BWFD 2025, Day 1

TODAY:

1. FORMAL DEMOGRAPHY ⟺ MODELS
2. MOST IMPORTANT EQN IN DEMOGRAPHY
3. GROWTH MODELS BASED ON THAT EQN
4. FORMAL RELATIONSHIPS
5. PROJECTION
6. CONSEQUENCES OF PROJ. W/ FIXED RATES

DEMOGRAPHY IS STUDY OF POPS, CHARACTERISTICS, AND HOW THEY CHANGE

FORMAL DEMOG → HOW THEY CHANGE ⟺ MODELS

GEORGE BOX: "ALL MODELS ARE WRONG
SOME ARE USEFUL"

SAM KARLIN: "THE PURPOSE OF MODELS IS NOT TO FIT
THE DATA BUT TO SHARPEN THE QUESTIONS."

BALANCING EQUATION

FERTILITY MORTALITY MIGRATION

$$POP_{TODAY} = POP_{YESTERDAY} + BIRTHS_y - DEATHS_y + \cancel{IMMIGRATION}_y - \cancel{OUTMIGRATION}_y + \cancel{ADJUSTMENTS}_y + \cancel{ERRORS}_y$$

APPLIED STATISTICAL

$$POP_{TODAY} = POP_{YESTERDAY} + B_y - D_y$$

$$\underbrace{POP_{2025}}_{\text{"STOCK"}} = \underbrace{POP_{2024}}_{\text{"STOCK"}} + \underbrace{B_{2024} - D_{2024}}_{\text{FLOW VARIABLES}}$$

$$K_{2025} = K_{2024} + B_{2024} - D_{2024}$$ ← ADDITION

$$= K_{2024} \cdot \left(1 + \frac{B_{2024}}{K_{2024}} - \frac{D_{2024}}{K_{2024}}\right)$$ ← MULTIPLICATION

$$= K_{2024} \cdot \left(1 + b_{2024} - d_{2024}\right)$$

$$= K_{2024} \cdot (1 + \underset{\uparrow}{b_{2024}} - \underset{\uparrow}{d_{2024}})$$

CRUDE BIRTH RATE CRUDE DEATH RATE

$$= K_{2024} (1 + r_{2024})$$

$r_{2024} = b_{2024} - d_{2024}$

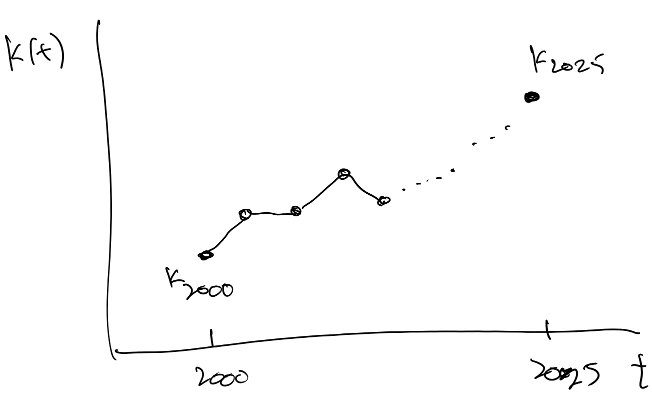
CRUDE RATE OF NATURAL INCREASE

$b_t - d_t = r_t$

$$K_{2025} = K_{2024}(1 + r_{2024})$$
$$K_{2024} = K_{2023}(1 + r_{2023})$$
$$\vdots$$
$$K_{2001} = K_{2000}(1 + r_{2000})$$
$$K_{2025} = K_{2000} \underbrace{(1+r_{2000})(1+r_{2001})(1+r_{2002})\cdots(1+r_{2024})}_{25 \text{ TERMS}}$$

$$K_{t_2} = K_{t_1} \underbrace{(1+r_{t_1})(1+r_{t_1+1})\cdots(1+r_{t_2-1})}_{t_2 - t_1 \text{ TERMS}} = \boxed{K_{t_1} \prod_{i=t_1,t_2} (1+r_i)}$$

MULTIPLICATIVE OR GEOMETRIC GROWTH



Q: WHAT WAS THE AVERAGE r BETWEEN 2000 AND 2025?

$$K_{2025} = K_{2000}(1+r_{2000})(1+r_{2001})\cdots(1+r_{2024})$$

$$K_{2025} = K_{2000} \underbrace{(1+r_{2000})(1+r_{2001})\dots(1+r_{2024})}$$

HMMM. IF WE WANT THE AVG RATE r

$$K_{2025} = K_{2000} \underbrace{(1+r_{AVG})(1+r_{AVG})(1+r_{AVG})\dots(1+r_{AVG})}_{25 \text{ TERMS}}$$

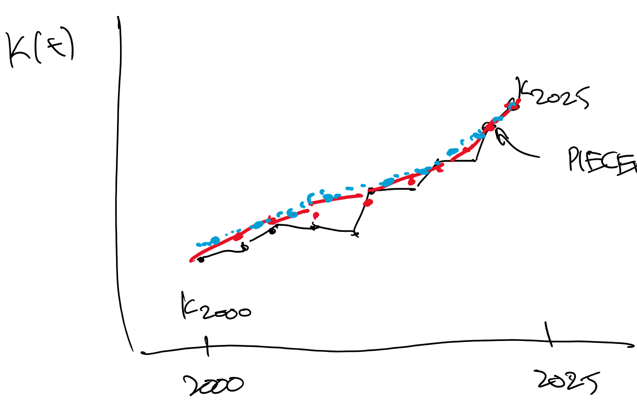
$$K_{2025} = K_{2000}(1+r_{AVG})^{25}$$

$$\boxed{K_{t_2} = K_{t_1}(1+r)^{t_2-t_1}}$$

$$K_{2025} = K_{2000}(1+r)^{25}$$

$$\frac{K_{2025}}{K_{2000}} = (1+r)^{25} \Rightarrow \left(\frac{K_{2025}}{K_{2000}}\right)^{\frac{1}{25}} = 1+r$$

$$\Rightarrow \boxed{r = \left(\frac{K_{2025}}{K_{2000}}\right)^{\frac{1}{25}} - 1}$$



SO IF WE LOOKED AT SMALLER INTERVALS THAN ANNUAL, (SAY MONTHLY) THEN THE GROWTH CURVE BECOMES SMOOTHER.

$(1+r)^{25}$: ANNUAL

$\left(1+\frac{r}{12}\right)^{300}$: MONTHLY

$\left(1+\frac{r}{365}\right)^{9000}$: DAILY

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

n	$\left(1+\frac{1}{n}\right)^n$
1	2
2	2.25
5	
10	
100	

$$K_{2025} = K_{2000} e^{25r}$$

$$\boxed{K_t = K_0 e^{rt}}$$ EXPONENTIAL GROWTH

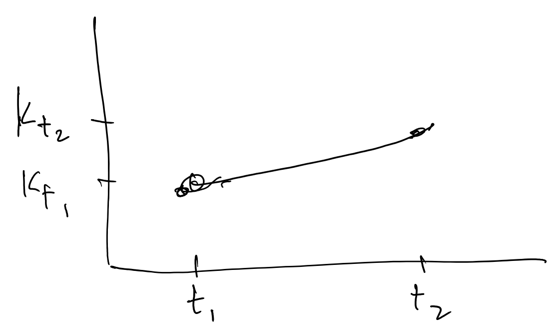
$$\boxed{K_t = K_0 (1+r)^t}$$ GEOMETRIC GROWTH

IF r IS SMALL OR t IS SHORT THEN GEOMETRIC AND EXPONENTIAL GROWTH WILL BE CLOSE.

$r \cdot t$

YOU GET TO CHOOSE (FOR HUMAN POPULATIONS)

QUICK EXAMPLE: DOUBLING TIMES.



FOR A GIVEN r HOW LONG TO DOUBLE IN SIZE?

$$K_{t_2} = 2K_{t_1}$$

IT'S EASIER TO USE EXPONENTIAL MODEL

$$K_{t_2} = 2\cancel{K_{t_1}} = \cancel{K_{t_1}} e^{r(t_2-t_1)}$$

$$2 = e^{r(t_2-t_1)}$$

HMMM. THE DOUBLING TIME DOESN'T DEPEND ON POP SIZE, IT ONLY DEPENDS ON r.

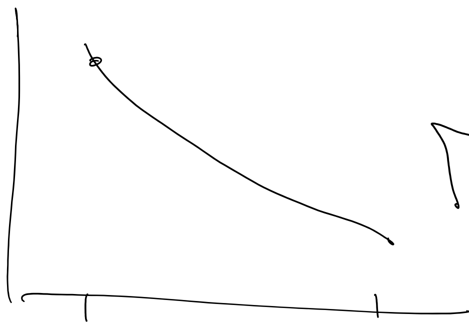
$$\log_e 2 = r(t_2-t_1)$$

$$.6931.. = r(t_2-t_1)$$

$$\frac{.6931}{r} = (t_2-t_1) = \text{DOUBLING TIME} \approx \frac{70}{r\%}$$

$-(t_2 - t_1)$ $r\%$

SUPPOSE $r < 0$.



$$\log \tfrac{1}{2} = r(t_2 - t_1)$$

HALVING TIME

$\boxed{\text{HALF - LIFE}}$

LONG - TERM GROWTH IN HUMAN POPULATIONS

FOR MUCH OF HUMAN HISTORY r WAS VERY LOW, VERY CLOSE TO ZERO.

$b \sim d$

IF A POP HAS BEEN AT $r = 0$ FOR A LONG TIME, WE CALL THAT A STATIONARY POPULATION $(b = d)$

$$d = \text{DEATH RATE} = \frac{\text{DEATHS / YEAR}}{\text{POP}}$$

THE RECIPROCAL OF THE RATE IS THE AVERAGE INTERVAL BETWEEN EVENTS $\Big\}$ *

$$\frac{1}{d} = \frac{\#\ \text{YEARS}}{\text{DEATH}} = \text{AVG INTERVAL BETW. BIRTH AND DEATH}$$

$= \text{AVG AGE AT DEATH}$

$= \text{EXPECTATION OF LIFE AT BIRTH}$

$= e_0$

$$\frac{1}{d} = e_0$$

$$\frac{1}{d} = e_0$$

IN A STATIONARY POP, $b = d$

$$\frac{1}{b} = e_0$$

$$\boxed{b \cdot e_0 = 1}$$ STATIONARY POPULATION IDENTITY

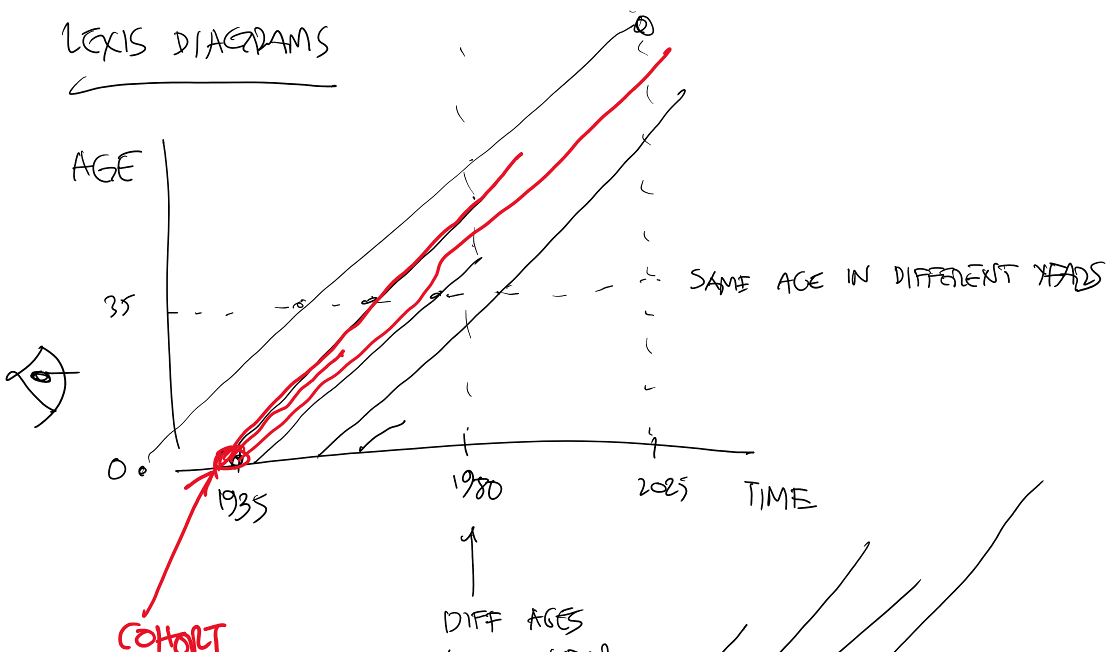
IN THE US, JUST PRIOR TO COVID, $e^{US}_{0,2019} = 79$

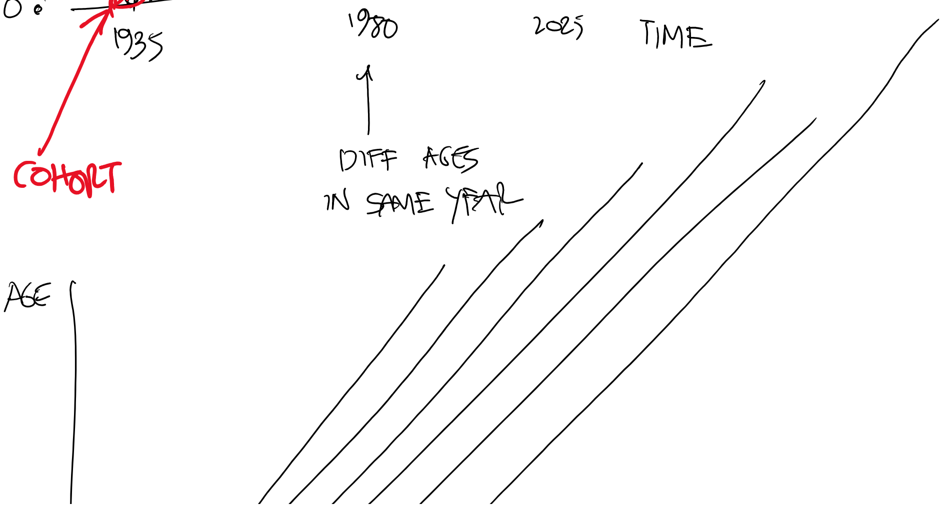
r IS LOW.

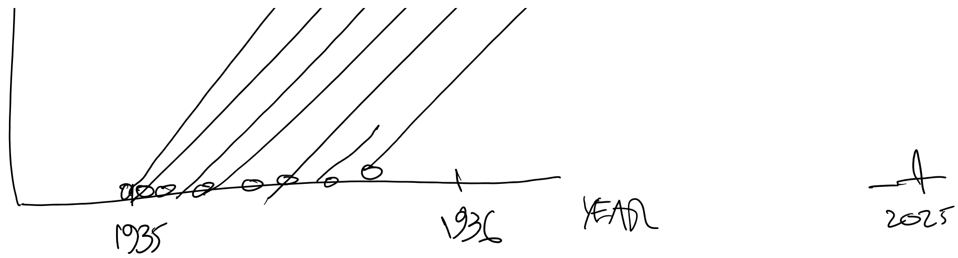
IF $b \sim \frac{1}{e_0} \sim \frac{1}{80} = .0125 = 12.5/1000$

IN 2019, US, $b = 12/1000$

LEXIS DIAGRAMS

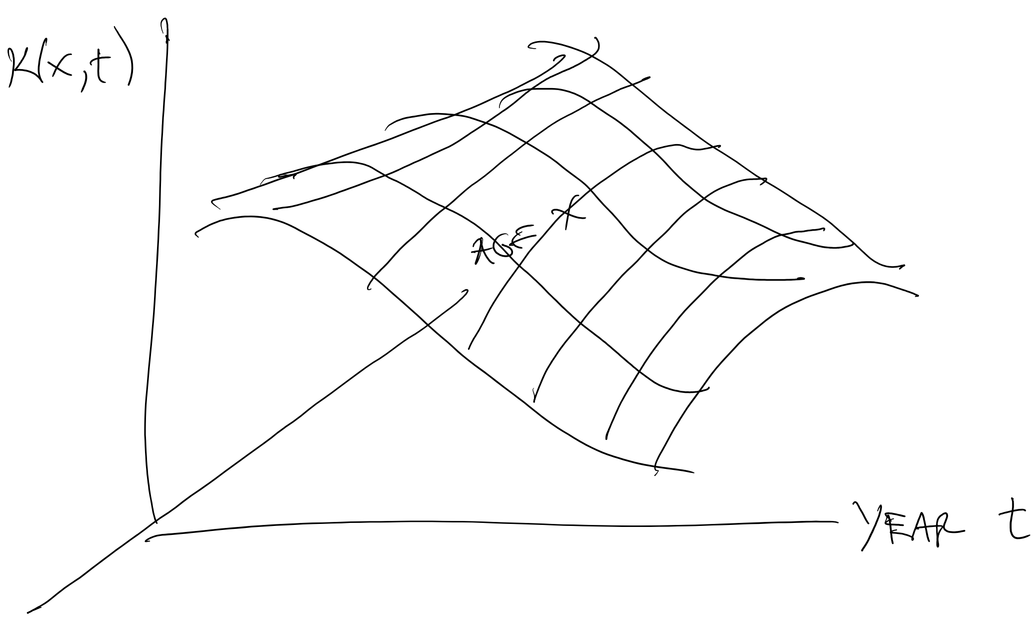




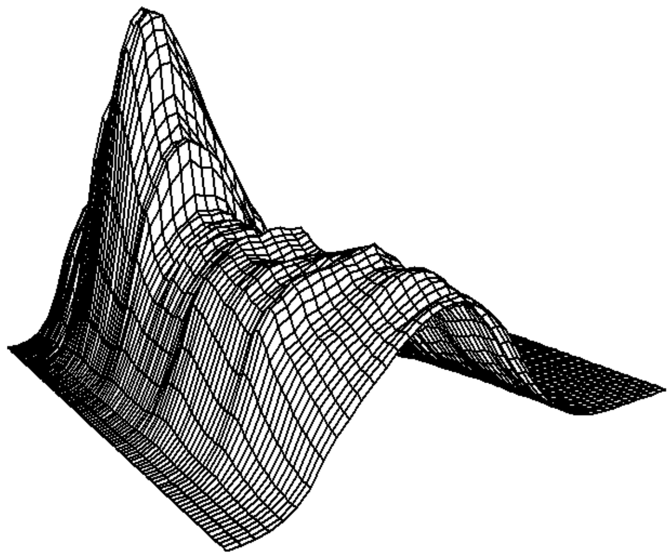


ADD UP THE LINE LENGTHS
FOR EVERYONE BORN IN 1935.
= TOTAL PERSON YEARS LIVED

DIVIDE $\frac{\text{TPYL}}{\text{TOTAL BIRTHS}}$ = AVG AGE AT DEATH
= LIFE EXPECTATION AT BIRTH



ASFR, USA 1933-2023

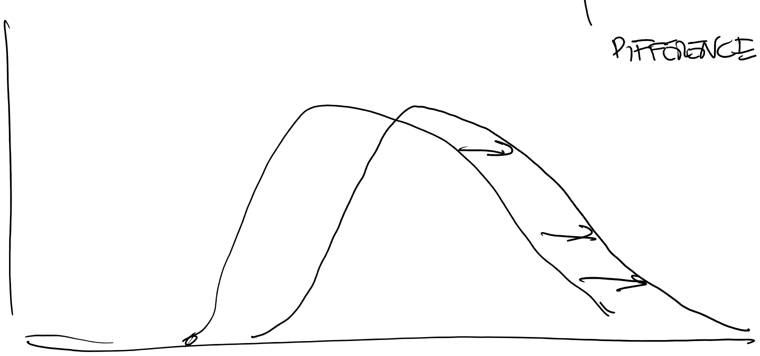


TYPICAL SHAPE IS SKEWED

$$\int f(x)\, dx = TFR$$

$$\sum {}_n f_x = TFR$$

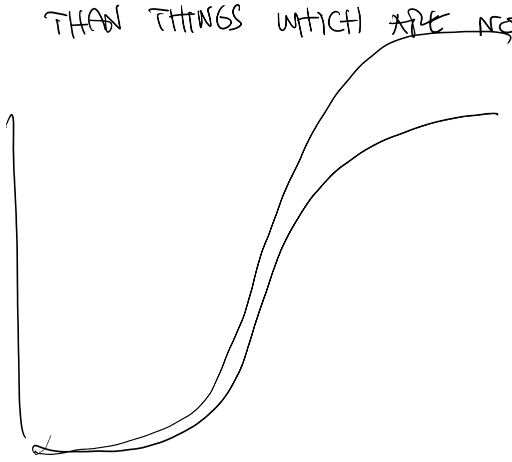
DIFFERENCE IN QUANTUM



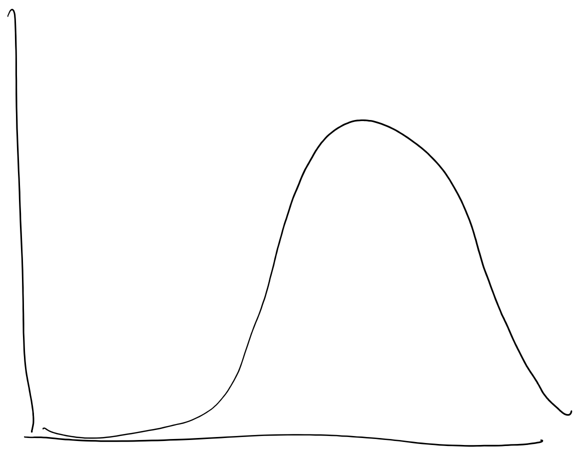
DIFFERENCE IN TIMING

↖ DIFFERENCE IN TIMING
TEMPO

✱ → THINGS WHICH ARE MONOTONIC ~~ARE~~ EASIER TO MODEL
THAN THINGS WHICH ~~ARE~~ NON-MONOTONIC

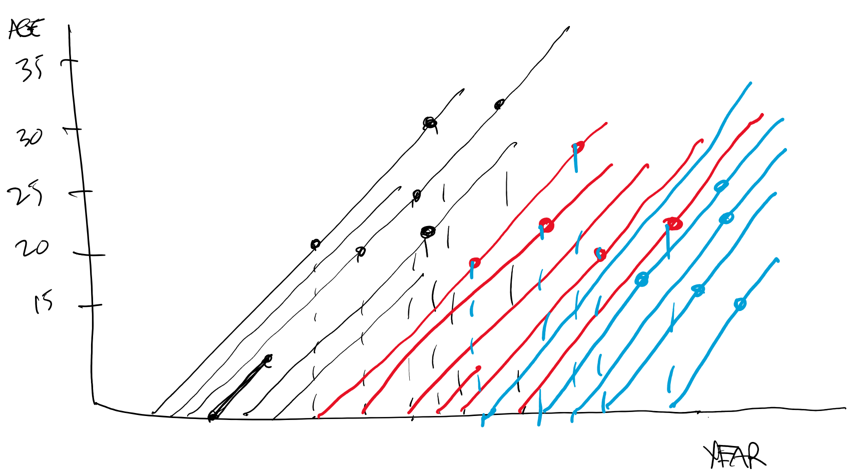


MODELING THIS



EASIER THAN THIS

GROWTH AND REPRODUCTION



WOMEN ONLY

ONLY LOOK AT FEMALES,
IGNORE MALES

TRACK FOR EACH COHORT
COUNT "DAUGHTERS" FOR EACH WOMAN

CALCULATE THE RATIO OF "DAUGHTERS" IN NEXT GENERATION TO "MOTHERS" IN PREVIOUS GENERATION.

ANDERSON & MAY (1990)

$$\frac{\text{RED}}{\text{BLACK}} \quad \frac{\text{BLUE}}{\text{RED}} = \text{NET REPRODUCTION RATIO} = \text{NRR}$$

ID EPI = BASIC REPRO NUMBER $= R_0$

$$\frac{\text{POP NEXT GENERATION}}{\text{POP THIS GENERATION}} = \frac{K t_2}{K t_1} = e^{r(t_2 - t_1)} = \text{NRR}$$

$\text{NRR} = e^{rG}$ WHERE G IS THE GENERATION LENGTH.

$\log \text{NRR} = rG$

AS IT HAPPENS, CALCULATING G IS HARD.

CLAIM: G IS CLOSE TO AVG AGE OF THE FERTILITY SCHEDULE.

$f(x)$ ANY WEIGHTED MEAN HAS THE FORM

$$\frac{\int x\, w(x)\, dx}{\int w(x)\, dx}$$

$$\frac{\sum \left(x + \frac{n}{2}\right) {}_n w_x}{\sum {}_n w_x}$$

$$\frac{\int x f(x)\, dx}{\int f(x)\, dx} \quad \text{OR} \quad \frac{\sum \left(x + \frac{n}{2}\right) {}_n f_x}{\sum {}_n f_x}$$

${}_n f_x$ THE FERTILITY RATE BETWEEN AGES x AND $x+n$

MEAN AGES COMMON IN FORMAL DEMOGRAPHY

① $$\frac{\int x f(x)\, dx}{\int f(x)\, dx} = \bar{x} \text{ or } A_m$$

$$\int f(x)\,dx =$$

(2) $$\frac{\int x f(x)\, l(x)\,dx}{\int f(x)\, l(x)\,dx} = \mu$$

(3) $$\frac{\int x f(x)\, l(x)\, e^{-rx}\,dx}{\int f(x)\, l(x)\, e^{-rx}\,dx} = A_r$$

(4) $$\frac{\int x\, l(x)\, e^{-rx}\,dx}{\int l(x)\, e^{-rx}\,dx}$$

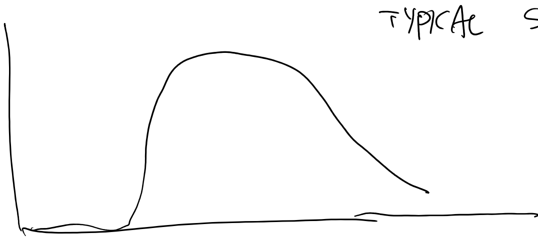
FOR COMPLETENESS BUT WE WON'T TALK ABOUT IT IN THIS WORKSHOP

CLAIM: G IS CLOSE TO $\bar{x}$ OR A_m

ALSO CLOSE TO μ

$NRR = e^{rG} \sim e^{r\bar{m}}$ AND THIS IS MUCH EASIER TO CALCULATE.

TYPICAL SHAPE OF ASFR



MEAN AGE FALLS IN A SMALL RANGE

MOST NATIONAL POPS, $\bar{x} \in [25, 33]$

[QUARTER, THIRD]

1934 BIRTH COHORT US HIGH FERTILITY COHORT

NRR $= 1.49$ $r = .0157 = 1.57\%$ $M = 25.8$ $G = 25.4$

$$\frac{\log NRR}{G} = r \qquad \frac{\log 1.49}{25.4} \qquad \frac{\log 1.49}{25.8}$$

$r = .0157$ $r = .0155$

NOT BAD

<u>NRR</u> TFR $= \int f(x)\,dx$ = ALL SEXES OF BABIES NOT JUST FEMALES

· EXCLUDES MORTALITY OF MOTHERS

GROSS REPRODUCTION RATIO : EXCLUDES MORTALITY OF MOTHERS

INCLUDES ONLY FEMALE BABIES .

TFR → GRR → NRR

TFR → GRR $\qquad TFR \cdot F_{FAB} = GRR$

$\boxed{F_{FAB} = 0.4886}$

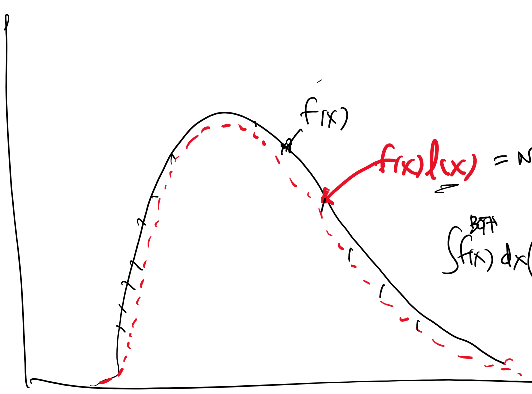
$F_{MAB} = 0.5114$

SRB = 1.05

$l(x) = p(a)$

$f(a) = m(a)$

$\phi(a) =$



f(x)l(x) = NET MATERNITY FUNCTION $= \phi(x)$

$$\int^{\text{BOTH}} f(x)\,dx\,(.4886) = \int^{\text{FEMALES}} f(x)\,dx > \int^{\text{FEMALE FEMALES}} f(x)\,l(x)\,dx$$

<u>CLAIM</u> IS THAT

$$\int f(x)\,l(x)\,dx \approx \int f(x)\,dx \cdot l(\bar{m})$$
$$\cdot\, l(\mu)$$

$$\boxed{NRR \approx GRR \cdot l(\mu)}$$ COCKTAIL PARTY APPROXIMATION

SUMMARY:

$$NRR = e^{rG} \approx e^{r\bar{m}} \text{ or } e^{r\mu}$$

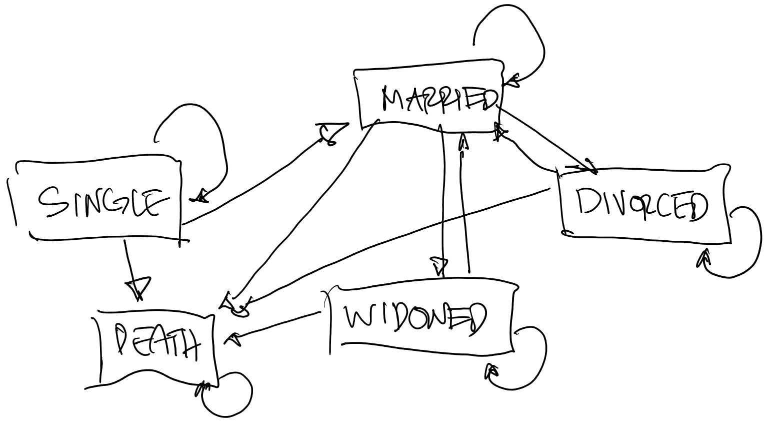
$$NRR = GRR \cdot l(\mu) \text{ or } l(\bar{m})$$

$$GRR = TFR \cdot F_{FAB}$$ (DEFAULT IS 0.4886)

PUTTING THINGS TOGETHER : PROJECTION

MARITAL STATUS
- SINGLE NEVER-MARRIED
- MARRIED
- WIDOWED
- DIVORCED
- DEAD

EXCLUSIVE
COMPLETE
PROJECTION PERIODS ARE SHORT ENOUGH SO YOU CAN TRANSITION AT MOST ONE STATUS



TIME $\emptyset$

TIME 1 \ TIME $\emptyset$	S	M	W	DIV	DEAD
S		0	0	0	0
M					0
W	0			0	0
DIV	0		0		0
DEAD					1.0

$= A$

0'S ARE <u>STRUCTURAL</u> ZEROES

FILL THE REST WITH TRANSITION MULTIPLIERS

K_0 = COLUMN VECTOR AT TIME 0 WITH COUNTS OF PEOPLE IN EACH STATUS

$$K_1 = A_0 \cdot K_0$$

$$K_2 = A_1 \cdot K_1 = A_1 \cdot A_0 \cdot K_0$$

$$K_3 = A_2 K_2 = \underline{A_2} \cdot \underline{A_1} \cdot \underline{A_0} \cdot K_0$$

IF $A_2 = A_1 = A_0$

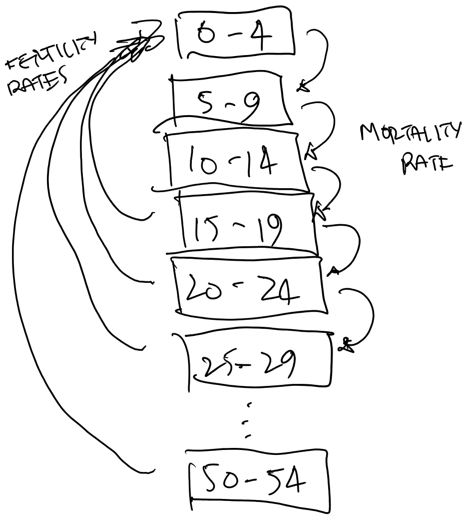
$$K_t = \underbrace{A \cdot A \cdot A \cdots A}_{t} \cdot K_0 = A^t K_0$$

MATRIX MULTIPLICATION NOT SCALAR.

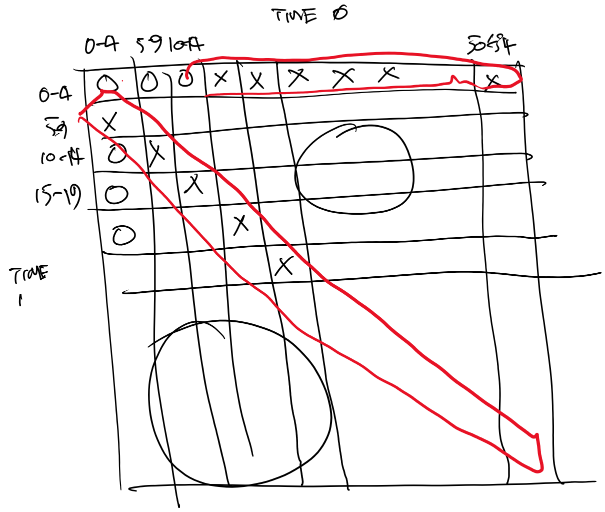
$\boxed{\text{SPECIAL CASE OF TRANSITION MATRICES}}$

MUCH EASIER IF OUR PROJ STEP WAS SAME AS AGE GROUP WIDTH

$\boxed{6-4}$



MUCH EASIER IF OUR PROJ STEP WAS SAME AS AGE GROUP WIDTH



LOT OF STRUCTURAL ZEROES IN THIS MATRIX.

LESLIE MATRIX

NON ZEROES ON TOP ROW AND SUB-DIAGONAL

11×11

SO, WE DID A DEMONSTRATION OF <u>LAB 2: PROJECTION</u>

WE FOUND OUT THAT THE LESLIE MATRIX CONTAINS LONG RUN IMPLICIT INFORMATION ABOUT LONG-RUN GROWTH AND LONG-RUN AGE STRUCTURE.

EARLIER, WE SAW THAT b, d CONTAINED ENOUGH INFORMATION TO DETERMINE r IN THE CRUDE RATE BALANCING EQUATION.

NOW WE'RE SEEING THAT IN THE AGE-STRUCTURED WORLD, $f(x)$ AND $l(x)$ (OR ${}_nf_x$ AND ${}_nL_x$) CAN DO THE SAME THING.

SHORT-RUN GROWTH IS VARIABLE BUT EVENTUALLY ATTAINS EQUILIBRIUM. SAME WITH AGE STRUCTURE.

THE EQUILIBRIUM GROWTH RATE AND AGE STRUCTURE ARE <u>INDEPENDENT</u>

THE EQUILIBRIUM GROWTH RATE AND AGE STRUCTURE ARE INDEPENDENT OF THE INITIAL POP. (THIS IS CALLED ERGODICITY)

THE EQUILIBRIUM DESCRIBES A STABLE POPULATION.

A STATIONARY POPULATION IS A STABLE POPULATION WHERE $r = 0$.

$\boxed{\text{EIGENDECOMPOSITION}}$

A IS A LESLIE MATRIX

ALL LESLIE MATRICES (AND MANY OTHERS) CAN BE DECOMPOSED INTO A SPECIAL FORM.

$$A = U \Lambda U^{-1}$$

Λ IS DIAGONAL

U IS ORTHOGONAL

THIS DECOMPOSITION IS HANDY BECAUSE

$$K_T = \underbrace{A \cdot A \cdot A \cdot A}_{t \text{ TIMES}} \cdot K_0$$

$$A \cdot A = (U \Lambda U^{-1})(U \Lambda U^{-1})$$

$$= U \Lambda \cdot \Lambda U^{-1} = U \Lambda^2 U^{-1}$$

$$\underbrace{A \cdot A \cdot A \cdot A}_{t \text{ TIMES}} = U \Lambda^t U^{-1}$$

$$A^t K_0 = U \Lambda^t U^{-1} K_0$$

Λ IS COMPRISED OF EIGENVALUES

U IS COMPRISED OF EIGENVECTORS

Λ CONTAINS EIGENVALUES

ONE EIGENVALUE HAS LARGEST SIZE "LEADING" EIGENVALUE
IT GIVES GROWTH RATE

THE EIGENVECTOR ASSOCIATED WITH LEADING EIGENVALUE GIVES STABLE AGE STRUCTURE

λ_0 IS THE LEADING EIGENVALUE, THEN $\lambda_0 = e^{r \cdot \text{TIME STEP}}$

$$\frac{\log \lambda_0}{\text{TIME STEP}} = r$$

IN OUR CASE TIME STEP = 5 $\qquad \dfrac{\log \lambda_0}{5} = r$

BWFD Day 2

Tuesday, June 3, 2025

$\boxed{\text{SOME COMMENTS FROM YESTERDAY}}$

$$\lambda_0 = e^{r \cdot \text{TIME INTERVAL}}$$

$$NRR = e^{rG}$$

DOUBLING TIMES IF GIVEN A GROWTH r

$$\cancel{K_0} e^{rt} = K_1 = 2 \cancel{K_0} \Rightarrow e^{r t_{DOUBLING}} = 2$$

$$\Rightarrow \log 2 = r\, t_{DOUBLE}$$

$$\Rightarrow \frac{.6931}{r} = t_{DOUBLING}$$

$$\frac{\log 3}{r} = t_{tripling}$$

BUT WE DON'T HAVE TO LOOK ONLY AT NUMBERS LIKE 2 OR 3. IT WORKS FOR ODDBALL NUMBERS LIKE 1.37

$$\frac{\log 1.37}{r} = t_{\times 1.37}$$

$$\frac{\log NRR}{r} = \text{time TO MULTIPLY BY A FACTOR OF THE } NRR$$

$$= G$$

$$e^{rG} = NRR$$

$$\lambda_0 = e^{r \cdot \overset{n}{(\text{TIME STEP})}} = e^{r5}$$

$$\frac{\log \lambda_0}{n} = r$$

COMPARATIVE STATICS

WE CAN NOW USE OUR TOOLS TO LOOK AT DYNAMICS AND THE PATH TO EQUILIBRIUM.

EIGENDECOMPOSITION

$$A = U \Lambda U^{-1}$$

$$A \cdot \underline{k}_0 = k_1$$

$n \times 1$

$$k_0' A = k_1'$$

$1 \times n$

FOR THE LEADING EIGENVALUE

RIGHT HAND EIGENVECTOR GIVES US THE STABLE EQUILIBRIUM AGE STRUCTURE

LEFT-HAND EIGENVECTOR GIVES US REPRODUCTIVE VALUE

(SEE EXERCISE #4)

TODAY:

- MOST IMPORTANT EQUATION IN FORMAL DEMOGRAPHY
- SOME CONSEQUENCES OF CHANGING RATES ← WHY WESTERN ECONOMIES ARE FREAKED OUT ABOUT SLOWER POP GROWTH
- COHORT AND PERIOD FERTILITY (AFTER LUNCH)
- DEMONSTRATIONS

RELATIONSHIPS AMONG DEMOGRAPHIC VARIABLES IN LONG RUN EQUILIBRIUM

LOOK ONLY AT ONE-SEX: (TYPICALLY, FEMALES)

$\left.\begin{matrix} f(x) \\ l(x) \end{matrix}\right\}$ FEMALE RATES FOR FEMALE OFFSPRING (SO CAN AVOID 0.4886)

$k(x,t)$ = WOMEN AGE x AT TIME t

${}_nk_x(t)$ = WOMEN AT TIME t IN AGE GROUP x TO $x+n$

$k(30, 2025)$ = # WOMEN AGE 30 IN YEAR 2025

$K(30, 2025)$ = # WOMEN AGE 30 IN YEAR 2025

= SURVIVORS OF GIRL BABIES BORN IN 1995

$= K(0, 1995) \cdot l(30)$

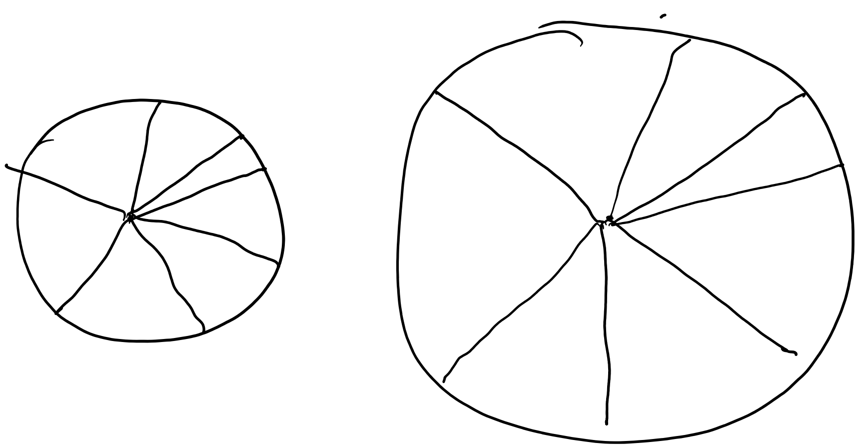
$$K(x,t) = \underbrace{K(0, t-x)}_{B(t-x)} \cdot l(x) = B(t-x)\, l(x)$$

$B(t-x)$
BIRTHS IN YEAR $t-x$

IN OUR STABLE EQUILIBRIUM WORLD WITH FIXED $F(x)$, $l(x)$

WE KNOW 1) THE TOTAL POP GROWS AT RATE r

2) THE EQUILIBRIUM AGE STRUCTURE IS FIXED



PIE GROWS AT r

RELATIVE SIZES OF SLICES IS FIXED

SO EACH SLICE ALSO GROWS AT r.

SO ALL AGE GROUPS GROW AT r.

70 Y.O.'S GROW AT r

60 Y.O'S GROW AT r

BIRTHS GROW AT r

$$B(t) = (1+r)\,B(t-1) \quad \text{or} \quad \boxed{B(t) = B(t-x)\,e^{rx}}$$

$$\boxed{B(t)\,e^{-rx} = B(t-x)}$$

EQUIVALENT

$$B(2025) = B(1995)\,e^{30r}$$

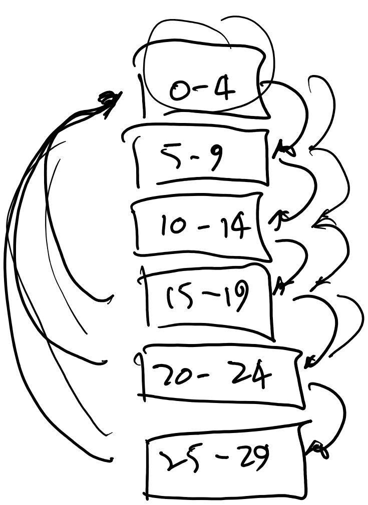
$$B(1995) = B(2025)\,e^{-30r}$$

$$K(x,t) = \underbrace{K(0,t-x)}_{B(t-x)}\,l(x) = B(t-x)\,l(x)$$

$$= \underline{B(t)}\,e^{-rx}\,l(x)$$

$$K(x,2025) = B(2025)\,e^{-rx}\,l(x)$$

WE HAVE IS A TYPE OF EQUATION THAT RELATES BIRTHS TODAY TO POP AGE STRUCTURE TODAY.



ABOVE!

$$K(x,t) = B(t)\,e^{-rx}\,l(x)$$

$$K(t) = \int K(x,t)\,dx = \text{TOTAL POP ALL AGES AT TIME } t.$$

$$\frac{K(x,t)}{K(t)} = \text{PROPORTION AGE } x = \frac{B(t)\,e^{-rx}\,l(x)}{K(t)}$$

$$\frac{B(t)}{K(t)} = b = \text{BIRTH RATE.}$$

$$\boxed{C(x) = \begin{array}{c}\text{PROPORTION AGE } x\\ \text{IN THE STABLE}\end{array} = b \cdot e^{-rx}\,l(x)}$$ EULER 1760

$$\boxed{C(x) = \text{PROPORTION AGE } x \text{ IN THE STABLE EQUILIBRIUM} = b \cdot e^{-rx} \, l(x)}$$ EULER 1760

$$\int c(x)\,dx = 1 = \int b\, e^{-rx} l(x)\, dx = b \int e^{-rx} l(x)\, dx$$

$1 = b \int e^{-rx} l(x)\, dx$ IN THE STABLE POP GROWING AT r.

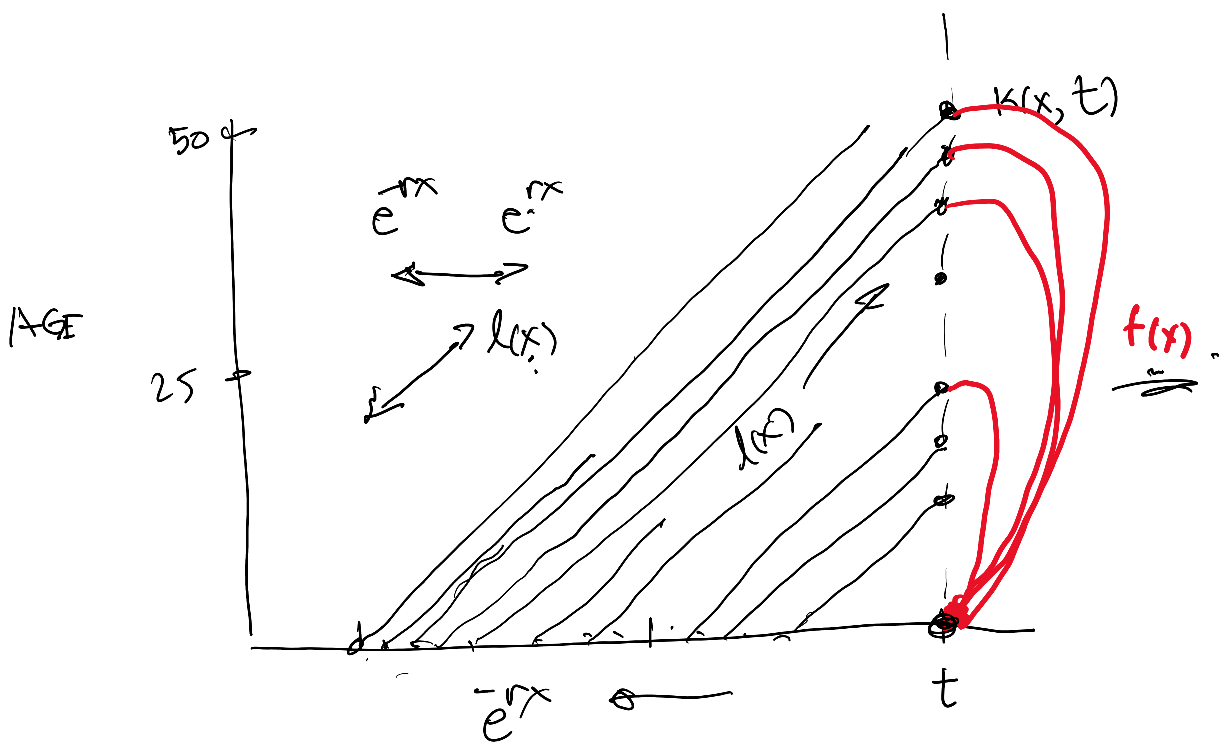
WHEN $r = 0$ THEN

$1 = b \int l(x)\, dx$. WHAT IS $\int l(x)\, dx$?

e_0

$1 = b \cdot e_0$ WHEN $r = 0$

STATIONARY POP IDENTITY FROM YESTERDAY.



$$B(t) = \int K(x,t) \cdot f(x)\, dx$$
$$= \int B(t-x)\, l(x) f(x)\, dx$$
$$= \int B(t) e^{-rx} l(x) f(x)\, dx$$

$$\underbrace{B(t)}_{\text{BIRTHS TODAY}} = \int \underbrace{B(t) e^{-rx}}_{\text{PAST BIRTHS}} \cdot \underbrace{l(x)}_{\text{SURVIVORS OF PAST BIRTHS}} \cdot \underbrace{f(x)}_{\text{FERTILITY OF PAST BIRTHS WHO SURVIVED TO ~~AGE~~ } x} \cdot dx$$

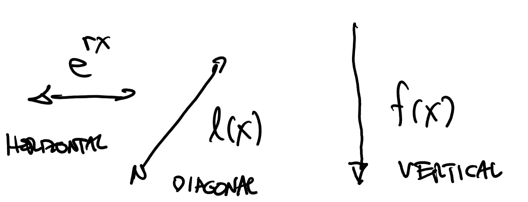
RENEWAL EQUATION
THAT RELATES BIRTHS TODAY TO
BIRTHS IN THE PAST.

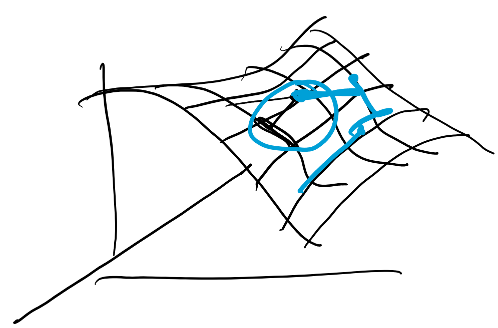
$$B(t) = \int B(t) e^{-rx} l(x) f(x)\, dx$$

$$\cancel{B(t)} = \cancel{B(t)} \int e^{-rx} l(x) f(x)\, dx$$

$$\boxed{\int e^{-rx} l(x) f(x)\, dx = 1}$$

* LOTKA - EULER EQUATION





HORIZONTAL MOVES BY e^{rx}

DIAGONAL MOVES BY $l(x)$

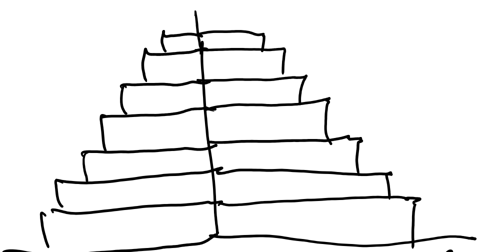
VERTICAL MOVES BY $f(x)$

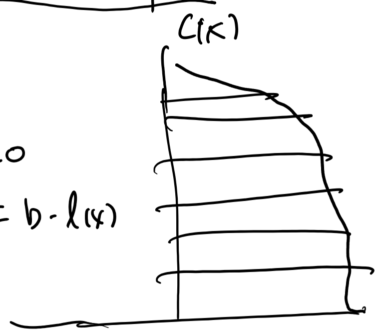
WE CAN GET FROM ANY POINT ON LEXIS SURFACE TO ANY OTHER POINT.

SUPPOSE WE'RE IN THIS STABLE EQUILIBRIUM WORLD. $r > 0$

$c(x)$ = AGE STRUCTURE = $b\, l(x)\, e^{-rx}$

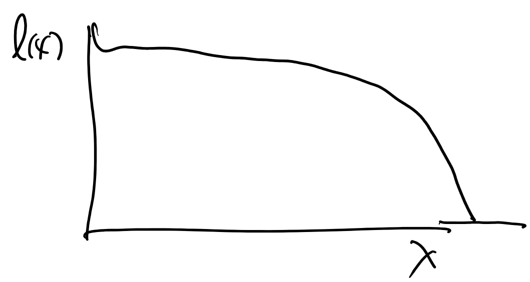
AGE PYRAMIDS





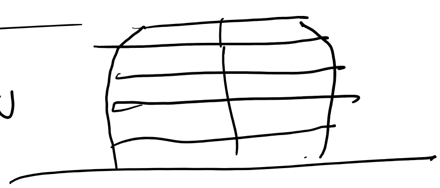
WHEN $r=0$

THE $c(x) = b \cdot l(x)$



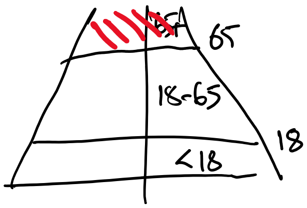
$_nL_x$ STATIONARY POP VECTOR

$r < 0$ THEN



... OLD-AGE DEPENDENCY RATIO WHEN r CHANGES?

WHAT HAPPENS TO OLD-AGE DEPENDENCY RATIO WHEN r CHANGES?



KNOW AGE PYRAMID CHANGES WITH r.

$$\alpha = \frac{\text{POP } 65+}{\text{POP } 18\text{-}64}$$

$$c(x) = b\, l(x) e^{-rx}$$

$$b = \frac{1}{\int l(x) e^{-rx}}$$

$$\boxed{\text{So } c(x) = \frac{l(x) e^{-rx}}{\int l(x) e^{-rx}\, dx}}$$

$$\left[{}_nC_x = \frac{{}_nL_x e^{-rx}}{\sum {}_nL_x e^{-rx}} \right]$$

DISCRETE VERSION IF YOU EVER ACTUALLY NEEDED TO CALCULATE THIS FOR MONEY.

$$\alpha = \frac{\int_{65}^{\infty} \cancel{b}\, l(x) e^{-rx}\, dx}{\int_{18}^{64} \cancel{b}\, l(x) e^{-rx}\, dx} = \frac{\int_{65}^{\infty} l(x) e^{-rx}\, dx}{\int_{18}^{64} l(x) e^{-rx}\, dx}$$

WHAT HAPPENS TO α WHEN r CHANGES? $\frac{d\alpha}{dr}$

IT TURNS OUT THAT IT IS OFTEN EASIER TO ~~LOOK~~ AT $\frac{d \log \alpha}{dr}$

$$\log \alpha = \log \int_{65}^{\infty} l(x) e^{-rx}\, dx - \log \int_{18}^{64} l(x) e^{-rx}\, dx$$

$$\frac{d \log \alpha}{dr} = \frac{d}{dr} \log \int_{65}^{\infty} l(x) e^{-rx}\, dx - \frac{d}{dr} \log \int_{18}^{64} l(x) e^{-rx}\, dx$$

$$= \frac{\frac{d}{dr} \int_{65}^{\infty} l(x) e^{-rx}\, dx}{\int_{65}^{\infty} l(x) e^{-rx}\, dx} - \frac{\frac{d}{dr} \int_{18}^{64} l(x) e^{-rx}\, dx}{\int_{18}^{64} l(x) e^{-rx}\, dx}$$

6)

$$\frac{\qquad}{\int_{65}^{\infty} l(x)e^{-rx}\,dx} \qquad \frac{18}{\int_{18}^{64} l(x)e^{-rx}\,dx}$$

$$= \frac{\int_{65}^{\infty} -x\, l(x)e^{-rx}\,dx}{\int_{65}^{\infty} l(x)e^{-rx}\,dx} - \frac{\int_{18}^{64} -x\, l(x)e^{-rx}\,dx}{\int_{18}^{64} l(x)e^{-rx}\,dx}$$

$$= \frac{\int_{18}^{64} x\, l(x)e^{-rx}\,dx}{\int_{18}^{64} l(x)e^{-rx}\,dx} - \frac{\int_{65}^{\infty} x\, l(x)e^{-rx}\,dx}{\int_{65}^{\infty} l(x)e^{-rx}\,dx}$$

WEIGHTED MEAN AGE OF WORKING PEOPLE

IN DEVELOPED COUNTRIES

40 ISH

WEIGHTED MEAN AGE OF PEOPLE 65+

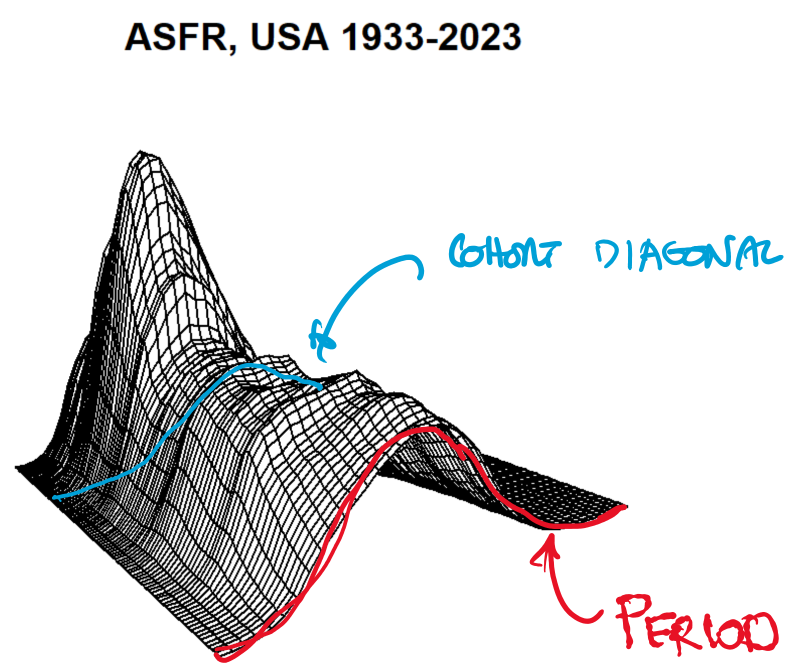
75 ISH

-35

A DECREASE IN r OF 1% MEANS AN INCREASE IN α OF $\sim 35\%$ *

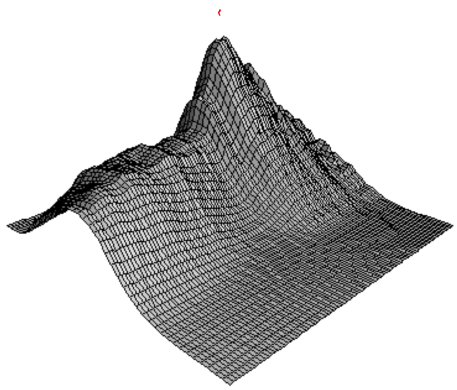
THIS IS WHY DEVELOPED COUNTRIES ARE FREAKING OUT

REVISITING PERIODS AND COHORTS

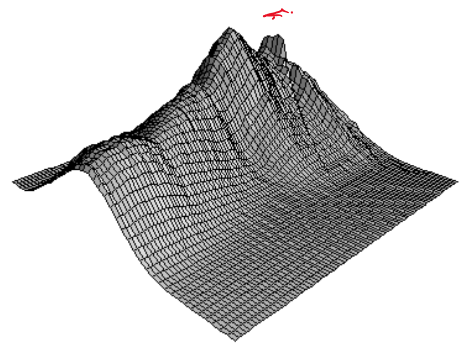


SURFACE PLOTS FOR DIFFERENT COUNTRIES
(SAME SCALE, SAME PERSPECTIVE)

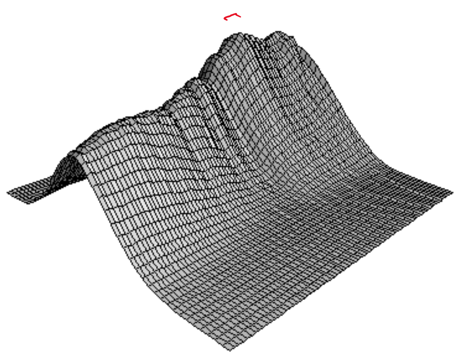
USA ASFR, 1933-2023, Ages 12-55



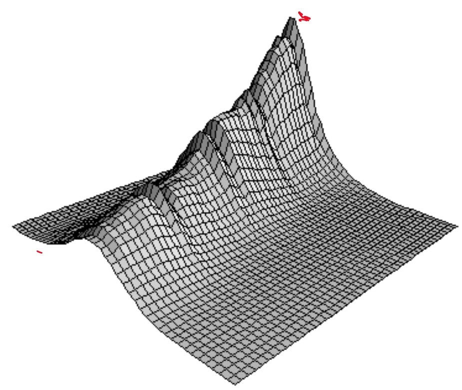
England&Wales ASFR, 1938-2022, Ages 12-55



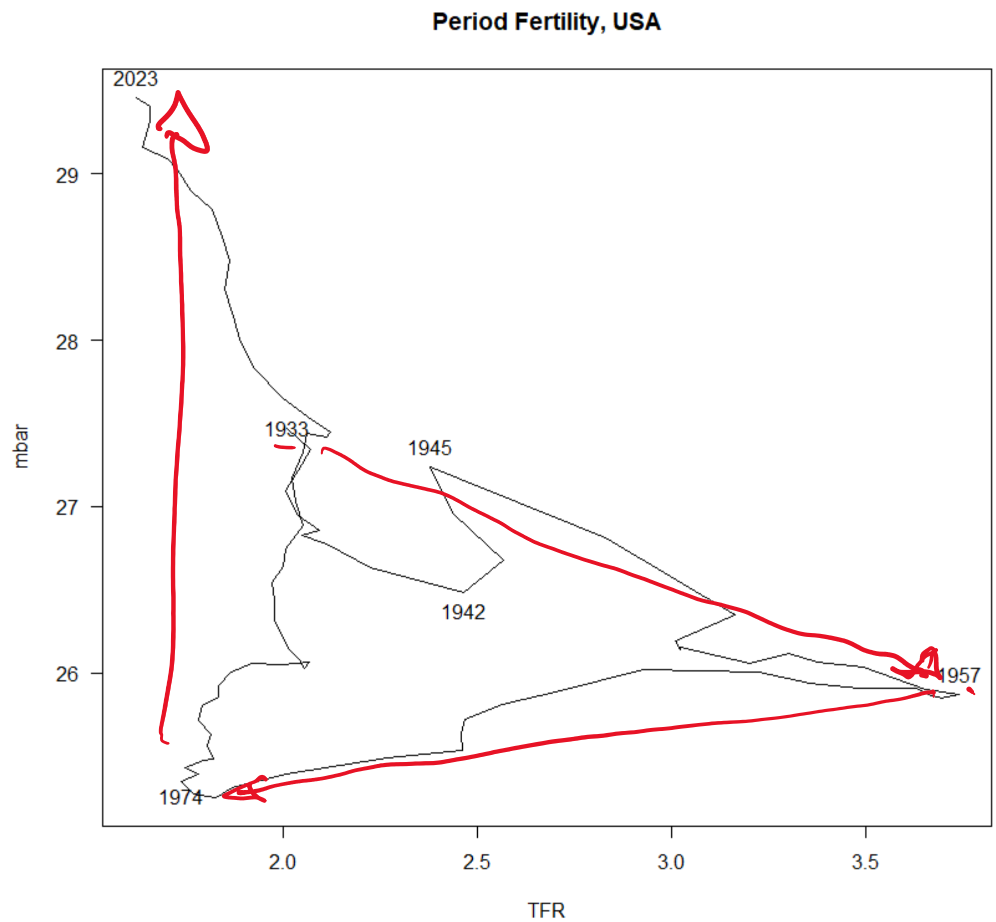
France ASFR, 1946-2022, Ages 12-55



Taiwan ASFR, 1976-2023, Ages 12-55

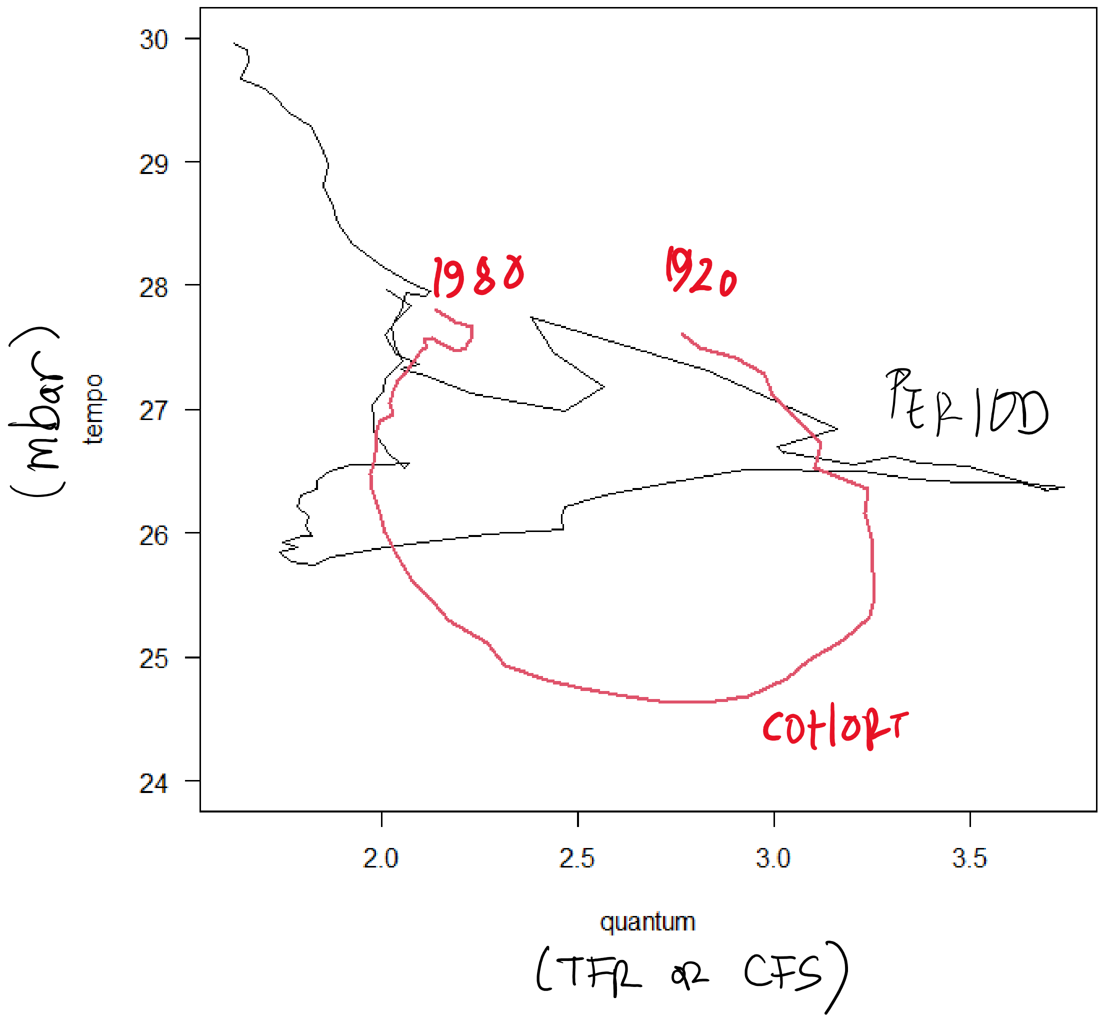


TEMPO - QUANTUM PHASE PLOTS



Period Fertility, USA
2023
1933
1945
1942
1974
1957
29
28
27
26
mbar
2.0
2.5
3.0
3.5
TFR

PERIOD AND COHORT PHASE PLOTS



```
# download the whole damn ASFR data from the HFD humanfertility.org
# and unzip it to get asfrRR.txt

dat <- read.table("asfrRR.txt", skip=2, header=T)

# Fix the age variable

dat$Age[dat$Age=="12-"] <- "12"
dat$Age[dat$Age=="55+"] <- "55"
dat$Age <- as.numeric(dat$Age)

# Add a cohort variable
dat$Cohort <- dat$Year - dat$Age

# get some country files
USA <- dat[dat$Code=="USA",]  # 1933-2023, 91 years
SWE <- dat[dat$Code=="SWE",]  # Longest time series 1891-2023, 133 years
TWN <- dat[dat$Code=="TWN",]  # Highest single-year rate 1976-2023, 48 years
FRA <- dat[dat$Code=="FRATNP",] # 1946-2022, 77 years
EW <- dat[dat$Code=="GBRTENW",] # 1938-2022, 85 years


# persp() takes a matrix, so need to re-shape asfr's into a matrix

# 44 age categories from 12 to 55.
# number of years differ across countries, so need to check

summary(USA)
asfr.USA <- matrix(USA$ASFR, nrow=91, ncol=44, byrow=T)
asfr.FRA <- matrix(FRA$ASFR, nr=77, nc=44, byrow=T)
asfr.TWN <- matrix(TWN$ASFR, nr=48, nc=44, byrow=T)
asfr.EW <- matrix(EW$ASFR, nr=85, nc=44, byrow=T)
asfr.SWE <- matrix(SWE$ASFR, nr=133, nc=44, byrow=T)

# alternatively, plot3d() in package rgl takes "long format" data, so could
# also do that.

persp(asfr.USA)

# OK, let's change the viewpoint and add some labels

persp(asfr.USA, theta=135, phi=20, shade=.5, box=F, scale=F, expand=2, main="USA ASFR, 1933-2023, Ages 12-55")


# seems to work. Let's plot US, GB, FRA, and TWN on one plot. Scale them the same way

par(mfrow=c(2,2), mai=rep(.1,4), omi=rep(.5, 4))

persp(asfr.USA, zlim=c(0,0.29), theta=135, phi=20, shade=.5, box=F, scale=F, expand=2, main="USA 1933-2023")
persp(asfr.EW, zlim=c(0,0.29), theta=135, phi=20, shade=.5, box=F, scale=F, expand=2, main="England&Wales 1938-2022")
persp(asfr.FRA, zlim=c(0,0.29), theta=135, phi=20, shade=.5, box=F, scale=F, expand=2, main="France 1946-2022")
persp(asfr.TWN, zlim=c(0,0.29), theta=135, phi=20, shade=.5, box=F, scale=F, expand=2, main="Taiwan 1976-2023")
title("ASFR, ages 12-55", outer=T)


# TFR and CFR

tapply(USA$ASFR, USA$Year, sum) -> USA.tfr
tapply(USA$ASFR, USA$Cohort, sum) -> USA.cfs

# mean age of the period asfr fertility schedule

tapply(USA$ASFR*((12:55)+0.5), USA$Year, sum) -> y
y / USA.tfr -> USA.mbar.period
```

```
rm(y)

# plot tfr vs. mbar

plot(USA.tfr, USA.mbar.period, las=1, main="Period Fertility, USA", xlab="TFR", ylab="mbar", type="l")

# add some date labels using identify()
# identify(USA.tfr, USA.mbar, names(USA.tfr))

tapply(USA$ASFR * seq(12.5, 55.5, by=1), USA$Cohort, sum) -> y
y/USA.cfs -> USA.mbar.cohort
rm(y)
# We only want the cohorts from 1920 to 1980.
# Those are the cohorts 43 through 103

USA.cfs <- USA.cfs[43:103]
USA.mbar.cohort <- USA.mbar.cohort[43:103]

plot(USA.tfr, USA.mbar.period, las=1, xlab="TFR or CFS", ylab="mbar", xlim=c(1.6, 3.7), ylim=c(24,30), type="l")
lines(USA.cfs, USA.mbar.cohort, col=2)

# Didn't expect that
```