CENTER FOR RISK MANAGEMENT RESEARCH

The Decision to Lever

Robert M. Anderson, University of California at Berkeley
Stephen W. Bianchi, University of California at Berkeley
Lisa R. Goldberg, University of California at Berkeley

University of California
Berkeley

The Decision to Lever

Robert M. Anderson[*]   Stephen W. Bianchi[†]
Lisa R. Goldberg[‡]

University of California at Berkeley

November 17, 2013[§]

Abstract

We provide an exact performance attribution for levered strategies. The attribution includes some familiar elements: return magnification due to leverage, borrowing costs, trading costs, and the variance drag; while familiar, these elements are sometimes downplayed in backtests of levered strategies. In addition, we find empirically that the covariance between return of the source portfolio and leverage plays an important role in determining the cumulative return of levered strategies. This covariance is highly unstable over horizons of three to five years. In our examples, risk parity levered to the volatility of a traditional 60/40 portfolio, and bonds levered to the volatility of stocks, the covariances were negative over our 84-year horizon (1929–2012); as a consequence, fixed-leverage versions outperformed the volatility-targeting versions of those strategies. We find no evidence that traditional fully-invested strategies such as 60/40 underperformed risk parity strategies with comparable volatility.

Key terms: Leverage; source portfolio; trading cost; magnified source return; excess borrowing return; risk parity; pension fund; fixed leverage; dynamic leverage; volatility target

1 Introduction

Even among the most conservative and highly regulated investors such as US public pension funds, the use of levered investment strategies is widespread and growing.[1] In the period since the financial crisis, strategies such as risk parity that explicitly lever holdings of publicly traded securities have emerged as candidates for these investment portfolios.[2]

In the single-period Capital Asset Pricing Model (CAPM), the market portfolio is the unique portfolio of risky assets that maximizes the Sharpe ratio. Leverage serves only as a means to travel along the efficient frontier. Both excess return and volatility scale linearly with leverage, and a rational investor will lever or de-lever the market portfolio in accordance with his or her risk tolerance.

Empirically, certain low-volatility portfolios exhibit higher Sharpe ratios than does the market portfolio,[3] which suggests that levering a low-volatility source portfolio could deliver an attractive risk-return tradeoff. However, market frictions and the correlations that arise in multi-period models make the relationship between the realized return of a levered portfolio and the Sharpe ratio of its source portfolio both nuanced and complex. Levered portfolios tend to have substantially higher transaction costs than traditional strategies.[4]

We develop an exact performance attribution for levered strategies that takes into account market frictions. Specifically, we show that there are five important elements to cumulative return. The first element is the return to the fully invested portfolio to be levered, which we call the *source portfolio*. The second element is the expected return to the source in excess of the borrowing rate, amplified by leverage minus one. We call the sum of these terms the *magnified source return*, and it represents the performance of a levered strategy in an idealized world. In the real world, the magnified source return is enhanced or diminished by the covariance between leverage and excess borrowing return, which is the third element of cumulative return of a levered strategy. The covariance term is highly unstable on medium horizons of three to five years and can serve to make a levered strategy seem particularly enticing or disappointing, depending on past performance. Prospectively, the covariance term appears to be unforecastable over a medium horizon, so it adds considerable noise to returns. The fourth and fifth elements, the cost of trading and the variance drag, are familiar to many investors. We penalize trading according to a linear model and we estimate the variance drag, which is effectively the difference between arithmetic and geometric return, using a formula that is adapted from Booth and Fama (1992).

Section 2 provides the foundation for our performance attribution, which is derived in

---

[1]See, for example, Kozlowski (2013).

[2]Sullivan (2010) discusses the risks that a pension fund incurs by employing a levered strategy.

[3]See, for example, Anderson et al. (2012).

[4]We acknowledge that investment returns are often reported gross of fees and transaction costs. That practice may be reasonable in comparing strategies with roughly equal fees or transaction costs, but it is entirely inappropriate in comparing strategies with very different fees or transaction costs.

Subsection 2.1 and illustrated in Subsection 2.2. Our basic example is a risk parity strategy that targets a fixed volatility. In Section 3, we consider more risk parity strategies, two that target fixed levels of leverage and one that targets conditional volatility. The four risk parity strategies we consider lever a common source portfolio, so it is straightforward to compare the attributions of the strategies. We find that along a number of different dimensions, dynamically levered risk parity strategies underperform fixed levered benchmarks (whose covariance terms are zero by construction).

Following the concluding remarks in Section 4 are a number of appendices that support our main narrative. Appendix A provides a detailed overview of the literature on low-risk investing and leverage. Appendix B describes the data in enough detail to allow researchers to replicate our results. Appendix C describes our linear trading model. Appendix D derives our approximation of geometric return from simple period return. As illustrated in our empirical examples, this approximation has a high degree of accuracy in practical situations. Finally, Appendix E describes a second family of levered strategies in which the source portfolio is a bond index and the target volatility is determined by an equity index. This example is qualitatively similar to the risk parity example considered in the main text. However, the results are more dramatic in the bonds-to-stocks example since the leverage is higher on average and more volatile.

2 The Impact of Leverage on the Return to an Investment Strategy

Leverage magnifies return, but that is only one facet of the impact that leverage has on an investment strategy. Leverage requires financing and exacerbates turnover, thereby incurring transaction costs. It amplifies the variance drag on cumulative return due to compounding. When leverage is dynamic, it can add substantial noise to strategy return. We provide an exact attribution of the cumulative return to a levered strategy that quantifies these effects.

A *levered strategy* is built from a fully invested *source portfolio*, presumably chosen for its desirable risk-adjusted returns, and a *leverage rule*. An investor has a certain amount of capital, L. The investor chooses a leverage ratio λ, borrows $(\lambda - 1)L$, and invests λL in the source portfolio.[5]

In what follows, we assume $\lambda > 1$.

---

[5]Leverage may be achieved through explicit borrowing. It may also be achieved through the use of derivative contracts, such as futures. In these derivative contracts, the borrowing cost is implicit rather than explicit, but it is real and is typically at a rate higher than the T-Bill rate. For example, Naranjo (2009) finds that the implicit borrowing cost using futures is approximately the applicable LIBOR rate, applied to the notional value of the futures contract.

2.1 Attribution of Arithmetic and Geometric Return

In the absence of market frictions, the relationship between the single-period return to a levered portfolio, $r^{\mathbf{L}}$, and to its source portfolio, $r^{\mathbf{S}}$, is given by:

$$r^{\mathbf{L}} = \lambda r^{\mathbf{S}} - (\lambda - 1)r^f, \tag{1}$$

where r^f is the risk-free rate. Note that the excess return is given by:

$$r^{\mathbf{L}} - r^f = \lambda \left(r^{\mathbf{S}} - r^f \right) \tag{2}$$

so that excess return and volatility scale linearly in λ for $\lambda \geq 0$, just as in the single-period CAPM. The only difference is that the source portfolio need not be the market portfolio. It follows that:

$$r^{\mathbf{L}} = r^{\mathbf{S}} + (\lambda - 1) \left(r^{\mathbf{S}} - r^f \right). \tag{3}$$

Formula (3) shows that the levered portfolio with $\lambda > 1$ will outperform the source portfolio when the source return exceeds the risk-free rate, but not otherwise.

When the borrowing rate, r^b, exceeds the risk-free rate, the relationship between the return to a levered portfolio and the return to its source is:

$$r^{\mathbf{L}} = r^{\mathbf{S}} + (\lambda - 1) \left(r^{\mathbf{S}} - r^b \right). \tag{4}$$

Note that the bar for leverage to have a positive impact on return has gotten higher: the excess borrowing return, $r^{\mathbf{S}} - r^b$, must be positive. Note also that the excess borrowing return of the levered strategy is

$$r^{\mathbf{L}} - r^b = \lambda \left(r^{\mathbf{S}} - r^b \right) \tag{5}$$

It is the excess *borrowing* return and volatility that scale linearly in leverage, for $\lambda \geq 1$. The Sharpe ratio is a *decreasing* function of leverage for $\lambda \geq 1$.

The expected return to a levered strategy is estimated by taking the expectation of Formula (4) over multiple periods:

$$E\left[r^{\mathbf{L}}\right] = E\left[r^{\mathbf{S}}\right] + E\left[\lambda - 1\right] E\left[r^{\mathbf{S}} - r^b\right] + \operatorname{cov}(\lambda, r^{\mathbf{S}} - r^b) \tag{6}$$

We use the term *magnified source return* to denote the sum of the first two terms on the right side of Formula (6). That formula shows that the expected return to a levered strategy is equal to the magnified plus a covariance correction. We fine empirically that, even when the correlation between leverage and excess borrowing return is quite small, the covariance correction can be substantial in relation to the magnified source return.

We can interpret the expectation and covariance in Formula (6) in two ways: prospectively and retrospectively. Prospectively, they represent the expectation and covariance

under the true probability distribution. Retrospectively, they represent the realized mean and realized covariance of the returns.[6]

Also important over multiple periods is the cost of trading, which imposes a drag $r^{\mathbf{TC}}$ on any strategy: To take account of this effect, we extend Formula (6):

$$
\begin{aligned}
E\left[r^{\mathbf{L}}\right] &= E\left[r^{\mathbf{S}}\right] + E\left[\lambda - 1\right] E\left[r^{\mathbf{S}} - r^{b}\right] + \mathrm{cov}(\lambda, r^{\mathbf{S}} - r^{b}) - E\left[r^{\mathbf{TC}}\right] \\
&= E\left[r^{\mathbf{S}}\right] + E\left[\lambda - 1\right] E\left[r^{\mathbf{S}} - r^{b}\right] + \mathrm{cov}(\lambda, r^{\mathbf{S}} - r^{b}) - \left(E\left[r^{\mathbf{TCS}}\right] + E\left[r^{\mathbf{TCL}}\right]\right) \quad (7)
\end{aligned}
$$

where $r^{\mathbf{TC}}$ is expressed as a sum of trading costs due to turnover in the source portfolio and trading costs due to leverage-induced turnover:

$$
r^{\mathbf{TC}} = r^{\mathbf{TCS}} + r^{\mathbf{TCL}}.
$$

Estimates of $r^{\mathbf{TC}}$ and its components rely on assumptions about the relationship between turnover and trading cost. We assume cost depends linearly on the fraction of the portfolio that turns over, and we use Formulas (14) and (15) to estimate $r^{\mathbf{TC}}$ in our empirical studies. More information is in Appendix C.

Formula (7) is based on arithmetic expected return, which does not correctly account for compounding. The correction for compounding imposes a variance drag on cumulative return that affects strategies differentially; for any given source portfolio, the variance drag is quadratic in leverage. If the levered strategy has high volatility, the variance drag may be substantial.

If we have monthly returns for months $t = 0, 1 \ldots, T - 1$ the realized geometric average of the monthly returns is:

$$
G[r] = \left(\prod_{t=0}^{T-1} (1 + r_t)\right)^{1/T} - 1 \quad (8)
$$

where r_t is the arithmetic return in month t. Given two strategies, the one with the higher realized geometric average will have higher realized cumulative return. In Appendix D, we show that the following holds to a high degree of approximation:[7]

$$
G\left[r\right] \sim (1 + E\left[r\right]) e^{-\frac{\mathrm{var}(r)}{2}} - 1 \quad (9)
$$

Note that the correction depends only on the realized variance of return.[8] Booth and Fama (1992) provide a correction for compounding based on continuously compounded

---

[6]Note that we take the realized covariance, obtained by dividing by the number of dates, rather than the realized sample covariance, which would be obtained by dividing by one less than the number of dates. We use the realized covariance because it makes Formula (6) true.

[7]The magnitude of the error is estimated following Formula (19). Note that E and G denote realizations of the average arithmetic and average geometric return, respectively. $\mathrm{var}(r)$ denotes the realized variance of r, rather than the realized sample covariance.

[8]In an earlier version of this paper, we indicated, incorrectly, that both the level and the variability of volatility determine the magnitude of the variance drag.

return; our correction for the geometric average of monthly returns in Formula (9) is slightly simpler.

Thus, in comparing the realized returns of strategies, the magnified source return of the levered strategy must be adjusted for three factors that arise only in the multi-period setting: the covariance correction, the variance drag, and trading costs.[9]

2.2 Empirical Example: Performance Attribution of a Levered Risk Parity Strategy

We demonstrate the utility of the performance attribution detailed above in the context of UVT (11.59%), a risk parity strategy that is rebalanced monthly and levered to an unconditional volatility target of 11.59%.[10] The source portfolio is unlevered risk parity based on two asset classes, US Equity and US Treasury Bonds. The target volatility of 11.59% is the realized volatility of the target portfolio, a 60/40 fixed-mix, between January 1929 and December 2012.[11] Foresight is required in order to set this target: the volatility of the 60/40 strategy is not known until the end of the period.[12] The information required to replicate our strategy is in Appendix B.

Figure 1 shows the magnified source return and the realized cumulative return to UVT (11.59%), as well as the realized cumulative return to its source portfolio (fully invested risk parity) and target (60/40 fixed mix). All computations assume leverage is financed at the Eurodollar deposit rate. The realized cumulative returns are also based on the assumption in addition that trading is penalized according to the linear model described in Appendix C, and take into account the covariance correction and variance drag on cumulative return. The magnified source return of UVT (11.59%) easily beats the cumulative return of both the source and the target; however, the realized cumulative return of UVT (11.59%) is well below the realized cumulative return of the 60/40 target portfolio (with essentially equal volatility (11.58%) and only slightly better than *unlevered*

---

[9]Note that the source and target portfolios may incur their own trading costs, as well as benefit from volatility pumping. The performance attribution of Formula (7) uses the source return and magnified source return, *gross* of trading costs. When we report historical arithmetic returns to the source and target portfolio, we report these net of trading costs, and inclusive of any benefit from volatility pumping. When we report cumulative returns to the source and target portfolios, we report these net of the variance drag.

[10]The leverage is chosen so that the volatility, gross of trading costs, is exactly 11.59%. When trading costs are taken into account, the realized volatility is slightly lower: 11.54%.

[11]UVT (11.59%) is constructed in effectively the same way as the levered risk parity strategy in Asness et al. (2012), with one main difference. They levered risk parity to match the volatility of the market, which had higher volatility than 60/40. They then compared the return of their levered strategy to the return of the market and the return of 60/40. Since 60/40 is a common default strategy for pension and endowment portfolios, while the market is not, 60/40 is a more appropriate volatility target; we are grateful to Patrice Boucher for this insight.

[12]Normalization is discussed further in Section 3.

risk parity source portfolio, which has much lower volatility (4.20%).[13]
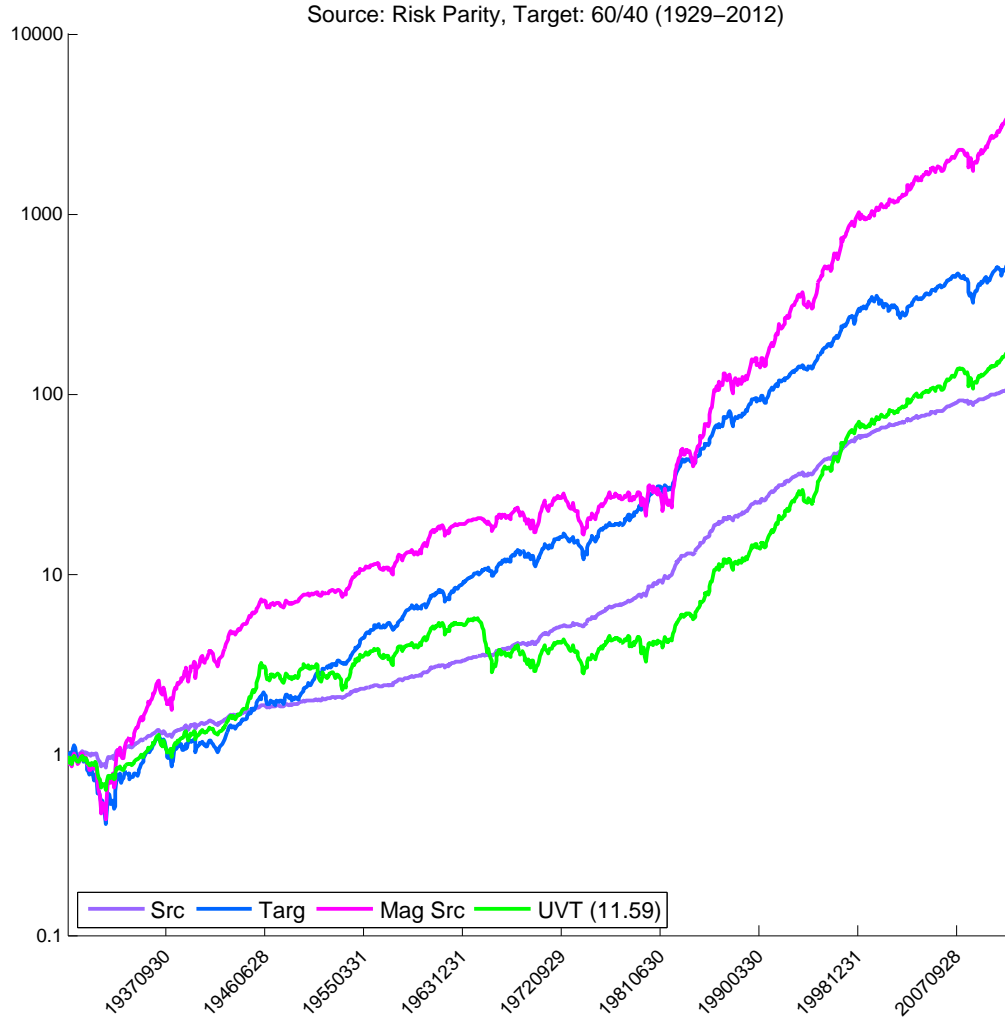
Figure 1: Magnified source return (in magenta) and realized cumulative return (in light green) for UVT (11.59%) (risk parity unconditionally levered to a target volatility of 11.59%) over the period 1929–2012. For comparison, we also plot the realized cumulative return of the volatility target (60/40 fixed mix, in blue) and the source (fully invested risk parity, in lavender).

The return decomposition Formulas (7) and (9) provide a framework for analyzing the performance of UVT (11.59%). Table 1 provides the required information. Consider first the magnified source return. The source portfolio had an annualized arithmetic return of 5.75% gross of trading costs.[14] Leverage added an extra 3.97% to annualized return

---
[13]The volatilities are reported in Table 2.

[14]Trading costs subtract only 7 basis points per year from the source return.

from the magnification term, the average excess borrowing return to the source portfolio multiplied by average leverage minus one. The annualized magnified source return is thus 9.72%. However, the covariance between leverage and excess borrowing return reduced the annualized return by 1.84%, trading costs by 96 basis points, and variance drag by a further 48 basis points. Together, these three effects eat up 3.28% of the 3.97%, or 82.6%, of the contribution of leverage to the magnified source return.

Table 1: Performance Attribution

		UVT (11.59%)
Sample Period: 1929-2012 Source: Risk Parity, Target: 60/40 r^b = 3M-EDR, with trading costs		
$\mathbb{E}[\mathbf{r^S}]$ (gross of trading costs)		**5.75**
$\mathbb{E}[\lambda - 1]$	2.66	
$\mathbb{E}[r^S - r^b]$	1.49	
$\mathbb{E}[\lambda - 1] \cdot \mathbb{E}[r^S - r^b]$		3.97
$\mathbb{E}[\mathbf{r^S}] + \mathbb{E}[\mathbf{\lambda - 1}] \cdot \mathbb{E}[\mathbf{r^S - r^b}]$		**9.72**
$\sigma(\lambda)$	7.7212	
$\sigma(r^S - r^b)$	4.2219	
$\rho(\lambda, r^S - r^b)$	-0.0566	
$\mathbf{Cov}(\lambda, r^S - r^b)$		-1.84
$-\mathbb{E}[r^{TCS}]$		-0.07
$-\mathbb{E}[r^{TCL}]$		-0.96
$\mathbb{E}[\mathbf{r^L}]$		**6.85**
$(1 + \mathbb{E}[r^L]/1200)^{12}$	1.0707	
$\exp(-\sigma^2_{rL}/2)$	0.9934	
$[(1 + \mathbb{E}[r^L]/1200)^{12} \cdot \exp(-\sigma^2_{rL}/2) - 1] \cdot 100 - \mathbb{E}[r^L]$		-0.48
Approximation Error		0.00
$\mathbb{G}[\mathbf{r^L}]$		**6.37**

Table 1: Performance attribution of the realized geometric return of the levered strategy UVT(11.59%) in terms of its source portfolio, risk parity, over the period January 1929–December 2012. The performance attribution is based on Formulas (7) and (9). Borrowing is at the Eurodollar deposit rate and trading costs are based on the linear model in Appendix C. Arithmetic returns are estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and are annualized by $(1 + G[r])^{12} - 1$.

3 Benchmarks for a Levered Strategy

3.1 Fully Invested Benchmarks

There would be no rational reason to pursue a levered strategy if traditional, fully-invested strategies were to offer superior risk-return characteristics. Table 2 gives annualized arithmetic and geometric return, volatility and Sharpe ratio to UVT (11.59%), its source, and its target. UVT (11.59%) had annualized geometric return only 63 basis points higher than the source portfolio, unlevered risk parity.[15] At the same time, the source portfolio had a much lower volatility (4.20%). As a result, UVT (11.59%) had a Sharpe ratio of 0.29, compared to 0.52 for unlevered risk parity. Note that the high Sharpe ratio of unlevered risk parity is obtained at the cost of low expected return. Risk parity is attractive only if the high Sharpe ratio can be preserved under leverage, but UVT (11.59%) fails in this regard.

60/40 and UVT (11.59%) had essentially equal volatilities, but 60/40 delivered an annualized geometric return of 7.77% and had a Sharpe ratio of 0.40. The analogous figures were 6.37% and .29 for UVT (11.59%). UVT (11.59%) also had a negative skew (-0.43) compared to a slightly positive skew for 60/40. In its favor, UVT (11.59%) does have a substantially smaller excess kurtosis than 60/40.

---

[15]Note that the annualized geometric return of the source portfolio, 5.74% slightly exceeds 5.68%, the annualized arithmetic return of the source portfolio, net of trading costs. This is an artifact of the annualization procedures for arithmetic and geometric return. The source portfolio has monthly arithmetic return of 47.3 basis points, net of transaction costs. The latter is annualized by multiplying by 12: $12 \times 0.473\% = 5.68\%$. Annualized geometric return takes into account compounding: $1.00473^{12} - 1 = 5.83\%$. The variance drag reduces this by 9 basis points to 5.74%. Note that the variance drag on the source return is much smaller than the variance drag on the levered portfolios, because the source portfolio is so much less volatile and the variance drag is quadratic in volatility.

Table 2: Historical Performance

Sample Period: 1929-2012 Source: Risk Parity, Target: 60/40 r^b = 3M-EDR	Arithmetic Total Return	Geometric Total Return	Average Leverage	Volatility	Arithmetic Excess Return	Sharpe Ratio	Skewness	Excess Kurtosis
60/40	8.18	7.77	1.00	11.58	4.69	0.40	0.19	7.44
Risk Parity	5.68	5.74	1.00	4.20	2.20	0.52	0.05	4.92
UVT (σ = 11.59%)	6.85	6.37	3.66	11.54	3.36	0.29	-0.43	2.23

Table 2: Annualized arithmetic and geometric returns, volatility and Sharpe ratio, of UVT (11.59%) (UVT (11.59%) (risk parity levered to an unconditional volatility target of 11.59%, the realized volatility of 60/40), the source portfolio (unlevered risk parity), and the volatility target (60/40) over the period 1929–2012. Arithmetic returns are estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and are annualized by $(1 + G[r])^{12} - 1$. Volatility is measured from monthly returns and annualized by multiplying by $\sqrt{12}$. Sharpe ratios are calculated using annualized excess return and annualized volatility.

3.2   Fixed Leverage Benchmarks

Like any volatility targeting strategy, UVT (11.59%) is dynamically levered. However, as we saw in Section 2.2, the covariance between leverage and excess borrowing return diminished annualized arithmetic return by 1.84%. Deeper insight into this cost is provided in Table 1, which decomposes these covariances into products of correlation and standard deviations. Note that the magnitude of the correlation between leverage and excess borrowing return is small: -0.056. A small change in that correlation could have flipped the sign and turned a negative contribution into a positive contribution. Figure 2 shows rolling 36-month estimates of the correlation between leverage and excess borrowing return, and indicates that the sign of the correlation flips easily; at investment horizons of three to five years, the main effect of the covariance term appears to be to add noise to the returns. This may give pause to investment managers whose jobs or bonuses are tied to performance relative to benchmarks.
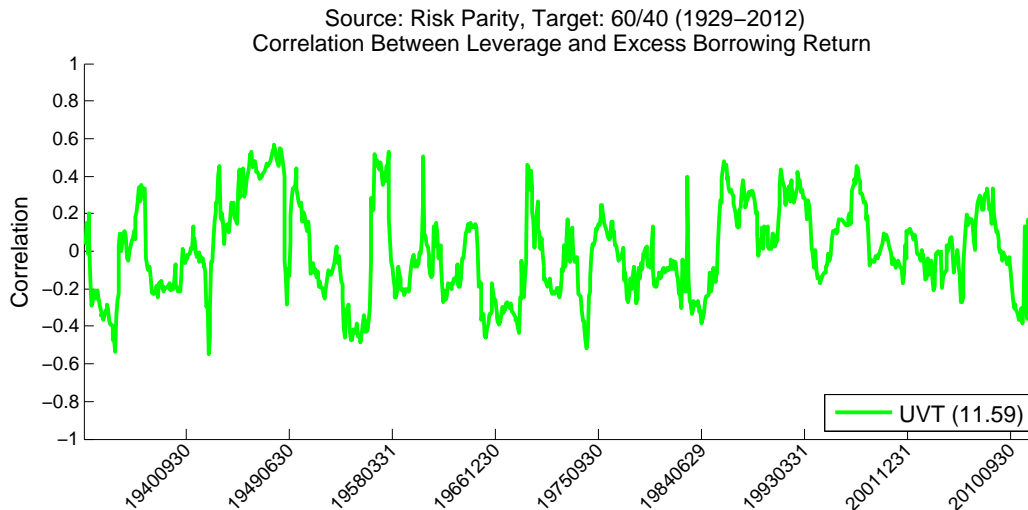
Figure 2: Correlation of excess borrowing return and leverage for UVT (11.59%), risk parity levered to match the realized volatility of 60/40 fixed mix over the period 1929–2012. Correlation is computed from monthly data using a trailing 36-month window.

An investor who wants to lever but does not want exposure to the unstable correlation between leverage and excess borrowing return can opt for a fixed leverage target (FLT). When leverage is fixed, the covariance between leverage and excess borrowing return must be zero. FLT (3.69) matches the average leverage of UVT (11.59%), but has higher volatility, while FLT(2.75) matches the volatility of UVT (11.59%) but has lower leverage.

A conditional volatility targeting strategy, CVT, levers fully invested risk parity so that the projected volatility (based on the previous 36 months returns) equals the volatility of the target 60/40 over the previous 36 months.[16]

Table 3 provides performance attributions for all four strategies. The covariance term plays a substantial role in the cumulative return ranking of the strategies. Note also that the two FLT strategies incur lower trading costs than UVT (11.59%) and CVT.[17] As a result, the geometric returns of FLT (3.69), FLT (2.75) and CVT outperform the geometric return of UVT (11.59%) by 192, 125 and 66 basis points, respectively.

---

[16]CVT was introduced in Anderson et al. (2012), and criticized by Asness et al. (2013).

[17]As discussed in Section 3.3 below, even maintaining a fixed leverage requires trading. It is possible in principle that the trading needed to adjust leverage to meet a volatility target could offset some of the trading required to maintain fixed leverage, but this strikes us as unlikely in typical situations.

Table 3: Performance Attribution

Sample Period: 1929-2012 Source: Risk Parity, Target: 60/40 r^b = 3M-EDR, with trading costs	UVT (11.59%)		FLT (3.69)		FLT (2.75)		CVT	
$\mathbb{E}[\mathbf{r^S}]$ (gross of trading costs)		**5.75**		**5.75**		**5.75**		**5.75**
$\mathbb{E}[\lambda - 1]$	2.66		2.69		1.75		2.31	
$\mathbb{E}[r^S - r^b]$	1.49		1.49		1.49		1.49	
$\mathbb{E}[\lambda - 1] \cdot \mathbb{E}[r^S - r^b]$		3.97		4.02		2.61		3.45
$\mathbb{E}[\mathbf{r^S}] + \mathbb{E}[\mathbf{\lambda-1}] \cdot \mathbb{E}[\mathbf{r^S-r^b}]$		**9.72**		**9.77**		**8.37**		**9.20**
$\sigma(\lambda)$	7.7212		0.0000		0.0000		5.0791	
$\sigma(r^S - r^b)$	4.2219		4.2219		4.2219		4.2219	
$\rho(\lambda, r^S - r^b)$	-0.0566		0.0000		0.0000		-0.0299	
$\mathbf{C}ov(\lambda, r^S - r^b)$		-1.84		0.00		0.00		-0.64
$-\mathbb{E}[r^{TCS}]$		-0.07		-0.07		-0.07		-0.07
$-\mathbb{E}[r^{TCL}]$		-0.96		-0.51		-0.27		-0.93
$\mathbb{E}[\mathbf{r^L}]$		**6.85**		**9.19**		**8.03**		**7.56**
$(1 + \mathbb{E}[r^L]/1200)^{12}$	1.0707		1.0959		1.0833		1.0783	
$\exp(-\sigma^2_{r^L}/2)$	0.9934		0.9881		0.9934		0.9926	
$[(1 + \mathbb{E}[r^L]/1200)^{12} \cdot \exp(-\sigma^2_{r^L}/2) - 1] \cdot 100 - \mathbb{E}[r^L]$		-0.48		-0.91		-0.41		-0.53
Approximation Error		0.00		0.01		0.01		0.00
$\mathbb{G}[\mathbf{r^L}]$		**6.37**		**8.29**		**7.62**		**7.03**

Table 3: Performance attribution of the realized geometric return of the levered strategies UVT (11.59%), FLT (3.69), FLT (2.75), and CVT in terms of their common source portfolio, risk parity, over the period January 1929–December 2012. FLT (3.69) has constant leverage 3.69, matching the average leverage of UVT (11.59%), while FLT (2.75) has constant leverage 2.75 chosen to match the volatility of UVT (11.59%). The performance attribution is based on Formulas (7) and (9). Borrowing is at the Eurodollar deposit rate and trading costs are based on the linear model in Appendix C. Arithmetic returns are estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and are annualized by $(1 + G[r])^{12} - 1$.

3.3 Attributes of Levered Strategies

The parameters of the UVT and two FLT levered strategies were set with foresight. The dynamically levered strategy UVT (11.59%) is based on the volatility of a 60/40 fixed mix between January 1929 and December 2012. That volatility is known only at period end even though it is used to make leverage decisions throughout the period. The FLT (3.69) leverage is set to match the average leverage of UVT (11.59%) and the FLT (2.75) leverage is set so that the volatility matches the volatility of UVT (11.59%).

CVT, introduced in Section 3.2, does not rely on future information to set leverage.[18] As a result, its realized volatility fails to match the realized volatility of the target. At each monthly rebalancing, CVT is levered to match the volatility of the 60/40 fixed mix; both volatilities are estimated using a 36-month rolling window.

---

[18]The foresight in the definitions of UVT and the two FLT strategies allow them to exactly match their volatility or leverage targets, gross of trading costs. Since CVT does not rely on foresight, it will not exactly match the realized target volatility, gross of trading costs. Both UVT and CVT volatility and FLT leverage are further affected by trading costs.

All else equal, UVT, FLT and CVT call for additional investment in the source portfolio when its price rises. A decline in the value of the source portfolio reduces the net value of the levered portfolio, while keeping the amount borrowed constant; leverage has increased, and rebalancing requires selling the source portfolio to return to leverage λ. Similarly, an increase in the value of the source portfolio results in taking on more debt and using the proceeds to buy more of the source portfolio. In this sense, the UVT, FLT and CVT with $\lambda > 1$ are momentum strategies. UVT, FLT and CVT respond differently to changes in asset volatility; see Table 4.

Table 4: Strategy Responses to Changes in Market Conditions

| | Response: | | |
Trigger	FLT	UVT	CVT
Increase in Target Volatility	no change	no change	↑ leverage
Increase in Source Volatility	no change	↓ leverage	↓ leverage
Increase in Price of Source	buy source	buy source	buy source

Table 4: Responses of levered strategies to changes in market conditions.

3.4 Historical Performance of the Various Levered and Fully Invested Strategies

Table 5 summarizes the historical performance of the source (unlevered risk parity), volatility target (fully invested 60/40) and the four levered strategies UVT (11.59%), FLT (3.69), FLT (2.75) and CVT. Unlevered risk parity has the highest Sharpe ratio (0.52), but its low expected return makes it unattractive. The two FLT strategies and 60/40 have nearly identical Sharpe ratios (.37, .39 and .40); FLT (3.69) has higher return and higher volatility than 60/40, while FLT(2.75) closely matches 60/40 in return and volatility. UVT (11.59%) and CVT have lower Sharpe ratios (0.29 and 0.33, respectively), primarily because of the effect of the negative covariance term and trading costs. UVT and CVT also have negative skew (-0.43 and -0.41, respectively). In its favor, UVT (11.59%) has the smallest excess kurtosis, while CVT and 60/40 are roughly tied for the highest excess kurtosis.

Table 5: Historical Performance

Sample Period: 1929-2012 Source: Risk Parity, Target: 60/40 r^b = 3M-EDR	Arithmetic Total Return	Geometric Total Return	Average Leverage	Volatility	Arithmetic Excess Return	Sharpe Ratio	Skewness	Excess Kurtosis
60/40	8.18	7.77	1.00	11.58	4.69	0.40	0.19	7.44
Risk Parity	5.68	5.74	1.00	4.20	2.20	0.52	0.05	4.92
UVT ($\sigma = 11.59\%$)	6.85	6.37	3.66	11.54	3.36	0.29	-0.43	2.23
FLT ($\lambda = 3.69$)	9.19	8.29	3.69	15.53	5.70	0.37	-0.01	4.77
FLT ($\lambda = 2.75$)	8.03	7.62	2.75	11.57	4.54	0.39	0.00	4.80
CVT	7.56	7.03	3.31	12.21	4.07	0.33	-0.41	7.13

Table 5: Annualized arithmetic and geometric returns, volatility and Sharpe ratio, of the volatility target (60/40), the source portfolio (unlevered risk parity), UVT (11.59%) (risk parity levered to an unconditional volatility target of 11.59%, the realized volatility of 60/40), FLT (3.69) (risk parity levered to constant 3.69, the average leverage of UVT (11.59%)), FLT (2.75) (risk parity levered to match the volatility of UVT (11.59%), and CVT (risk parity conditionally levered to match the current volatility of 60/40), over the period 1929–2012. Arithmetic returns are estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and are annualized by $(1+G[r])^{12}-1$. Volatility is measured from monthly returns and annualized by multiplying by $\sqrt{12}$. Sharpe ratios are calculated using annualized excess arithmetic return and annualized volatility.

In Table 6, we summarize the ranking of these six strategies under three investment criteria: realized geometric return, Sharpe ratio, and the probability of meeting an 8% return target. That probability is determined using a bootstrapping procedure. We took the returns of each of our strategies in the 1008 months in our 84-year period. We drew 10,000 bootstraps, each of which sampled 1008 monthly returns with replacement, and calculated the probability that the annual geometric return of a bootstrap exceeded 8%. Note that both FLT (3.69) and FLT (2.75) outperform CVT on all three criteria, while CVT outperforms UVT (11.59%) on all three criteria. Unlevered risk parity has the highest Sharpe ratio but the worst geometric return and the worst probability of meeting the 8% return target.

Table 6: Ranking of Strategies by Three Criteria

	Geometric Return	Sharpe Ratio	P(Meeting 8% Return Target)
Source (Risk Parity)	6	1	6
Target (60/40)	2	2	2
UVT (11.59%)	5	6	5
FLT (3.69)	1	4	1
FLT (2.75)	3	3	3
CVT	4	5	4

Table 6: Ranking of Strategies by Geometric Return, Sharpe ratio, and the probability of meeting an 8% return target. That probability was determined by bootstrapping. Each bootstrap sampled 1008 months with replacement. We carried out 10,000 bootstraps, and calculated the probability that the annual geometric return exceeded 8%.

We have seen that the covariance term, which appears to be unpredictable at short to medium time horizons, is sufficiently large to have a notable effect on realized return. This does not rule out the possibility that the covariance terms could be systematically positive or negative when computed over long horizons, such as 25 or 50 years. This will be the subject of future research.

Finally, we note that the history provides no evidence that any of these levered risk parity strategies will systematically outperform 60/40.

4   Conclusion

In this article, we develop a platform that supports both backward-looking performance attribution and forward-looking investment decisions concerning levered strategies. Specifically, in Formula (7), we express the difference between arithmetic expected return to a levered strategy portfolio and its source portfolio as a sum of four terms:

$$E\left[r^{\mathbf{L}}\right] = E\left[r^{\mathbf{S}}\right] + E\left[\lambda - 1\right] E\left[r^{\mathbf{S}} - r^{b}\right] + \mathrm{cov}(\lambda, r^{\mathbf{S}} - r^{b}) - \left(E\left[r^{\mathbf{TCS}}\right] + E\left[r^{\mathbf{TCL}}\right]\right).$$

The first two terms, whose sum we have called magnified source return, are the ones that most easily come to mind in the context of a levered strategy. However, as we have shown empirically, other factors have a material effect on the cumulative return to a levered strategy. These are: covariance of leverage with the excess borrowing return, trading costs and compounding effects.

Formula (7) accounts for both the covariance term and transaction costs. However, it neglects the effect of compounding, which imposes a variance drag on cumulative return that is not captured in arithmetic expected return. If the levered strategy has high volatility, the variance drag may be substantial. Hence a more accurate decision rule depends

on geometric expected return in Formula (9):

$$G[r] \sim (1 + E[r])e^{-\frac{\text{var}(r)}{2}} - 1.$$

In this paper, we examine the realized performance of four levered strategies: two fixed leverage targeting (FLT), one unconditional volatility targeting (UVT) and one conditional volatility targeting (CVT). Some scholars have expressed the view that CVT is a poor strategy compared to UVT;[19] this view is not supported by the results reported in Tables 3, 5 and 6. In fact, it is the leverage that is implicitly determined by the volatility targets in UVT and CVT, and not the volatility itself, that interacts with the return to the source portfolio to determine strategy performance. In our 1929–2012 period, CVT outperforms UVT. As noted in Section 3.4, future work will explore whether the sign of the covariance term might be predictable at longer horizons.

Our analysis provides no evidence that any of the levered risk parity strategies systematically outperforms 60/40 over the long horizon.

Risk parity performed relatively well over the period 2008–2012, which featured Fed-supported interest rates that are extraordinarily low by historical standards. As quantitative easing comes to an end, the cost of funding a levered strategy will rise dramatically, and historical precedent suggests that the impact may well be amplified by declines in asset prices. These considerations should be incorporated in any decision to lever low-risk portfolios whenever interest rates are unusually low.

Appendices

A   Related Literature

A.1   CAPM

Finance continues to draw heavily on the Capital Asset Pricing Model (CAPM) developed in Treynor (1962), Treynor and Black (1976), Sharpe (1964), Lintner (1965b), Lintner (1965a), Mossin (1966), and extended in Black and Litterman (1992).[20] Here, leverage is a means to adjust the level of risk in an efficient portfolio and nothing more. In contrast, Markowitz (2005) illustrates another facet of leverage in the context of a market composed of three coconut farms. In this disarmingly simple example, some investors are leverage-constrained and others are not. The market portfolio is mean-variance inefficient; as a result, no mean-variance investor chooses to hold it, and expected returns of assets do not depend linearly on market betas.

---

[19]See Asness et al. (2013).

[20]A history of the CAPM elucidating Jack Treynor's role in its development is in French (2003).

A.2 Measurement of Risk and Nonlinearities

An impediment to a clear understanding of leverage may be the way we measure its risk. Standard risk measures such as volatility, value at risk, expected shortfall, and beta scale linearly with leverage. But as we know from the collapse of Long Term Capital, the relationship between risk and leverage can be non-linear; see, for example, Jorion (2000). Föllmer and Schied (2002) and Föllmer and Schied (2011, Chapter 4) describe risk measures that penalize leverage in a super-linear way. Recent experience suggests that these measures may be useful in assessing the risk of levered strategies.

One contribution of this paper is to explain how the interaction between leverage and market frictions creates specific nonlinearities in the relationship between leverage and return. Understanding these specific nonlinearities provides a practical framework to guide the decision on whether and how to lever.

A.3 Motivations for Leverage

If investors are overconfident in their predictions of investment returns, they may find leverage attractive because it magnifies the returns when times are good, and because they underestimate the risk of bad outcomes.[21]

Perfectly rational investors may also be attracted to leverage by the low risk anomaly, the apparent tendency of certain low-risk portfolios to have higher risk-adjusted return than high-risk portfolios. An investor who believes in the low risk anomaly will be tempted to lever low-risk portfolios, in the hope of achieving high expected returns at acceptable levels of risk.

In a CAPM world, investors with below-average risk aversion will choose to lever the market portfolio.[22] The low risk anomaly provides a rational argument for investors with typical risk aversion to use leverage. Indeed, the low risk anomaly is arguably the only rational argument for an investor to use leverage in an investment portfolio composed of publicly traded securities.[23] Differences in risk aversion could explain some investors choosing higher expected return at the price of higher volatility, but there is little reason for a rational investor to choose *leverage* unless the source portfolio being levered offers superior *risk-adjusted* returns, at a volatility *below* the investor's risk tolerance.

---

[21]A positive relationship between overconfident CEOs and firm leverage is documented in Malmendier et al. (2011). Shefrin and Statman (2011) identify excessive leverage taken by overconfident bankers as a contributor to the global financial crisis.

[22]Note, however, that the market portfolio in CAPM includes bonds and other risky asset classes, rather than just stocks. Levered strategies include the use of margin, and futures and other derivatives, to assemble levered equity-only portfolios, which behave quite differently from levered portfolios in CAPM.

[23]There are, of course, other rational arguments for using leverage in other contexts. The leverage provided by a mortgage may be the only feasible way for a household to buy a house, which provides a stream of consumption benefits and tax advantages in addition to facilitating an investment in the real estate market. Companies leverage their shareholder equity with borrowing to finance operations, for a variety of reasons, including differences in risk aversion, informational asymmetries, and tax implications.

A.4   Levered Low-Risk Strategies

Low-risk investing refers to a diverse collection of investment strategies that emphasize low beta, low idiosyncratic risk, low volatility or downside protection. The collection of low-risk strategies includes broad asset allocations, but it also includes narrower strategies restricted to a single asset class. An early reference to low-risk investing is Markowitz (1952) who comments that a minimum-variance portfolio is mean-variance optimal if all assets returns are uncorrelated and have equal expectations. But low-risk strategies typically require leverage in order to meet expected return targets. In an exploration of this idea, Frazzini and Pedersen (2013) echo some of the conclusions in Markowitz (2005), and they complement theory with an empirical study of an implicitly levered equity risk factor that is long low-beta stocks and short high-beta stocks. This factor descends from Black et al. (1972), which provides evidence that the CAPM may not properly reflect market behavior.

A.5   Empirical Evidence on Levered Low-Risk Investing

There is a growing empirical literature indicating that market frictions may present investors from harvesting the returns promised by a frictionless analysis of levered low-risk strategies. Anderson et al. (2012) show that financing and trading costs can negate the abnormal profits earned by a levered risk parity strategy in a friction-free market. Li et al. (2013) and Fu (2009) show that market frictions may impede the ability to scale up the return of low-risk strategies through leverage.[24]

Asset allocation that is based on capital weights has a long and distinguished history; see, for example Graham (1949) and Bogle (2007). However, rules-based strategies that allocate risk instead of, or in addition to, capital are of a more recent vintage. Risk-based investing is discussed in Lörtscher (1990), Kessler and Schwarz (1996), Qian (2005), Clarke et al. (2011), Shah (2011), Sefton et al. (2011), Clarke et al. (2013), Anderson et al. (2012), Cowan and Wilderman (2011), Bailey and de Prado (2012), Goldberg and Mahmoud (2013) and elsewhere. Strategies that target volatility are also gaining acceptance, although the literature is still sparse. Goldsticker (2012) compares volatility targeting strategies to standard allocations such as fixed-mix, and finds that the relative performance of the strategies is period dependent.

A.6   The Effect of Leverage on Markets

Another important question is the extent to which leverage may *contribute* to market instability. See, for example, Brunnermeier and Pedersen (2009), Adrian and Shin (2010) and Geanakoplos (2010). We do not address that question here, as we restrict our analysis

---

[24]Ross (2004) provides an example of the limits to arbitraging mispricings of interest-only strips of mortgage backed securities.

to the effect of leverage on the return of investment strategies, taking the distribution of the underlying asset returns as given.

A.7   Arithmetic versus Geometric Return

Despite the large literature on the importance of compounding to investment outcomes, analyses of investment strategies are often based on arithmetic expected return. Background references on compounding and geometric return include Fernholz (2002) and MacLean et al. (2011). Perold and Sharpe (1988) discuss how the interplay among volatility, rebalancing and compound return causes a fully-invested fixed-mix or portfolio-insurance strategy to behave differently from a buy-and-hold strategy with the same initial mix. Booth and Fama (1992) work out the relationship between the compound return to a fixed-mix portfolio and its constituents, and their results are applied to portfolios that include commodities in Willenbrock (2011). Markowitz (2012) compares six different mean-variance approximations to geometric return.

B   Data

The results presented in this paper are based on CRSP stock and bond data from January of 1929 through December of 2012. The aggregate stock return is the CRSP value weighted market return (including dividends) from the table *Monthly Stock–Market Indices (NYSE/AMEX/NASDAQ)* – variable name *vwretd*. The aggregate bond return is the face value outstanding (cross-sectionally) weighted average of the unadjusted return for each bond in the *CRSP Monthly Treasury (Master)* table. In this table, the variable name for the unadjusted return is *retnua* and for the face value outstanding is *iout1r*. All bonds in the table are used, provided the values for both *retnua* and *iout1r* are not missing.

The proxy for the risk-free rate is the *USA Government 90-day T-Bills Secondary Market* rate, provided by Global Financial Data (`http://www.globalfinancialdata.com`), covering the period from January of 1929 through December of 2012. The proxy for the cost of financing leverage is the *U.S. 3-Month Euro-Dollar Deposit* rate, downloaded from the Federal Reserve (`http://www.federalreserve.gov/releases/h15/data.htm`). The 3-Month Euro-Dollar Deposit data is available from January of 1971 through December of 2012. Prior to January of 1971, a constant of 60 basis points is added to the 90-day T-Bill rate.[25] Trading costs are calculated using the procedure described in Appendix C. We assume the cost of trading is 100 basis points from 1926 to 1955, 50 basis points from 1956 to 1970, and 10 basis points from 1971 onward.

The process for constructing unlevered, UVT and CVT risk parity is exactly as detailed in Anderson et al. (2012). In the bonds-to-stocks example in Appendix E, the

---

[25]The average difference between the 90-day T-Bill Rate and the 3-Month Euro-Dollar Deposit Rate from 1971 through 2012 is 102 basis points. So our estimate of 60 basis points is relatively conservative.

construction of CVT is the same, but, the process for constructing UVT is slightly simpler; here, we choose a fixed volatility target (say, 11.59% per year) and rebalance to that target each period.

Anderson et al. (2012), following Asness et al. (2012), used the volatility of the market as the target for risk parity. Here, we use the volatility of 60/40 as the target, because it provides a more appropriate comparison to traditional strategies used by institutional investors. The return of UVT is particularly sensitive to the volatility target.

C   Trading Costs

We estimate the drag on return that stems from the turnover-induced trading required to maintain leverage targets in a strategy that levers a source portfolio S.

At time t, the strategy calls for an investment with a leverage ratio of λ_t. We make the harmless assumption that the value of the levered strategy at t, denoted L_t, is \$1.[26] Then the holdings in the source portfolio, or assets, are $A_t = \lambda_t$. The debt at time t is given by $D_t = \lambda_t - 1$.

We need to find holdings A_{t+1} in the portfolio at time $t + 1$ that are consistent with the leverage target λ_{t+1}. This turns out to be a fixed point problem since the trading costs must come out of the investor's equity. Between times t and $t + 1$, the value of the source portfolio changes from S_t to S_{t+1} and the strategy calls for rebalancing to achieve leverage λ_{t+1}. Just prior to rebalancing, the value of the investment is

$$A'_t = \lambda_t(1 + r_t^{\mathbf{S}}), \tag{10}$$

the liability has grown to $D'_t = (\lambda_t - 1)(1 + r_t^b)$ and the investor's equity is:

$$
\begin{aligned}
L'_t &= A'_t - D'_t \\
&= \lambda_t(1 + r_t^{\mathbf{S}}) - (\lambda_t - 1)(1 + r_t^b).
\end{aligned} \tag{11}
$$

Note that in Formulas (10) and (11), we use the source return $r_t^{\mathbf{S}}$ *gross* of trading costs in the source portfolio.

Let $w_t = (w_{t1}, \ldots, w_{tn})^\top$ denote the vector of relative weights assigned to the n asset classes in the source portfolio at time t, so that $\sum_{i=1}^{n} w_{ti} = 1$ for all t. Just prior to rebalancing, the weights have changed to $w'_t = (w'_{t1}, \ldots, w'_{tn})^\top$, where $w'_{ti} = \frac{w_{ti}(1+r_i^t)}{1+r_t^{\mathbf{S}}}$. At time $t + 1$, the strategy is rebalanced according to its rules, which produces holdings of $A_{t+1}w_{t+1}$ in the n asset classes. We let $x_t = (x_{t1}, \ldots, x_{tn})^\top$ denote the vector of dollar amounts of the changes in value due to rebalancing, so that:

$$x_t = A_{t+1}w_{t+1} - A'_t w'_t. \tag{12}$$

---

[26]This assumption is harmless in a linear model of trading costs, which we develop here. It would be inappropriate for a realistic model of market impact.

If we assume a linear model, the cost of trading x_t is $\kappa\|x_t\|_1 = \sum_{i=1}^{n} |x_{ti}|$ for some $\kappa \geq 0$. The cost reduces the investor's equity to:

$$
\begin{aligned}
L_{t+1} &= L_t' - \kappa\|x_t\|_1 \\
&= \lambda_t(1 + r_t^{\mathbf{S}}) - (\lambda_t - 1)(1 + r_t^b) - \kappa\|x_t\|_1.
\end{aligned}
\tag{13}
$$

Now let

$$
\begin{aligned}
g(\alpha) &= \frac{\alpha}{L_{t+1}} - \lambda_{t+1} \\
&= \frac{\alpha}{L_t' - h(\alpha)} - \lambda_{t+1},
\end{aligned}
$$

where $g(\alpha)$ denotes the leverage implied by holding αw_{t+1} in the n assets, taking into account the effect of trading costs on equity L_{t+1}, minus the desired leverage. Assuming that g is defined on the whole interval $[0, \lambda_{t+1} L_t']$, it is continuous, $g(0) = -\lambda_{t+1} < 0$, and $g(\lambda_{t+1} Lt')$, so by the Intermediate Value Theorem, there exists α_{t+1} such that $g(\alpha_{t+1}) = 0$.[27] The value of α_{t+1} can readily be found by a bisection algorithm, which worked well in all of the empirical situations studied in this paper.[28]

We set $A_{t+1} = \alpha_{t+1}$, so the holdings of the n assets are given by $A_{t+1} w_{t+1} = \alpha_{t+1} w_{t+1}$. The reduction in return due to trading costs is given by:

$$
r^{\mathbf{TC}} = \kappa\|\alpha_{t+1} w_{t+1} - A_t' w_t'\|_1.
\tag{14}
$$

We compute the trading cost incurred by the source portfolio, $E[r^{\mathbf{TCS}}]$ in the same way and define the trading cost due to leverage by

$$
E[r^{\mathbf{TCL}}] = E[r^{\mathbf{TC}}] - E[r^{\mathbf{TCS}}].
\tag{15}
$$

D   Geometric Return

In order to analyze the effects of compounding, Booth and Fama (1992) express continuously compounded return in terms of arithmetic return. We have chosen to analyze the effects of compounding using the geometric average of monthly returns. Our Formula (18) for the geometric average of monthly returns is somewhat simpler than the formula for continuously compounded return in Booth and Fama (1992). Both derivations rely on the second-order Taylor expansion approximation of the logarithm.

Let L_t denote the equity in a strategy at month t, where $t = 0, 1, \ldots, T$.

---

[27]Typically, α_{t+1} is uniquely determined; if not, choose the largest value satisfying the equation.

[28]If there is *no* α_{t+1} such that $g(\alpha) = 0$, it means the equity of the strategy is so low that the transaction costs in getting to the desired leverage wipe out the equity. We do not observe such severe drawdown in our empirical examples, but clearly it would be possible with extreme leverage or a very volatile source portfolio.

The correct ranking of realized strategy performance, taking compounding into account, is given by $G[r]$, the geometric average of the monthly returns, minus one:

$$
\begin{aligned}
G\left[r\right] &= \left(\frac{L_T}{L_0}\right)^{1/T} - 1 \\[2mm]
&= \left[\prod_{t=0}^{T-1} \frac{L_{t+1}}{L_t}\right]^{1/T} - 1 \\[2mm]
&= \left[\prod_{t=0}^{T-1} \left(1 + r_t\right)\right]^{1/T} - 1 \quad (16)
\end{aligned}
$$

Because the logarithm is strictly increasing, $\log\left(1 + G\left[r\right]\right)$ induces exactly the same ranking of realized strategy returns as $G[r]$. It is a different ranking than the one induced by $E[r]$ and $\log\left(1 + E\left[r\right]\right)$, requiring a correction term involving $\operatorname{var}(r)$:

$$
\begin{aligned}
\log\left(1 + G\left[r\right]\right) &= \frac{1}{T}\sum_{t=0}^{T-1}\log\left(1 + r_t\right) \\[2mm]
&\sim \frac{1}{T}\sum_{t=0}^{T-1}\left(r_t - \frac{\left(r_t\right)^2}{2}\right) \quad (17) \\[2mm]
&= \frac{1}{T}\sum_{t=0}^{T-1} r_t - \frac{1}{T}\sum_{t=0}^{T-1}\frac{\left(r_t\right)^2}{2} \\[2mm]
&= E\left[r\right] - \frac{\operatorname{var}\left(r\right) + \left(E\left(r\right)\right)^2}{2} \\[2mm]
&\sim \log\left(1 + E\left[r\right]\right) - \frac{\operatorname{var}\left(r\right)}{2} \quad (18) \\[2mm]
G\left[r\right] &\sim \left(1 + E\left[r\right]\right) e^{-\frac{\operatorname{var}(r)}{2}} - 1 \quad (19)
\end{aligned}
$$

Formulas (17) and (18) approximate the logarithm by its quadratic Taylor polynomial. When $r_t > 0$, the Taylor series for logarithm is alternating and decreasing in absolute value for $|r_t| < 1$, so the error in the approximation of $\log\left(1 + r_t\right)$ in Formula (17) is negative and bounded above in magnitude by $|r^t|^3/3$ for each month t. When $r_t < 0$, the error is positive and may be somewhat larger than $|r^t|^3/3$. Since the monthly returns are both positive and negative, the errors in months with negative returns will substantially offset the errors in months with positive returns, so the errors will tend not to accumulate over time. The approximation error in annual geometric return is at most one basis point in our risk parity examples (see Table 3) and ten basis points in our levered bond examples (see Table 7).

22

E Levering Bonds to Stocks

A previous version of this article used a different example to illustrate the decomposition of cumulative return to a levered strategy, which we report briefly here.[29] The source portfolio is a US bond index and the target is a US equity index.[30] The discrepancy between the source and target volatilities tends to be larger when levering a bond index to have equity-like volatility than in a risk parity strategy. As a result, the leverage is larger on average and more volatile, and the results are more dramatic. We updated the example, UVT (18.93%) to correspond to the normalizations in this paper. Over the period January 1929–December 2012, the magnification effect raised annual return by 6.35%. However, the covariance between leverage and excess borrowing return diminished annualized arithmetic return by 4.96%, leverage-induced trading costs by 3.42% year, and the volatility drag by 2.61%, for total leverage costs of 10.99%, or 173% of the leverage benefits. The annualized geometric return of UVT (18.93%) was 0.44%, compared to the 5.14% return of the source portfolio (bonds) and the 9.00% return of the target portfolio (stocks). Although the magnified source return of higher-leverage FLT (8.72) comfortably exceeded that of FLT (6.74), the trading costs and volatility drag reversed the ranking on geometric return. Both FLT(8.72) and FLT (6.74) outperformed CVT, which in turn outperformed UVT (18.93%). Further details are in Tables 7 and 8.

---

[29]The previous version can be found at `http://riskcenter.berkeley.edu/working-papers/documents/Anderson2031-01.pdf`.

[30]This example is technically simpler than risk parity because the source portfolio does not require rebalancing. As a result, the source incurs no trading costs.

Table 7: Performance Attribution

	UVT (18.93%)		FLT (8.72)		FLT (6.74)		CVT	
$\mathbb{E}[\mathbf{r^S}]$ (gross of trading costs)		**5.08**		**5.08**		**5.08**		**5.08**
$\mathbb{E}[\lambda - 1]$	7.72		7.72		5.74		5.82	
$\mathbb{E}[r^S - r^b]$	0.82		0.82		0.82		0.82	
$\mathbb{E}[\lambda - 1] \cdot \mathbb{E}[r^S - r^b]$		6.35		6.35		4.72		4.79
$\mathbb{E}[\mathbf{r^S}] + \mathbb{E}[\mathbf{\lambda - 1}] \cdot \mathbb{E}[\mathbf{r^S - r^b}]$		**11.43**		**11.43**		**9.80**		**9.87**
$\sigma(\lambda)$	20.4298		0.0000		0.0000		12.2539	
$\sigma(r^S - r^b)$	3.2711		3.2711		3.2711		3.2711	
$\rho(\lambda, r^S - r^b)$	-0.0742		0.0000		0.0000		-0.0263	
$\mathbf{Cov}(\lambda, r^S - r^b)$		-4.96		0.00		0.00		-1.06
$-\mathbb{E}[r^{TCS}]$		0.00		0.00		0.00		0.00
$-\mathbb{E}[r^{TCL}]$		-3.42		-2.24		-1.29		-2.71
$\mathbb{E}[\mathbf{r^L}]$		**3.05**		**9.19**		**8.51**		**6.10**
$(1 + \mathbb{E}[r^L]/1200)^{12}$	1.0310		1.0958		1.0885		1.0627	
$\exp(-\sigma_{rL}^2/2)$	0.9752		0.9604		0.9762		0.9805	
$[(1 + \mathbb{E}[r^L]/1200)^{12} \cdot \exp(-\sigma_{rL}^2/2) - 1] \cdot 100 - \mathbb{E}[r^L]$		-2.51		-3.94		-2.25		-1.90
Approximation Error		-0.10		-0.07		-0.01		-0.07
$\mathbb{G}[\mathbf{r^L}]$		**0.44**		**5.18**		**6.25**		**4.13**

Table 7: Performance attribution of the realized geometric return of the levered strategies UVT (18.93%), FLT (8.72), FLT (6.54), and CVT in terms of their common source portfolio, a US bond index, over the period January 1929–December 2012. UVT is levered each month so that the predicted volatility, based on a 36-month rolling window, is 18.93%, the volatility of stocks. FLT (8.72) has constant leverage 8.72 matching the average leverage of UVT (18.93%), while FLT (6.54) has constant leverage 6.54 and realized volatility equal to the volatility of UVT (18.93%). The performance attribution is based on Formulas (7) and (9). Borrowing is at the Eurodollar deposit rate and trading costs are based on the linear model in Appendix C. Arithmetic returns are estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and are annualized by $(1 + G[r])^{12} - 1$.

Table 8: Historical Performance

Sample Period: 1929-2012 Source: Bonds, Target: Stocks r^b = 3M-EDR	Arithmetic Total Return	Geometric Total Return	Average Leverage	Volatility	Arithmetic Excess Return	Sharpe Ratio	Skewness	Excess Kurtosis
Stocks	10.43	9.00	1.00	18.93	6.95	0.37	0.18	7.46
Bonds	5.08	5.14	1.00	3.26	1.59	0.49	0.03	4.74
UVT ($\sigma = 18.93\%$)	3.05	0.44	8.72	22.47	-0.43	-0.02	-0.58	4.81
FLT ($\lambda = 8.72$)	9.19	5.18	8.72	28.50	5.70	0.20	-0.10	4.70
FLT ($\lambda = 6.74$)	8.51	6.25	6.74	22.02	5.02	0.23	-0.08	4.67
CVT	6.10	4.13	6.82	19.91	2.61	0.13	-0.68	6.71

Table 8: Annualized arithmetic and geometric returns, volatility and Sharpe ratio, of the volatility target (US equity), the source portfolio (a US bond index), UVT (18.93%) (US bonds levered to an unconditional volatility target of 18.93%, the realized volatility of US equity), FLT (8.72) (US bonds levered to constant 8.72, the average leverage of UVT (18.93%)), FLT (6.54) (US bonds levered to match the volatility of UVT (18.93%), and CVT (US Bonds conditionally levered to match the current volatility of US equity), over the period 1929–2012. Arithmetic returns are estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and are annualized by $(1 + G[r])^{12} - 1$. Volatility is measured from monthly returns and annualized by multiplying by $\sqrt{12}$. Sharpe ratios are calculated using annualized excess return and annualized volatility.

References

Tobias Adrian and Hyun Song Shin. Liquidity and Leverage. *J. Finan. Intermediation*, 19:418–437, 2010.

Robert M. Anderson, Stephen W. Bianchi and Lisa R. Goldberg. Will My Risk Parity Strategy Outperform? *Financial Analysts Journal*, 68(6):75–93, 2012.

Clifford Asness, Andrea Frazzini and Lasse Heje Pedersen. Leverage Aversion and Risk Parity. *Financial Analysts Journal*, 68(1):47–59, 2012.

Clifford Asness, Andrea Frazzini and Lasse Heje Pedersen. Will My Risk Parity Strategy Outperform?: A Comment. *Financial Analysts Journal*, 69(2):12–15, 2013.

David H. Bailey and Marcos López de Prado. Balanced Baskets: A New Approach to Trading and Hedging Risks. *Journal of Investment Strategies*, 1(4):21–62, 2012.

Fischer Black, Michael C. Jensen and Myron Scholes. The Capital Asset Pricing Model: Some Empirical Tests. In Michael C. Jensen, editor, *Studies in the Theory of Capital Markets*, pages 79–121. Praeger Publishers Inc., 1972.

Fisher Black and Robert Litterman. Global Portfolio Optimization. *Financial Analysts Journal*, September:28–43, 1992.

John C. Bogle. *The Little Book of Common Sense Investing: The Only Way to Guarantee Your Fair Share of Stock Market Returns (Little Books. Big Profits)*. Wiley, 2007.

David G. Booth and Eugene F. Fama. Diversification Returns. *Financial Analysts Journal*, 48(3):26–32, 1992.

Markus K. Brunnermeier and Lasse Heje Pedersen. Market Liquidity and Funding Liquidity. *Review of Financial Studies*, 22(6):2201–2238, 2009.

Roger Clarke, Harindra De Silva and Stephen Thorley. Minimum Variance Portfolio Composition. *Journal of Portfolio Management*, 37(2):31–45, 2011.

Roger Clarke, Harindra De Silva and Stephen Thorley. Minimum Variance, Maximum Diversification and Risk Parity: An Analytic Perspective. *Journal of Portfolio Management*, 39(3):39–53, 2013.

David Cowan and Sam Wilderman. Rethinking Risk: What the Beta Puzzle Tells us About Investing, 2011. GMO White Paper.

E. Robert Fernholz. *Stochastic Portfolio Theory*. Springer, New York, 2002.

Hans Föllmer and Alexander Schied. Convex Risk Measures and Trading Constraints. *Finance and Stochastics*, 6:429–427, 2002.

Hans Föllmer and Alexander Schied. *Stochastic Finance: An Introduction in Discrete Time*. Walter de Gruyter, third edition, 2011.

Andrea Frazzini and Lasse Heje Pedersen. Betting Against Beta. forthcoming in *Journal of Financial Economics*, 2013.

Craig French. The Treynor Capital Asset Pricing Model. *Journal of Investment Management*, 1(2):60–72, 2003.

Fangjian Fu. Idiosyncratic Risk and the Cross-Section of Expected Stock Returns. *Journal of Financial Economics*, 91:24–37, 2009.

John Geanakoplos. The Leverage Cycle, 2010. Cowles Foundation Discussion Paper No.. 1715R.

Lisa R. Goldberg and Ola Mahmoud. Risk Without Return. *Journal of Investment Strategies*, 2(2):111–120, 2013.

Ralph Goldsticker. Volatility Targeting May Miss the Mark, 2012. BNY Mellon Asset Management.

Benjamin Graham. *The Intelligent Investor*. Collins, 1949.

Philippe Jorion. Risk Management Lessons From Long-Term Capital Management. *European Financial Management*, 2000.

Christoph Kessler and Günter Schwarz. Investment management for private investors at SBC - The benchmark, 1996. Swiss Bank Corporation / The Global 2nd Quarter.

Rob Kozlowski. Kansas City Employees pension fund commits to BlackRock for Risk Parity. *Pensions & Investments,* June 20, 2013.

Xi Li, Rodney Sullivan and Luis Garcia-Feijoo. The Limits to Arbitrage Revisited: The Low Risk Anomaly. forthcoming in *Financial Analysts Journal*, 2013.

John Lintner. Securities, Prices, Risk and Maximal Gains from Diverification. *The Journal of Finance*, 20(4):13–37, 1965a.

John Lintner. The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. *Review of Economics and Statistics*, 47(1):587–615, 1965b.

Rudolf Lörtscher. Investment Policy at SBC - basic premises and model, 1990. Swiss Bank Corporation / The Global 10/90.

Leonard C. MacLean, Edward O. Thorp and William T. Ziemba. *The Kelly Capital Growth Investment Criterion.* World Scientific, 2011.

Ulrike Malmendier, Geoffrey Tate and Jon Yan. Overconfidence and Early-Life Experiences: The Effect of Managerial Traits on Corporate Financial Policies. *Journal of Finance*, 66(5):1687–1783, 2011.

Harry Markowitz. Portfolio Selection. *Journal of Finance*, VII(1):77–91, 1952.

Harry Markowitz. Mean-Variance Approximations to the Geometric Mean. *Annals of Financial Economics*, 7(1), 2012.

Harry M. Markowitz. Market Efficiency: A Theoretical Distinction and So What? *Financial Analysts Journal*, September/October:17–30, 2005.

Jan Mossin. Equilibrium in a Capital Asset Market. *Econometrica*, 34(4):768–783, 1966.

Lorenzo Naranjo. Implied Interest Rates in a Market with Frictions, 2009. Working paper, New York University.

Andre F. Perold and William F. Sharpe. Dynamic Strategies for Asset Allocation. *Financial Analysts Journal*, 44(1):16–27, 1988.

Edward Qian. Risk Parity Portfolios: Efficient Portfolios Through True Diversification, 2005. PanAgora.

Stephen Ross. *Neoclassical Finance.* Princeton University Press, 2004.

James Sefton, David Jessop, Giuliano De Rossi, Claire Jones and Heran Zhang. Low-Risk Investing, 2011. UBS Investment Research.

Ronnie R. Shah. Understanding Low Volatility Strategies: Minimum Variance, 2011. Dimensional Fund Advisors.

William Sharpe. Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk. *Journal of Finance*, 19(3):425–442, 1964.

Hersh Shefrin and Meir Statman. Behavioral Finance in the Financial Crisis: Market Efficiency, Minsky, and Keynes, 2011. Santa Clara University.

Rodney Sullivan. Speculative Leverage: A False Cure for Pension Woes. *Financial Analysts Journal*, 66(3):6–8, 2010.

Jack Treynor. Toward a Theory of Market Value of Risky Assets, 1962. Presented to the MIT Finance Faculty Seminar in 1962, finally published in 1999 in *Asset Pricing and Portfolio Performance*, Robert J. Korajczyk (editor) London: RiskBooks, 15–22.

Jack L. Treynor and Fischer Black. Capital Market Theory and the Pricing of Financial Securities. In Stewart C. Myers, editor, *Modern Developments in Financial Management*, pages 310–327. Praeger Publishers, 1976.

Scott Willenbrock. Diversification Return, Portfolio Rebalancing, and the Commodity Return Puzzle. *Financial Analysts Journal*, 67(4):42–49, 2011.