Reversing Ryder: Notes on Period-tempo Translation

Joshua R. Goldstein

June 9, 2023

Abstract

Ryder translated from cohort fertility to period fertility. In this note, I show that it is also possible to do the reverse, taking period fertility levels and changes in timing and translating them to cohorts. The exact translation relationship assume linearly changing age-specific rates, but it can still be useful as an approximation when change is not exactly linear. An advantage of the resulting translation is that all of the period quantities can be observed and that inferences can be made about still incomplete cohorts.

Introduction

Ryder introduced the fundamental relationships between period and cohort total fertility rates using cohort as the "fundamental" dimension **?**. This meant that he was expressing period fertility as a "distorted" version of cohort fertility, with the difference being due to cohort changes in fertility timing.

Bongaarts and Feeney took an exclusively period perspective, showing how changes in period timing can influence period fertility. But they emphasized that their resulting tempo adjusted fertility rate was *not* meant to be interpreted as a cohort fertility rate.

In this note, I show that it is possible to use Ryder's logic, in reverse, and to derive – under the assumption of linearly changing fertility rates – the cohort total fertility rate from the observed period fertility rate and the observed rate of change in period fertility timing.

The assumption of linear change can either be taken literally, or as a 1st order approximation of more general, non-linear changes.

Definitions

- Linearly changing age-specific fertility rates

$$f(a,t) = f(a,0) + f'(a)t \tag{1}$$

  (Note: $f'(a)$ is constant with no time argument)

- Period TFR

$$TFR_p(t) = \sum_a f(a,t) \tag{2}$$

- Cohort TFR for the cohort born in year c

$$TFR_c(c) = \sum_a f(a, c+a) \tag{3}$$

  (Note: age-specific fertility $f(a, c+a)$ is indexed by time. But now the sum over age influences both age and time.)

- Period mean age of childbearing

$$\mu_p(t) = \frac{\sum_a af(a,t)}{\sum_a f(a,t)} \tag{4}$$

The result

$$TFR_c(t - \mu_p(t)) = TFR_p(t) \times \left[1 + \mu'_p(t)\right] \tag{5}$$

In words, the total fertility of the cohort born a generation ago is equal to today's period fertility increased by the rate of change in the period mean age childbirth.

An feature of this formula is that it expresses the potentially still unobserved cohort total fertility in terms of observed period quantities.

Derivation

To simplify the notations, all summations are are over age a and the time derivative $f'(a)$ is used to denote the time-constant derivative $f'(t,a)$.

Write

$$\begin{aligned}
TFR_c(t-\mu_p(t)) &= \sum f(a, t-\mu_p(t)+a) \\
&= \sum f(a,0) + \sum [t-\mu_p(t)+a] f'(a) \\
&= \sum f(a,0) + t\sum f'(a) + \sum [a-\mu_p(t)] f'(a) \\
&\text{Using the definition of period TFR,} \\
&= TFR_p(t) + \sum \left[a-\mu_p(t)\right] f'(a) \qquad (6)
\end{aligned}$$

We will see that the right-most summation can be expressed in terms of the rate of change in the period mean.

The time derivative of the period mean is

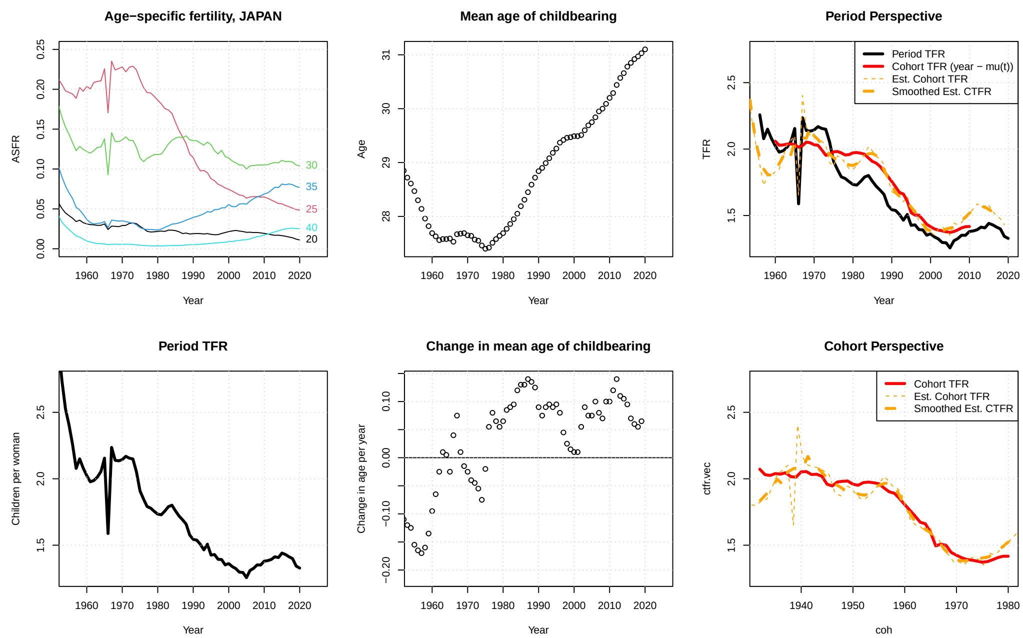
$$\begin{aligned}
\mu_p'(t) &= \frac{d}{dt}\frac{\sum af(a,t)}{\sum f(a,t)} \\
&\text{Applying quotient rule,} \\
&= \frac{\sum af'(a)\sum f(a,t) - \sum af(a,t)\sum f'(a}{\left(\sum f(a,t)\right)^2} \\
&\text{Simplifying,} \\
&= \frac{\sum af'(a) - \mu_p(t)\sum f'(a)}{TFR_p(t)} \\
&\text{Combining terms,} \\
&= \frac{\sum [a-\mu_p(t)] f'(a,t)}{TFR_p(t)}
\end{aligned}$$

Multiplying both sides by the period total fertility rate gives

$$\sum [a-\mu_p(t)] f'(a,t) = TFR_p(t)\mu_p'(t). \qquad (7)$$

Substituting into equation (6) then gives our desired result:

$$TFR_c(t-\mu_p(t)) = TFR_p(t) \times \left[1+\mu_p'(t)\right]$$



An illustrative application

As an illustration we look at fertility in Japan since 1960, which has long-term decline in the period fertility rate and also saw a temporarly shock to fertility in the Fire Horse year of 1967. Like many countries the timing of fertility fell until the early 1970s, after which it has seen a steady rise at about one-tenth of a year of age per calendar year.

The right-most panels show the "reverse Ryder" translation. From the period perspective, the translated cohort estimates for those born in year $t - \mu_p(t)$ are plotted in year t in order to allows easy comparision of the period and cohort TFRs. We see that the appropriately off-set cohort TFR was somewhat smaller than the period TFR before 1970 and then consistently larger than the period TFR ever since. Furthermore we can see that estimated cohort TFR based on equation (5) generally tracks well with the observed cohort TFR. The figure shows both the estimated cohort TFR (in the light orange dashed line) and a five-year moving average "smoothed" estimate of the cohort TFR. The translation equation tracks the Firehorse year, giving a very inaccurate estimate of cohort TFR in 1967, as we would expect since it this is such a strong violation of the linearity assumption in age specific fertility. On the other hand, the smoothed estimate tracks the actual cohort TFR quite well.

The panel marked "cohort perspective" allows a clearer view of the comparison between observed and estimated cohort TFR. Here we can see the departure during the firehorse years more clearly and the generally close match of the observed and estimated cohort TFR across the entire period.

This example illustrates that it is possible for the linear translation equation to be fairly accurate even when fertility rates are not changing exactly linearly. Of course, larger departures from linearity could result in considerably less accuracy.

History

Ryder's result appears in several early publications but is formalized in Ryder (1964).

Keyfitz (1985) was the first as far as I know to present the result given here, where the cohort TFR is given in terms of observable period moments. Indeed, Keyfitz tell us that the expansion of the cohort in terms of period moemtns is "useful because it can incorporate the latest period information to suggest how current cohorts are likely to be completed."

Yi and Land's paper is highly suggestive of the result here, with the title "Adjusting Period Tempo Changes in an Extension of Ryder's Basic Translation Equation." However, the extension they consider is to assume that the entire cohort schedule is shifting linearly, such that $f(a, c+a) = f_0(a+s(c))$ and the result they find, $CTFR = TFR(1+r_c)$, refers to changes in the mean age of the cohort schedule.[1]

Keilman (2000) presents general formula for translation from cohort to period and vice-versa, based on the earlier work of Yntema. The approach taken is somewhat different. Instead of – as Ryder did – writing Taylor approximations of each age-specific fertility rate, Keilman and Yntema perform a double approximation for both terms of the product of total fertility and the age-specific proportions. The results obtained differ and, with $(4a')$ and $(4b')$ involving an additional term for ....

[1] Contrary to Yi and Land's suggestion, it seems from the analysis here that Ryder's linearized approach and that here do not imply "contant quantum" but rather linearly changing quantum.

A small extension

Cohorts are defined along the Lexis surface in such a way that a year in age equals a year of calendar time. One can imagine TFRs defined which sum over age along other paths. For example, one could consider diagonals that intersected the period at the period mean age but which had slope θ, such that $TFR_\theta(t - \mu_p(t)/\theta) = \sum f(a, t - \mu_p(t)/\theta + a/\theta)$ In this case

$$TFR_\theta(t - \mu_p(t)/\theta) = TFR_p(t)[1 - \mu_p'(t)/\theta] \tag{8}$$

I don't see any application for this, but it shows you can pivot around the period mean age. In the extreme case, as the diagonal becomes vertical, theta becomes very large, and both sides of the equation will be equal to the period TFR.