

Submodular Risk Allocation¹

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October 24, 2017

¹The opinions expressed in this presentation do not necessarily reflect the views of the U.S. Department of the Treasury, Office of Financial Research, or their staff.

We study the problem of allocating transactions or other individual sources of risk to portfolios in order to minimize a sum of risk-based costs for the portfolios.

Our investigation is motivated by changes in the over-the-counter (OTC) derivatives markets. Prior to the financial crisis, the market for OTC derivatives was largely unregulated, and it operated as a diffuse network of bilateral contracts between market participants. In 2009, regulatory authorities from the G-20 countries agreed to reforms that have reshaped the market.

The reform program seeks to reduce the systemic risk from OTC derivatives. Our study is motivated by two main elements of the reform program:

- ▶ All standardised OTC derivatives should be cleared through central counterparties (CCPs).
- ▶ Non-centrally-cleared derivatives should be subject to higher capital and margin requirements.

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- ▶ Non-centrally-cleared derivatives should be subject to higher capital and margin requirements.

The second of these elements was designed and implemented to create a cost incentive in favor of central clearing.

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- ▶ The collateral cost associated with each portfolio increases with the risk of the portfolio. A dealer will often have some flexibility in choosing the CCP through which to clear a contract or deciding to trade the contract bilaterally.
- ▶ To minimize its total collateral costs, the dealer needs to allocate trades to portfolios in a way that minimizes the sum of risk-based costs over the portfolios.

Consider a dealer with a set of trades S with a counterparty. Suppose the dealer is limited to two trading channels, for example, one CCP and the bilateral market.

For any $A \subseteq S$, let $F(A)$ and $G(S \setminus A)$ denote the collateral costs associated with assigning portfolio A to the first channel and $S \setminus A$ to the other channel.

The dealer's optimization problem is then

$$\min_{A \subseteq S} \{F(A) + G(S \setminus A)\}. \quad (1)$$

Regulators have sought to increase bilateral margin requirements to incentivize greater use of central clearing, i.e. increasing costs under G to encourage allocation to F .

See Duffie et al. (2015), Ghamami and Glasserman (2017), and the references therein for details on OTC derivatives market reforms and empirical examinations of their impact.

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- ▶ The cost associated with a portfolio is proportional to its risk. So, F and G represent measures of portfolio risk.
- ▶ We interpret submodularity as a strong version of the notion that diversification reduces risk. *Under submodularity, the marginal change in risk from adding an asset to a portfolio decreases with the addition of another asset.*

- ▶ We first investigate the submodularity of portfolio risk measures with emphasis on standard deviation, viewed as set functions defined over a finite set of assets. Despite the vast literature on properties of risk measures growing out of Artzner et al. (1999), the submodular case has received little prior attention.

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- ▶ Under submodularity, problem (1) leads to several important properties studied by Edmonds (1970), Fujishige (2005) and Schrijver (2003). *These results provide a dual characterization of (1) through which an individual cost may be attributed to each trade for each of the two portfolios. This representation provides the dealer with a margin attribution for each trade under an optimal allocation.*

- ▶ When two or more dealers seek to allocate overlapping sets of trades, their optimal allocations may conflict. Differences in their cost attributions characterize payments between dealers that would reconcile their conflicting allocations. Our framework provides a rigorous mechanism for the ad hoc practice of decomposing portfolio-level “valuation adjustments” into trade-level adjustments.

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- ▶ We also compare systemwide costs in a market with multiple dealers having potentially conflicting allocation preferences for their shared trades. We compare the sum of individually optimal costs, the optimal systemwide cost, and costs under a sequential protocol. *We use the structure of cost attribution vectors to bound cost differences across these scenarios.*

- ▶ A dealer has a fixed set of trades to allocate, represented by a set of jointly distributed random variables $S = \{X_1, X_2, \dots, X_N\}$. Interpret X_i as the change in value of a derivatives contract over a period of 1-10 days. Assume that every X_i has mean zero.

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- ▶ A margin function F assigns a margin requirement $F(A)$ to any subset $A \subseteq S$ of trades. F is submodular if

$$F(A \cap B) + F(A \cup B) \leq F(A) + F(B), \quad \forall A, B \subseteq S.$$

Or, equivalently

$$F(A \cup \{i, j\}) - F(A \cup \{i\}) \leq F(A \cup \{j\}) - F(A), \quad \forall i, j \in S \setminus A, i \neq j. \quad (2)$$

SUBMODULARITY OF STANDARD DEVIATION

For any $A \subseteq S$, let

$$\sigma(A) = \text{Standard Deviation} \left(\sum_{i \in A} X_i \right).$$

Identify $A \subseteq S$ with the vector $x \in \{0, 1\}^N$ satisfying $x_i = 1$ if $i \in A$ and $x_i = 0$ if $i \notin A$. This vector is denoted by x_A . We write

$$\sigma(A) = \sqrt{x_A^\top \Sigma x_A} \quad \text{and} \quad \sigma(x) = \sqrt{x^\top \Sigma x},$$

where Σ denote the covariance matrix of the trades X_1, \dots, X_N . An individual element $i \in S$ is identified with the unit vector e_i . With this notation, condition (2) for submodularity becomes

$$\sigma(x + e_i + e_j) - \sigma(x + e_i) \leq \sigma(x + e_j) - \sigma(x), \quad \forall x, x + e_i + e_j \in \{0, 1\}^N.$$

Write D_σ for the diagonal matrix with the σ_i on the diagonal to represent Σ as

$$\Sigma = D_\sigma R D_\sigma,$$

where R is a correlation matrix. If Σ is positive definite, then R is uniquely determined. For $0 \leq \lambda \leq 1$, let

$$\Sigma_\lambda = D_\sigma(\lambda R + (1 - \lambda)I)D_\sigma,$$

where I is the $N \times N$ identity matrix.

PROPOSITION

Let Σ be a covariance matrix.

- (I) *If Σ is diagonal, then Σ is submodular.*
- (II) *If R is the matrix of all 1s, then Σ is submodular.*
- (III) *For any correlation matrix R , there is a $\lambda' > 0$ such that Σ_λ is submodular for all $0 \leq \lambda < \lambda'$.*

We treat $\sigma(\cdot)$ as a function on the vertices of the unit hypercube. Such a function can be submodular even if its natural extension to the interior of the unit hypercube is not.

This distinction is important in light of the following result.

PROPOSITION

Suppose Σ is positive definite, and let $\sigma(w) = \sqrt{w^\top \Sigma w}$, for all $w \in [0, 1]^N$. Then σ is submodular throughout $[0, 1]^N$ if and only if either Σ is diagonal or $N = 2$.

A sufficient condition for submodularity of σ can be formulated through submodularity and monotonicity of variance. Write σ_{ij} for the ij -entry of Σ

PROPOSITION

Suppose Σ satisfies the following two conditions:

- (I) $\sigma_{ij} \leq 0$, for all distinct $i, j = 1, \dots, N$;
- (II) $\sigma_i^2 \geq -2 \sum_{j \neq i} \sigma_{ij}$, for all $i = 1, \dots, N$.

Then Σ is submodular, and $\sigma(\cdot)$ is monotone increasing on $\{0, 1\}^N$.

Without the factor of 2, and under the sign restriction in (i), condition (ii) would be the familiar diagonal dominance condition for positive semidefiniteness of Σ . In particular, a matrix that satisfies (i) and (ii) is an M -matrix.

SUBMODULARITY OF σ : EXCHANGEABILITY

The random variables X_1, \dots, X_N are called exchangeable if their joint distribution is invariant under permutations of the random variables. The covariance matrix of exchangeable random variables takes the form

$$\Sigma = \begin{pmatrix} \sigma^2 & \sigma^2\rho & \cdots & \sigma^2\rho \\ \sigma^2\rho & \sigma^2 & & \sigma^2\rho \\ \vdots & & \ddots & \vdots \\ \sigma^2\rho & \sigma^2\rho & \cdots & \sigma^2 \end{pmatrix}, \quad -1/(N-1) \leq \rho \leq 1,$$

and Σ is positive definite if the bounds on ρ are strict.

PROPOSITION

The covariance matrix of exchangeable random variables is submodular.

PROPOSITION

(i) Suppose that for some $v \in \mathbb{R}_{++}^N$ and some $a \in \mathbb{R}$,

$$\Sigma = \text{diag}(v) + avv^\top.$$

If Σ is positive semidefinite then it is submodular. (ii) Suppose that for some $v, w \in \mathbb{R}_+^N$ and some $a, b \geq 0$

$$\Sigma = \text{diag}(v + w) + avv^\top + bww^\top.$$

If

$$4a(|w| + b|w|^2) \leq 1 \quad \text{and} \quad 4b(|v| + a|v|^2) \leq 1,$$

then Σ is submodular. $|v|$ denotes the sum of absolute values of the entries of v .

The exchangeable case becomes a special case of this result (for $\rho \neq 1$) by taking $a = \rho/(1 - \rho)^2\sigma^2$ and $v_i = (1 - \rho)\sigma^2$, $i = 1, \dots, N$.

Proof of (i). Given $\Sigma = \text{diag}(v) + avv^\top$ for $v \in \mathbb{R}_{++}^N$ and $a \in \mathbb{R}$, we take advantage of the fact that *we require submodularity only on the vertices of the hypercube*, i.e. for $x \in \{0, 1\}^N$,

$$\sigma^2(x) = \sum_i v_i x_i^2 + a(v^\top x)^2 = (v^\top x) + a(v^\top x)^2,$$

so σ depends on x only through $v^\top x$.

Write $\sigma(x) = g(v^\top x)$, where $g(s) = \sqrt{s + as^2}$ is concave on $[0, \infty)$. If $x + e_i + e_j \in \{0, 1\}^N$, $i \neq j$, let $s_1 = v^\top x$, $s_2 = v^\top(x + e_i)$, $s_3 = v^\top(x + e_j)$, and $s_4 = v^\top(x + e_i + e_j)$. Suppose $s_2 \leq s_3$. As $v_i, v_j \geq 0$, we have $s_1 \leq s_2$ and $s_3 \leq s_4$, and we also have $s_2 - s_1 = s_4 - s_3$. Concavity of g therefore implies that $g(s_4) - g(s_3) \leq g(s_2) - g(s_1)$.

Submodularity is also preserved by diagonal deviations from exchangeability that satisfy the bound in the following result.

PROPOSITION

For $\xi \in \mathbb{R}_+^N$, let $\Sigma_\xi = \text{diag}(\xi) + \Sigma$, where Σ denotes the covariance matrix of exchangeable random variables defined earlier. Then Σ_ξ is submodular if either (i) $-1/(2N-1) \leq \rho \leq 0$ or (ii) $0 < \rho < 1$ and

$$|\xi| \equiv \sum_{i=1}^N \xi_i \leq \frac{\sigma^2}{4} \frac{(1-\rho)^2}{\rho}.$$

A function f on the lattice of subsets 2^S is called *normalized* if $f(\emptyset) = 0$. Assume that F and G are normalized and nonnegative. For a normalized submodular function f , define the *submodular polyhedron*

$$P(f) = \{x \in \mathbb{R}^N : \sum_{i \in A} x_i \leq f(A), \forall A \subseteq S\},$$

recalling that $N = |S|$, and its *base polyhedron*,

$$B(f) = \{x \in P(f) : \sum_{i \in S} x_i = f(S)\}.$$

PROPOSITION (EDMONDS (1970), FUJISHIGE (2005), AND SCHRIJVER (2003).)

If F and G are normalized and submodular, then

$$\min_{A \subseteq S} \{F(A) + G(S \setminus A)\} = \max \left\{ \sum_{i=1}^N x_i \wedge y_i : x \in B(F), y \in B(G) \right\}. \quad (3)$$

If x and y solve the problem on the right, then $A_0 = \{i : x_i < y_i\}$ and $A_1 = \{i : x_i \leq y_i\}$ are optimal for the problem on the left.

The literature on convex games deals with the problem of allocating a supermodular value function among multiple players, which can be recast as a problem of allocating submodular costs.

In most applications of convex games, cost decomposition is the main objective. In our setting, it is an intermediate step in characterizing the optimal allocation of trades to portfolios.

We interpret the previous Proposition as follow. *The dealer chooses an attribution $x \in B(F)$ and an attribution $y \in B(G)$. Each trade i is allocated to the channel for which its attributed charge (x_i or y_i) is smaller. A rule that maximizes the total attributed margin charge $\sum_i x_i \wedge y_i$ is optimal.*

A core attribution can be constructed as follows. Consider any permutation i_1, \dots, i_N of the trades, and allocate to each trade its incremental margin charge under this permutation:

$$x_{i_1} = F(\{i_1\}), x_{i_2} = F(\{i_1, i_2\}) - F(\{i_1\}), \dots, x_{i_N} = F(S) - F(S \setminus \{i_N\}) \quad (4)$$

Under this rule, the sum of x_i telescopes, so the full amount $F(S)$ is decomposed.

Shapley (1971) showed that this attribution is in the core; in fact, x is an extreme point of the convex set $B(F)$, and all extreme points of $B(F)$ are of this form. Taking the equally weighted average over all attributions (4) yields the Shapley value.

Suppose that S consists of all trades between two dealers. To focus on differences in credit quality, suppose that dealers face costs

$$h_1(A) = F(A) + G_1(S \setminus A), \quad h_\theta(A) = F(A) + G_\theta(S \setminus A),$$

for some normalized submodular functions G_1 and G_θ on 2^S .

Suppose that the *strong map* relation $G_\theta \rightarrow G_1$ holds, i.e. for all $A \subseteq B \subseteq S$,

$$G_\theta(B) - G_\theta(A) \geq G_1(B) - G_1(A). \quad (5)$$

$G_\theta \rightarrow G_1$ implies $h_1 \rightarrow h_\theta$.

Suppose A_1 minimizes h_1 and A_θ minimizes h_θ . Then $A_1 \cap A_\theta$ also minimizes h_1 , and $A_1 \cup A_\theta$ also minimizes h_θ . In particular, if either h_1 or h_θ has a unique minimizer, then $A_1 \subseteq A_\theta$. These statements follow from Topkis (1998).

Suppose that $G_\theta = \theta G_1$, which satisfies (5) for $\theta \geq 1$ if G is monotone increasing. It follows from Fujishige and Nagano (2009) that there exist $x \in B(F)$ and $y \in B(G_1)$ such that

$$\min_{A \subseteq S} \{F(A) + \theta G_1(S \setminus A)\} = \sum_{i=1}^N \min(x_i, \theta y_i), \quad \text{for all } \theta \geq 1. \quad (6)$$

Dealer θ associates a collateral cost of θy_i for trading bilaterally, whereas dealer 1 associates a collateral cost of y_i for the same trade.

The difference $(\theta - 1)y_i$ provides a rigorous foundation for computing CVA. The decomposition in (6) provides a mechanism for computing trade-level CVA consistent with an overall portfolio of trades, and consistent for both parties.

Consider a market with K dealers and a set of trades $S = \{1, \dots, N\}$ among these dealers. Denote by A_k , $k = 1, \dots, K$, the set of trades in which dealer k participates. Write F_k and G_k for dealer k 's margin functions, defined on all subsets of A_k . Extend these functions to all subsets of S by setting, for any $B \subseteq S$,

$$F_k(B) = F_k(B \cap A_k), \quad G_k(B) = G_k(B \cap A_k).$$

Assume that F_k and G_k are normalized and submodular on 2^{A_k} , and the same therefore holds for their extensions to 2^S . The functions

$$F = \sum_{k=1}^K F_k, \quad G = \sum_{k=1}^K G_k$$

are then also normalized and submodular.

If dealer k could make its allocation decision in isolation, it would incur an optimal cost of

$$c_k = \min_{A \subseteq A_k} F_k(A) + G_k(A_k \setminus A) = \max_{x^k \in B(F_k), y^k \in B(G_k)} \sum_{i \in A_k} x_i^k \wedge y_i^k. \quad (7)$$

Write c_{tot} for the sum $c_1 + \dots + c_K$ of these individual costs.

The systemwide optimum solves

$$c_{sys} = \min_{A \subseteq S} F(A) + G(S \setminus A).$$

LEMMA

The systemwide optimal cost satisfies

$$c_{sys} = \max \left\{ \sum_{i=1}^N \left(\sum_{k=1}^K x_i^k \right) \wedge \left(\sum_{k=1}^K y_i^k \right) : x^k \in B(F_k), y^k \in B(G_k) \right\}. \quad (8)$$

ATTRIBUTING THE OPTIMAL SYSTEMWIDE COST TO INDIVIDUAL TRADES

The dealers can cooperate to achieve c_{sys} by submitting their trade sets and margin functions to a central planner. The planner announces charges

$$\frac{1}{2} \sum_{k=1}^K x_i^k \quad \text{and} \quad \frac{1}{2} \sum_{k=1}^K y_i^k$$

for clearing or not clearing trade i . Based on these charges, each dealer makes its own decisions and contributes the corresponding margin charge. Under this mechanism, each dealer would make the systemwide optimal allocation.

In the related setting of *trade compression*, dealers cooperate by submitting information about their derivatives positions to a third party, which finds cycles in the network of contracts that can be eliminated without changing dealers' net positions.

Define

$$D(F, G) = \min \left\{ \frac{1}{2} \sum_{i=1}^N \sum_{k=1}^K |x_i^k - y_i^k| : x \in B(F), y \in B(G) \right\}$$

as a measure of deviation between the cost functions F and G .

PROPOSITION

The systemwide optimal cost satisfies

$$c_{sys} \leq c_{tot} + D(F, G).$$

We interpret $D(F, G)$ as the potential additional cost, over all trades and dealers, resulting from deviations from the systemwide cost attributions $x \in B(F)$ and $y \in B(G)$.

“SEQUENTIAL” OPTIMIZATION

Let $C_k = A_1 \cup \dots \cup A_k$ denote the cumulative set of trades in which the first k dealers participate.

The first dealer clears a set $B_1 \subseteq A_1$ of trades and incurs a cost $\bar{c}_1 = c_1$. Once dealer k has made its allocation decision, $k = 1, \dots, K - 2$, dealer $k + 1$ solves

$$\begin{aligned} \bar{c}_{k+1} = & \min_{B_{k+1} \subseteq A_{k+1} \setminus C_k} F_{k+1}(B_{k+1} \cup B_k \cup \dots \cup B_1) \\ & + G_{k+1}((C_k \cup A_{k+1}) \setminus (B_{k+1} \cup B_k \cup \dots \cup B_1)). \end{aligned}$$

The process terminates at the first k_o for which $C_{k_o} = S$, which occurs at $k_o \leq K - 1$. If $k \geq k_o$, then no trades remain to be allocated by dealer $k + 1$, and \bar{c}_{k+1} is evaluated with $B_{k+1} = \emptyset$.

The total cost under this protocol is $c_{seq} = \bar{c}_1 + \dots + \bar{c}_K$.

“SEQUENTIAL” OPTIMIZATION

For any $C \subseteq S$ and $k = 1, \dots, K$, define

$$\delta_k(C) = \max \left\{ \sum_{i \in C \cap A_k} |x_i^k - y_i^k| : x^k \in B(F_k), y^k \in B(G_k) \right\}.$$

We will use the δ_k to bound c_{seq} . The allocation of trades in C is constrained and cannot be chosen by dealer k . If x_i and y_i measure cost attributions for trades $i \in C$, then the additional cost faced by the dealer as a result of having the allocation of i fixed should be bounded by $|x_i - y_i|$.

PROPOSITION

The systemwide optimal cost satisfies

$$c_{seq} \leq c_{tot} + \sum_{k=2}^K \delta_k(C_{k-1}).$$

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- ▶ We have analyzed conflicting allocation decisions by the parties to a set of trades. Optimal attribution vectors yield trade-specific valuation adjustments to reconcile conflicting preferences between two parties.
- ▶ We also analyzed total systemwide costs in a market with multiple dealers. We have compared decentralized costs, systemwide optimal costs, and costs under a sequential protocol.

For computational purposes, one can cast

$$\min_{A \subseteq S} \{F(A) + G(S \setminus A)\}$$

as a submodular minimization problem. If G is submodular on 2^S , then so is the function defined by $A \mapsto G(S \setminus A)$. Submodularity is also preserved by addition, so H defined by

$H(A) = [F(A) + G(S \setminus A)] - G(S)$ is submodular. Subtracting $G(S)$ in this definition normalizes H to satisfy $H(\emptyset) = 0$. Solving the above problem is equivalent to minimizing the submodular function H over $A \subseteq S$.

Strongly polynomial algorithms for submodular function minimization are discussed in Fujishige (2005) and Schrijver (2003).

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