The Temporal Dimension of Drawdown

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ABSTRACT. Multi-period measures of risk account for the path that the value of an investment portfolio takes. The most widely used such path-dependent indicator of risk is drawdown, which is a measure of decline from a historical peak in cumulative returns. In the context of probabilistic risk measures, the focus has been on one particular dimension of drawdown, its magnitude, and not on its temporal dimension, its duration. In this paper, the concept of temporal path-dependent risk measure is introduced to capture the risk associated with the temporal dimension of a stochastic process. We analyze drawdown duration, which measures the length of excursions below a running maximum, and liquidation stopping time, which denotes the first time drawdown duration exceeds a subjective liquidation threshold, in the context of coherent temporal path-dependent risk measures and show that they, unlike drawdown magnitude, do not satisfy any of the axioms for coherent risk measures. Despite its non-coherence, we illustrate through an empirical example some of the insights gained from analyzing drawdown duration in the investment process and discuss the challenges of path-dependent risk estimation in practice.

1. INTRODUCTION

Single-period measures of risk do not account for the path an investment portfolio takes. Since investment funds do not hold static positions, measuring the risk of investments should ideally be defined over random paths rather than random single-period gains or losses. Mathematically, a path-dependent measure of risk is a real valued function $\rho : \mathcal{R}^\infty \to \mathbb{R}$ on the space of stochastic processes $\mathcal{R}^\infty$ representing cumulative returns over a path of fixed length. Most existing path-dependent risk measures are essentially a measure of the magnitude of investment loss or gain. However, by moving from the single-period to the multi-period framework, a second dimension becomes manifest, namely that of time. This temporal dimension to a stochastic process has traditionally not been incorporated into the probabilistic theory of risk measures pioneered by Artzner et al. (1999).

For a given time horizon $T \in (0, \infty)$, we define a *temporal risk measure* to be a path-dependent risk measure $\rho : \mathcal{R}^\infty \to \mathbb{R}$, which first maps a stochastic process $X \in \mathcal{R}^\infty$ to a random time τ, that is a random variable taking values in the time interval $[0, T]$. This random variable τ is meant to summarize a certain temporal behavior of the process X that we are interested in. Then, a real-valued risk functional, such as deviation or tail mean, is applied to τ, describing a feature of its distribution.

In this paper, our focus is on one of the most widely quoted indicators of multi-period risk: drawdown, which is the decline from a historical peak in net asset value or cumulative return. In the event of a large drawdown, conventional single-period risk diagnostics, such as volatility or Expected Shortfall, are irrelevant and liquidation under unfavorable market conditions after

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an abrupt market decline may be forced. Since the notion of drawdown inherently accounts for the path over a given time period, it comes equipped with two dimensions: a size dimension (drawdown magnitude) and a temporal dimension (drawdown duration).

While the magnitude component of drawdown has been extensively studied in the academic literature and is regularly used by the investment community, the temporal dimension, its duration, has not received the same kind of attention. In particular, even though it is a widely quoted performance measure, a generally accepted mathematical methodology for forming expectations about future duration does not seem to exits in practice.

Our purpose is to analyze the properties of the temporal dimension of drawdown in the context of coherent measures of risk developed by Artzner et al. (1999), and to study its practicality in terms of usability in the investment process. To this end, we formulate the temporal dimension of drawdown as a temporal risk measure $\rho : \mathcal{R}^\infty \to \mathbb{R}$, which essentially yields a methodology for forming expectations about future potential temporal risk. We then analyze the coherency properties of drawdown duration, which measures the length of excursions below a running maximum, and liquidation stopping time, which denotes the first time drawdown duration exceeds a subjective liquidation threshold, in the context of temporal path-dependent risk measures.

We show that, unlike the magnitude of drawdown, its duration and stopping time do not satisfy any of the axioms for coherent risk measures. In practice, non-coherence implies, amongst others, that linear attribution to temporal risk is not supported, and that convex minimization of temporal risk is not applicable. Despite this limited use, we argue, however, that temporal risk should not be ignored in the risk management process of investment funds; it encapsulates a good diagnostic measure of a dimension of risk that is paramount but traditionally not incorporated in the risk management process. To support this viewpoint, we include an empirical study showing that duration (i) is not necessarily correlated to conventional risk metrics, (ii) captures temporal dependence in asset returns, and (iii) is strongly related to the magnitude dimension of drawdown risk.

1.1. **Synopsis.** We start in Section 2 by reviewing the probabilistic theory of single-period and path-dependent risk measures in a continuous-time setting. We then introduce the notion of temporal path-dependent risk, which measures the risk of the temporal dimension of a stochastic process over a path of finite length. The risk associated with drawdown magnitudes is reviewed in Section 3. We generalize the discrete-time setup of Goldberg and Mahmoud (2014) to the continuous-time framework and recall some of the properties which make drawdown amenable to the investment process. A measure of drawdown risk aversion is also introduced. Section 4 includes the analysis of duration as a temporal path-dependent risk measure. Duration captures the time it takes a stochastic process to reach a previous running maximum for the first time. We prove that duration risk measures are not coherent in the sense of Artzner et al. (1999). We then analyze liquidation stopping time (LST) as a temporal risk measure in Section 5. LST denotes the first time that drawdown duration exceeds a subjectively set liquidation threshold. Again, it is shown that LST risk is not coherent. Section 6 includes an analysis of temporal path-dependent risk in practice. We discuss the practical impact of non-coherence, empirically study the distribution of drawdown duration, and discuss the challenges of path-dependent risk estimation in practice. We conclude in Section 7 with a summary of our findings.

1.2. **Background literature.** We summarize work related to the probabilistic theory of path-dependent risk measures, and to the theoretical and practical analysis of drawdown magnitude and duration.

1.2.1. *Path-dependent risk measures.* The seminal work of Artzner et al. (1999) introducing coherent risk measures is centered around single-period risk, where risk is measured at the beginning of the period and random loss or gain is observed at the end of the period. In Artzner et al. (2002, 2007), the framework of Artzner et al. (1999) is generalized to discrete-time multi-period models, and in Cheridito et al. (2004, 2005) representation results for coherent and convex risk measures were developed for continuous-time stochastic models. Riedel (2004) defines the concept of dynamic risk measure, where dynamic risk assessment consists of a sequence of risk mappings and is updated as time evolves to incorporate new information. Such measures come equipped with a notion of dynamic consistency, which requires that judgements based on the risk measure are not contradictory over time (see also Bion-Nadal (2008, 2009) and Fasen and Svejda (2012)). Dynamic risk measures have been studied extensively over the past decade; see Föllmer and Penner (2006), Cheridito et al. (2006), Klöppel and Schweizer (2007), and Fritelli and Rosazza Gianin (2004), amongst others.

We point out that the focus of the studies mentioned above is on the magnitude of losses and gains and not on the temporal behavior of the underlying process. To our knowledge, the notion of path-dependent risk measure capturing the temporal dimension of risk has not been formally developed in the academic literature.

1.2.2. *Drawdown magnitude.* The analytical assessment of drawdown magnitudes has been broadly studied in the literature of applied probability theory. To our knowledge, the earliest use of the Laplace transform on the maximum drawdown of a Brownian motion appeared in Taylor (1975), and it was shortly afterwards generalized to time-homogenous diffusion processes by Lehoczky (1977). Douady et al. (2000) and Magdon-Ismail et al. (2004) derive an infinite series expansion for a standard Brownian motion and a Brownian motion with a drift, respectively. The discussion of drawdown magnitude was extended to studying the frequency rate of drawdown for a Brownian motion in Landriault et al. (2015b). Drawdowns of spectrally negative Lévy processes were analyzed in Mijatovic and Pistorius (2012). The notion of drawup, the dual of drawdown measuring the maximum cumulative gain relative to a running minimum, has also been investigated probabilistically, particularly in terms of its relationship to drawdown. For example, Zhang and Hadjiliadis (2010) derive the probability that a drawup precedes a drawdown of equal units in a random walk and a Brownian motion model and discuss applications in risk management and finance. See also Hadjiliadis and Vecer (2006) and Pospisil et al. (2009) for more studies of drawup and drawdown.

Reduction of drawdown in active portfolio management has received considerable attention in mathematical finance research. Grossman and Zhou (1993) considered an asset allocation problem subject to drawdown constraints; Cvitanic and Karatzas (1995) extended the same optimization problem to the multi-variate framework; Chekhlov et al. (2003, 2005) developed a linear programming algorithm for a sample optimization of portfolio expected return subject to constraints on their drawdown risk measure CDaR, which, in Krokhmal et al. (2003), was numerically compared to shortfall optimzation with applications to hedge funds in mind; Carr et al. (2011) introduced a new European style drawdown insurance contract and derivative-based drawdown hedging strategies; and most recently Cherney and Obloj (2013), Sekine (2013), Zhang et al. (2013) and Zhang (2015) studied drawdown optimization and drawdown insurance under various stochastic modeling assumptions. Zabarankin et al. (2014) reformulated the necessary optimality conditions for a portfolio optimization problem with drawdown

in the form of the Capital Asset Pricing Model (CAPM), which is used to derive a notion of drawdown beta. More measures of sensitivity to drawdown risk were introduced in terms of a class of drawdown Greeks in Pospisil and Vecer (2010).

In the context of probabilistic risk measurement, Chekhlov et al. (2003, 2005) develop a quantitative measure of drawdown risk called Conditional Drawdown at Risk (CDaR), and Goldberg and Mahmoud (2014) develop a measure of maximum drawdown risk called Conditional Expected Drawdown (CED). Both risk measures, CDaR and CED, are deviation measures (Rockafellar et al. (2002, 2006)).

1.2.3. *Drawdown duration.* The notion of drawdown duration has not been previously studied in the context of coherent risk measures. However, it has been considered in terms of its probabilistic properties. In Zhang and Hadjiliadis (2012), the probabilistic behavior of drawdown duration is analyzed and the joint Laplace transform of the last visit time of the maximum of a process preceding the drawdown, its, and the maximum of the process under general diffusion dynamics is derived. More recently, Landriault et al. (2015a) consider derive the duration of drawdowns for a large class of Levy processes and find that the law of duration of drawdowns qualitatively depends on the path type of the spectrally negative component of the underlying Levy process.

2. Path-Dependent Risk Measures

2.1. Single-period risk. In classical risk assessment, uncertain portfolio outcomes over a fixed time horizon are represented as random variables on a probability space. A risk measure maps each random variable to a real number summarizing the overall position in risky assets of a portfolio. Formally, for the probability space $(\Omega, \mathcal{F}, P)$, let $L^0(\Omega, \mathcal{F}, P)$ be the set of all random variables on $(\Omega, \mathcal{F})$. A *risk measure* is a real-valued function $\rho \colon \mathcal{M} \to \mathbb{R}$, where $\mathcal{M}$ is a convex cone.

One of the most widely used measures of single-period risk is volatility, or the standard deviation of portfolio return, which was introduced in Markowitz (1952). However, Markowitz himself was not satisfied with volatility, since it penalizes gains and losses equally, and he proposed semideviation, which penalizes only losses, as an alternative. Over the past two decades, risk measures that focus on losses, such as Value-at-Risk (VaR) and Expected Shortfall (ES), have increased in popularity, both in the context of regulatory risk reporting and in downside-safe portfolio construction. An axiomatic approach to (loss-based) risk measures was initiated by the landmark research of Artzner et al. (1999). They specified a number of properties that a good risk measure should have, with particular focus on applications in financial risk management. Their main focus is the class of *monetary* such measures, which can translate into capital requirement, hence making risk directly useful to regulators. Here, the risk $\rho(L)$ of a financial position L is interpreted as the minimal amount of capital that should be added to the portfolio positions (and invested in a risk-free manner) in order to make them acceptable:

Definition 2.1 (Monetary Risk Measure). A risk measure $\rho : \mathcal{M} \to \mathbb{R}$ is called *monetary* if it satisfies the following two axioms:

  (A1) Translation invariance: For all $L \in \mathcal{M}$ and all constant almost surely $C \in \mathcal{M}$,
      $\rho(L + C) = \rho(X) - C$.

(A2) Monotonicity: For all $L_1, L_2 \in \mathcal{M}$ such that $L_1 \leq L_2$, $\rho(L_1) \geq \rho(L_2)$.[1]

A monetary risk measure is *coherent* if it is convex and positive homogenous:

Definition 2.2 (Coherent Risk Measure). A risk measure $\rho : \mathcal{M} \to \mathbb{R}$ is called *coherent* if it is monetary and satisfies the following two axioms:

(A3) Convexity: For all $L_1, L_2 \in \mathcal{M}$ and $\lambda \in [0,1]$, $\rho(\lambda L_1 + (1-\lambda)L_2) \leq \lambda \rho(L_1) + (1-\lambda)\rho(L_2)$.

(A4) Positive homogeneity: For all $L \in \mathcal{M}$ and $\lambda > 0$, $\rho(\lambda L) = \lambda \rho(L)$.

Since coherent measures of risk were introduced by Artzner et al. (1999), several other classes of risk measures were proposed, most notably *convex* measures[2] (Föllmer and Schied (2002, 2010, 2011)) and *deviation* measures (Rockafellar et al. (2002, 2006)).

2.2. **Continuous-time path-dependent risk.** Single-period measures of risk do not account for the return path an investment portfolio takes. Since investment funds do not hold static positions, measuring the risk of investments should ideally be defined over random processes rather than random variables. Indeed, the most widely used and quoted such measure of multi-period risk is drawdown, which is inherently path-dependent.

Mathematically, path-dependent risk measures are defined over return paths, formalized via continuous-time stochastic processes, rather than single-period losses or gains. We use the general setup of Cheridito et al. (2004) for continuous-time path dependent risk. Continuous-time cumulative returns, or equivalently net asset value processes, are represented by essentially bounded càdlàg processes (in the given probability measure) that are adapted to the filtration of a filtered probability space. More formally, for a time horizon $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$ be a filtered probability space satisfying the usual assumptions, that is the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, $(\mathcal{F}_t)$ is right-continuous, and $\mathcal{F}_0$ contains all null-sets of $\mathcal{F}$. For $p \in [1, \infty]$, $(\mathcal{F}_t)$-adapted càdlàg processes lie in the Banach space

$$\mathcal{R}^p = \{X : [0,T] \times \Omega \to \mathbb{R} \mid X \ (\mathcal{F}_t)\text{-adapted càdlàg process }, \|X\|_{\mathcal{R}^p}\},$$

which comes equipped with the norm

$$\|X\|_{\mathcal{R}^p} := \|X^*\|_p$$

where $X^* = \sup_{t \in [0,T]} |X_t|$.

All equalities and inequalities between processes are understood throughout in the almost sure sense with respect to the probability measure $\mathbb{P}$. For example, for processes X and Y, $X \leq Y$ means that for $\mathbb{P}$-almost all $\omega \in \Omega$, $X_t(\omega) \leq Y_t(\omega)$ for all t.

Definition 2.3 (Continuous-time path-dependent risk measure). A continuous-time *path-dependent risk measure* is a real-valued function $\rho : \mathcal{R}^\infty \to \mathbb{R}$.

Analogous to single period risk, a path-dependent risk measure $\rho : \mathcal{R}^\infty \to \mathbb{R}$ is *monetary* if it satisfies the following axioms:

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[1] All equalities and inequalities between random variables and processes are understood in the almost sure sense with respect to the probability measure P. For example, for processes X_T and Y_T, $X_T \leq Y_T$ means that for P-almost all $\omega \in \Omega$, $X_t(\omega) \leq Y_t(\omega)$ for all $t \in T$.

[2] In the larger class of convex risk measures, the conditions of subadditivity and positive homogeneity are relaxed. The positive homogeneity axiom, in particular, has received some criticism since its introduction. For example, it has been suggested that for large values of the multiplier λ concentration risk should be penalized by enforcing $\rho(\lambda X) > \lambda \rho(X)$.

- Translation invariance: For all $X \in \mathcal{R}^\infty$ and all constant almost surely $C \in \mathcal{R}^\infty$, $\rho(X + C) = \rho(X) - C$.
- Monotonicity: For all $X, Y \in \mathcal{R}^\infty$ such that $X \leq Y$, $\rho(X) \leq \rho(Y)$.

It is positive homogenous of degree one if for all $X \in \mathcal{R}^\infty$ and $\lambda > 0$, $\rho(\lambda X) = \lambda \rho(X)$; and it is convex if for all $X, Y \in \mathcal{R}^\infty$ and $\lambda \in [0, 1]$, $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y)$. A monetary (path-dependent) risk measure that is both positive homogenous and convex is called *coherent*.

To every path-dependent risk measure $\rho : \mathcal{R}^\infty \to \mathbb{R}$ we associate its risk acceptance family $\mathcal{A}_\rho$, a concept introduced by Drapeau and Kupper (2013) for single-period risk measures, generalizing the notion of risk acceptance set of Artzner et al. (1999). For any *risk level* $m \in \mathbb{R}$, the *risk acceptance set of level m* is the subset $\mathcal{A}_\rho^m \subseteq \mathcal{R}^\infty$ of those paths that have a risk smaller than m. More formally,

$$\mathcal{A}_\rho^m = \{X \in \mathcal{R}^\infty : \rho(X) \leq m\},$$

and $\mathcal{A}_\rho = (\mathcal{A}_\rho^m)_{m \in \mathbb{R}}$ is the *risk acceptance family* corresponding to ρ. The notion of risk acceptance family is a major instrument for robust representation results of risk measures, and is often used to derive structural properties of risk measures and to model certain economic principles of risk.

2.3. **Path-dependent temporal risk measures.** Most existing path-dependent risk measures, such as drawdown, are in essence a measure of the magnitude of investment loss or gain. One may, however, be interested in the risk associated with the temporal dimension of the underlying stochastic process. For example in the fund management industry, historical values of the time it takes to regain a previous maximum ("peak-to-peak"), or the length of time between a previous maximum and a current low ("peak-to-trough") are frequently quoted alongside drawdown values. However, a generally accepted mathematical methodology for forming expectations about future potential such temporal risks does not seem to exist. We will later study one such path-dependent risk measure, duration, as a *temporal risk measure*.[3]

Fix a time horizon $T \in (0, \infty)$. Given a stochastic process $X \in \mathcal{R}^\infty$, a *random time* τ is a random variable on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as X, taking values in the time interval $[0, T]$. We say that X_τ denotes the state of the process X at random time τ. Random times can be thought of as elements of the space $\mathcal{V}_T \subset L_0(\Omega, \mathcal{F}, \mathbb{P})$ of real-valued random variables $\tau : \Omega \to [0, T]$.

In the probabilistic study of stochastic processes, typical examples of random time include the *hitting time*, which is the first time at which a given process hits a given subset of the state space, and the *stopping time*, which is the time at which a given stochastic process exhibits a certain behavior of interest, a concept we will discuss briefly in Section 5 in relation to drawdown processes.

Given a stochastic process, one can hence construct a corresponding random time summarizing a certain temporal behavior we are interested in. A temporal risk measure is then simply a real-valued function describing a feature of the distribution of this random time.

Definition 2.4 (Temporal risk measure). For a given time horizon $T \in (0, \infty)$, a *temporal risk measure* is a path-dependent risk measure $\rho : \mathcal{R}^\infty \to \mathbb{R}$ that can be decomposed as

$$\mathcal{R}^\infty \xrightarrow{f_T} \mathcal{V}_T \xrightarrow{\rho_T} \mathbb{R},$$

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[3]The notion of temporal risk we introduce is not to be confused with that of Machina (1984), which, in the context of economic utility maximizing preferences, captures the idea of delayed risk as opposed to immediately resolved risk when choosing amongst risky prospects.

where $f_T : \mathcal{R}^\infty \to \mathcal{V}_T$ is a transformation mapping a stochastic process to a random time, and $\rho_T : \mathcal{V}_T \to \mathbb{R}$ is a real-valued risk functional.

3. Drawdown Magnitude

Conventional risk metrics, such as volatility, Value-at-Risk and Expected Shortfall, measure one period losses and are evidently not path-dependent. The notions of drawdown and drawup, however, inherently account for a path over a given time period. Drawdown measures the decline in value from the running maximum (high water mark) and drawup measures the rise in value from the running minimum (low water mark) of a stochastic process typically representing the net asset value or cumulative return of an investment. From a probabilistic viewpoint, drawdowns and drawups occur at the first hitting times of the drawdown and drawup processes to the high and low water marks, respectively.

Drawdown, in particular, is one of the most frequently quoted indicators of downside risk in the fund management industry. In the event of a large drawdown, common single-period risk diagnostics are irrelevant and liquidation under unfavorable market conditions after an abrupt market decline may be forced. Drawup, on the other hand, can be viewed as an upside performance measure. For example, a rally of an investment process can be defined in terms of drawups, which is simply the difference between the present value of a process and its historical minimum. Moreover, insurance contracts protecting against drawdowns preceding drawups have been studied by Zhang et al. (2013). Such insurance contracts expire early if a drawup event occurs prior to a drawdown. From the investor's perspective, when a drawup is realized, there is little need to insure against a drawdown. Therefore, this drawup contingency automatically stops the premium payment and is an attractive feature that will potentially reduce the cost of drawdown insurance.

We review the concept of drawdown magnitude as a path-dependent probabilistic risk measure by generalizing the discrete-time setup of Goldberg and Mahmoud (2014) to the continuous-time framework, and discuss some of its properties that make drawdown amenable to the investment process. We then analyze the properties of the dual notion of drawup. Finally, a measure of drawdown risk aversion is introduced combining the measures of drawdown and drawup.

3.1. **Drawdown.** For a horizon $T \in (0, \infty)$, the *drawdown process* $D^{(X)} := \{D_t^{(X)}\}_{t \in [0,T]}$ corresponding to a stochastic process $X \in \mathcal{R}^\infty$ is defined by

$$D_t^{(X)} = M_t^{(X)} - X_t \, ,$$

where

$$M_t^{(X)} = \sup_{u \in [0,t]} X_u$$

is the running maximum of X up to time t.

The drawdown process associated with a given stochastic process has some practically intuitive properties. Clearly, a constant deterministic process does not experience any changes in value over time, implying that the corresponding drawdown process is zero. Moreover, any constant shift in a given process does not alter the magnitude of its drawdowns, and any constant multiplier of the stochastic process affects the drawdowns by the same multiplier. However, drawdown magnitudes are not preserved under monotonicity, which means that processes that can be ordered according to their magnitudes do not necessarily imply the same or opposite ordering on the drawdown magnitudes. Finally, a practically important property is convexity. Indeed, a convex combination of two processes results in a drawdown process that is smaller in

magnitude than the average standalone drawdowns of the underlying processes. We formally verify these properties next.

Lemma 3.1 (Properties of drawdown). *Given the stochastic process $X \in \mathcal{R}^\infty$, let $D^{(X)}$ be the corresponding drawdown process for a fixed time horizon T. Then:*

(1) *For all constant deterministic processes $C \in \mathcal{R}^\infty$, $D^{(C)} = 0$.*
(2) *For constant deterministic $C \in \mathcal{R}^\infty$, $D^{(X+C)} = D^{(X)}$.*
(3) *For $\lambda > 0$, $D^{(\lambda X)} = \lambda D^{(X)}$.*
(4) *For $Y \in \mathcal{R}^\infty$ and $\lambda \in [0,1]$, $D^{(\lambda X + (1-\lambda)Y)} \le \lambda D^{(X)} + (1-\lambda)D^{(Y)}$.*

Proof. See Appendix A   □

Remark 3.2. Note that it is not generally the case that for $Y \in \mathcal{R}^\infty$ for which $X \le Y$, either $D^{(X)} \le D^{(Y)}$ or $D^{(X)} \ge D^{(Y)}$. The only thing $X \le Y$ implies is that $M^{(X)} \le M^{(Y)}$. However, since at any point in time within the horizon the magnitude of a drop from peak is not specified, one cannot form a conclusion about the magnitude order of the corresponding drawdown processes. To see this more formally, note that under either of the assumptions that $D^{(X)} > D^{(Y)}$ or $D^{(X)} < D^{(Y)}$, we always get $X > Y$.

In practice, the use of the maximum drawdown as an indicator of risk is particularly popular in the universe of hedge funds and commodity trading advisors, where maximum drawdown adjusted performance measures, such as the Calmar ratio, the Sterling ratio and the Burke ratio, are frequently used.

Definition 3.3 (Maximum drawdown). Within a fixed time horizon $T \in (0, \infty)$, the *maximum drawdown* of the stochastic process $X \in \mathcal{R}^\infty$ is the maximum drop from peak to trough of X in $[0,T]$, and hence the largest amongst all drawdowns $D_t^{(X)}$:

$$\mu(X) = \sup_{t \in [0,T]} \{D_t^{(X)}\}.$$

Alternatively, maximum drawdown can be defined as the random variable obtained through the following transformation of the underlying stochastic process X:

$$\mu(X) = \sup_{t \in [0,T]} \sup_{s \in [t,T]} \{X_s - X_t\} \ .$$

We refer to its expectation $\mathbb{E}[\mu(X)]$ as the mean maximum drawdown.

The tail of the maximum drawdown distribution, from which the likelihood of a drawdown of a given magnitude can be distilled, could be of particular interest in practice. The drawdown risk measure defined in Goldberg and Mahmoud (2014) is the tail mean of the maximum drawdown distribution. At confidence level $\alpha \in [0,1]$, the *Conditional Expected Drawdown* $\mathrm{CED}_\alpha : \mathcal{R}^\infty \to \mathbb{R}$ is defined to be the path-dependent risk measure mapping the process X to the expected maximum drawdown $\mu(X)$ given that the maximum drawdown threshold at α is breached. More formally,

$$\mathrm{CED}_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{DT}_u\left(\mu(X)\right) du,$$

where DT_α is a quantile of the maximum drawdown distribution:

$$\mathrm{DT}_\alpha\left(\mu(X)\right) = \inf\left\{m \mid \mathbb{P}\left(\mu(X) > m\right) \le 1 - \alpha\right\}$$

If the distribution of $\mu(X)$ is continuous, then CED_α is equivalent to the tail conditional expectation:

$$\text{CED}_\alpha(X) = \mathbb{E}\left(\mu(X) \mid \mu(X) > \text{DT}_\alpha\left(\mu(X)\right)\right).$$

CED has sound mathematical properties making it amenable to the investment process. Indeed, it is a degree-one positive homogenous risk measure, so that it can be attributed to factors, and convex, so that it can be used in quantitative optimization.

Proposition 3.4 (Goldberg and Mahmoud (2014)). *For a given confidence level $\alpha \in [0, 1]$, Conditional Expected Drawdown $\text{CED}_\alpha : \mathcal{R}^\infty \to \mathbb{R}$ is a degree-one positive homogenous and convex path-dependent measure of risk, that is $\text{CED}_\alpha(\lambda X) = \lambda\text{CED}_\alpha(X)$ for $\lambda > 0$ and $\text{CED}_\alpha(\lambda X + (1 - \lambda)Y) \leq \lambda\text{CED}_\alpha(X) + (1 - \lambda)\text{CED}_\alpha(Y)$ for $\lambda \in [0, 1]$.*

3.2. **Drawdown aversion.** In Markowitz's Modern Portfolio Theory, risk aversion is measured as the additional marginal reward an investor requires to accept additional risk, where risk is measured as standard deviation of the return on investment, that is the square root of its variance. In advanced portfolio theory, different other types of risk can be taken into consideration. Consider taking drawdown as the underlying risk measure for constructing a Markowitz-type portfolio. Instead of minimizing drawdown risk, it may be of interest to balance out drawdown against drawup.

Dual to the notion of drawdown, a drawup describes the increase of a stochastic process from its running minimum. For a given time horizon $T \in (0, \infty)$, the *drawup process* $U^{(X)} := \{U_t^{(X)}\}_{t \in [0,T]}$ corresponding to a stochastic process $X \in \mathcal{R}^\infty$ is defined as

$$U_t^{(X)} = X_t - m_t^{(X)} ,$$

where

$$m_t^{(X)} = \inf_{u \in [0,t]} X_u$$

is the running minimum of X up to time t.

The following properties can be derived using the same arguments in the proof of Lemma 3.1. Note, in particular, the concave-like property of drawup — a convex combination of two stochastic processes results in a drawup process that is greater in magnitude than the average standalone drawups of the underlying processes. This makes drawup valuable in terms of quantitative optimization in practice.

Lemma 3.5 (Properties of drawup). *Given the stochastic process $X \in \mathcal{R}^\infty$, let $U^{(X)}$ be the corresponding drawup process for a fixed time horizon T. Then:*

(1) *For all constant deterministic processes $C \in \mathcal{R}^\infty$, $U^{(C)} = 0$.*
(2) *For constant deterministic $C \in \mathcal{R}^\infty$, $U^{(X+C)} = U^{(X)}$.*
(3) *For $\lambda > 0$, $U^{(\lambda X)} = \lambda U^{(X)}$.*
(4) *For $Y \in \mathcal{R}^\infty$ and $\lambda \in [0, 1]$, $U^{(\lambda X + (1-\lambda)Y)} \geq \lambda U^{(X)} + (1 - \lambda)U^{(Y)}$.*

Remark 3.6. As is the case with drawdown, it is not generally the case that for $Y \in \mathcal{R}^\infty$ for which $X \leq Y$, either $U^{(X)} \leq U^{(Y)}$ or $U^{(X)} \geq U^{(Y)}$.

The *maximum drawup* is the maximum increase from trough to peak in $[0, T]$, and hence the largest amongst all drawups $U_t^{(X)}$, defined by

$$\nu(X) = \sup_{t \in [0,T]} \{U_t^{(X)}\},$$

while the *average drawup* is defined as

$$\upsilon(X) = \frac{1}{T} \int_0^T U_s^{(X)} ds.$$

We can then construct path-dependent real-valued measures, such as mean, deviation or tail conditional expectation, on top of the random variables $\nu(X)$ and $\upsilon(X)$ representing the idea of upside or profit. Denote by ρ_U an arbitrary such measure.

We now formulate an optimization problem seeking the lowest drawdown for a given level of drawup and a given level of tolerance towards drawdown risk. For a given portfolio $X \in \mathcal{R}^\infty$, if we take Conditional Expected Drawdown $\mathrm{CED}_\alpha(X)$ (for $\alpha \in [0, 1]$) as the risk to be minimized and a selected measure ρ_U of drawup as the performance metric to be maximized, we obtain the following generalization of Markowitz's mean-variance approach to portfolio construction:

$$\min_{\mathbf{w}} \quad \gamma \mathrm{CED}_\alpha(X; \mathbf{w}) - \rho_U(X; \mathbf{w}) \,,$$

where $\mathbf{w}$ is a vector of portfolio weights, and $\gamma \geq 0$ can be thought of as a *drawdown aversion* coefficient. For small γ, aversion to drawdown is low and the penalty from the contribution of drawdown risk is also small, leading to more risky portfolios. Conversely, when γ is large, portfolios with more exposure to drawdown become more highly penalized. One may then gradually increase γ from zero and for each instance solve the optimization problem until the desired drawdown risk profile is reached.

4. Drawdown Duration

For a fixed time horizon T, our main object of interest is now the temporal dimension of drawdown. One such dimension is the duration of the drawdown process $D^{(X)}$ corresponding to the price process X, which measures the length of exursions of X below a running maximum. Commonly referred to as *Time To Recover (TTR)* in the fund management industry, the duration captures the time it takes to reach a previous running maximum of a process for the first time.

4.1. **Measuring duration.** As before, fix a time horizon $T \in (0, \infty)$ and let $D^{(X)} = \{D_t^{(X)}\}_{t \in [0,T]}$ be the drawdown process corresponding to the stochastic process $X \in \mathcal{R}^\infty$, and $M^{(X)} = \{M_t^{(X)}\}_{t \in [0,T]}$ be the running maximum of X.

Definition 4.1 (Peak time). The *peak time process* $G^{(X)} = \{G_t^{(X)}\}_{t \in [0,T]}$ is defined by

$$G_t^{(X)} = \sup \left\{ s \in [0, t] : X_s = M_s^{(X)} \right\}.$$

In words, $G_t^{(X)}$ denotes the last time X was at its peak, that is the last time it coincided with its maximum $M^{(X)}$ before t.

Note that $G^{(X)}$ is necessarily non-decreasing, consists of only linear subprocesses (more specifically, as a function of t, linear intervals $\{G_t^{(X)}\}_{t \in [r,s]}$ for $r < s$ are either the identity or a constant), and has jump discontinuities (under the realistic assumption that the underlying process X is not monotonic). Moreover, the process $G^{(X)}$ is invariant under constant shifts or scalar multiplication of the underlying process. Similar to the drawdown process, one can show that the peak time process associated to a stochastic process X does not preserve monotonicity. In other words, the peak time process corresponding to a process of larger value need

not be larger. However, unlike the drawdown process which preserves convexity, peak time is not preserved under either convexity or concavity. This means that a convex combination of two processes does not result in a peak time process that is consistently smaller or greater in magnitude than the average standalone peak times of the underlying processes. We formalize these properties next.

Lemma 4.2 (Properties of peak time). *Given stochastic processes $X \in \mathcal{R}^\infty$, let $G^{(X)}$ be the corresponding peak time process for a fixed time horizon T. Then:*

(1) *For all constant deterministic processes $C \in \mathcal{R}^\infty$, $G_t^{(C)} = t$ for all $t \in [0, T]$.*
(2) *For constant deterministic $C \in \mathcal{R}^\infty$, $G_t^{(X+C)} = G_t^{(X)}$ for all $t \in [0, T]$.*
(3) *For $\lambda > 0$, $G_t^{(\lambda X)} = G_t^{(X)}$ for all $t \in [0, T]$.*

Proof. See Appendix B.

$\square$

Remark 4.3. Note also that peak time is not necessarily preserved under monotonicity, that is $X \leq Y$ does not necessarily imply either $G_t^{(X)} \leq G_t^{(Y)}$ or $G_t^{(X)} \geq G_t^{(Y)}$ for all $t \in [0, T]$. Intuitively, the last time a process coincides with its running maximum is independent of the magnitude of the process. Moreover, peak time does not necessarily exhibit either quasiconvex- or quasiconcave-like behavior, that is for $\lambda \in [0, 1]$, $G_t^{(\lambda X + (1-\lambda)Y)}$ is not necessarily either greater than $\min\{G_t^{(X)}, G_t^{(Y)}\}$ or smaller than $\max\{G_t^{(X)}, G_t^{(Y)}\}$ for all $t \in [0, T]$. We construct a simple example showing that for $\lambda \in [0, 1]$, $G_t^{(\lambda X + (1-\lambda)Y)}$ is not necessarily greater than $\min\{G_t^{(X)}, G_t^{(Y)}\}$ for all $t \in [0, T]$. Fix a time $t \in [0, T]$ and, without loss of generality, let $G_t^{(X)} = t_0$ and $G_t^{(Y)} = t_1$ with $t_0 < t_1 \leq t$. Let $G_t^{(\lambda X + (1-\lambda)Y)} = t^*$ and we examine what happens if $t^* < t_0 = \min\{G_t^{(X)}, G_t^{(Y)}\}$. In this case, we have by definition that $(\lambda X + (1-\lambda)Y)_s < M_s^{(\lambda X + (1-\lambda)Y)}$ for all $s \in (t^*, t]$. In particular, at t_0, we have

$$\lambda X_{t_0} + (1-\lambda)Y_{t_0} < M_{t_0}^{(\lambda X + (1-\lambda)Y)} = \lambda M_{t_0}^{(X)} + (1-\lambda)M_{t_0}^{(Y)} = \lambda X_{t_0} + (1-\lambda)M_{t_0}^{(Y)},$$

implying that $Y_{t_0} < M_{t_0}^{(Y)}$. Note that we do not have information about where Y is relative to its running maximum $M^{(Y)}$ before time t_1. This means that if $Y_{t_0} < M_{t_0}^{(Y)}$, then $t^* < \min\{G_t^{(X)}, G_t^{(Y)}\}$, and on the other hand, if $Y_{t_0} = M_{t_0}^{(Y)}$, then $t^* \geq \min\{G_t^{(X)}, G_t^{(Y)}\}$. One can construct a similar argument to show that $G_t^{(\lambda X + (1-\lambda)Y)}$ is not necessarily smaller than $\max\{G_t^{(X)}, G_t^{(Y)}\}$.

Probabilistically, the trajectory of the process X between its peak time $G_t^{(X)}$ and its recovery time $L_t = \sup\{s \in [t, T] : X_s = M_s^{(X)}\}$ is the excursion of X at its running maximum, which straddles time t. If $X < M^{(X)}$ during this excursion, we say that X is in drawdown or *underwater*. We are now interested in $t - G_t^{(X)}$, which is the duration of this excursion.

Definition 4.4 (Drawdown Duration). Given a process $X \in \mathcal{R}^\infty$ and time horizon T, the *duration process* $\delta^{(X)} = \{\delta_t^{(X)}\}_{t \in [0,T]}$ associated with X is defined by

$$\delta_t^{(X)} = t - G_t^{(X)}.$$

Lemma 4.5 (Properties of drawdown duration). *Given a process $X \in \mathcal{R}^\infty$ and time horizon T, the duration process $\delta^{(X)} = \{\delta_t^{(X)}\}_{t \in [0,T]}$ satisfies the following properties on $[0, T]$:*

(1) *For all constant deterministic processes $C \in \mathcal{R}^\infty$, $\delta^{(C)} = 0$.*
(2) *For all $X \in \mathcal{R}^\infty$ and all constant deterministic processes $C \in \mathcal{R}^\infty$, $\delta^{(X+C)} = \delta^{(X)}$.*

(3) *For all $X \in \mathcal{R}^\infty$ and $\lambda > 0$, $\delta^{(\lambda X)} = \delta^{(X)}$.*

Proof. All properties are immediate consequences of Lemma 4.2   □

Remark 4.6. Similar to peak time, drawdown duration is not necessarily preserved under monotonicity, that is $X \leq Y$ does not necessarily imply either $\delta^{(X)} \leq \delta^{(Y)}$ or $\delta^{(X)} \geq \delta^{(Y)}$, and that drawdown duration does not necessarily exhibit either convex- or concave-like behavior, that is for $\lambda \in [0, 1]$, $\delta^{(\lambda X + (1-\lambda)Y)}$ is not necessarily either greater or smaller than $\lambda \delta^{(X)} + (1 - \lambda)\delta^{(Y)}$.

Of particular interest now is the maximum time spent underwater within a fixed time horizon T, independent of the magnitude of the actual drawdown experienced by the process X during this time interval.

Definition 4.7 (Maximum duration). Given a process $X \in \mathcal{R}^\infty$ and time horizon T, let $\delta^{(X)}$ be the duration process corresponding to X. The *maximum duration* of the stochastic process X is the real valued random variable defined by

$$\delta_{\max}^{(X)} = \sup_{t \in [0,T]} \{\delta_t^{(X)}\}.$$

We refer to its expectation $\mathbb{E}[\delta_{\max}^{(X)}]$ as the mean maximum duration.

Maximum duration is clearly a random time defined on the same probability space as X and taking values in the interval $[0, T]$.

Remark 4.8 (Duration of Maximum Drawdown). We point out that our notion of maximum duration $\delta_{\max}^{(X)}$ differs from the duration or length of the deepest excursion below the maximum, $\mu(X)$, within the given path. Suppose that the maximum drawdown of the process X occurred between times $\tau_p \in [0, T)$ (the "peak") and $\tau_r \in [\tau_p, T]$ (the "recovery"), where we assume for the sake of illustration that τ_r is defined, that is recovery indeed occurs within the given horizon. Note that there must be a point in time $\tau_b \in (\tau_p, \tau_r)$ where X was at its minimum (the "bottom") during the interval (τ_p, τ_r). The time at which the minimum of X within the interval in which the maximum drawdown occurred is given by

$$\tau_b = \inf\{t \in [0, T] : \mu(X) = \sup_{t \in [0,T]} D_t\}.$$

Then τ_p is the last time X was at its maximum before τ_b:

$$\tau_p = \sup\{t \in [0, \tau_b] : X_t = M_t\},$$

and τ_r is the first time X coincides again with its rolling maximum:

$$\tau_r = \inf\{t \in [\tau_b, T] : X_t = M_t\}.$$

Given a process $X \in \mathcal{R}^\infty$, the duration of the maximum drawdown of X is then the random variable defined by $\delta_\mu^{(X)} = \tau_r - \tau_p$. It is a straightforward exercise to show that $\delta_\mu^{(X)}$ satisfies the same properties that maximum duration $\delta_{\max}^{(X)}$ satisfies.

4.2. **Duration risk.** Even though, in a given horizon, only a single maximum duration is realized along any given path, it is beneficial to consider the distribution from which the maximum duration is taken. By looking at this distribution, one can form reasonable expectations about the expected length of drawdowns for a given portfolio over a given investment horizon. Mathematically, the maximum duration $\delta_{\max}^{(X)}$ of a given stochastic process X is a random variable, whose distribution we can describe using path-dependent temporal risk measures, as introduced in Definition 2.4.

Definition 4.9 (Measures of duration risk). We define the following path-dependent temporal measures of risk $\rho : \mathcal{R}^\infty \to \mathbb{R}$ describing the distribution of the maximum duration $\delta_{\max}^{(X)}$ associated with a stochastic process $X \in \mathcal{R}^\infty$:

(1) *Duration Deviation:* $\sigma_\delta : \mathcal{R}^\infty \to \mathbb{R}$ is defined by
$$\sigma_\delta(X) = \sigma\left(\delta_{\max}^{(X)}\right) \ ,$$
where σ is the standard deviation.

(2) *Duration Quantile:* For confidence level $\alpha \in [0,1]$, $Q_{\delta,\alpha} : \mathcal{R}^\infty \to \mathbb{R}$ is defined by
$$Q_{\delta,\alpha}(X) = q_\alpha\left(\delta_{\max}^{(X)}\right) = \inf_{d \in \mathbb{R}}\left\{\mathbb{P}(\delta_{\max}^{(X)} < d) \leq 1 - \alpha\right\}.$$

(3) *Conditional Expected Duration:* For confidence level $\alpha \in [0,1]$, $\mathrm{CE}_{\delta,\alpha} : \mathcal{R}^\infty \to \mathbb{R}$ is defined by
$$\mathrm{CE}_{\delta,\alpha}(X) = \mathrm{TM}_\alpha\left(\delta_{\max}^{(X)}\right) = \frac{1}{1-\alpha}\int_\alpha^1 Q_{\delta,u}(X)du$$
For continuous $\delta(X)$, the above amounts to
$$\mathrm{CE}_{\delta,\alpha}(X) = \mathbb{E}\left[\delta_{\max}^{(X)} \mid \delta_m^{(X)} < Q_{\delta,\alpha}(X)\right].$$

Note that each of these path-dependent risk measures is temporal in the sense of Definition 2.4. Indeed, in each case, the path-dependent risk measure $\rho : \mathcal{R}^\infty \to \mathbb{R}$ is the composite of a risk functional (deviation, quantile, tail mean) applied to the maximum duration with the transformation $f_T : \mathcal{R}^\infty \to \mathcal{V}_T$ mapping a stochastic process $X \in \mathcal{R}^\infty$ to its corresponding maximum duration $\delta_{\max}^{(X)}$.

Proposition 4.10. *None of the path-dependent duration risk measures $\sigma_\delta, Q_\delta, \mathrm{CE}_\delta : \mathcal{R}^\infty \to \mathbb{R}$ satisfies any of the coherence axioms of risk measures (that is monotonicity, translation-invariance, degree-one positive homogeneity, and convexity).*

This result is once again a direct corollary of the properties of the duration process illustrated in Lemma 4.5. Consider for example duration deviation. The standard deviation functional is a deviation risk measure (see Rockafellar et al. (2002) and Rockafellar et al. (2006)) when applied to the (essentially single-period) distribution of the maximum duration. However, deviation-like properties are not preserved any longer when the functional is applied to the stochastic process X. As an example, consider the (deviation and coherence) property of degree-one positive homogeneity. Standard deviation is positive homogenous on the space of maximum duration, that is we do indeed have $\sigma(\lambda\delta_{\max}^{(X)}) = \lambda\sigma(\delta_{\max}^{(X)})$. However, when applied to X, deviation is degree-zero positive homogenous, since $\sigma(\delta_{\max}^{(\lambda X)}) = \sigma(\delta_{\max}^{(X)})$. Similarly, the tail mean functional TM_α is coherent independent of the distribution it is applied to (see Acerbi and Tasche (2002a,b)). However, all coherence properties are lost when moving from the space of the maximum duration random variable to the underlying process X. As before, these properties are lost because of the inherent non-coherent-like behavior of the corresponding duration process $\delta^{(X)}$.

5. Stopping Time

In probability theory, a stopping time (also known as Markov time) is a random time whose value is interpreted as the time at which a given stochastic process exhibits a certain behavior of interest. A stopping time is generally defined by a stopping rule, a mechanism for deciding whether to continue or stop a process on the basis of the present position and past events, and

which will almost always lead to a decision to stop at some finite time. It is thus completely determined by (at most) the total information known up to a certain time. For continuous-time stochastic processes, stopping time is formally defined with respect to a filtration representing the information available up to a given point in time; that is, given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in I}, \mathbb{P})$, a random variable $\tau : \Omega \to I$ is a stopping time if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in I$.

In our context of drawdowns of investments, it may be of interest to calculate the probability that a process stays underwater for a period longer than a certain subjective acceptance threshold. No matter what the magnitude of the loss is, if the time to recover exceeds this threshold, one may be forced to liquidate. To this end, we define the liquidation stopping time.

Definition 5.1 (Liquidation stopping time). Given a process $X \in \mathcal{R}^\infty$ over a time horizon $T \in (0, \infty)$ with corresponding duration process $\delta^{(X)}$, denote by $l \in (0, T]$ a subjectively set liquidation threshold. The *liquidation stopping time* (LST) τ_L is defined by

$$\tau_L^{(X)} = \inf\{t \in [0, T] : \delta_t^{(X)} \geq l\}.$$

The stopping time τ_L hence denotes the first time the drawdown duration $\delta^{(X)}$ exceeds the pre-specified liquidation threshold l; that is the first time the process X has stayed underwater for a consecutive period of length greater than l. It essentially specifies a rule that tells us when to exit a trade. Note that the decision to "stop" at time τ_L can only depend (at most) on the information known up to that time and not on any future information.

From a probabilistic viewpoint, the liquidation stopping time τ_L can be identified with Parisian stopping times for a zero barrier, which was studied in Chesney et al. (1997) and Loeffen et al. (2013). Stopping times were also introduced in actuarial risk theory in Dassios and Wu (2009), where the process X models the surplus of an insurance company with initial capital, and the stopping time of an excursion is referred to as Parisian ruin time.

Lemma 5.2 (Properties of liquidation stopping time). *Given a process $X \in \mathcal{R}^\infty$ and time horizon T, the liquidation stopping time $\tau_L^{(X)}$ satisfies the following properties on $[0, T]$:*

   (1) *For all constant deterministic processes $C \in \mathcal{R}^\infty$, $\tau_L^{(C)} = \infty$.*
   (2) *For all $X \in \mathcal{R}^\infty$ and all constant deterministic processes $C \in \mathcal{R}^\infty$, $\tau_L^{(X+C)} = \tau_L^{(X)}$.*
   (3) *For all $X \in \mathcal{R}^\infty$ and $\lambda > 0$, $\tau_L^{(\lambda X)} = \tau_L^{(X)}$.*

Proof. To see the first property, note that $\delta^{(C)} = 0$, which implies that $\tau_L^{(C)} = \inf(\emptyset)$. The remaining properties are once again immediate consequences of Lemma 4.5. $\square$

Remark 5.3. Note that liquidation stopping time is not necessarily preserved under monotonicity, that is $X \leq Y$ does not necessarily imply either $\tau_L^{(X)} \leq \tau_L^{(Y)}$ or $\tau_L^{(X)} \geq \tau_L^{(Y)}$. Moreover, liquidation stopping time does not necessarily exhibit either convex- or concave-like behavior, that is for $\lambda \in [0, 1]$, $\tau_L^{(\lambda X + (1-\lambda)Y)}$ is not necessarily either greater or smaller than $\lambda \tau_L^{(X)} + (1 - \lambda)\tau_L^{(Y)}$.

Just like maximum duration, liquidation stopping time is a random time essentially sharing the same non-coherent-like properties. For example, property (5) of Lemma 5.2 implies that by dividing investments amongst two assets, an investor does not necessarily obtain a liquidation stopping time that is either smaller or greater than the weighted average of the standalone liquidation stopping times of the underlying assets. This is in contrast to drawdown, where we are guaranteed the diversification promoting reduction in overall portfolio drawdown. However, it may still be of interest to consider the distribution of the liquidation stopping time, which can be described via path-dependent temporal risk functionals.

Definition 5.4 (Measures of stopping time risk)**.** We define the following path-dependent measures of risk $\rho : \mathcal{R}^\infty \to \mathbb{R}$ describing the distribution of the liquidation stopping time (LST) $\tau_L^{(X)}$:

(1) *LST Deviation:* $\sigma_{LST} : \mathcal{R}^\infty \to \mathbb{R}$ is defined by

$$\sigma_{LST}(X) = \sigma\left(\tau_L^{(X)}\right).$$

(2) *LST Quantile:* For confidence level $\alpha \in [0,1]$, $Q_{LST,\alpha} : \mathcal{R}^\infty \to \mathbb{R}$ is defined by

$$Q_{LST,\alpha}(X) = q_\alpha\left(\tau_L^{(X)}\right) = \inf_{d\in\mathbb{R}}\left\{\mathbb{P}(\tau_L^{(X)} < d) \leq 1 - \alpha\right\}.$$

(3) *Conditional Expected LST*: For confidence level $\alpha \in [0,1]$, $\mathrm{CE}_{LST,\alpha} : \mathcal{R}^\infty \to \mathbb{R}$ is defined by

$$\mathrm{CE}_{LST,\alpha}(X) = \mathrm{TM}_\alpha\left(\tau_L^{(X)}\right) = \frac{1}{1-\alpha}\int_\alpha^1 Q_{LST,u}(X)du$$

For continuous $\kappa_L(X)$, the above amounts to

$$\mathrm{CE}_{LST,\alpha}(X) = \mathbb{E}\left[\tau_L^{(X)} \mid \kappa_L^{(X)} < Q_{LST,\alpha}(X)\right].$$

Proposition 5.5. *None of the path-dependent measures of liquidation stopping time risk* $\sigma_{LST}, Q_{LST}, \mathrm{CE}_{LST} : \mathcal{R}^\infty \to \mathbb{R}$ *satisfies any of the coherence axioms of risk measures (that is monotonicity, translation-invariance, degree-one positive homogeneity, and convexity).*

6. Temporal Risk in Practice

6.1. **Impact of non-coherence in practice.** We have shown that none of the drawdown, duration or stopping time risk measures are monetary as they do not satisfy the translation invariance and monotonicity axioms. Both of these axioms were originally introduced as desirable properties for risk measures under the assumption that the risk of a position represents the amount of capital that should be added so that it becomes acceptable to the regulator. From a regulatory viewpoint, translation invariance means that adding the value of any guaranteed position to an existing portfolio simply decreases the capital required by that guaranteed amount. Monotonicity essentially states that positions that lead to higher losses should require more risk capital. Our interest throughout this work not being the regulatory reporting framework but rather the mathematical formalization of temporal risk in a way that is amenable to risk analysis and management, which is of particular interest to the asset management community, the impact of the non-monetariness of drawdown and duration risk in practice is limited.

The first dimension of path-dependent risk, drawdown magnitude, nevertheless satisfies two theoretically and practically important properties, namely convexity and positive homogeneity. Convexity enables investors to allocate funds in such a way that minimizes drawdown risk, while positive homogeneity ensures that the overall drawdown risk of a portfolio can be linearly decomposed into additive subcomponents representing the individual factor contributions to drawdown risk. The temporal risk dimensions, drawdown duration and stopping time, are neither convex nor degree-one positive homogenous. This implies on the one hand that linear attribution to random time risk is not supported, and on the other hand that the favorable convex optimization theory is not applicable. Temporal risk hence seems to have limited practical application in the investment process.

	Volatility	$ES_{0.9}$	$CED_{0.9}$	$\mathbb{E}[\delta_m^{(X)}]$	σ_δ	$CE_{\delta,0.9}$
US Equity	18.35%	2.19%	47%	456	489	1323
US Bonds	5.43%	0.49%	29%	976	590	1070

TABLE 1. Single-period and path-dependent risk statistics for daily US Equity and US Bond Indices over the period between January 1973 and December 2013, inclusive. Expected Shortfall (ES), Conditional Expected Drawdown (CED) and Conditional Expected Duration (CE_δ) are calculated at the 90% confidence level. $\mathbb{E}[\delta_m^{(X)}]$ and σ_δ are the mean maximum duration and deviation of maximum duration, respectively.

However, we argue that temporal risk should not be ignored as it encapsulates a potentially useful diagnostic measure of a dimension of risk that is paramount but traditionally not incorporated in the risk management process within the investment management industry. We next show simple diagnostics that can help illuminate the temporal drawdown dimension in practice. More specifically, we (i) summarize the empirical distribution of drawdown duration and compare it with the traditional single period risk measures of volatility and expected shortfall; (ii) analyze the way in which path-dependent magnitude and temporal risk measures account for serial correlation; and (iii) investigate the relationship between the magnitude and time dimensions of path-dependent risk. Finally, we discuss the issue of path-dependent risk forecasting in practice. In particular, good forecasting accuracy evidently relies on the soundness of the underlying risk model used to form expectations, a challenge we will briefly address at the end of this Section.

6.2. **Empirical analysis of temporal risk.** We study historical values of duration risk based on daily data for two asset classes: US Equity and US Government Bonds.[4] Summary statistics for the two asset classes are shown in Table 1.

6.2.1. *Duration distribution.* Maximum drawdown distributions are generally positively skewed independent of the underlying risk characteristics, which implies that very large drawdowns occur less frequently than smaller ones (see Burghardt et al. (2003) and Goldberg and Mahmoud (2014)). This is not necessarily the case for the distribution of maximum duration. Figure 1 displays the empirical duration distribution of US Equity and US Government Bonds over the 40-year period 1973–2013 using daily data. Positive skewness is pronounced for US Equity with a value of 1.3, while it is less noticeable for US Bonds at a value of 0.4. Moreover, note from Table 1 that Conditional Expected Duration is larger for US Equity than it is for US Bonds, which is consistent with the stylized fact that equities are riskier than bonds. Indeed along volatility, shortfall and drawdown, US Equity is consistently riskier than US Bonds. On the other hand, consistent with Figure 1, both average duration and duration deviation are considerably larger for the less risky fixed income asset than for the more risky equities asset.

6.2.2. *Temporal risk and serial correlation.* Goldberg and Mahmoud (2014) argue that one advantage of looking at drawdown rather than return distributions lies in the fact that drawdown is inherently path dependent by showing that drawdown risk captures temporal dependence to

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[4]The data were obtained from the Global Financial Data database. We took the daily time series for the S&P 500 Index and the USA 10-year Government Bond Total Return Index covering the 40-year period between January 1973 and December 2013, inclusive.
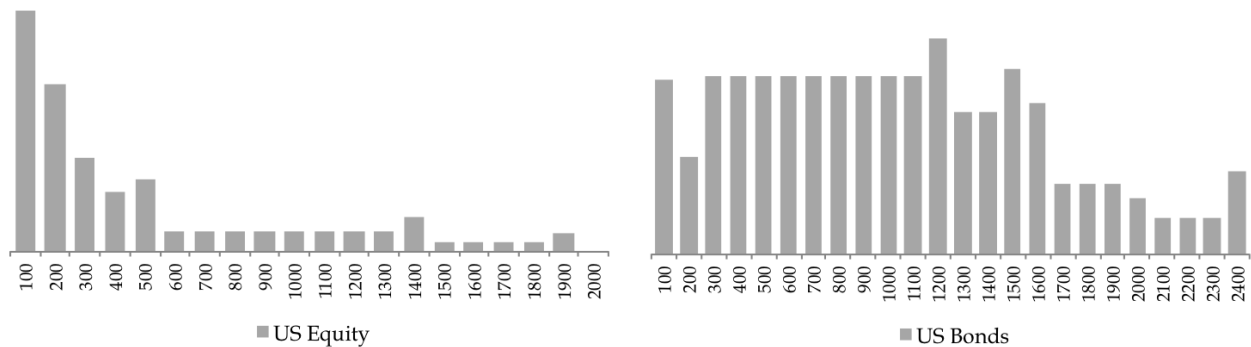
FIGURE 1. Empirical distribution of maximum duration for daily US Equity and US Bond Indices over the period between January 1973 and December 2013, inclusive. The distribution is generated empirically as follows. From the historical time series of returns, we generate return paths of a fixed 6-month length ($n = 180$ trading days) using a one-day rolling window. The maximum duration within each path is then calculated. This means that consecutive paths overlap. The advantage is that for a daily return time series of length T, we obtain a maximum duration series of length $T - n$, which for large T and small n is fairly large, too.

a greater degree than traditional one-period risk measures; this path dependency makes them more sensitive to serial correlation. We next show that temporal risk measures capture temporal dependence in asset returns. We use Monte Carlo simulation to generate an autoregressive AR(1) model:

$$r_t = \kappa r_{t-1} + \epsilon_t,$$

with varying values for the autoregressive parameter κ, where ϵ is fixed to be Gaussian with variance 0.01. We then calculate volatility, Expected Shortfall, Conditional Expected Drawdown, and Conditional Expected Duration of each simulated autoregressive time series and list the results as a function of κ (see Table 2). Both single-period risk measures are affected by the increase in the value of the autoregressive parameter. However, the increase is steeper for the two path dependent risk measures. We next use maximum likelihood to fit the same AR(1) model to the daily time series of US Equity and US Government Bonds on a 6-month rolling basis to obtain a time series of estimated κ values for each asset. The correlations of the time series of κ with the time series of 6-month rolling volatility, Expected Shortfall, Conditional Expected Drawdown, and Conditional Expected Duration are shown in Table 3. For both assets, the correlation with the autoregressive parameter is relatively large for drawdown and duration compared to volatility and shortfall, and highest for duration. Moreover, single-period and drawdown risk is consistently higher for US Equity than it is for US Bonds, whereas the opposite is true for duration risk.

6.2.3. *Magnitude versus time.* Even though duration is theoretically defined independent of the drawdown magnitude, there is a close relationship between the temporal and the size dimensions of a cumulative drop in portfolio value. Figure 2 display the daily time series of drawdown magnitude and its duration for each of US Equity and US Government Bonds. Clearly, a drawdown's magnitude is positively correlated to its duration. Therefore, even though some smaller drawdowns can stay under water a long period of time, empirically larger drawdowns tend to come with an extended duration. In practice, minimizing the convex risk of drawdown magnitude may in fact lead to a lower overall (non-convex) duration risk.

Kappa	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Volatility	0.07	0.07	0.08	0.09	0.10	0.11	0.11	0.12	0.12
$ES_{0.9}$	0.16	0.16	0.17	0.17	0.18	0.19	0.19	0.20	0.21
$CED_{0.9}$	0.15	0.16	0.18	0.21	0.22	0.29	0.34	0.38	0.39
$CE_{\delta,0.9}$	291	328	350	339	411	467	487	504	496

TABLE 2. Volatility, 90% Expected Shortfall, 90% Conditional Expected Drawdown, and 90% Conditional Expected Duration of a Monte Carlo simulated AR(1) model (with 10,000 data points) for varying values of the autoregressive parameter κ.

	Volatility	$ES_{0.9}$	$CED_{0.9}$	$CE_{\delta,0.9}$
US Equity	0.45	0.52	0.70	0.81
US Bonds	0.32	0.39	0.67	0.85

TABLE 3. For the daily time series of each of US Equity and US Government Bonds, correlations of estimates of the autoregressive parameter κ in an AR(1) model with the values of the four risk measures (volatility, 90% Expected Shortfall and 90% Conditional Expected Drawdown, and 90% Conditional Expected Duration) estimated over the entire period (1973–2013).
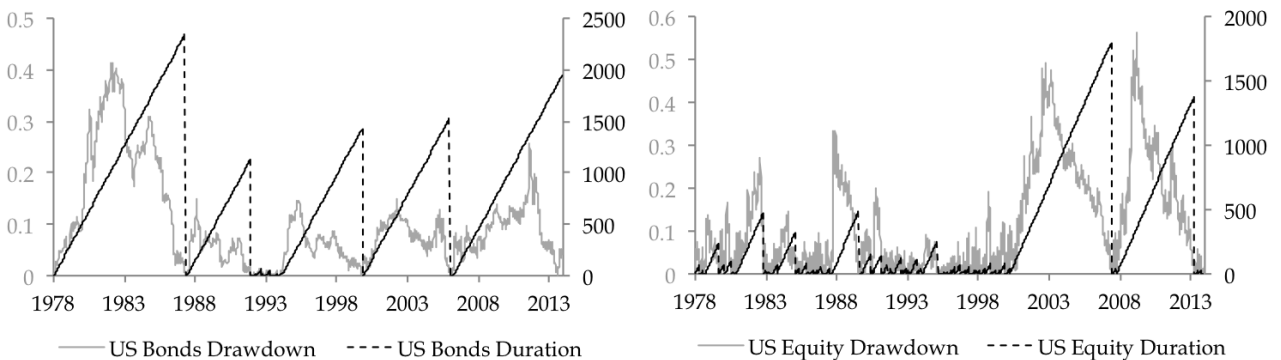

FIGURE 2. Daily time series of historical drawdown (with scale in percentages on the left-hand side) and duration (with scale in trading days on the right-hand side) for US Bonds and US Equity over the period 1978–2013.

6.3. **Path-dependent risk estimation.** Our study of mathematically sound notions of the magnitude and duration of path-dependent risk measures in the context of probabilistic risk measures essentially yields a methodology for forming expectations about future potential drawdown risk. With this mathematical setup for drawdown and duration risk measures in place, a natural ensuing question is: how do we use this formalism to make statements about future path-dependent expectations in practice? The answer lies beyond the scope of this article, and we conclude by briefly pointing towards the need for and challenges in developing a sound path-dependent risk model.

The key to accurate path-dependent risk forecasts is a realistic scenario generation process representing the underlying returns. Consider for example the most basic parametric Gaussian model. Despite the evidence that the Gaussian viewpoint does not yield a realistic representation of asset returns, relying on the normality assumption continues to be standard in quantitative risk measurement and reporting. There are parametric alternatives to the normal model that account for heavy tails and skewness of portfolio returns. However, the challenges of developing a flexible, robust, multi-horizon parametric model that is diverse enough to be applied to a wide range of portfolios have led to the popularity of historical simulation. We refer the reader to Cont (2001) for a summary of difficulties arising from parametric modelling of equity time series.

Historical simulation, on the other hand, certainly presents challenges of its own. The methodology assumes that the past accurately represents the future, while market conditions change over time. Moreover, the data required for historical simulation may not be available. Recently developed assets may have insufficient history, and external events and economic dynamics may lead to the insignificance of an asset's history.

These issues need to be addressed in an economically sound path-dependent risk model. Compared to single-period risk measures, path-dependency introduces additional challenges. In particular, models that account for this inherent temporal dependency tend to be more complicated to estimate, simulate, and backtest in practice. On the other modeling spectrum, consider the simplest form of empirical estimation: random sampling. Such a methodology fails to account for a notion of memory in time series of returns. Memory is a stylized fact incorporating the ideas that serial autocorrelation increases during turbulent markets, volatilities change over time, and high volatility regimes have a tendency to occur immediately following large drawdowns. Alternatives to random sampling, such as the moving block bootstrap introduced by Künsch (1989), have certainly been developed in statistical theory. A large literature on backtesting such parametric and empirical models in the context of forecasting path-dependent risk measures does, however, not seem to exist.

7. Conclusion

In this paper, we analyzed the temporal dimension of one of the most widely quoted indicators of multi-period risk: drawdown, which is the decline from a historical peak in net asset value or cumulative return. In the event of a large drawdown, conventional single-period risk diagnostics, such as volatility or Expected Shortfall, are irrelevant and liquidation under unfavorable market conditions after an abrupt market decline may be forced. In particular, we studied the properties of the temporal dimension of drawdown in the context of coherent measures of risk developed by Artzner et al. (1999). To this end, we formulated the temporal dimension of drawdown as a temporal risk measure $\rho : \mathcal{R}^\infty \to \mathbb{R}$ and analyzed the coherency properties of drawdown duration, which measures the length of excursions below a running maximum, and liquidation stopping time, which denotes the first time drawdown duration exceeds a subjective liquidation threshold, in the context of temporal path-dependent risk measures. We concluded that neither of these temporal path-dependent risk measures satisfies the axioms for coherent risk measures. In practice, non-coherence implies, amongst others, that linear attribution to temporal risk is not supported, and that convex minimization of temporal risk is not applicable. Despite this limited use, we argued, however, that temporal risk should not be ignored in the risk management process of investment funds; it encapsulates a good diagnostic measure of a dimension of risk that is paramount but traditionally not incorporated in the risk management process. To support this viewpoint, we provided an empirical study showing that duration

is not necessarily correlated to conventional risk metrics, captures serial correlation in asset returns, and is strongly related of the magnitude dimension of drawdown. Finally, we included a brief discussion on the challenges in developing a mathematically and economically sound path-dependent risk model enabling forecasting such risks in practice.

APPENDIX A. PROOF OF LEMMA 3.1

Lemma 3.1 (Properties of drawdown). Given the stochastic process $X \in \mathcal{R}^\infty$, let $D^{(X)}$ be the corresponding drawdown process for a fixed time horizon T. Then:

(1) For all constant deterministic processes $C \in \mathcal{R}^\infty$, $D^{(C)} = 0$.
(2) For constant deterministic $C \in \mathcal{R}^\infty$, $D^{(X+C)} = D^{(X)}$.
(3) For $\lambda > 0$, $D^{(\lambda X)} = \lambda D^{(X)}$.
(4) For $Y \in \mathcal{R}^\infty$ and $\lambda \in [0,1]$, $D^{(\lambda X + (1-\lambda)Y)} \leq \lambda D^{(X)} + (1-\lambda)D^{(Y)}$.

Proof. (1) Since $M^{(C)} = C$, we immediately get $D^{(C)} = M^{(C)} - C = 0$.

(2) Since for $t \in [0,T]$, $M_t^{(X+C)} = \sup_{u \in [0,t]}(X+C)_u = \sup_{u \in [0,t]}(X)_u + C = M_t^{(X)} + C$, we have $D^{(X+C)} = M^{(X+C)} - X - C = M^{(X)} + C - X - C = M^{(X)} - X = D^{(X)}$.

(3) For $\lambda > 0$, we have for $t \in [0,T]$, $M_t^{(\lambda X)} = \sup_{u \in [0,t]}(\lambda X)_u = \lambda \sup_{u \in [0,t]}(X)_u = \lambda M_t^{(X)}$, and therefore $D^{(\lambda X)} = \lambda M^{(X)} - \lambda X = \lambda D^{(X)}$.

(4) For $\lambda \in [0,1]$, we clearly have $M^{(\lambda X + (1-\lambda)Y)} \leq \lambda M^{(X)} + (1-\lambda)M^{(Y)}$ by properties of the supremum, and therefore $D^{(\lambda X + (1-\lambda)Y)} = M^{(\lambda X + (1-\lambda)Y)} - \lambda X - (1+\lambda)Y \leq \lambda M^{(X)} + (1-\lambda)M^{(Y)} - \lambda X - (1+\lambda)Y = \lambda D^{(X)} + (1-\lambda)D^{(Y)}$. $\square$

APPENDIX B. PROOF OF LEMMA 4.2

Lemma 4.2 (Properties of peak time). Given stochastic processes $X \in \mathcal{R}^\infty$, let $G^{(X)}$ be the corresponding peak time process for a fixed time horizon T. Then:

(1) For all constant deterministic processes $C \in \mathcal{R}^\infty$, $G_t^{(C)} = t$ for all $t \in [0,T]$.
(2) For constant deterministic $C \in \mathcal{R}^\infty$, $G_t^{(X+C)} = G_t^{(X)}$ for all $t \in [0,T]$.
(3) For $\lambda > 0$, $G_t^{(\lambda X)} = G_t^{(X)}$ for all $t \in [0,T]$.

Proof. (1) Since for all $t \in [0,T]$, $M_t^{(C)} = C$, $G_t^{(C)} = \sup\{s \in [0,t] : C_s = C_s\} = t$.

(2) Because the running maximum process $M^{(X+C)}$ corresponding to $X+C$ satisfies $M_t^{(X+C)} = \sup_{u \in [0,t]}(X+C)_u = \sup_{u \in [0,t]} X_u + C = M_t^{(X)} + C$ for every t, we have $G_t^{(X+C)} = \sup\{s \in [0,t] : (X+C)_s = M_s^{(X+C)}\} = \sup\{s \in [0,t] : X_s + C = M_s^{(X)} + C\} = \sup\{s \in [0,t] : X_s = M_s^{(X)}\} = G_t^{(X)}$.

(3) Similarly, since for $\lambda > 0$, $M_t^{(\lambda X)} = \lambda M_t^{(X)}$ for every t, we have $G_t^{(\lambda X)} = \sup\{s \in [0,t] : (\lambda X)_s = M_s^{(\lambda X)}\} = \sup\{s \in [0,t] : \lambda(X_s) = \lambda M_s^{(X)}\} = \sup\{s \in [0,t] : (X)_s = M_s^{(X)}\} = G_t^{(X)}$.

$\square$

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