STAT151A Quiz 4

Please write your full name and email address here:

Also, please put your intials on each page in case the pages get separated.

You have 30 minutes for this quiz.

There are three questions, each weighted equally..

There are extra pages at the end if you need more space for solutions.

Question 1

Suppose that

$$\hat{\boldsymbol{\beta}} \sim \mathcal{N}\left(\begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}2 & 1.9 & 0\\1.9 & 1 & 0\\0 & 0 & 1\end{pmatrix}\right).$$

Let $\boldsymbol{v} = (1, -1, 0)$ and $\boldsymbol{u} = (1, 0, -1)$, so that

$$\boldsymbol{v}^\intercal \hat{\boldsymbol{\beta}} = \hat{\beta}_1 - \hat{\beta}_2 \quad \text{and} \quad \boldsymbol{u}^\intercal \hat{\boldsymbol{\beta}} = \hat{\beta}_1 - \hat{\beta}_3,$$

the differences between the first component of $\hat{\boldsymbol{\beta}}$ and the other components.

(a) What is $\mathbb{E}\left[\boldsymbol{v}^\intercal \hat{\boldsymbol{\beta}}\right]$?

(b) What is $\mathbb{E}\left[\boldsymbol{u}^\intercal \hat{\boldsymbol{\beta}}\right]$?

(c) What is $\mathrm{Var}\left(\boldsymbol{v}^\intercal \hat{\boldsymbol{\beta}}\right)$?

(d) What is $\mathrm{Var}\left(\boldsymbol{u}^\intercal \hat{\boldsymbol{\beta}}\right)$?

(e) What is $\mathrm{Cov}\left(\hat{\beta}_1, \hat{\beta}_2\right)$?

(f) What is $\mathrm{Cov}\left(\hat{\beta}_1, \hat{\beta}_3\right)$?

(g) What is the distribution of $\boldsymbol{v}^\intercal \hat{\boldsymbol{\beta}}$?

(h) What is the distribution of $\boldsymbol{u}^\intercal \hat{\boldsymbol{\beta}}$?

Solutions

Question 2

Let us assume that $\boldsymbol{Y} = \boldsymbol{X}\beta_x + \boldsymbol{A}\beta_a + \boldsymbol{B}\beta_b + \boldsymbol{\varepsilon}$, where $\mathbb{E}\left[\boldsymbol{\varepsilon}|\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{B}\right] = \mathbf{0}$. (These are the "linear expecation" assumptions for lecture.) Here, $\boldsymbol{X}$, $\boldsymbol{A}$, and $\boldsymbol{B}$ are all $N \times 1$.

Assume that

- $\boldsymbol{X}^\intercal \boldsymbol{A} = 0$ ($\boldsymbol{X}$ is orthogonal to $\boldsymbol{A}$)
- $\boldsymbol{X}^\intercal \boldsymbol{B} = 0.5N$ ($\boldsymbol{X}$ is not orthogonal to $\boldsymbol{B}$)
- $\boldsymbol{A}^\intercal \boldsymbol{B} = 0$ ($\boldsymbol{A}$ is orthogonal to $\boldsymbol{B}$)
- $\boldsymbol{X}^\intercal \boldsymbol{X} = \boldsymbol{A}^\intercal \boldsymbol{A} = \boldsymbol{B}^\intercal \boldsymbol{B} = N$. (The regressors are standardized.)

(a)

Let $\hat{\boldsymbol{\gamma}} = (\hat{\gamma}_x, \hat{\gamma}_a, \hat{\gamma}_b)^\intercal$ be the estimate from the regression $\boldsymbol{Y} \sim \boldsymbol{X}\gamma_x + \boldsymbol{A}\gamma_a + \boldsymbol{B}\gamma_b$. What is $\mathbb{E}\left[\hat{\boldsymbol{\gamma}}|\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{B}\right]$?

(b)

Now, let $\hat{\boldsymbol{\gamma}}' = (\hat{\gamma}'_x, \hat{\gamma}'_a)^\intercal$ be the estimate from the regression $\boldsymbol{Y} \sim \boldsymbol{X}\gamma'_x + \boldsymbol{A}\gamma'_a$. What is $\mathbb{E}\left[\hat{\boldsymbol{\gamma}}'|\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{B}\right]$?

(c)

Now, let $\hat{\boldsymbol{\gamma}}'' = (\hat{\gamma}''_x, \hat{\gamma}''_b)^\intercal$ be the estimate from the regression $\boldsymbol{Y} \sim \boldsymbol{X}\gamma''_x + \boldsymbol{B}\gamma''_b$. What is $\mathbb{E}\left[\hat{\boldsymbol{\gamma}}''|\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{B}\right]$?

(d)

Does $\hat{\gamma}_x = \hat{\gamma}'_x$? Does $\hat{\gamma}_x = \hat{\gamma}''_x$? In a single sentence, explain intuitively why they are or are not equal.

Solutions

Question 3

Suppose that one thousand children were administered an IQ test at age 10, and again at age 14. Let the age 10 measurements be denoted a_n and the age 18 measurements denoted n.

Viewing these IQ tests as a draw from all IQ tests administered in a given year, the scores are standardized so that $\mathbb{E}[a_n] = \mathbb{E}[n] = 100$ and $\sqrt{\mathrm{Var}(a_n)} = \sqrt{\mathrm{Var}(n)} = 15$.

The IQ tests can be considered a noisy measure of an inate capacity to perform well on the test. That is, the exact same student can be expected to receive different scores on an IQ test, even if nothing about the student changes from one test to the next.

Suppose we run the regression $n \sim \beta_0 + a_n \beta_a$ and measure $\hat{\beta}_a = 0.8$ and $\hat{\beta}_0 = 100$ so that

$$\hat{b}_n = 100 + 0.8(a_n - 100) = 20 + 0.8a_n.$$

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(a)

Evaluate $\hat{b}_n$ for the following a_n values:

- $a_n = 80$
- $a_n = 100$
- $a_n = 120$

(b)

When $a_n = 120$, you should have found that $\hat{b}_n < a_n$. Does this imply that, on average, high–performing children because less intelligent between the ages of 10 and 14?

Briefly explain your answer with reference to a concept from class. A single sentence is enough.

(c)

Suppose we replaced the IQ tests with a longer assessment that has less chance variability. Let these measurements be denoted a'_n and $'_n$, and consider the regression $'_n \sim \beta'_0 + a'_n \beta'_a$.

Assuming that the scores are again standardized so that $\mathbb{E}[a'_n] = \mathbb{E}['_n] = 100$ $\sqrt{\mathrm{Var}(a'_n)} = \sqrt{\mathrm{Var}('_n)} = 15$, how do you expect $\hat{\beta}'_a$ to compare with $\hat{\beta}_a$?

Solutions

Extra space for answers (indicate clearly which problem you are working on)

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