STAT151A Quiz 1 (Jan 30th)

Please write your full name and email address:

For this quiz, we'll consider the linear model $y_n = \beta_1 z_n + \beta_2 w_n + \varepsilon_n$.

Note that there is no intercept, and instead are two scalar regressors, z_n and w_n.

Recall that the inverse of a 2x2 matrix is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

You have 20 minutes for this quiz.

There are three parts, (a), (b), and (c), each weighted equally..

(a)

Write the set of equations

$$y_n = \beta_1 z_n + \beta_2 w_n + \varepsilon_n$$

for $n \in \{1, \ldots, N\}$ in matrix form. That is, let $\boldsymbol{X}$ denote an $N \times 2$ matrix, $\boldsymbol{Y}$ and $\boldsymbol{\varepsilon}$ length–N column vectors, and $\boldsymbol{\beta} = (\beta_0, \beta_1)^\intercal$ a length–2 column vector. Then express the matrices $\boldsymbol{Y}$, $\boldsymbol{X}$, and $\boldsymbol{\varepsilon}$ in terms of the scalars y_n, z_m, w_n, and ε_n so that $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ is equivalent to the set of regression equations.

(b)

Define the following quantities:

$$\overline{z} := \frac{1}{N}\sum_{n=1}^{N} z_n \quad \overline{w} := \frac{1}{N}\sum_{n=1}^{N} w_n \quad \overline{y} := \frac{1}{N}\sum_{n=1}^{N} y_n$$

$$\overline{ww} := \frac{1}{N}\sum_{n=1}^{N} w_n^2 \quad \overline{zw} := \frac{1}{N}\sum_{n=1}^{N} z_n w_n \quad \overline{zz} := \frac{1}{N}\sum_{n=1}^{N} z_n^2 \quad \overline{wy} := \frac{1}{N}\sum_{n=1}^{N} w_n y_n \quad \overline{zy} := \frac{1}{N}\sum_{n=1}^{N} z_n y_n.$$

In terms of these quantities and N alone, write expressions for $\boldsymbol{X}^\intercal \boldsymbol{X}$, $\boldsymbol{X}^\intercal \boldsymbol{Y}$, and $(\boldsymbol{X}^\intercal \boldsymbol{X})^{-1}$.

(c)

Now, for only this part of the quiz, assume that $\overline{wz} = 0$. Under this assumption, write an expression for the least squares solution $\hat{\beta}$ which minimizes

$$\hat{\beta} := \underset{\beta}{\operatorname{argmin}} \sum_{n=1}^{N} (y_n - \beta_1 z_n - \beta_2 w_n)^2.$$