STAT151A Homework 3: Due February 23rd

Your name here

Normal intervals

For these problems, assume I give you a computer program that can compute the function $\Phi(z) = \mathbb{P}\left(\tilde{z} \le z\right)$ where $\tilde{z}$ is a standard scalar-valued random variable.

Let $\tilde{x}$ denote a scalar-valued $N(\mu, \sigma^2)$ random variable. Using only $\Phi(z)$ and elementary arithmetic, construct functions that evaluate the following:

(a)

$a \mapsto \mathbb{P}\left(\tilde{x} \le a\right)$

(b)

$b \mapsto \mathbb{P}\left(\tilde{x} \ge b\right)$

(c)

$a, b \mapsto \mathbb{P}\left(b \le \tilde{x} \le a\right)$

(d)

$a \mapsto \mathbb{P}\left(|\tilde{x}| \le a\right)$

(e)

$a \mapsto \mathbb{P}\left(|\tilde{x}| \ge a\right)$

(f)

$a \mapsto \mathbb{P}\left(|\tilde{x}| > a\right)$

(g)

$a \mapsto \mathbb{P}\left(|\tilde{x}| = a\right)$

Multivariate CLT

Let $\tilde{\boldsymbol{x}}_n$ denote an IID sequence of random variables in $\mathbb{R}^P$ (not necessarily normal), each with zero mean and finite covariance matrix $\boldsymbol{\Sigma}$. Let $\boldsymbol{a} \in \mathbb{R}^P$ denote a fixed vector.

(a)

Using the univariate CLT, find the limiting distribution of

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} \boldsymbol{a}^\intercal \tilde{\boldsymbol{x}}_n.$$

(b)

Using the multivariate CLT and the continuous mapping theorem, find the limiting distribution of

$$\boldsymbol{a}^\intercal \left(\frac{1}{\sqrt{N}} \sum_{n=1}^{N} \tilde{\boldsymbol{x}}_n \right).$$

(c)

Now, suppose that $P = 2$ and

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Note that we can write

$$\tilde{\boldsymbol{x}}_n = \begin{pmatrix} \tilde{\boldsymbol{x}}_{n1} \\ \tilde{\boldsymbol{x}}_{n2} \end{pmatrix},$$

where $\tilde{\boldsymbol{x}}_{n1}$ and $\tilde{\boldsymbol{x}}_{n2}$ are scalars. Find the limiting distributions of each of the following expressions:

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} \tilde{\boldsymbol{x}}_{n1} \rightarrow ?$$

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} \tilde{\boldsymbol{x}}_{n2} \rightarrow ?$$

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} (\tilde{\boldsymbol{x}}_{n1} + \tilde{\boldsymbol{x}}_{n2}) \rightarrow ?$$

(This result demonstrates why it's not enough to only look at the marginal distribution of the vector components when using a multivariate CLT.)

Valid covariance matrices

Suppose I were to tell you that the vector-valued random variable $\boldsymbol{x}$ has a covariance matrix $\mathrm{Cov}(\boldsymbol{x}) = \boldsymbol{\Sigma}$ where $\boldsymbol{\Sigma}$ is not positive semi-definite (i.e., $\boldsymbol{\Sigma}$ has at least one negative eigenvalue). Show that, if this were true, you could construct a scalar-valued random variable with *negative* variance, which is impossible.

(It follows from this argument every covariance matrix must be postive semi-definite.)