Dynamic models of fertility change

Berkeley Summer Workshop in Formal Demography

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From: *Amplified changes: an analysis of four dynamic fertility models,* in Dynamic Demographic Analysis, ed. R. Schoen, Springer (2016)

June 7, 2023

Plan for the day

Part 1

Modeling fertility, the Lexis surface, and Ryder's demographic translation

Part 2

Period shift and Cohort shift fertility models

LUNCH

Part 3

Ron Lee's moving target model with period shocks

HIKE

1: Why model fertility?

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- Natural processes follow mathematical laws
- Models can approximate reality
- Models have predictive capabilities
- Models are interpretive tools

  "*Model-building is the reverse of the typical demographic posture. Instead of taking a set of data and milking them for information about the determinant vital processes . . ., the model builder starts with those processes, and shows what consequences flow from them.*" - *N. Ryder*

1: A static model of fertility

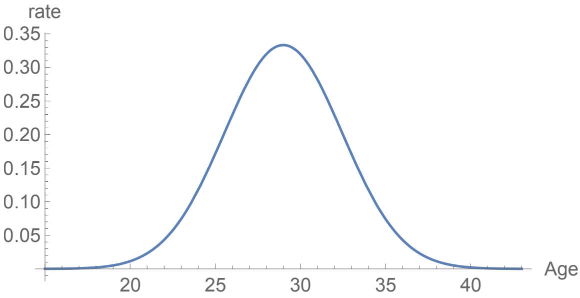

Figure: Fertility $f_0(a)$

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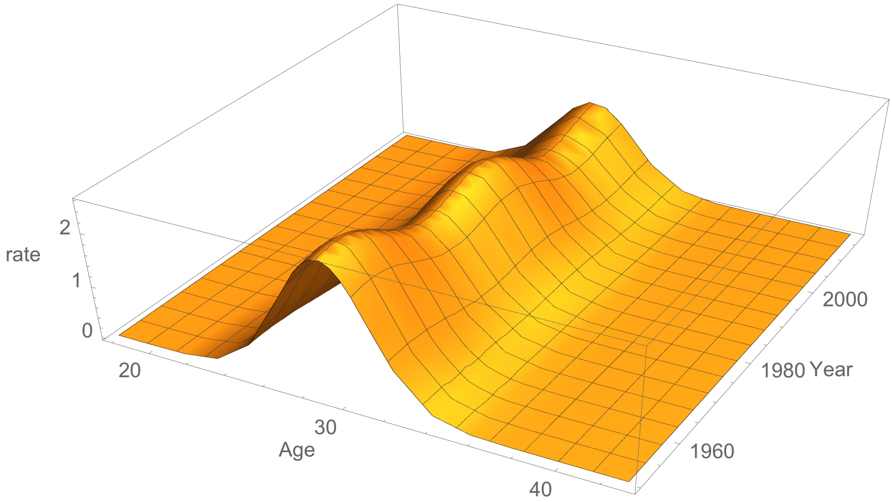

Figure: Fertility $f(a,t) = f_0(a)q(t)$

1: The Lexis surface

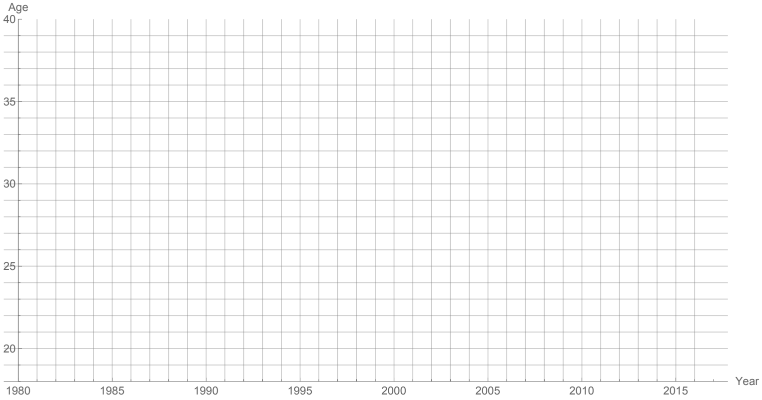

Figure: Lexis surface

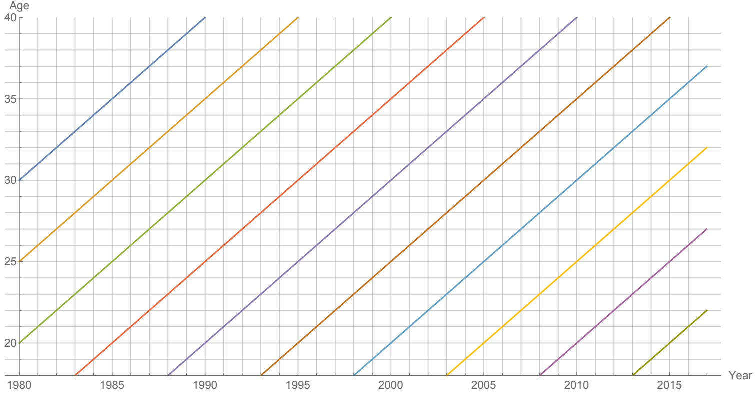

Figure: Lexis surface with cohorts

1: Lexis Surface with Age, Period and Cohort effects

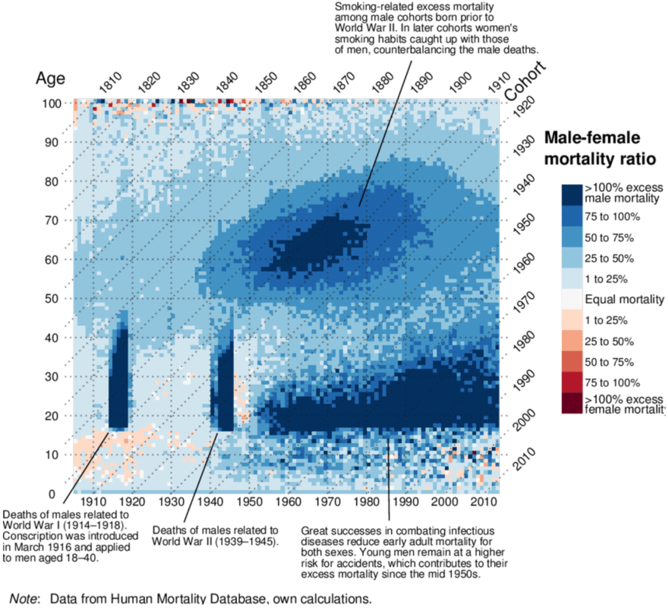

Figure: Schöley & Willekens

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- $\mu_c =$ cohort mean age of childbearing; $\mu_p =$ period mean age of childbearing.

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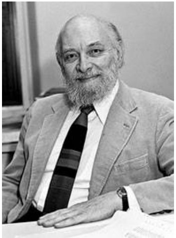
Ryder was interested in translating between measurements of cohorts and measurements of periods. Cohorts are "most suited to the analysis of determinants," while periods are "most suited to the analysis of consequences."



Model: $f(a,t) = f_0(a) + f_1(a)t + f_2(a)t^2 + \cdots + f_n(a)t^n$.

- From this model, Ryder derives a translation formula to express period moments in terms of derivatives of cohort moments, and vice versa.
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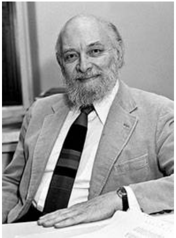

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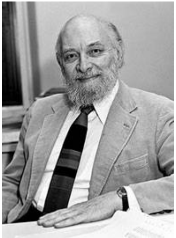

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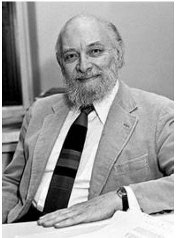

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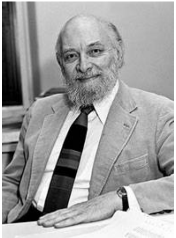

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$$TFR(t) = \frac{CTFR(t - \mu_p(t))}{1 + \mu_p'(t)}.$$

Exercise 1

Goals:

- Apply Ryder's formula to our made-up pre-workshop data
- Apply Ryder's formula to real US data

Period Paramount

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2: A different perspective

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". . . period is unambiguously the prime source of variation in fertility rates."

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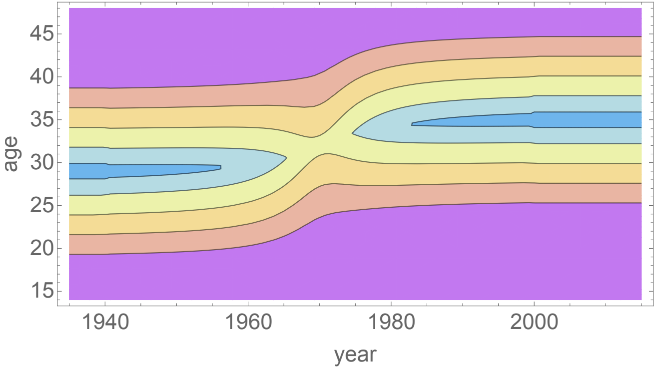

Figure: Contour plot of $f(a,t)$ with no quantum change and 6 years of postponement starting in 1940 and ending in 2000

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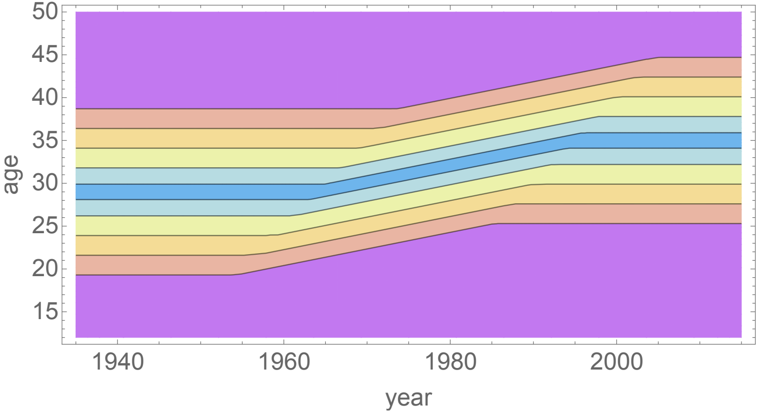

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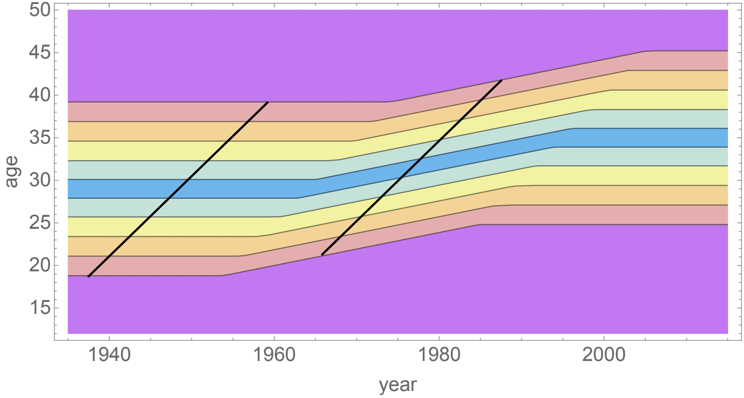

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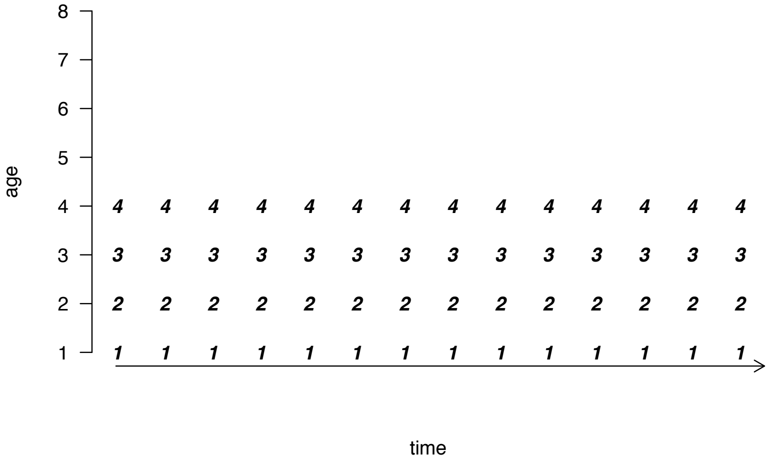
A movie of cohort shifts



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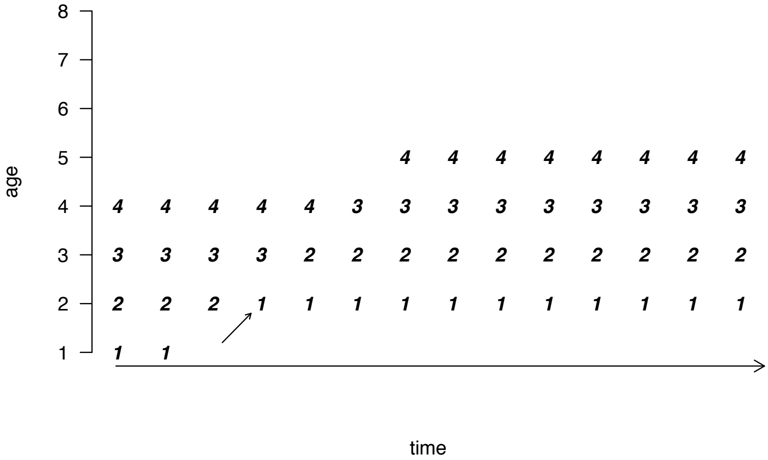

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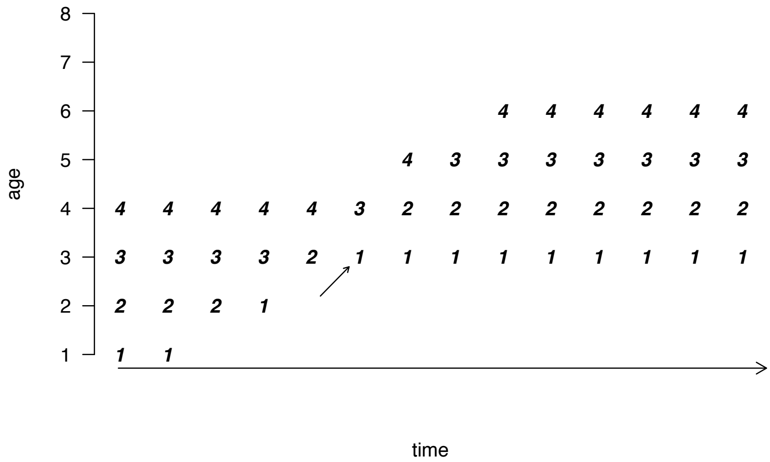

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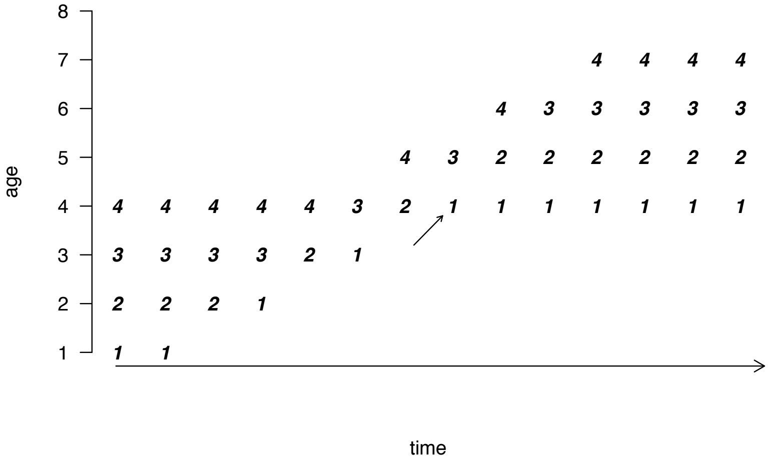

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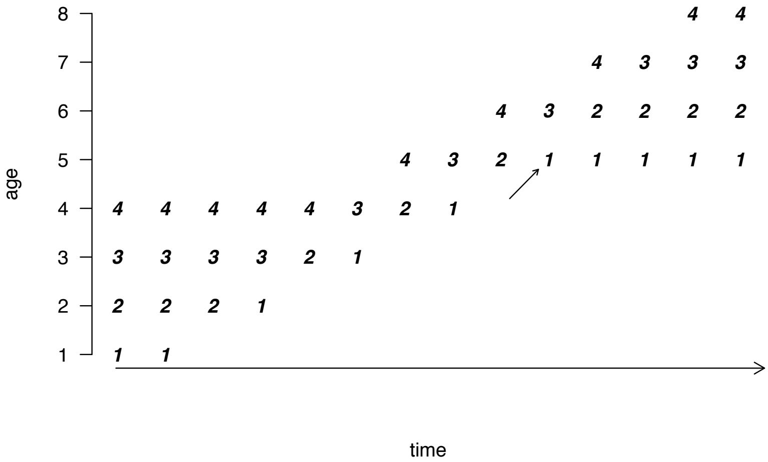

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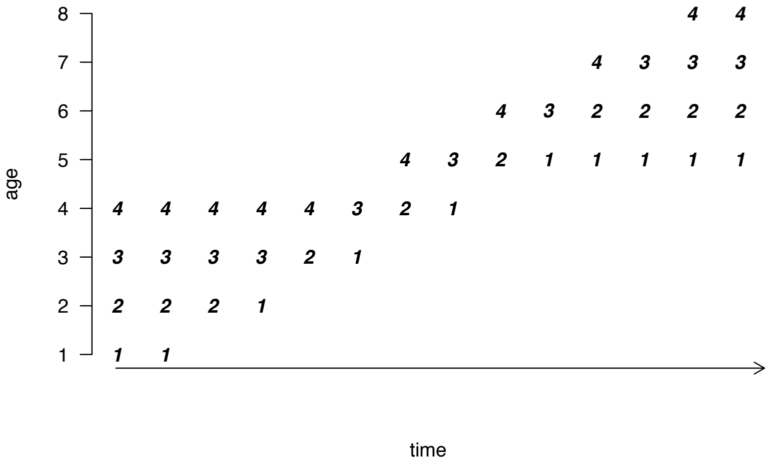

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Cohort-shift model: *TFR* can continue to evolve for decades as past fertility decisions play out over a lifetime.

Consequently, the cohort-shift model can be used to project future *TFR*, provided we make assumptions about future quantum effects.

Exercise 2

Goal: Use Bongaarts-Feeney to see if recent declines in fertility in Finland and Korea might be due to accelerating postponement

3: Lee's moving target model

Assumptions and notation:

- Desired completed family size D changes with year, and influences all ages equally.
- $F(a,c) = \int_0^a g(x,c)dx =$ achieved fertility at age a of the cohort born in year c.
- A cohort's fertility in a given year depends on the gap between its achieved fertility, and the value of D for that year. We call this gap *additional desired fertility*.
- The birth rate is a fixed fraction, α, of additional desired fertility.

Model: $g(a,c) = \alpha\left[D(a+c) - F(a,c)\right]$

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$$g(a,c) = \alpha e^{-\alpha a}\left[\int_0^a e^{\alpha u} D'(u+c)du + D(c)\right].$$

Alternatively,

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This implies that $\dfrac{\partial}{\partial a} g(a,c) = \alpha\left[D'(a+c) - g(a,c)\right]$.

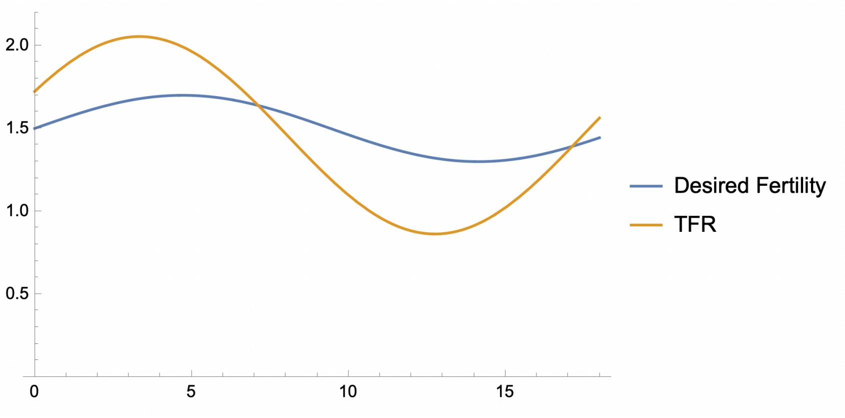
This differential equation can be solved to express fertility rates in terms of a history of changing desired family sizes:

$$g(a,c) = \alpha e^{-\alpha a}\left[\int_0^a e^{\alpha u} D'(u+c)du + D(c)\right].$$

Alternatively,

$$g(a,c) = \alpha D(a+c) - \alpha e^{-\alpha a}\int_0^a \alpha e^{\alpha u} D(u+c)du.$$

3: Desired Fertility and TFR



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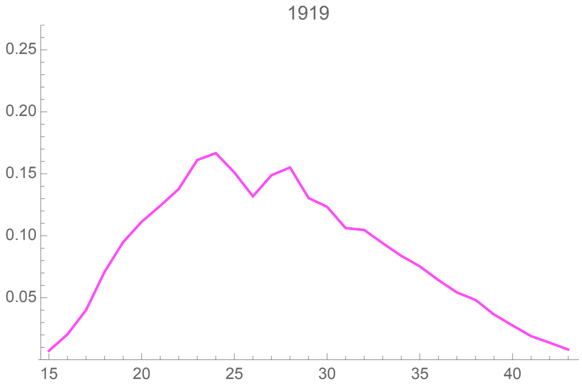
$$g(a,c) = \alpha\left[D(a+c) - F(a,c)\right]$$
$$f(a,t) = \alpha\left[D(t) - F(a,t-a)\right]$$

45 U.S. cohorts



Summary

1. Why model fertility?
2. The Lexis surface
3. Ryder's demographic translation
4. The period-shift model
5. The cohort-shift model
6. Comparing shift models
7. Lee's moving target model
8. 45 U.S. cohorts