CENG 382 - Analysis of Dynamic Systems
20221
Take Home Exam 1 Solutions

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1. (a) A difference equation is called linear if it is in the following form.

$$a_n(k)y(k+n) + a_{n-1}(k)y(k+n-1) + ... + a_0(k)y(k) = g(k) \tag{1}$$

y(k+2) is missing in this system. Therefore, it is a non-linear system.

If $\forall i$, $a_i(k)$ does not depend on k, then the system is time invariant.
None of the coefficients of the system does not depend on k. So, the system is time invariant.

It is a forced system since $g(k) = 5k + 8$.

(b) A differential equation is called linear if it is in the following form.

$$a_n(t)\frac{d^n x}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + ... + a_0(t)x = g(t) \tag{2}$$

This system is obeys this rule. Therefore, it is a linear system.

The system is time variant since $a_1(t) = -(t-1)^2$ depends on t.

It is a unforced system since $g(t) = 0$.

(c) This system does not obey the rule that I mentioned in equation (2). Due to $y^2(t)$ term, the system is a non-linear system.

The system is time invariant since none of the coefficients does not depend on t.

It is a forced system since $g(t) = 3$.

2. (a) We can re-write the equation as,

$$x'(t) = S\Lambda S^{-1}x(t) + b$$

In order to do variable change, multiply every term with S^{-1}.

$$S^{-1}x'(t) = \Lambda S^{-1}x(t) + S^{-1}b$$

Let's say that,

$$u = S^{-1}x$$

Then,

$$u' = S^{-1}x'$$

The equation becomes,

$$u'(t) = \Lambda u(t) + S^{-1}b$$

In order to find Λ, we need to find eigenvalues of A.

$$det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -2 \\ -5 & \lambda + 1 \end{vmatrix} = \lambda^2 - \lambda - 12 = 0$$

When we find the roots of the equation, we get eigenvalues of A.

$$\lambda_1 = 4 \qquad \lambda_2 = -3$$

Then we can create Λ as,

$$\Lambda = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}$$

Now we need to find S. First of all, we need to find eigenvectors of these eigenvalues.

i. For $\lambda_1 = 4$

$$\begin{bmatrix} 2 & 2 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

ii. For $\lambda_1 = -3$

$$\begin{bmatrix} -5 & 2 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

We can create S using v_1 and v_2

$$S = \begin{bmatrix} 1 & 2 \\ 1 & -5 \end{bmatrix}$$

We can find S^{-1} by using this formula below,

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{3}$$

$$S^{-1} = \begin{bmatrix} \frac{5}{7} & \frac{2}{7} \\ \\ \frac{1}{7} & \frac{-1}{7} \end{bmatrix}$$

We found every component we need, now we can solve the differential equation.
Our equation become,

$$u'(t) = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix} u(t) + \begin{bmatrix} \frac{5}{7} & \frac{2}{7} \\ \\ \frac{1}{7} & \frac{-1}{7} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

After calculation of $S^{-1}b$,

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} \frac{9}{7} \\ \frac{-1}{7} \end{bmatrix}$$

The general solution of these equations are,

$$u(t) = e^{at}(u_0 + \frac{b}{a}) - \frac{b}{a} \tag{4}$$

We need to find u_0,

$$S^{-1}x_0 = u_0 = \begin{bmatrix} \frac{5}{7} & \frac{2}{7} \\ \\ \frac{1}{7} & \frac{-1}{7} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{7} \\ \\ \frac{-3}{7} \end{bmatrix} \tag{5}$$

Now, we need to find $u_1(t)$ and $u_2(t)$, then we need to change the variable to x again.

$$u_1'(t) = e^{4t}(\frac{-1}{7} + \frac{9}{28}) - \frac{9}{28} = \frac{5}{28}e^{4t} - \frac{9}{28}$$

$$u_2'(t) = e^{-3t}(\frac{-3}{7} + \frac{1}{21}) - \frac{1}{21} = \frac{-8}{21}e^{-3t} - \frac{1}{21}$$

Then $u(t)$ becomes,

$$u(t) = \begin{bmatrix} \frac{5}{28}e^{4t} - \frac{9}{28} \\ \\ \frac{-8}{21}e^{-3t} - \frac{1}{21} \end{bmatrix}$$

We said that $u(t) = S^{-1}x(t)$ before, if we multiply each side with S, then we get:

$$Su(t) = x(t)$$

$$x(t) = \begin{bmatrix} 1 & 2 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} \frac{5}{28}e^{4t} - \frac{9}{28} \\ \\ \frac{-8}{21}e^{-3t} - \frac{1}{21} \end{bmatrix} = \begin{bmatrix} \frac{5}{28}e^{4t} - \frac{-16}{21}e^{-3t} - \frac{35}{84} \\ \\ \frac{5}{28}e^{4t} + \frac{40}{21}e^{-3t} + \frac{7}{84} \end{bmatrix}$$

(b) As $t \to \infty$, e^{4t} term beats the other terms. e^{-3t} will converge to 0 eventually.
We can say that the system is exploding in the direction of the eigenvector associated with $\lambda_1 = 4$.

$$x(t) \approx \frac{5}{28}e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3. As I stated before in the equation (4), the solution of this systems is,

$$e^{at}(x_0 + \frac{b}{a}) - \frac{b}{a}$$

$$x(t) = e^{-7t}(x_0 - \frac{5}{7}) + \frac{5}{7}$$

As $t \to \infty$, since $a < 0$, $e^{at} \to 0$. Therefore, the fixed point of the system is $\frac{5}{7}$ regardless of x_0. The fixed point is stable.

Also we can state that in order to find fixed point of the system, the derivative of x(t) must be equal to 0.

$$-7x(t) + 5 = 0$$
$$x(t) = 5/7$$

4. Let's define the following terms,

$$y_1(t) = x(t)$$
$$y_2(t) = x'(t)$$
$$y_3(t) = x''(t)$$

Then we can derive that,

$$y_1'(t) = y_2(t)$$
$$y_2'(t) = y_3(t)$$
$$y_3'(t) = f(y_1(t), y_2(t), y_3(t), t)$$

Then, we get,

$$y_3'(t) = -t^3 y_3(t) - (t+1)y_2(t) + y_1(t) + 2t + 1$$

After defining these terms, we can write this third order system as a system of first order equations.

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -(t+1) & -t^3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2t+1 \end{bmatrix}$$

And our system becomes,

$$y'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -(t+1) & -t^3 \end{bmatrix} y(t) + \begin{bmatrix} 0 \\ 0 \\ 2t+1 \end{bmatrix}$$

5. (a) i. For $x^1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

We need to start from 1 to k to find a solution.

$$x^1(0) = A(0)x^1(0) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x^1(1) = A(1)x^1(1) = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$x^1(2) = A(2)x^1(2) = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$$

$$x^1(3) = A(3)x^1(3) = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 14 \end{bmatrix}$$

We can see the pattern. The first index of resulting matrix is always 1. The second term is increasing with $\frac{(k+1)(k+2)}{2} - 1$.

Therefore,

$$x^1(k) = A(k-1)x^1(k-1) = \begin{bmatrix} 1 \\ \frac{(k+1)(k+2)}{2} - 1 \end{bmatrix}$$

ii. For $x^2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

We need to start from 1 to k to find a solution.

$$x^2(0) = A(0)x^2(0) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x^2(1) = A(1)x^2(1) = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The resulting matrix is always $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Therefore,

$$x^2(k) = A(k-1)x^2(k-1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The fundamental set of solutions is,

$$X(k) = \begin{bmatrix} 1 & 0 \\ \frac{(k+1)(k+2)}{2} - 1 & 1 \end{bmatrix}$$
$$X(k+1) = A(k)X(k)$$

Remember that, A set n linearly independent solutions, $x^1(k), ..., x^n(k)$ to x(k) = A(k)x(k) is called a fundamental set of solutions. In our case, these two solutions are linearly independent since the determinant of the X(k) is always equal to 1.

Also, we can show the linear independence of these two solutions by this formula,

$$\alpha_1 x^1(k) + \alpha_2 x^2(k) = 0$$

If this relation is equal to 0 $\forall k$ implies that α_1 and α_2 are zero.

$$1 * \alpha_1 + 0 * \alpha_2 = 0$$
$$\alpha_1 * (\frac{(k+1)(k+2)}{2} - 1) + \alpha_2 * 1 = 0$$

After solving these equations, α_1 and α_2 must be equal to 0. Therefore, the solutions are linearly independent.

(b) The definition of state transition matrix is,

$$\Phi(k,0) = A(k-1)A(k-2) \ \ ... \ \ A(0)$$

$$\Phi(k,0) = X(k)X^{-1}(0)$$

Then we put the $X(k)$ and $X^{-1}(0)$ in the equation above,

$$\Phi(k,0) = \begin{bmatrix} 1 & 0 \\ \frac{(k+1)(k+2)}{2} - 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{(k+1)(k+2)}{2} - 1 & 1 \end{bmatrix}$$

(c) Define,

$$\overline{x}(k) = X(k)X^{-1}(0)x(0)$$

$\overline{x}(k)$ is a linear combination of solutions. By linearity, $\overline{x}(k)$ is itself a solution.

The general solution is,

$$x(k) = \begin{bmatrix} 1 & 0 \\ \frac{(k+1)(k+2)}{2} - 1 & 1 \end{bmatrix} x(0)$$

We can re-write this equation as,

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} x_1(0) \\ (\frac{(k+1)(k+2)}{2} - 1)x_1(0) + x_2(0) \end{bmatrix}$$

As $k \to \infty$, $x_1(k)$ is $x_1(0)$; therefore, it is fixed at $x_1(0)$.

However, $x_2(k)$

i. if $x_1(0) > 0$, goes to ∞

ii. if $x_1(0) = 0$, fixed at $x_2(0)$

iii. if $x_1(0) < 0$, goes to $-\infty$