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Question 1

Part A)

Derivation of **Lame-Emden Equation**

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \tag{1}$$

$$\frac{dp(r)}{dr} = -G\frac{m(r)\rho(r)}{r^2} \tag{2}$$

$$\frac{1}{\rho}\frac{dp(r)}{dr} = -G\frac{m(r)}{r^2} \tag{3}$$

Then

$$\frac{d}{dr}\left(\frac{1}{\rho}\frac{dp(r)}{dr}\right) = G\frac{m}{r^3} - G\frac{1}{r^2}\frac{dm}{dr} \tag{4}$$

$$\frac{d}{dr}\left(\frac{1}{\rho}\frac{dp(r)}{dr}\right) = -\frac{2}{\rho * r}\frac{dp}{dr} - 4\pi G\rho \tag{5}$$

Multiply both sides with r^2, which results.

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp(r)}{dr}\right) = -4\pi G\rho \tag{6}$$

The equation of state (EOS) is the relation between P and ρ. Take an example from the ideal gas equation

$$PV = \frac{k_B}{\mu m_H}T\rho \tag{7}$$

Assume polytropic EOS

$$P = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}} \tag{8}$$

Where

$$\rho = \rho_c \theta^n \tag{9}$$

Which makes

$$P = K{\rho_c}^{1+\frac{1}{n}}\theta^{n+1} \tag{10}$$

Insert equaiton 10 into 6.

$$\frac{1}{r^2}\frac{d}{dr}(r^2 K{\rho_c}^{\frac{1}{n}}(n+1)\frac{d\theta}{dx}) = -4\pi G\rho_c\theta^n \tag{11}$$

Say that $r = \alpha\,\xi$, where

$$\alpha^2 = \frac{K{\rho_c}^{\frac{1}{n}-1}(n+1)}{4\pi G} \tag{12}$$

Then

$$\left(\frac{1}{\alpha^2}\frac{1}{\xi^2}\right)\frac{1}{\alpha}\frac{d}{d\xi}\left(\alpha^2\xi^2\alpha^2\frac{1}{\alpha}\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{13}$$

This is nothing but **Lame-Emden Equation:**

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) + \theta^n = 0 \tag{14}$$

Derivation of **Total Mass of the Star**

$$M = \int_0^R 4\pi r^2\rho dr \tag{15}$$

From $\rho = \rho_c\theta^n$ and $r = r_n\,\xi \rightarrow dr = r_n\,d\xi$

$$M = \int_0^R 4\pi r^2\rho dr = 4\pi\rho_c\int_0^R \theta^n r^2 dr = 4\pi\rho_c\frac{r^3}{\xi^3}\int_0^\xi \theta^n\,\xi^2 d\xi \tag{16}$$

Than $\frac{r^3}{\xi^3} = r_n^3$ and from Lane-Emden equation $\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) = -\theta^n \rightarrow \xi^2\theta^n = -\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi})$ wiht that

$$M = 4\pi\rho_c\frac{r^3}{\xi^3}\int_0^\xi -\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) = 4\pi\rho_c\,r^3(-\frac{1}{\xi}\frac{d\theta}{d\xi}) \tag{17}$$

Which makes

$$M = 4\pi R^3 (-\frac{\theta'(\xi)}{\xi_n})$$

(18)

Mass and Radius Proportionality.

The proportionality relation can be found by looking at central density(ρ_c) term power such that

$$R = C_1 \rho_c^{\frac{1-n}{2n}}$$

(19)

$$M = C_2 \rho_c^{\frac{3-n}{2n}}$$

(20)

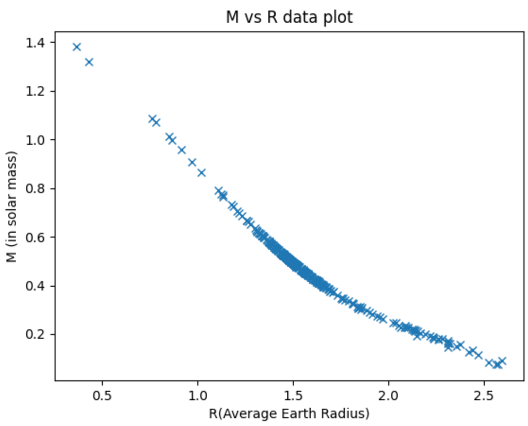
Which results

$$M \propto R^{\frac{3-n}{1-n}}$$

(21)

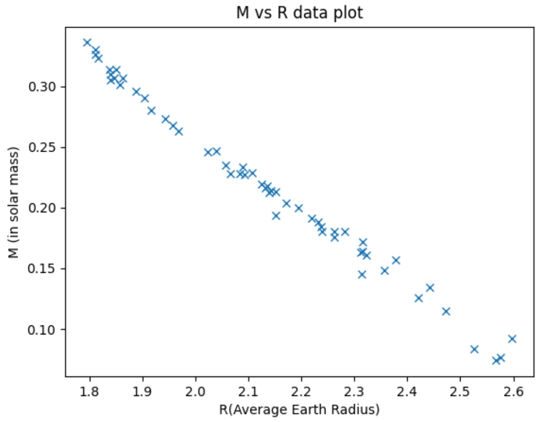
See the Mathematica file for other related derivations.

Part B)

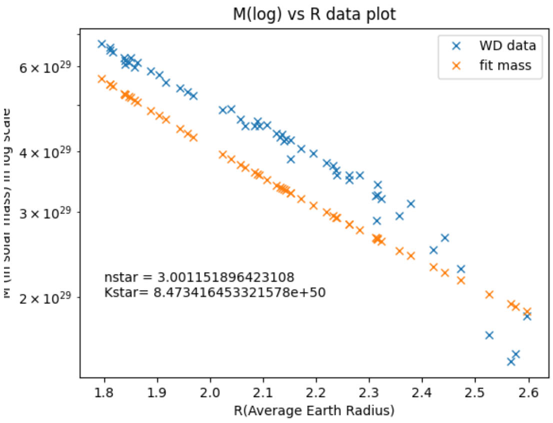


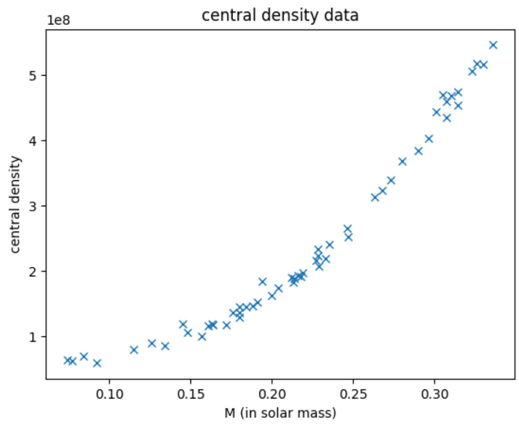
Part C)

I limit the M value to satisfy the low mass condition (M<0.34)

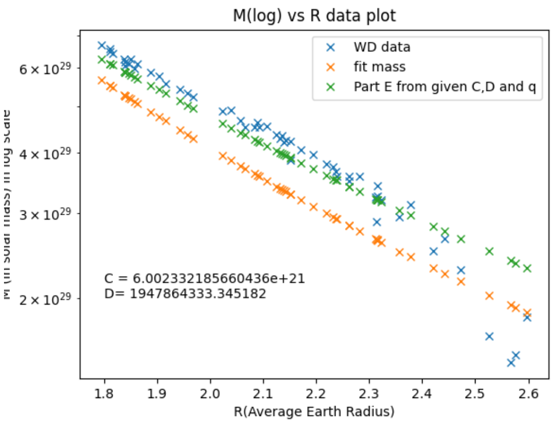


Then I make fit this data to find $n_* and\ K_*$





Part E)

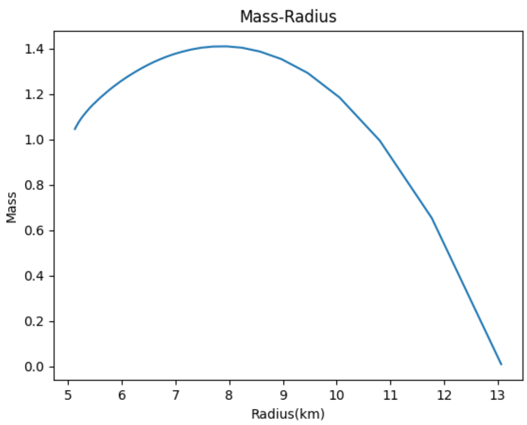


$$\boldsymbol{M_{ch} = 1.45832M}$$

$$\boldsymbol{M_{ch}^{data}} \approx 1.3\text{M}$$

Question 2

Part A)



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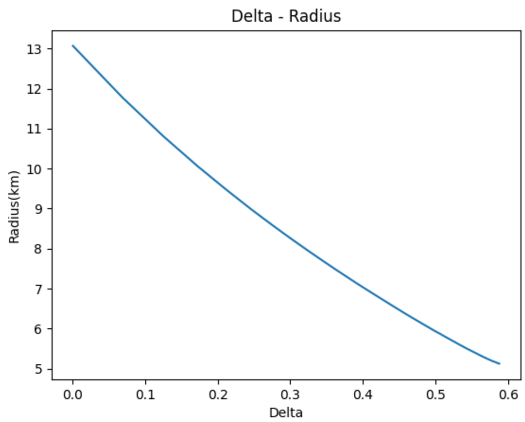
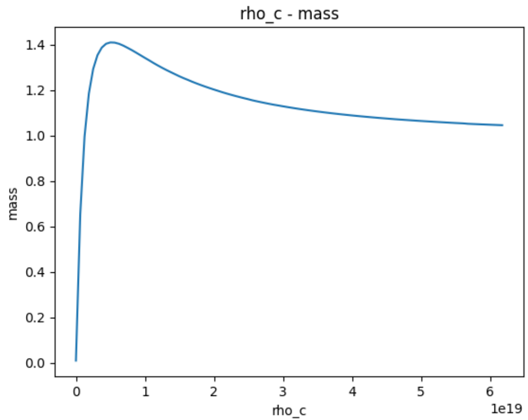
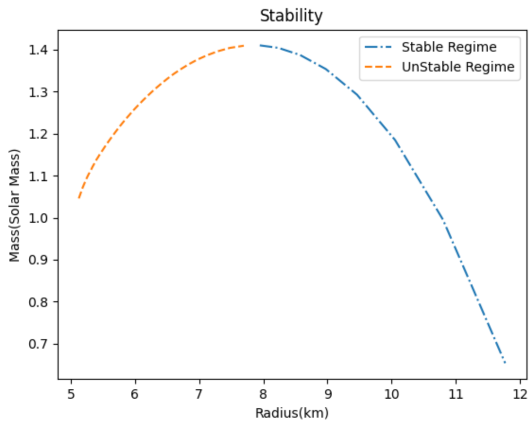


Part C)



Part D)



Part E)

See the Mathematica file