$.

$$z = \frac{(1+j)(\sqrt{3}-j)^2}{(\sqrt{3}+j)(\sqrt{3}-j)}$$

$$z = \frac{(1+j)(\sqrt{3}-j)^2}{3-j^2} = \frac{(1+j)(\sqrt{3}-j)^2}{4}$$

Note that, $j^2 = -1$.

$$z = \frac{(1+j)(3-2\sqrt{3}j+j^2)}{4} = \frac{(1+j)(2-2\sqrt{3}j)}{4}$$

From now on, we will find the polar form of $z_1 = 1 + j$ and $z_2 = 2 - 2\sqrt{3}j$. And, at the end, we will find the polar form of z to find the magnitude and the angle of z.

$$z_1 = (1 + j)$$

$$z_2 = (2 - 2\sqrt{3}j)$$

Firstly, we will find the magnitudes of these complex numbers. Then, we will find the angles of them.

$$r_1 = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$r_2 = \sqrt{2^2 + (-2\sqrt{3})^2} = 4$$

$$\Theta_1 = arctan(1/1) = \frac{\pi}{4}$$

$$\Theta_2 = arctan(\frac{2}{-2\sqrt{3}}) = -\frac{\pi}{3}$$

Finally, we got $z1$ and $z2$ respectively.

$$z_1 = \sqrt{2}e^{j\frac{\pi}{4}}$$

$$z_2 = 4e^{-j\frac{\pi}{3}}$$

Then, our equation becomes:

$$z = \frac{\sqrt{2}e^{j\frac{\pi}{4}} * 4e^{-j\frac{\pi}{3}}}{4}$$

$$z = \frac{\sqrt{2}e^{j\frac{3\pi}{12}} * 4e^{-j\frac{4\pi}{12}}}{4}$$

Finally, we got

$$z = \frac{4\sqrt{2}e^{-j\frac{\pi}{12}}}{4} = \sqrt{2}e^{-j\frac{\pi}{12}}$$

The magnitude of z is $\sqrt{2}$, and the angle is $\frac{-\pi}{12}$

(d) Firstly, we can write the $(1+j)^8$ as $((1+j)^2)^4$.
Also, we can write $(1+j)^2 = 1 + 2j + j^2 = 2j$
Then we got,

$$(1+j)^8 = (2j)^4 = 16j^4 = 16$$

Then, we put 16 instead of $(1+j)^8$.

$$z = -(1+j)^8 e^{j\frac{\pi}{2}} = -16e^{j\frac{\pi}{2}}$$

This is the polar form of z. Where magnitude of z is 16 and angle of z is $\frac{\pi}{2}$. Minus sign indicates the direction of the complex number.

2. (a) Firstly, the Energy formula of the discrete time signal in the range of $[n_1, n_2]$:

$$E = \sum_{n_1}^{n_2} |x[n]|^2$$

Now, we put the numbers and signal on this equation:

$$E = \sum_{-\infty}^{\infty} |nu[n]|^2$$

We can seperate the summation into two parts such as:

$$E = \sum_{k=-\infty}^{0} |ku[k]|^2 + \sum_{k=1}^{\infty} |ku[k]|^2$$

And, since u[n] is 0 when $n < 0$, the first part of this equation is 0.
Then, our energy formula looks like this:

$$E = \sum_{k=1}^{\infty} |ku[k]|^2$$

Since, u[n] is 1 when $n \geq 1$, we can discard it from the summation since it has no effect.

$$E = \sum_{k=1}^{\infty} |k|^2$$

Now, we put limit on this summation.

$$E = \lim_{m\to\infty} \sum_{k=1}^{m} |k|^2$$

Since $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, we got

$$E = \lim_{m\to\infty} \frac{m(m+1)(2m+1))}{6} = \infty$$

If a signal is Energy Signal, $E < \infty$ must be satisfied. However, we cannot say that this signal is energy signal.

Secondly, the Power formula of the discrete time signal is:

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Now, we will put the signal in this equation, and as we did before, we will seperate the summation into two parts:

$$P = \lim_{N\to\infty} \frac{1}{2N+1} (\sum_{n=-N}^{0} |nu[n]|^2 + \sum_{n=1}^{N} |nu[n]|^2)$$

Since $u[n]$ is 0 when $n < 0$ and when $n = 0$, $nu[n] = 0$, the first summation is 0. So, we can discard it.

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=1}^{N} |nu[n]|^2$$

And, since $u[n]$ is 1 when $n \geq 0$, it has no effect on the summation. So, we can discard it.

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=1}^{N} |n|^2$$

Since $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, we got:

$$P = \lim_{N\to\infty} \frac{N(N+1)(2N+1)}{2N+1} = \infty$$

By the definition of the Power Signals, $P \neq 0$ and $P < \infty$. However, for this signal, we cannot say it is a power signal.

(b) Firstly, the Energy formula of the continuous time signal in the range of $[t_1, t_2]$:

$$E = \int_{t_1}^{t_2} |x(t)|^2 \, dt$$

Now, we put the numbers and signal on this equation:

$$E = \int_{-\infty}^{\infty} |e^{-2t} u(t)|^2 \, dt$$

We can seperate this integral into two parts such as:

$$E = \int_{-\infty}^{0} |e^{-2t} u(t)|^2 \, dt + \int_{0}^{\infty} |e^{-2t} u(t)|^2 \, dt$$

And, since u(t) is 0 when $t < 0$, the first integral is 0.
Then, our energy formula looks like this:

$$E = \int_{0}^{\infty} e^{-4t} u(t)^2 \, dt$$

Since, u(t) is 1 when $t \geq 1$, we can discard it from the integral since it has no effect.

$$E = \int_{0}^{\infty} e^{-4t} \, dt$$

Now, we put limit on this integral.

$$E = \lim_{\tau\to\infty} \int_{0}^{\tau} e^{-4t} \, dt$$

Now, we take the integral:

$$E = \lim_{\tau\to\infty} [(-\frac{e^{-4\tau}}{4}) - (-\frac{1}{4})]$$

$$E = \lim_{\tau\to\infty} -\frac{e^{-4\tau}}{4} + \lim_{\tau\to\infty} \frac{1}{4}$$

The first limit equation is 0, and the second one is $\frac{1}{4}$ since it does not depend on τ.
So, we got:

$$E = \frac{1}{4}$$

If a signal is Energy Signal, $E < \infty$ must be satisfied. And this signal is Energy signal since it is less than infinity.

Secondly, the Power formula of the continuous time signal is:

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

Now, we will put the signal in this equation, and as we did before, we will seperate the integral into two parts:

$$P = \lim_{T\to\infty} \frac{1}{2T} \left(\int_{-T}^{0} |e^{-2t}u(t)|^2 \, dt + \int_{0}^{T} |e^{-2t}u(t)|^2 \, dt \right)$$

Since $u(t)$ is 0 when $n < 0$, the first integral is 0. So, we can discard it.

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T} |e^{-2t}u(t)|^2 \, dt$$

And, since $u(t)$ is 1 when $t \geq 0$, it has no effect on the integral. So, we can discard it.

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T} e^{-4t} \, dt$$

Now, we take the integral:

$$P = \lim_{T\to\infty} \frac{-\frac{e^{-4T}}{4} + \frac{1}{4}}{2T} = \lim_{T\to\infty} -\frac{1}{8e^{4T}T} + \lim_{T\to\infty} \frac{1}{8T} = 0$$

By the definition of the Power Signals, $P \neq 0$ and $P < \infty$. However, for this signal, we cannot say it is a power signal since it is 0.

3. Firstly, we will write x(t) as a partial function.

$$x(t) = \begin{cases} 0 & -3 < t \leq -1 \\ 2 + 2t & -1 \leq t \leq 0 \\ 2 & 0 \leq t \leq 1 \\ 4 - 2t & 1 \leq t \leq 2 \\ 0 & 2 \leq t < 3 \end{cases}$$

Now, we will apply the shift and time scale and time reverse operations to x(t) in order to find $x(-\frac{1}{3}t + 2)$.

(a) Time Reverse : $-t$

(b) Time Scale : $-\frac{1}{3}t$

(c) Time Shift : $-\frac{1}{3}t + 2$

$$x(-\frac{1}{3}t + 2) = \begin{cases} 0 & -3 < (-\frac{1}{3}t + 2) \leq -1 \\ 2 + 2(-\frac{1}{3}t + 2) & -1 \leq (-\frac{1}{3}t + 2) \leq 0 \\ 2 & 0 \leq (-\frac{1}{3}t + 2) \leq 1 \\ 4 - 2(-\frac{1}{3}t + 2) & 1 \leq (-\frac{1}{3}t + 2) \leq 2 \\ 0 & 2 \leq (-\frac{1}{3}t + 2) < 3 \end{cases}$$

$$x(-\frac{1}{3}t + 2) = \begin{cases} 0 & -3 < t \leq 0 \\ \frac{2}{3}t & 0 \leq t \leq 3 \\ 2 & 3 \leq t \leq 6 \\ 6 - \frac{2}{3}t & 6 \leq t \leq 9 \\ 0 & 9 \leq t < 15 \end{cases}$$

Finally, we will find $\frac{1}{2}x(-\frac{1}{3}t + 2)$. We need to divide the values in these pieces by 2.

$$\frac{1}{2}x(-\frac{1}{3}t + 2) = \begin{cases} 0 & -3 < t \leq 0 \\ \frac{1}{3}t & 0 \leq t \leq 3 \\ 1 & 3 \leq t \leq 6 \\ 3 - \frac{1}{3}t & 6 \leq t \leq 9 \\ 0 & 9 \leq t < 15 \end{cases}$$

Now, we can draw the signal $\frac{1}{2}x(-\frac{1}{3}t+2)$.

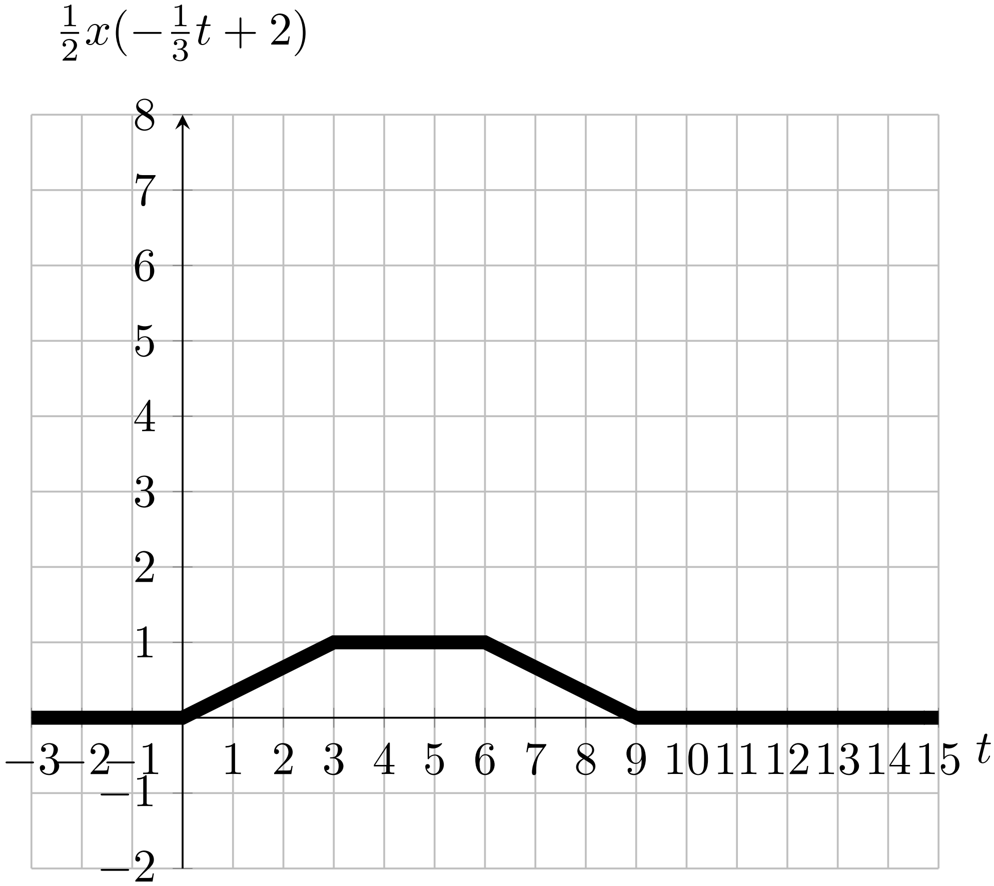

Figure 2: t vs. $\frac{1}{2}x(-\frac{1}{3}t+2)$.

4. (a) In order to find $x[-2n]$, we need to apply time reverse and time scale operations respectively. And, the plot of $x[-n]$:

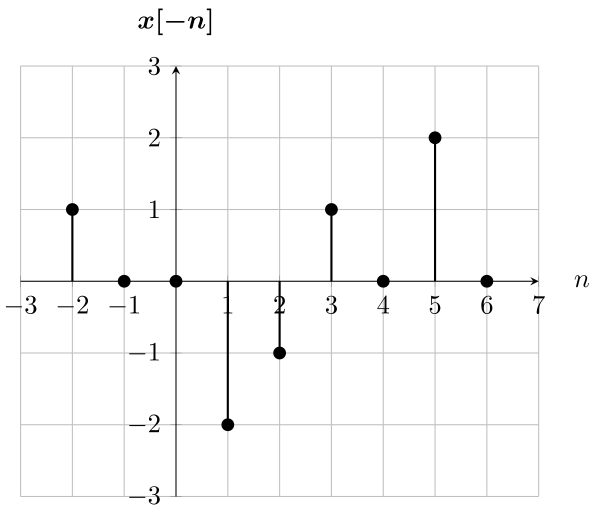

Figure 3: n vs. $x[-n]$.

Now, we apply time scale operation to $x[-n]$ to find $x[-2n]$.
Since it is a discrete time signal, the values, that do not have integer values for "n", will be discarded.

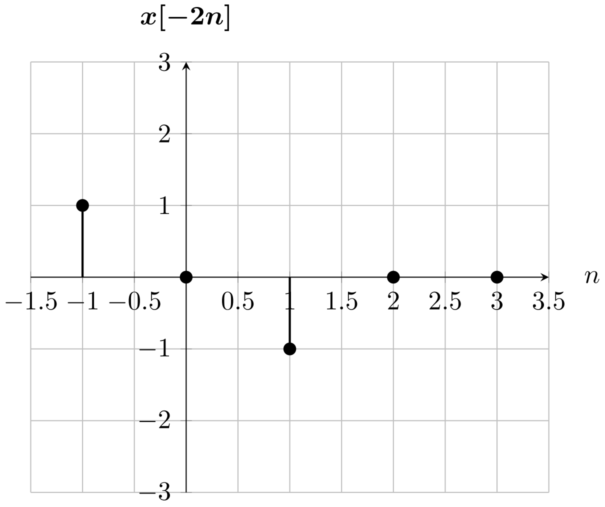

Figure 4: n vs. $x[-2n]$.

In order to find $x[n-2]$, we will apply time shift operation to the signal.

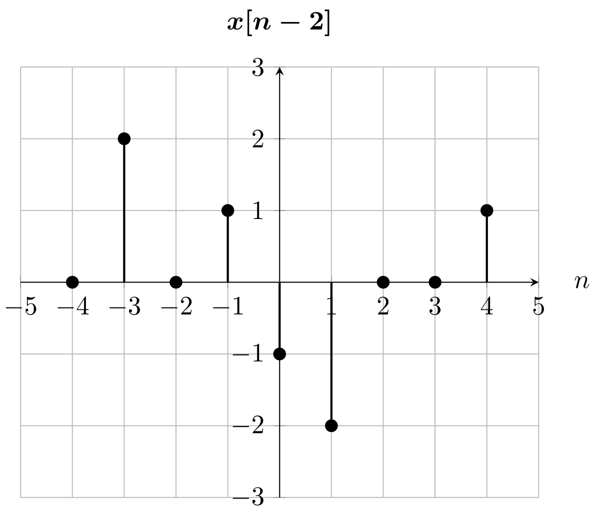

Figure 5: n vs. $x[n-2]$.

Finally, we will sum $x[-2n]$ and $x[n-2]$. And the plot is below.

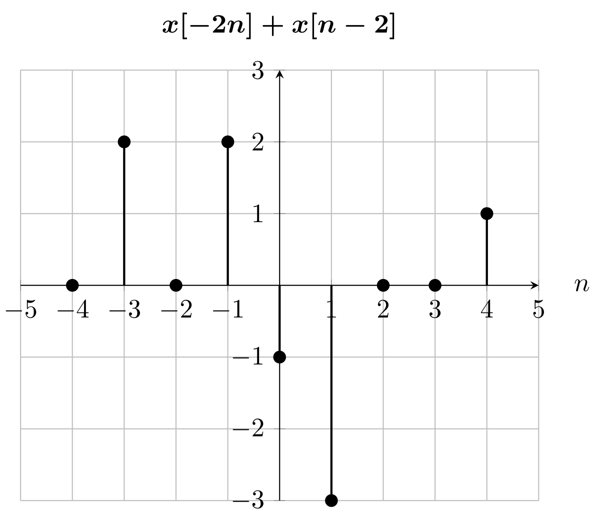

Figure 6: n vs. $x[-2n]+x[n-2]$.

(b) From the last graph, we can find the unit impulse function representation of $x[-2n] + x[n-2]$.

$$x[-2n] + x[n-2] = 2\delta[n+3] + 2\delta[n+1] - \delta[n] - 3\delta[n+1] + \delta[n-4]$$

5. (a) If $x(t)$ is periodic, $x(t) = x(t+T_0)$ must be satisfied.

$$x(t+T_0) = \frac{e^{j3(t+T_0)}}{-j} = \frac{e^{j3t+j3T_0}}{-j} = \frac{e^{j3t} * e^{j3T_0}}{-j}$$

So, we need:

$$\frac{e^{j3t}}{-j} = \frac{e^{j3t} * e^{j3T_0}}{-j}$$

Knowing that $\frac{1}{-j} = j$, e^{j3T_0} must be equal to 1.
By the Euler Formula, we have:

$$e^{j3T_0} = cos(3T_0) + jsin(3T_0)$$

The period of the complex exponential function should satisfy the following:

$$T_0 = k * \frac{2\pi}{|w_0|}$$

Then,

$$e^{j3T_0} = 1 = cos(3\frac{2\pi k}{3}) + jsin(3\frac{2\pi k}{3}) = cos(2\pi k)$$

When k is an integer, the imaginary part cancels and the real part becomes 1. Therefore, $x(t)$ is periodic with the fundamental period,

$$T = \frac{2\pi}{3}$$

(b) At first, we can convert the signal $x[n]$ to following version by using the equation $cos[\Theta - \frac{\pi}{2}] = sin[\Theta]$

$$x[n] = \frac{1}{2}sin[\frac{7\pi}{8}n] + 4sin[\frac{3\pi}{4}n]$$

If $x[n]$ is periodic, $x[n] = x[n+N_0]$ must satisfy for an integer N_0. So, we need:

$$\frac{1}{2}sin[\frac{7\pi}{8}n] + 4sin[\frac{3\pi}{4}n] = \frac{1}{2}sin[\frac{7\pi}{8}n + \frac{7\pi}{8}N_0] + 4sin[\frac{3\pi}{4}n + \frac{3\pi}{4}N_0]$$

For the above equation to hold, we need an integer N_0 that satisfies:

$$\frac{7\pi}{8}N_0 = 2\pi k \quad \frac{3\pi}{4}N_0 = 2\pi m$$

where k and m are integers. So, we have:

$$N_0 = \frac{16k}{7} = \frac{8m}{3}$$

The above equation holds when $k = 7$ and $m = 3$. Therefore, $x[n]$ is periodic with the fundamental period $N_0 = 16$.

6. (a) For a signal to be even, it must be symmetric according to the $y - axis$. The symmetry of the signal according to $y - axis$:

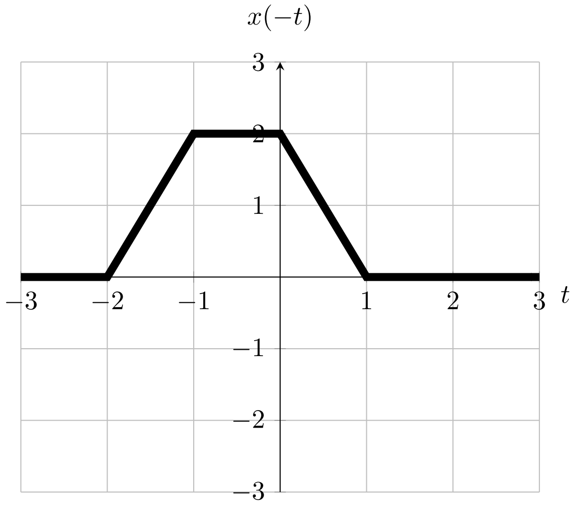

Figure 7: t vs. $x(-t)$.

As we can see from the graph, $x(t)$ is not symmetric to $y - axis$, so it is not even. Also, we can understand that by just the inequality of $x(1) \neq x(-1)$.

For a signal to be odd, it must be symmetric according to the origin. The symmetry of the signal according to origin:

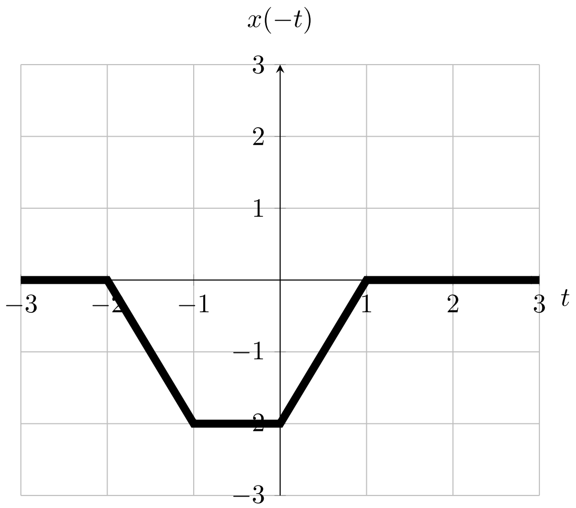

Figure 8: t vs. $-x(-t)$.

As we can see from the graph, $x(t)$ is not symmetric to origin, so it is not odd. Also, we can understand that by just the inequality of $x(1) \neq -x(-1)$.

(b) We can find the even composition of $x(t)$ in the following way:

$$Even\{x(t)\} = \frac{x(t) + x(-t)}{2}$$

So,

$$x(t) = \begin{cases} 0 & -3 < t \leq -1 \\ 2 + 2t & -1 \leq t \leq 0 \\ 2 & 0 \leq t \leq 1 \\ 4 - 2t & 1 \leq t \leq 2 \\ 0 & 2 \leq t < 3 \end{cases} \quad x(-t) = \begin{cases} 0 & -3 < t \leq -2 \\ 4 + 2t & -2 \leq t \leq -1 \\ 2 & 0 \leq t \leq -1 \\ 2 - 2t & 0 \leq t \leq 1 \\ 0 & 1 \leq t < 3 \end{cases}$$

So, the even part is:

$$Even\{x(t)\} = \frac{x(t) + x(-t)}{2} = \begin{cases} 0 & -3 < t \leq -2 \\ 2 + t & -2 \leq t \leq -1 \\ 2 + t & 0 \leq t \leq -1 \\ 2 - t & 0 \leq t \leq 1 \\ 2 - t & 1 \leq t \leq 2 \\ 0 & 2 \leq t < 3 \end{cases}$$

The plot is below:

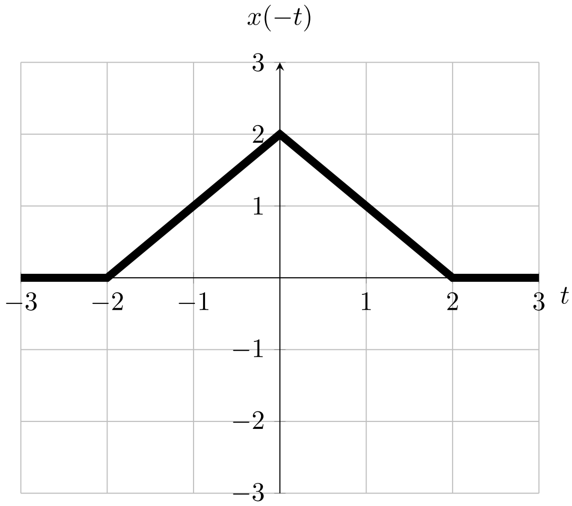

Figure 9: t vs. $\frac{x(t)+x(-t)}{2}$.

We can find the odd composition $x(t)$ in the following way:

$$Odd\{x(t)\} = \frac{x(t) - x(-t)}{2}$$

So, by using $x(t)$ and $x(-t)$ from even part, the odd part is:

$$Odd\{x(t)\} = \frac{x(t) - x(-t)}{2} = \begin{cases} 0 & -3 < t \leq -2 \\ -2 - t & -2 \leq t \leq -1 \\ t & 0 \leq t \leq -1 \\ t & 0 \leq t \leq 1 \\ 2 - t & 1 \leq t \leq 2 \\ 0 & 2 \leq t < 3 \end{cases}$$

The plot is below:

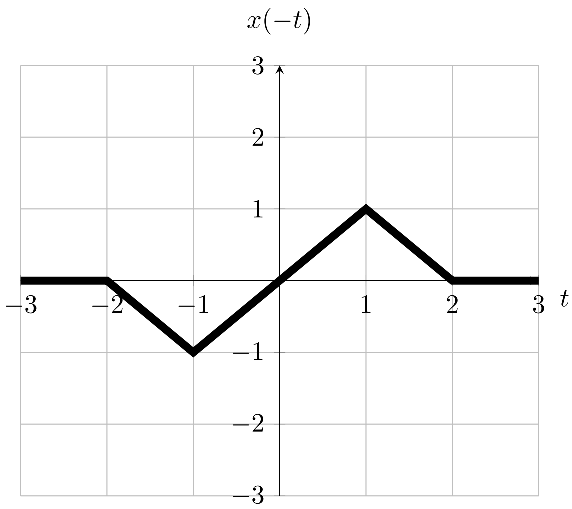

Figure 10: t vs. $\frac{x(t)-x(-t)}{2}$.

7. (a) Expression $x(t)$ in terms of unit step function:

$$x(t) = 3u(t+3) - 3u(t+1) + 2u(t-2) - 4u(t-4) + 3u(t-6)$$

(b) We can represent the $\frac{dx(t)}{dt}$ by scaling unit impulses at the points that $x(t)$ changes.

$$\frac{dx(t)}{dt} = 3\delta(t+3) - 3\delta(t+1) + 2\delta(t-2) - 4\delta(t-4) + 3\delta(t-6)$$

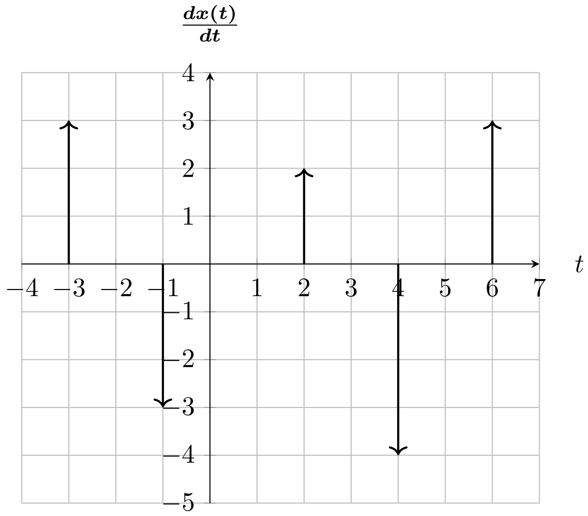

Figure 11: t vs. $x'(t)$.

8. (a)
- The system has memory. For example, $y[3] = x[4]$
- The system is stable. When we replace the input signal with a constant, output is also a constant.
- The system is causal. We can represent the system as:

$$y[n] = h(x[n-1]) = x[2(n-1)]$$

  Since $1 \geq 0$, the system is causal.
- For a system to be linear, superposition property must hold. Suppose we feed two inputs, x_1 and x_2, to the system. The outputs are:

$$x_1[n] \rightarrow y_1[n] = x_1[2n-2]$$
$$x_2[n] \rightarrow y_2[n] = x_2[2n-2]$$

  Output for the superposition of the two inputs, $x_3[n] = a_1x_1[n] + a_2x_2[n]$:

$$y_3[n] = x_3[2n-2] = a_1x_1[2n-2] + a_2x_2[2n-2] = a_1y_1[n] + a_2y_2[n]$$

  So, the system is linear.
- The system is invertible since it is one-to-one. We can find a unique inverse system which is:

$$y[\frac{n+2}{2}] = x[n]$$

- For a system to be time invariant, a time shift at the input generates the same time shift at the output. When we shift the input:

$$x[n-n_0] \rightarrow y[n] = x[2n-2n_0-2]$$

  The shift for the input does not give the same amount of shift at the output:

$$x[n-n_0] \rightarrow y[n] = x[2n-2n_0-2] \neq y[n-n_0] = x[2n-2-n_0]$$

  So, the system is not time-invariant.

(b)
- The system has memory. It does not only depends on the current t value. For example, $x(6) = 6 * x(2)$.
- The system is not stable. If we replace the input signal with a constant C, the output is $y(t) = t * 6$ which means it is not constant.
- The system is causal. We can represent the system as:

$$y(t) = h(x(t-2)) = x(\frac{t-2}{2})$$

  Since $2 \geq 0$, the system is causal.

- For a system to be linear, superposition property must hold. Suppose we feed two inputs, x_1 and x_2, to the system. The outputs are:

$$x_1(t) \rightarrow y_1(t) = t * x_1(\frac{t}{2} - 1)$$

$$x_2(t) \rightarrow y_2(t) = t * x_2(\frac{t}{2} - 1)$$

  Output for the superposition of the two inputs, $x_3(t) = a_1x_1(t) + a_2x_2(t)$:

$$y_3(t) = a_1y_1 + a_2y_2 = a_1(t * x_1(\frac{t}{2} - 1)) + a_2(t * x_2(\frac{t}{2} - 1)) = t * x_3(\frac{t}{2} - 1) = t * (a_1x_1(\frac{t}{2} - 1) + a_2x_2(\frac{t}{2} - 1))$$

  So, the system is linear.
- Because of coefficient t, there may be same outputs for different input signals. For example, suppose that for $t = 10, x(4) = 16$ and for $t = 16, x(7) = 10$. In this case, we can find $y(10) = y(16) = 160$. So, the system is not invertible.
- For a system to be time invariant, a time shift at the input generates the same time shift at the output. When we shift the input:

$$x(t - t_0) \rightarrow y(t) = x(\frac{t}{2} - \frac{t_0}{2} - 1)$$

  The shift for the input does not give the same amount of shift at the output:

$$x(t - t_0) \rightarrow y(t) = t * x(\frac{t}{2} - \frac{t_0}{2} - 1) \neq y[t - t_0] = (t - t_0) * (\frac{t}{2} - 1 - t_0)$$

  So, the system is not time-invariant.