Approximate data deletion and replication with the Bayesian influence function

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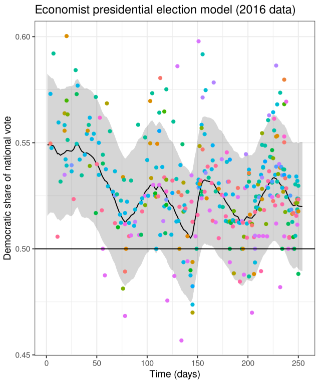
Oct 2nd, 2023

UC Berkeley

Economist 2016 Election Model [Gelman and Heidemanns, 2020]



A time series model to predict the 2016 US presidential election outcome from polling data.

Model:

- $X = x_1, \ldots, x_N =$ Polling data ($N = 361$).
- $\theta =$ Lots of random effects (day, pollster, etc.)
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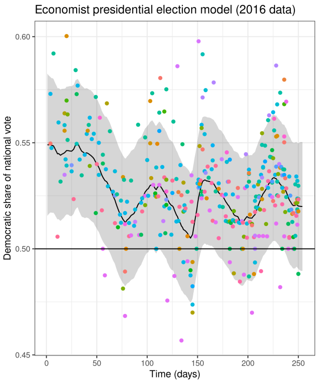

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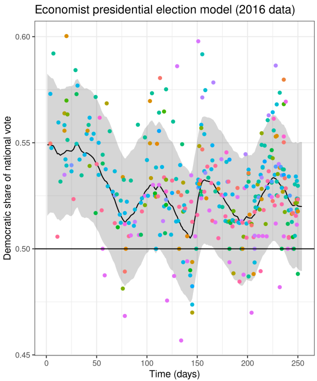

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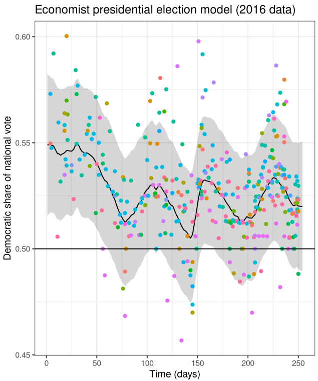

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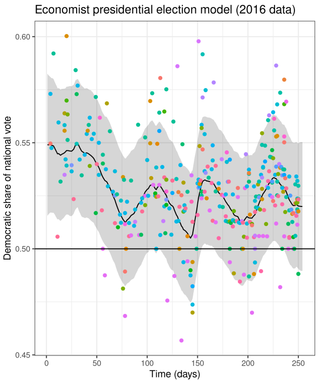

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Problem: Each MCMC run takes about 10 hours (Stan, six cores).

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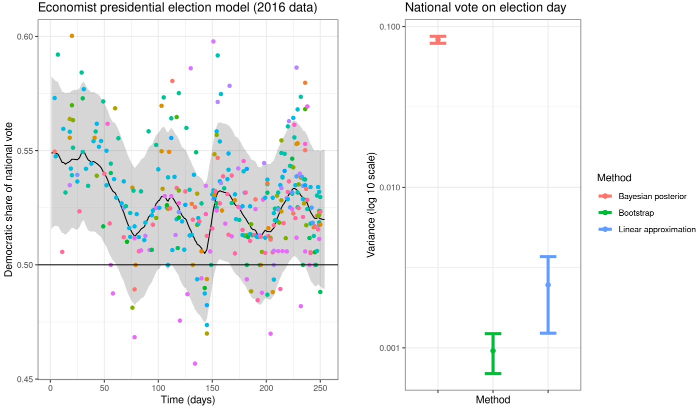

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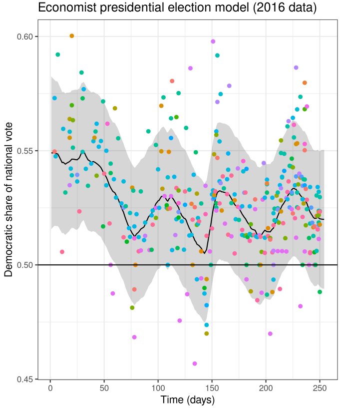

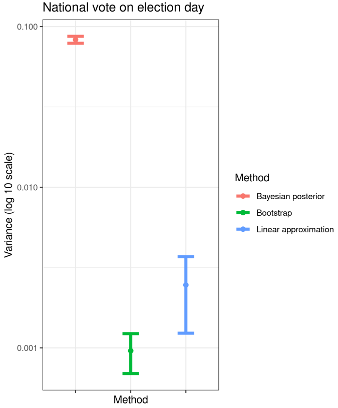

Compute time for 100 bootstraps: 51 days

Compute time for the linear approximation: Seconds
(But note the approximation has some error)

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  - Even for parameters which concentrate, even as $N \to \infty$
- A trick question, and some implications of different weightings.

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Augment the problem with *data weights* $w_1, \ldots, w_N$. We can write $\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)]$.

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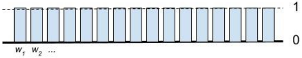

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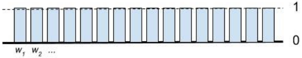

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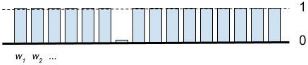

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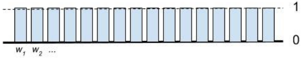

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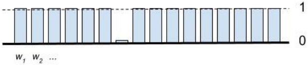

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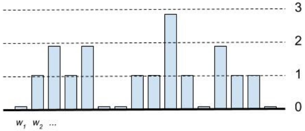

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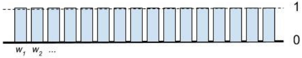

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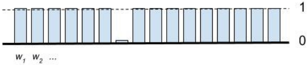

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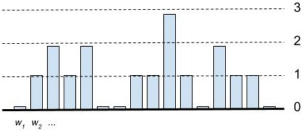

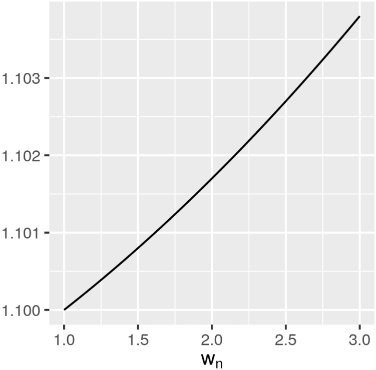

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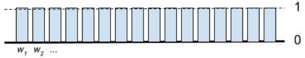

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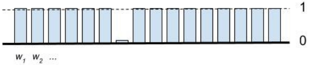

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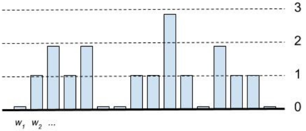

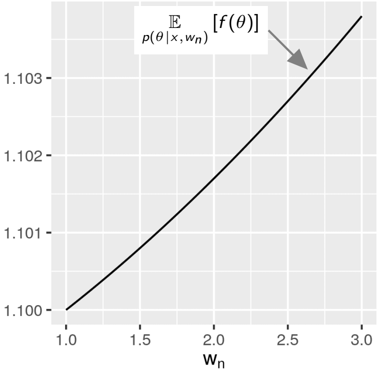

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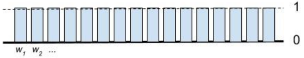

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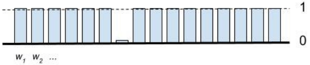

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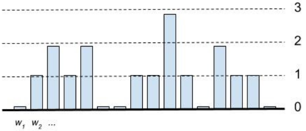

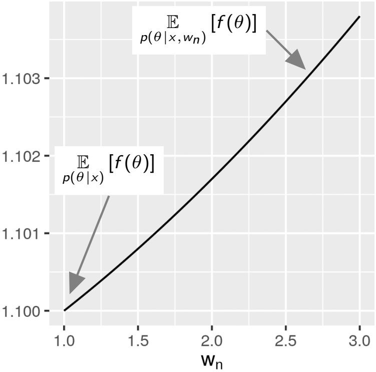

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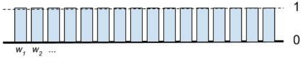

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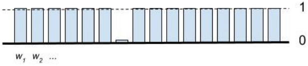

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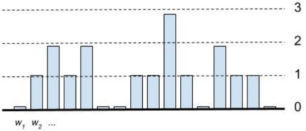

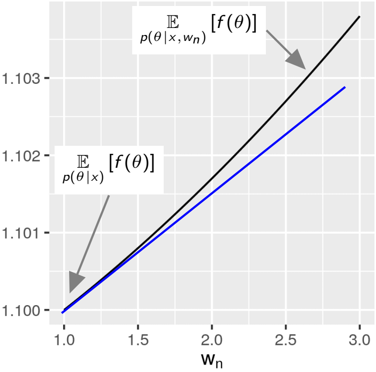

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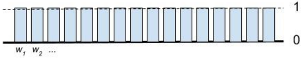

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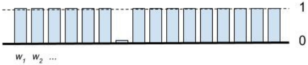

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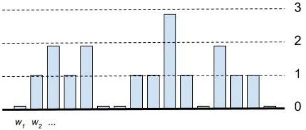

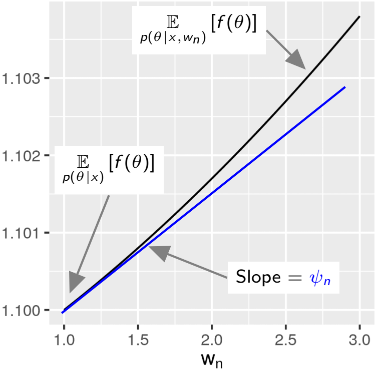

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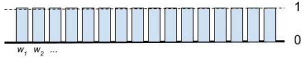

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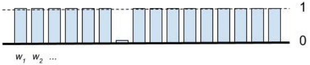

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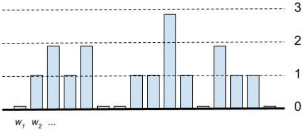

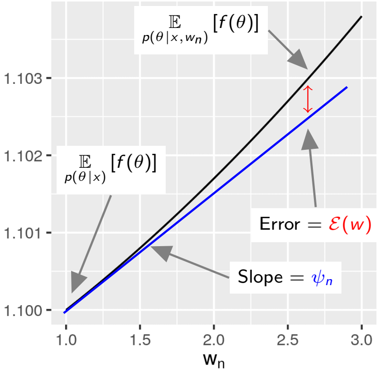

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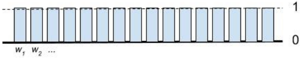

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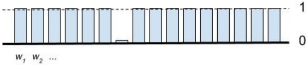

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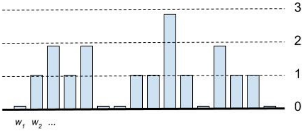

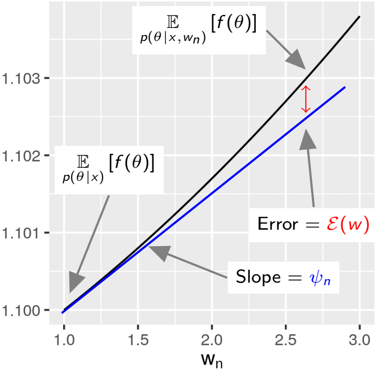

The re-scaled slope $N\psi_n$ is known as the “influence function” at data point x_n.

$$\mathbb{E}_{p(\theta|X,w)}[f(\theta)] - \mathbb{E}_{p(\theta|X)}[f(\theta)] = \sum_{n=1}^{N} \psi_n (w_n - 1) + \mathcal{E}(w)$$

Data re-weighting.

How can we use the approximation?

Assume the slope is computable and error is small.

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$$\text{LOO CV loss} = \frac{1}{N}\sum_{n=1}^{N} \underset{p(\theta|x,w_{(-n)})}{\mathbb{E}}[f_n(\theta)] \approx \frac{1}{N}\sum_{n=1}^{N} \underset{p(\theta|x)}{\mathbb{E}}[f_n(\theta)] - \frac{1}{N}\sum_{n=1}^{N} \psi_n$$

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Bootstrap. Draw bootstrap weights $w \sim p(w) = \mathrm{Multinomial}(N, N^{-1})$.

$$\text{Bootstrap variance} = \mathop{\mathrm{Var}}_{p(w)}\left(\mathop{\mathbb{E}}_{p(\theta|x,w)}[f(\theta)]\right) \approx \frac{1}{N^2}\sum_{n=1}^{N}\left(\psi_n - \frac{1}{N}\sum_{n'=1}^{N}\psi_{n'}\right)^2$$

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How can we use the approximation?

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$$\underset{p(\theta|X,w)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|X)}{\mathbb{E}}[f(\theta)] = \sum_{n=1}^{N} \psi_n (w_n - 1) + \mathcal{E}(w)$$

Cross validation. Let $w_{(-n)}$ leave out point n, and loss $f_n(\theta) = -\ell_n(\theta)$.

$$\text{LOO CV loss} = \frac{1}{N}\sum_{n=1}^{N} \underset{p(\theta|x,w_{(-n)})}{\mathbb{E}}[f_n(\theta)] \approx \frac{1}{N}\sum_{n=1}^{N} \underset{p(\theta|x)}{\mathbb{E}}[f_n(\theta)] - \frac{1}{N}\sum_{n=1}^{N}\psi_n$$

Bootstrap. Draw bootstrap weights $w \sim p(w) = \mathrm{Multinomial}(N, N^{-1})$.

$$\text{Bootstrap variance} = \underset{p(w)}{\mathrm{Var}}\left(\underset{p(\theta|x,w)}{\mathbb{E}}[f(\theta)]\right) \approx \frac{1}{N^2}\sum_{n=1}^{N}\left(\psi_n - \frac{1}{N}\sum_{n'=1}^{N}\psi_{n'}\right)^2$$

Influential subsets: Approximate maximum influence perturbation (AMIP).

Let $W_{(-K)}$ denote weights leaving out K points.

$$\max_{w\in W_{(-K)}}\left(\underset{p(\theta|x,w)}{\mathbb{E}}[f(\theta)] - \underset{p(\theta|x)}{\mathbb{E}}[f(\theta)]\right) \approx -\sum_{n=1}^{K}\psi_{(n)}.$$

Experessions for the slope and error

How to compute the slopes ψ_n? How large is the error $\mathcal{E}(w)$?

For simplicity, for the remainder of the presentation, we will consider a single weight.

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By exchanging integration and differentiation:

$$\psi_n = \left. \frac{\partial \underset{p(\theta|X,w_n)}{\mathbb{E}}[f(\theta)]}{\partial w_n} \right|_{w_n=1} = \underset{p(\theta|X)}{\mathbb{E}}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\right] \quad \textbf{Can estimate with MCMC!}$$

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By the mean value theorem, for some $\tilde{w}_n \in [0, w_n]$:

$$\mathcal{E}(w_n) = \frac{1}{2} \left. \frac{\partial^2 \mathop{\mathbb{E}}_{p(\theta|X,w_n)}[f(\theta)]}{\partial w_n^2} \right|_{w_n=\tilde{w}_n} (w_n - 1)^2 = \frac{1}{2} \mathop{\mathbb{E}}_{p(\theta|X,\tilde{w}_n)}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\bar{\ell}_n(\theta)\right](w_n - 1)^2$$

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The approximation is good if $\mathop{\mathbb{E}}_{p(\theta|X)}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\right] \gg \mathop{\mathbb{E}}_{p(\theta|X,\tilde{w}_n)}\left[\bar{f}(\theta)\bar{\ell}_n(\theta)\bar{\ell}_n(\theta)\right]$.

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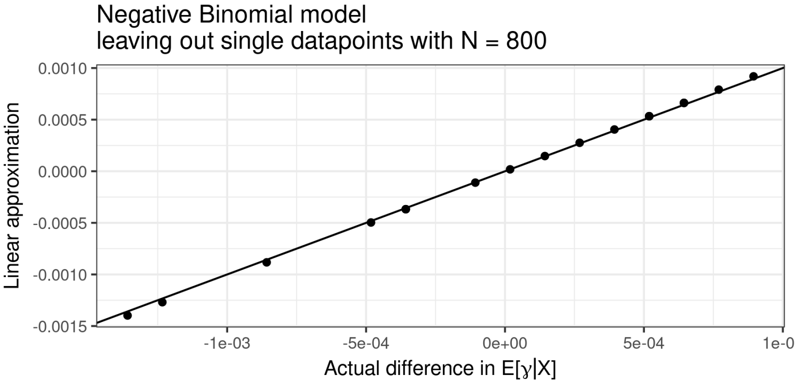
The map $w_n \mapsto N\left(\underset{p(\gamma|X,w_n)}{\mathbb{E}}[\gamma] - \underset{p(\gamma|X)}{\mathbb{E}}[\gamma]\right)$ becomes linear as $N \to \infty$.


Negative Binomial model
leaving out single datapoints with N = 800
0.0010
0.0005
0.0000
-0.0005
-0.0010
-0.0015
Linear approximation
-1e-03
-5e-04
0e+00
5e-04
1e-0
Actual difference in E[γ|X]

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- Parameter λ ("local") grows in dimension with N.
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Can we save the approximation when *some* parameters concentrate?
$\Rightarrow$ Does the residual vanish asymptotically for $w_n \mapsto \underset{p(\gamma|X,w_n)}{\mathbb{E}}[\gamma]$?

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We assume that $p(\gamma|X)$ concentrates but $p(\lambda|X)$ does not. By our series expansion:

$$\underset{p(\gamma,\lambda|X,w_n)}{\mathbb{E}}[\gamma] - \underset{p(\gamma,\lambda|X)}{\mathbb{E}}[\gamma] =$$

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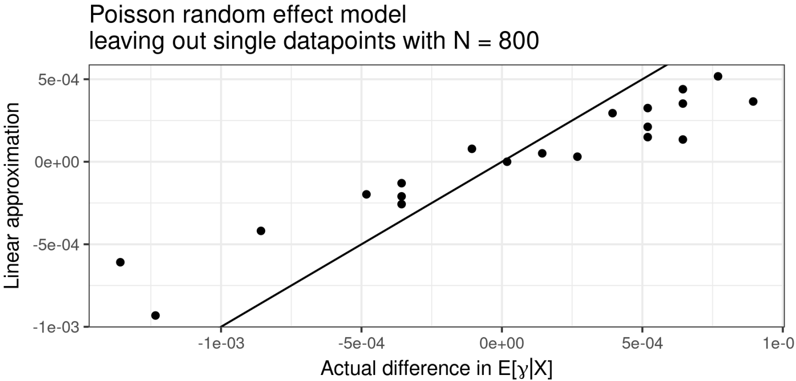
The map $w_n \mapsto N\left(\mathop{\mathbb{E}}_{p(\gamma|X,w_n)}[\gamma] - \mathop{\mathbb{E}}_{p(\gamma|X)}[\gamma]\right)$ remains non-linear as $N \to \infty$.

Experiments



A contradiction?

Negative binomial observations.

Asymptotically linear in w.

Poisson observations with random effects.

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With a constant regressor, Gamma REs, and one RE per observation,
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Is $\underset{p(\gamma|X,w)}{\mathbb{E}}[\gamma]$ linear in the data weights or not?

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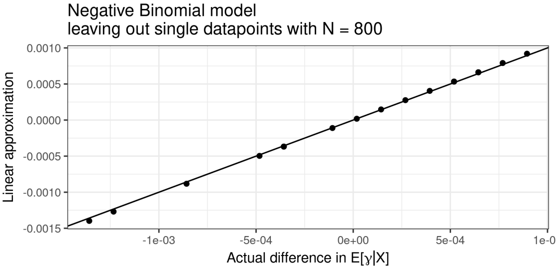
Trick question! We weight a log likelihood contribution, not a datapoint.

The two weightings are not equivalent in general.

Experimental results

Our results were actually computed on **identical datasets** with $G = N$ and $g_n = n$.

Approximation based on $\log p(x_n|\gamma)$.



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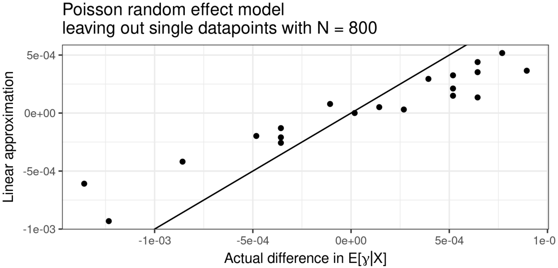

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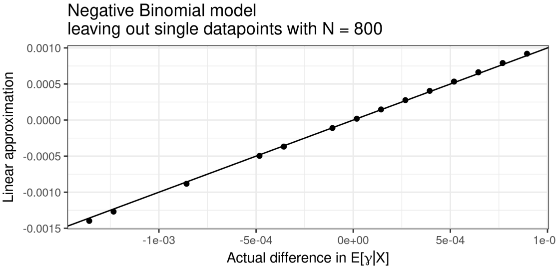

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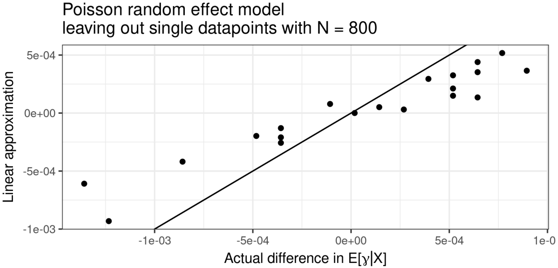

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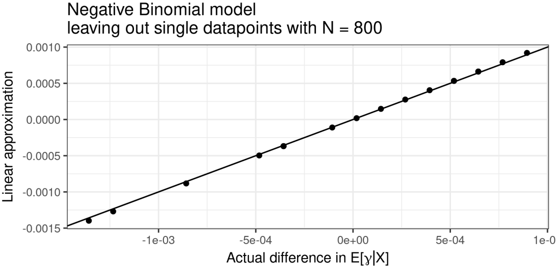
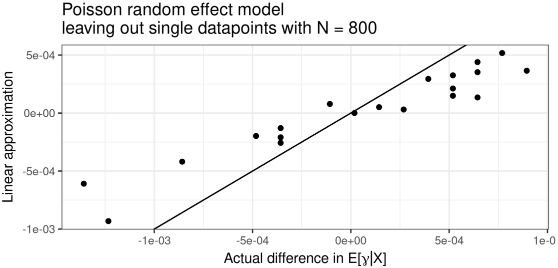


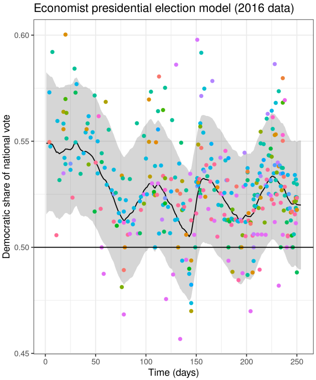
Approximation based on $\log p(x_n|\gamma, \lambda)$.

Computable from $\gamma, \lambda \sim p(\gamma, \lambda|X)$.

May still be useful when $p(\lambda|X)$ is *somewhat* concentrated.



Observations and consequences



Observations and consequences

- When $\log p(x_n|\gamma, \lambda)$ is the exchangeable unit, our results are problematic for
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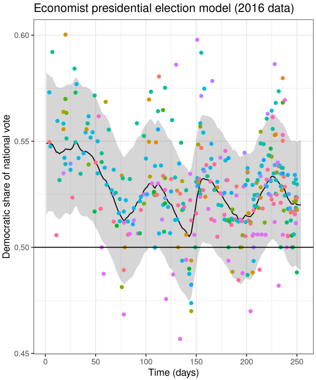
  - All of the above for Bayes-like optimization procedures (VB, the EM algorithm)



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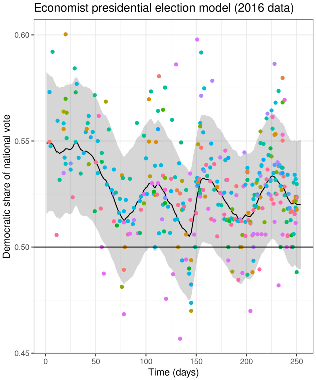

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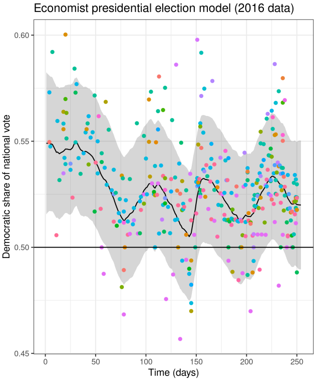

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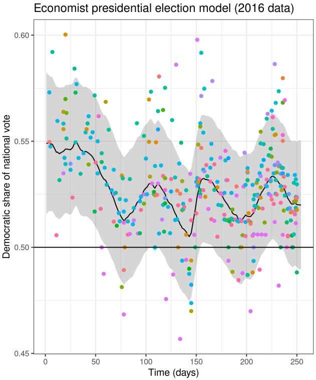
- There may be multiple ways to define "exchangable unit" in a given problem.
- But without nesting, $\log p(x_n|\gamma, \lambda)$ may be the natural model-free exchangeable unit.



T. Broderick, R. Giordano, and R. Meager. An automatic finite-sample robustness metric: When can dropping a little data make a big difference? *arXiv preprint arXiv:2011.14999*, 2020.

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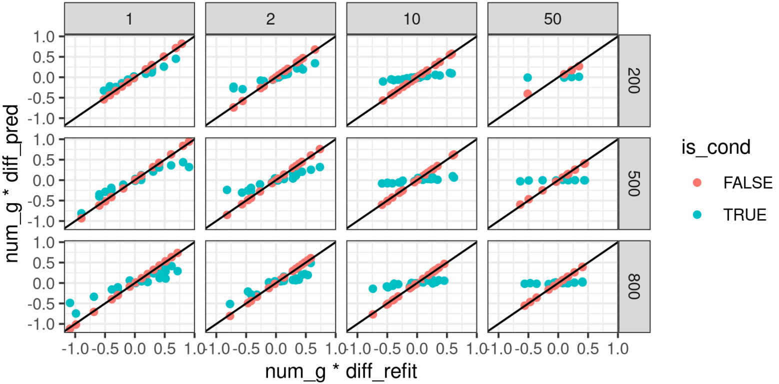
Supplemental slides

Non-equivalence of weighting (nonlinearity of marginalization)

Consider a single datapoint.

$$
\begin{aligned}
&\log p(x_n|\gamma, w_c) = \\
&\qquad \log\left(\int p(x_n|\gamma,\lambda,w_c)p(\lambda|\gamma)d\lambda\right) = \\
&\qquad \log\left(\int p(x_n|\gamma,\lambda)^{w_c} p(\lambda|\gamma)d\lambda\right) \neq \\
&\qquad \log\left(\int p(x_n|\gamma,\lambda)p(\lambda|\gamma)d\lambda\right)^{w_c} = \\
&\qquad w_c \log p(\lambda|\gamma)
\end{aligned}
$$

Extended experimental results



Extended experimental results