Public Key Cryptography (PKC) & RSA Cryptosystem

CS 411/507 - Cryptography

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Need for PKC

- Distribution and management of secret keys are difficult.
- Need for a secure distribution of secret keys,
  - a secure channel.
- In a networked environment, each pair of users has to have a different key;
  - resulting in too many keys in the system ($n \times (n-1)/2$ if there are n users)
- PKC solution was first proposed in 1976 by Diffie/Hellman

Principle & Protocol

- Every user has a pair of related keys
  - Public key
    - known to everyone in the system with assurance
  - Private key
    - known only to its owner
- Protocol
  1. Alice and Bob agrees on a PKC
  2. Bob sends his public key to Alice
  3. Alice encrypts her message with Bob's public key and sends the ciphertext to Bob
  4. Bob decrypts the ciphertext using his private key.

Hard Problems

- Integer factorization problem - IF (RSA)
- Discrete Logarithm problem - DL (Diffie-Helman, ElGamal)
- Elliptic curve discrete logarithm problem - ECDL (Elliptic Curve Cryptosystems)

Algorithm family	**Bit length**
Integer Factorization (IF)	2048/3072/7680/15360
Discrete Logarithm (DL)	2048/3072/7680/15360
Elliptic Curve Discrete Logarithm (ECDL)	224/256/384/512
Block cipher	112/128/192/256
Hash Functions (SHA-2 & SHA-3)	224/256/384/512

Table: Security levels of PKCs

See https://www.keylength.com/en/4/

PKC Applications

- Encryption/decryption
  - Only short messages are encrypted by the receiver's public key,
  - The receiver decrypts it by its private key.
- Digital signature
  - A message digest is encrypted by the message owner's private key
  - Anyone who knows the public key of the message owner can verify that the message and its origin are authentic.
- Key exchange

RSA

- Most popular PKC
- Invented by Rivest/Shamir/Adleman in 1977 at MIT.
- Its patent expired in 2000.
- Based on Integer Factorization problem
- Each user has public and private key pair.

RSA Encryption & Decryption

- Encryption done by using public key

  $$y = x^e \bmod n, \text{ where } x < n$$

- Decryption done by using private key

  $$x = y^d \bmod n$$

Fermat's Little Theorem

- If p is a prime and p does not divide a, then
$$a^{p-1} \equiv 1 \bmod p$$

Pierre de Fermat
(1601 or 1607 or 1608
- 12 January 1665)

Euler's Theorem

- If $gcd(a, n) = 1$, then

  $$a^{\phi(n)} \equiv 1 \bmod n$$

  where $\phi(n)$ is defined as the number of integers $1 \leq a \leq n$ such that $\gcd(a, n) = 1$ and called as Euler's ϕ-function.
- $\phi(p) = (p - 1)$

Leonhard Paul Euler
(15 April 1707 -
18 September 1783)

Important Principle

- Let a, n, x, y be integers with $n \geq 1$ and $gcd(a, n) = 1$.
  If $x \equiv y \bmod \phi(n)$ then
  $$a^x \equiv a^y \bmod n.$$
  Proof: $x = y + k \times \phi(n)$ from congruence relation.
  Then
  $a^x = a^{y+\phi(n)k} \equiv a^y \left(a^{\phi(n)}\right)^k \equiv a^y \times 1^k \equiv a^y \bmod n$
  In other words, if you work $\bmod n$ in the base, you should work $\bmod \phi(n)$ in the exponent.

RSA Setup Stage

1. Choose two large primes p and q
2. Compute $n = p \times q$
3. Compute $\Phi(n) = (p-1) \times (q-1)$
4. Choose a random integer, $0 < e < \Phi(n)$ with $\mathrm{gcd}(e, \Phi(n)) = 1$
5. Compute the inverse $d = e^{-1} \bmod \Phi(n)$,
   - i.e., $e \times d \equiv 1 \bmod \Phi(n)$,

- Public key: (e, n)
- Private key: (d, p, q)

RSA Encryption & Decryption

- Encryption done by using public key

  $y = x^e \bmod n$, where $x < n$

- Decryption done by using private key

  $x = y^d \bmod n$

Example

Alice

Bob

1. chooses $p = 3,\ q = 11$
2. $n = p \cdot q = 33$
3. $\Phi(n) = (p-1)(q-1) = 20$
4. Chooses $e = 3$; $gcd(3, 20) = 1$
5. Computes $d = e^{-1} \mod \Phi(n)$, $d = 7$
6. Sends (e, n) to Alice

Alice:

1. Message: $x = 4$
2. $y = x^e \bmod n = 31$
3. Sends y to Bob

Bob:

7. $x = y^d \bmod n = 4$

Why does RSA work?

- We want to show that $y^d \bmod n = x$.

- $y^d \bmod n \equiv (x^e \bmod n)^d \bmod n \equiv x^{ed} \mod n$
- $e \cdot d \equiv 1 \mod \Phi(n) \rightarrow e \cdot d = 1 + k \cdot \Phi(n)$
- $x^{ed} \bmod n \equiv x^{1+k\Phi(n)} \bmod n \equiv x^1 \cdot x^{k\Phi(n)} \bmod n$.
- If $x^{\Phi(n)} \equiv 1 \bmod n$
- Then, $x^1 \cdot x^{k\Phi(n)} \bmod n \equiv x \cdot (1)^k \bmod n \equiv x$.

Why does RSA work?

- Euler's theorem:

  If $\gcd(x, n) = 1$ then $x^{\Phi(n)} \equiv 1 \bmod n$
- What if $\gcd(x, n) \neq 1$ (i.e. $\text{gcd}(x, p \cdot q) \neq 1$)
- Assume x is a multiple of q $(x = k_1 q)$
  - $x^{k\Phi(n)} \bmod q = 0$
  - $x^{k\Phi(n)} = x^{k(p-1)(q-1)} = x^{\Phi(p)k(q-1)} \equiv 1^{k(q-1)} \bmod p \equiv 1 \bmod p$
- Using CRT,
  - $x^{k\Phi(n)} = (0 \times p \times (p^{-1} \bmod q) + 1 \times q \times (q^{-1} \bmod p)) \bmod n$
  - $x^{k\Phi(n)} = (q \times (q^{-1} \bmod p)) \bmod n = (1 + k_2 p) \bmod n$
  - $x \cdot x^{k\Phi(n)} = k_1 q \times (1 + k_2 p) \bmod n = x + k_1 k_2 pq \bmod n$

Computational Aspects

- Problem:
  - finding two large primes (> 1024 bits at least)
- Method:
  - pick a large integer and apply a primality test, which does not require factorization.
- Miller-Rabin Algorithm for primality testing
  - Input: n
  - Output:
    - "n is composite" $\rightarrow 100\%$ assurance
    - "n is probably prime" $\rightarrow$ prime with probability > 0.75
- Idea: Use this algorithm many times to get comfortable level of confidence about the primeness.

M-R Test: Method

- We repeat the TEST
- If, at any point, the TEST returns "composite", then n is determined to be nonprime.
- If the TEST returns "inconclusive" t times, then the probability that n is prime is at least $(1 - 4^{-t})$

Distribution of Primes 1/2

- Concern
  - how many integers are likely to be rejected before a prime number is found using a primality test.
- Prime Number Theorem: Let $\pi(x)$ be the $\#$ of primes less than x. Then

  $$\pi(x) \to x/\ln x$$

  - the primes near x are spaced, on the average, one every $(\ln x)$.
- Then, on average, one would have to test about (on the order of) $\ln x$ integers before a prime is found.

Distribution of Primes 2/2

- Example: $n = 2^{256}$, then the percentage of primes smaller than n is 0.56
- Example: $n = 2^{1024}$, then the percentage of primes smaller than n is 0.14
- Because of all even integers and all integers ending with digit 5 can be immediately rejected,
  - the exact number of the trials is $0.4 \times \ln(x)$.
- For 200-bit prime, the trial number on the average is
  - $0.4 \times \ln(2^{200}) \approx 55$
- For 512-bit prime, average trial number
  - $0.4 \times \ln(2^{512}) \approx 142$

Security of RSA 1/2

- Brute force attack
  - Given $y = x^e \bmod n$, try all possible keys d;
    - $0 < d < \Phi(n)$ to obtain $x = y^d \bmod n$.
  - In practice, the key space
    - $|K| = \Phi(n) \approx n > 2^t$ it is impossible apply brute force for even moderate values of t.
- Finding $\Phi(n)$
  - Given $n, e, y = x^e \bmod n$, find $\Phi(n)$ and compute $d = e^{-1} \bmod \Phi(n)$.
  - Computing $\Phi(n)$ is believed to be as difficult as factoring n.

Security of RSA 2/2

- Factoring n
  - Given $n, e, y = x^e \bmod n$, find p and q such that $n = p \cdot q$ and compute
  - $\Phi(n) = (p-1) \cdot (q-1)$
  - $d = e^{-1} \bmod \Phi(n)$
  - $x = y^d \bmod n$
  - Factoring n is the only practical approach.
  - We need efficient integer factorization algorithms

Factoring Algorithms

- Quadratic Sieve (QS): speed depends on the size of the modulus n. In 1994, RSA129 challenge (RSA with modulus of 129 digit (≈ 428 bits)) is broken by QS
- Elliptic Curve Method: speed depends on the size of the smallest factor of n, i.e. p or q.
- Number Field Sieve: Asymptotically better than QS. In 1999, RSA140 challenge (RSA with modulus of 140 digit (≈ 465 bits)) is broken by generalized number field sieve.

Factoring Algorithms

- The computational complexity of factoring algorithms

Algorithm	Complexity
Quadratic Sieve	$O(e^{(1+o(1))\sqrt{\ln(n)\ln(\ln(n))}})$
Elliptic Curve	$O(e^{(1+o(1))\sqrt{2\ln(p)\ln(\ln(p))}})$
Number Field Sieve	$O(e^{(1.92+o(1))(\ln(n))^{1/3}(\ln(\ln(p)))^{2/3}})$

Largest Number Factored So Far

- RSA-768 (768-bit modulus)
  - December 12, 2009 by T. Kleinjung, K. Aoki, J. Franke, A. K. Lenstra, E. Thomé, P. Gaudry, A. Kruppa, P. Montgomery, J. W. Bos, D. A. Osvik, H. te Riele, A. Timofeev, and P. Zimmermann
  - Method: NFS
  - The sieving effort is estimated to have taken the equivalent of 1500 years on a single 2.2 GHz Opteron CPU. (2000 years in total)
  - http://www.crypto-world.com/announcements/rsa768.txt
  - http://en.wikipedia.org/wiki/RSA-768#RSA-768
  - http://en.wikipedia.org/wiki/RSA_Factoring_Challenge

RSA Challenges

- The RSA Factoring Challenge was a challenge put forward by RSA Laboratories on March 18, 1991
- It was ended in 2007
- RSA-250 has 250 decimal digits (829 bits), and was factored in February 2020.

Challenge no	Approx. # of bits	Date	Prize
RSA-576	576	Dec 3, 2002	US$10.000
RSA-640	640	Nov 2, 2005	US$20.000
RSA-704	704	Jul 2, 2012	US$30.000
RSA-768	768	Dec 11, 2009	US$50.000
RSA-896	896	Not Yet	US$75.000
RSA-1024	1024	Not Yet	US$100.000
RSA-1536	1536	Not Yet	US$150.000
RSA-2048	2048	Not Yet	US$200.000

Modular Exponentiation

- $m^e \bmod n$
- Example: $2^{1234} \bmod 789$,
- Naïve method:
  - Compute 2^{1234} first
  - $(2.958112246080986290600446957161 \times 10^{371})$
  - then reduce the result modulo 789.
  - Is it practical (possible)?
- Practical method: Use binary expansion of the exponent.
- $1234 = (10011010010)_2$

Binary Left-to-Right Algorithm

Algorithm 1 Binary Left-to-Right Algorithm

Input: $1 < a < n$ and $e \geq 1 (e = e_{k-1}, \ldots, e_1, e_0)$

Output: $x \equiv a^e \bmod n$

```
x := 1
for i = k − 1 downto 0 do
    x := x × x mod n
    if e_i = 1 then
        x := x × a mod n
    end if
end for
return x mod n
```

Modular Exponentiation Example

$2^{1234} \bmod 789$, $1234 = (10011010010)_2$, $x = 1$

i	e_i	Squaring $x \cdot x$	Multiplication $2 \times x$
10	1	$x = 1 \cdot 1 = 1$	$x = 1 \cdot 2 = 2$
9	0	$x = 2 \cdot 2 = 4$	–
8	0	$x = 4 \cdot 4 = 16$	–
7	1	$x = 16 \cdot 16 = 256$	$x = 256 \cdot 2 = 512$
6	1	$x = 512 \cdot 512 = 196$	$x = 196 \cdot 2 = 392$
5	0	$x = 392 \cdot 392 = 598$	–
4	1	$x = 598 \cdot 598 = 187$	$x = 187 \cdot 2 = 374$
3	0	$x = 374 \cdot 374 = 223$	–
2	0	$x = 223 \cdot 223 = 22$	–
1	1	$x = 22 \cdot 22 = 484$	$x = 484 \cdot 2 = 179$
0	0	$x = 179 \cdot 179 = 481$	–

Binary Right-to-Left Algorithm

Algorithm 2 Binary Right-to-Left Algorithm

Input: $1 < a < n$ and $e \geq 1$
Output: $x \equiv a^e \mod n$

```
x := 1, y := a
while e ≠ 0 do
    if e is odd then
        x := x × y  mod n
    end if
    y := y × y  mod n
    e := e >> 1
end while
return x  mod n
```

Side-Channel Attacks

- Basic RSA operation
  - modular exponentiation

Algorithm 3 The binary left-to-right exponentiation algorithm

Input: $y, d = (d_{k-1}, d_{k-1}, \ldots, d_1, d_0), n, k = \lceil \log_2 n \rceil$
Output: $y^d \bmod n$

```
s := y
for i = k − 2 downto 0 do
    s := s × s mod n
    if d_i = 1 then
        s := s × y mod n
    end if
end for
return s
```

Example: RSA Decryption

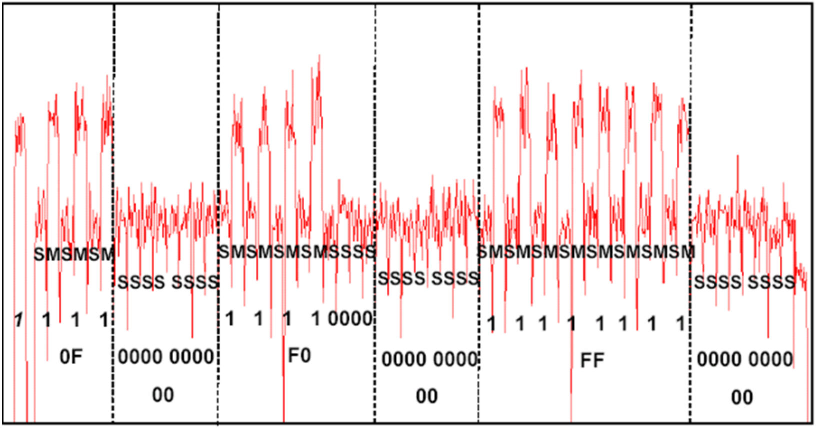
- $7^{560} \bmod 561$
- $d = (1000110000)$

Iteration	0	1	2	3	4	5	6	7	8	9
Exponent bits	1	0	0	0	1	1	0	0	0	0
Square	7	49	157	526	103	355	298	166	67	1
Multiply	7	-	-	-	160	241	-	-	-	-

- Assume that an adversary can observe the decryption of a ciphertext (or a signature) and record the power consumption
  - $y^d \bmod n$
- Attack Scenario: a smart card that relies on an external power
  - Power supplied by the reader

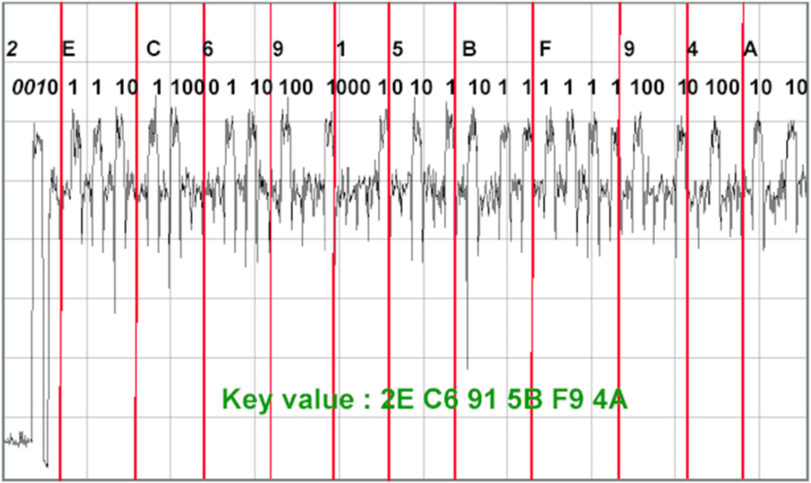
Simple Power Analysis (SPA) 1/2

- private key: d = 0F 00 F0 00 FF 00



Power Spectrum

- private key: $d = $ 2E C6 91 5B F9 4A



Algorithm 4 Double-and-Add Always Algorithm

Input: $y, d = (d_{k-1}, d_{k-1}, \ldots, d_1, d_0), n, k = \lceil \log_2 n \rceil$
Output: $y^d \bmod n$

```
s_0 := y, s_1 := 1
for i = k − 2 downto 0 do
    s_0 := s_0 × s_0 mod n
    s_1 := s_0 × y mod n
    b := d_i
    s_0 := s_b
end for
return s
```

Semantic Security

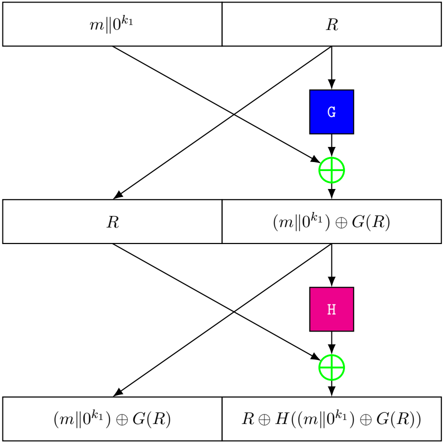
- A notion to describe the security of an encryption scheme
  - Goldwasser and Micali in 1982
- No (or negligable) information should be extracted from the ciphertext.
  - Semantically secure cryptosystem
- Message space can be small and known by any adversary.
  - There may be 2 message candidates (e.g., m_0 and m_1) in the worst case.
- Adversary should not figure out the message from ciphertext.
- Is RSA semantically secure?

RSA is not semantically secure

- Eve picks two arbitrary mesages $m_0, m_1 < n$, $m_0 \neq m_1$
- Eve is challenged to guess a uniformly randomly chosen $b \in \{0, 1\}$
  - Given $c = m_b^e \pmod{n}$
- Can she guess b correctly?
- As RSA is deterministic, yes she can
  - she computes $c_0 = m_0^e \pmod{n}$
  - if $c_0 = c_b$ then $b = 0$ else $b = 1$
- RSA in practice is probabilistic

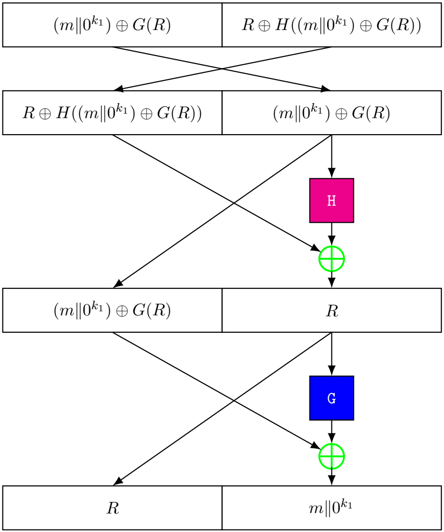
Optimal Asymmetric Encryption Padding (OAEP)

- Two rounds of Feistel Network
- $R \leftarrow \{0,1\}^{k_0}$
  - k_0-bit random number
- 2^{-k_0} and 2^{-k_1} are negligible
  - k_0 and k_1 are sufficiently large.
- $k = \lceil \log_2 n \rceil$
- G: $\{0,1\}^{k_0} \rightarrow \{0,1\}^{k-k_0}$
  - one-way function
  - G gets k_0-bit input and generates $(k-k_0)$-bit output
- $H : \{0,1\}^{k-k_0} \rightarrow \{0,1\}^{k_0}$



OAEP - Decoding

- Two rounds of Feistel Network
- $R \leftarrow \{0,1\}^{k_0}$
  - k_0-bit random number
- 2^{-k_0} and 2^{-k_1} are negligible
  - k_0 and k_1 are sufficiently large.
- $k = \lceil \log_2 n \rceil$
- `G`: $\{0,1\}^{k_0} \rightarrow \{0,1\}^{k-k_0}$
  - one-way function
  - `G` gets k_0-bit input and generates $(k-k_0)$-bit output
- $H : \{0,1\}^{k-k_0} \rightarrow \{0,1\}^{k_0}$



RSA-OAEP

- Specified in RFC8017
- `H` denotes the selected hash function
- L is optional label
- `MGF` denotes mask generating function.
  - `MGF`(Seed, $k - 2 \cdot hLen - 2$)
  - `MGF`(maskedDB, $hLen$)
- Encoded message (m') is used for encryption.
  - $y = m' \bmod n$