

Lecture Twenty Two

ARMA(p, q)

y_t is ARMA(p, q)

$$(y_t - \mu) - \phi_1(y_{t-1} - \mu) - \dots - \phi_p(y_{t-p} - \mu) = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}$$

$$y_t - \phi_0 - \phi_1 y_{t-1} - \dots - \phi_p y_{t-p} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}$$

$$\phi_0 = \mu(1 - \phi_1 - \dots - \phi_p) \quad \uparrow$$

ARMA(p, q)

$$\mu = \frac{\phi_0}{1 - \phi_1 - \dots - \phi_p}$$

$$y_t = \mu + \eta_t \text{ where } \eta_t \text{ is ARMA(p, q) with no } \mu \text{ or } \phi_0$$

ARMA(p, q) = AR(p) with MA(q) errors.

① $q=0 \rightarrow \text{ARMA}(p, 0) = \text{AR}(p)$

② $p=0 \rightarrow \text{ARMA}(0, q) = \text{MA}(q)$

Backshift Notation:

$$\rightarrow \phi(B)(y_t - \mu) = \theta(B)\varepsilon_t \quad \text{--- (ARMA)}$$

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$$

$$\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q$$

(Sometimes people use \downarrow instead of B)
 \uparrow
 Lag Operator

$$y_t - \mu = \frac{\theta(B)}{\phi(B)} \varepsilon_t$$

$$\phi(z) = (1 - a_1 z) \dots (1 - a_p z)$$

where roots of ϕ are $\frac{1}{a_1} \dots \frac{1}{a_p}$

$$= \frac{\theta(B)}{(1 - a_1 B) \dots (1 - a_p B)} \varepsilon_t$$

$$= \theta(B) [1 + a_1 B + (a_1 B)^2 + \dots] [1 + a_2 B + (a_2 B)^2 + \dots] \dots [1 + a_p B + (a_p B)^2 + \dots] \varepsilon_t$$

(If all $|a_j| < 1$, then the above is well-defined)

$$= \psi_0 \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \dots,$$

$$\sum |\psi_j| < \infty$$

$$\frac{1}{1 - x} = 1 + x + x^2 + \dots$$

$$\frac{\theta(z)}{\phi(z)} = \psi(z) = \psi_0 + \psi_1 z + \psi_2 z^2 + \dots$$

$$\boxed{\theta(z)} = (\psi_0 + \psi_1 z + \psi_2 z^2 + \dots) (1 - \phi_1 z - \dots - \phi_p z^p)$$

$$1 + \theta_1 z + \dots + \theta_q z^q$$

$$1 = \psi_0$$

$$\theta_1 = \psi_1 - \phi_1 \psi_0$$

$$\theta_2 = \psi_2 - \psi_1 \phi_1 - \phi_2 \psi_0$$

} ARMA₂MA

ARMA(p, q): Causal-Stationary Regime
(All roots of ϕ have modulus > 1)

$$y_t = \mu + \sum_{j=0}^{\infty} \boxed{\psi_j} \varepsilon_{t-j}$$

ACF & PACF

$h \geq 0$ $\{y_t\}$ STATIONARY time series model.

ACF(h) = correlation between y_t & y_{t+h}

PACF(h) = partial correlation between y_t & y_{t+h} after removing the effects of $y_{t+1}, \dots, y_{t+h-1}$

Facts

① MA(q) \rightarrow ACF(h) = 0 for $|h| > q$

② AR(p) \rightarrow PACF(h) = 0 for $|h| > p$

③ Given data y_1, \dots, y_n , get sample ACF(h) & sample PACF(h)

④ $ARMA(p, q)$: neither $ACF(h)$ nor $PACF(h)$ cut off after a finite lag.

$p \leq p_{\max}$, $q \leq q_{\max}$
 $ARMA(\hat{p}, \hat{q})$ for all \hat{p}, \hat{q} use Automatic model selection
 e.g. AIC or BIC.

Parameter Estimation in $ARMA(p, q)$

$ARIMA(\text{data}, \text{order} = (p, d, q))$

↓
 Statsmodels
 function

estimates

$\mu, \theta_1, \dots, \theta_q, \sigma^2$
 ϕ_1, \dots, ϕ_p

$p + q + 2$

Writing likelihood

Maximize log-likelihood

$MA(1)$

ARIMA uses Kalman filter
 to write the log likelihood

AIC & BIC

$(-2) \times \text{Maximized log-likelihood} + 2 (\# \text{ parameters})$

AIC
 ↑
 Akaike

Bayesian Information
 Criterion

(-2) * Maximized log-likelihood + (log n) (# parameters)
 ↓
 Bayesian Information Criterion (BIC)

Fit AR(2) to $\log y_t - \log y_{t-1}, t=2 \dots n$

$$\left\{ \begin{array}{l} \log y_{n+1} - \log y_n \\ \log y_{n+2} - \log y_{n+1} \\ \log y_{n+3} - \log y_{n+2} \\ \vdots \\ \log y_{n+99} - \log y_{n+98} \\ \log y_{n+100} - \log y_{n+99} \end{array} \right\} \quad \left\{ \begin{array}{l} \log y_{n+1} - \log y_n \\ \log y_{n+2} - \log y_n \\ \log y_{n+3} - \log y_n \\ \vdots \\ \log y_{n+100} - \log y_n \end{array} \right\}$$

ARIMA models

p, d, q

y_t is ARIMA(p, d, q)

if $(I-B)^d y_t$ is ARMA(p, q)

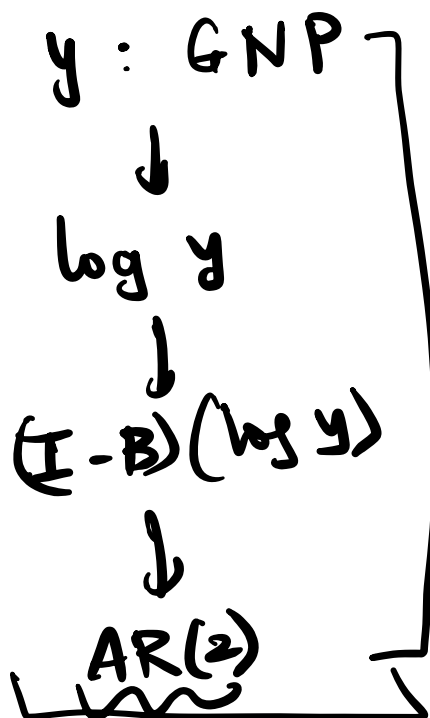
Difference
 y_t :

$$y_t - y_{t-1}$$

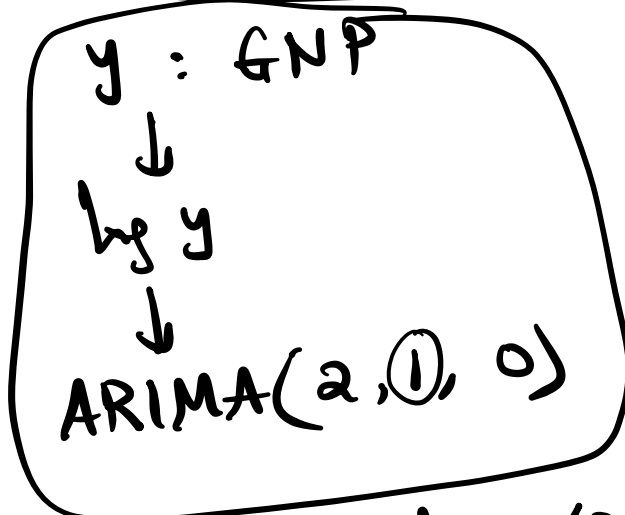
$$\nabla y_t = y_t - y_{t-1} = (I-B)y_t$$

$$\phi(B) \{ (I-B)^d y_t - \mu \} = \theta(B) \varepsilon_t$$

$$\begin{aligned} (I-B)y_t &= y_t - y_{t-1} \\ (I-B)^2 y_t &= (I-2B+B^2)y_t \\ &= y_t - 2y_{t-1} + y_{t-2} \end{aligned}$$



$$\begin{aligned}
 (I-B)(I-B)y_t &= (I-B)(y_t - y_{t-1}) \\
 &= (I-B)y_t - (I-B)y_{t-1} \\
 &= (y_t - y_{t-1}) - (y_{t-1} - y_{t-2})
 \end{aligned}$$



$\{ \text{arima}(\log y_t, \text{order} = (2, 1, 0),$
 $\text{trend} = 't') \}$

$d=1$

$\text{ARIMA}(p, 1, q)$

$$\phi(B)(\nabla y_t - \mu) = \theta(B)\varepsilon_t$$

$$\nabla y_t = \mu + \underbrace{\eta_t}_t$$

$\eta_t \sim \text{ARMA}(p, q)$
 with mean 0

$$y_1 - y_0 = \mu + \eta_1$$

$$y_2 - y_1 = \mu + \eta_2$$

$$y_n - y_{n-1} = \mu + \eta_n$$

$$y_t = y_0 + t\mu + (\eta_1 + \dots + \eta_t)$$

→ $y_t = \underbrace{y_0}_{\text{intercept}} + \underbrace{t\mu}_{\text{trend}} + \delta_t$ where $\nabla \delta_t = \text{ARMA}(p, q)$ with zero mean

↑
trend = 't' model ($d=1$)

↙ $y_t = y_0 + \delta_t$ where $\nabla \delta_t = \text{ARMA}(p, q)$ with zero mean

trend = 'c' Default : $\mu = 0$ when $d=1$ or more

S ARIMA
↓
Seasonal