LECTURE SIXTEEN

1. LAGGED REGRESSION
2. Neural Networks (RNN, LSTM)

LAGGED REGRESSION

or (AUTO REGRESSION)

ARIMA — A: Auto Regressive, R: Regressive, I: Integrated, MA: Moving Average

AR + MA + I = ARIMA

Auto Regression (AR)

Time series: $\boxed{y_1, \ldots, y_n}$

Forecasting: $\underbrace{y_{n+1}, y_{n+2}, \ldots, y_{n+k}}_{\text{we want to predict}}$ for some k

$$y = X\beta + \varepsilon$$

1. $$y_t = \beta_0 + \beta_1 t + \beta_2 (t-c)_+ + \varepsilon_t$$

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad X = \begin{bmatrix} 1 & t & (t-c)_+ \\ \vdots & & \\ 1 & & \end{bmatrix}$$

Models that we studied so far. X is made of t.

Auto Regression: X is made of past values of y.

AR(1) Model

Order

$$\boxed{y_t} = \phi_0 + \phi_1 \boxed{y_{t-1}} + \varepsilon_t,$$

$$\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2) \qquad t = 2, 3, \ldots, n$$

$$y = X\beta + \varepsilon$$

$$y = \begin{pmatrix} y_2 \\ \vdots \\ y_n \end{pmatrix} \quad X = \begin{bmatrix} 1 & y_1 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix} \quad \beta = \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}$$

AR(2)

$$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t,$$

$$t = 3, 4, \ldots, n$$

$$y = \begin{pmatrix} y_3 \\ \vdots \\ y_n \end{pmatrix} \quad X = \begin{bmatrix} 1 & y_2 & y_1 \\ \vdots & \vdots & \vdots \\ 1 & y_{n-1} & y_{n-2} \end{bmatrix}$$

AR(p)

$$y_t = \phi_0 + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t$$

$$t = p+1, \ldots, n$$

$$y = \begin{pmatrix} y_{p+1} \\ \vdots \\ y_n \end{pmatrix} \quad X = \begin{bmatrix} 1 & y_p & y_{p-1} & \cdots & y_1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & y_{n-1} & y_{n-2} & & y_{n-p} \end{bmatrix}$$

Unknown parameters: $\phi_0, \phi_1, \dots, \phi_p$ σ

Estimation & Inference:

y, X sm.OLS(y, x).fit()

$\hat{\phi}_0, \hat{\phi}_1, \dots, \hat{\phi}_p$: Least Squares

$$\min_{\phi_0, \dots, \phi_p} \sum_{t=p+1}^{n} \left(y_t - \phi_0 - \phi_1 y_{t-1} - \dots - \phi_p y_{t-p}\right)^2$$

$$\hat{\sigma}_{MLE} = \sqrt{\frac{1}{n-p} \sum_{t=p+1}^{n} \left(y_t - \hat{\phi}_0 - \dots - \hat{\phi}_p y_{t-p}\right)^2}$$

Forecasting: $y_{n+1}, y_{n+2}, \dots, y_{n+k}$

$$\hat{y}_{n+1} = \hat{\phi}_0 + \hat{\phi}_1 y_n + \hat{\phi}_2 y_{n-1} + \dots + \hat{\phi}_p y_{n+1-p}$$

$$\hat{y}_{n+2} = \hat{\phi}_0 + \hat{\phi}_1 \hat{y}_{n+1} + \hat{\phi}_2 y_n + \dots + \hat{\phi}_p y_{n+2-p}$$

$$\hat{y}_{n+i} = \hat{\phi}_0 + \hat{\phi}_1 \hat{y}_{n+i-1} + \dots + \hat{\phi}_p \hat{y}_{n+i-p}$$ $i = 1, 2, \dots$

$$\hat{y}_j = y_j \quad \text{if} \quad j \le n$$

History (How AR models were invented)

[Yule 1927]

[Sunspots data]

y_t : sunspots data

$$y_t = \beta_0 + \beta_1 \cos 2\pi f t + \beta_2 \sin 2\pi f t + \varepsilon_t$$

$$\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$$

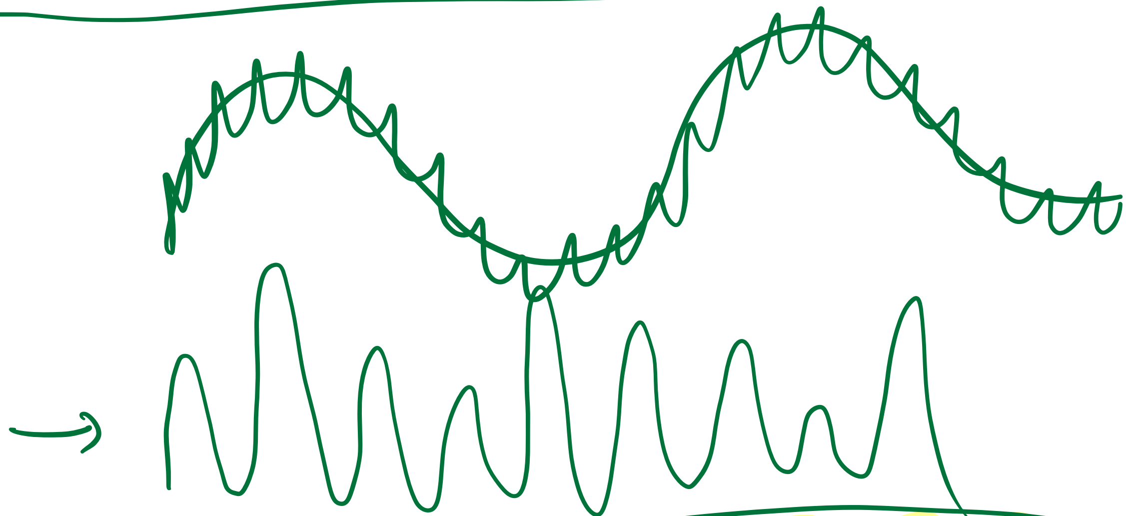
$$\boxed{f, \beta_0, \beta_1, \beta_2, \sigma}$$

$$f = \frac{1}{11}$$

$$\boxed{s(t) = \beta_0 + \beta_1 \cos 2\pi f t + \beta_2 \sin 2\pi f t}$$

$t = 1, 2, \ldots$

$$\boxed{\underline{\text{Model 1}} : \quad y_t = s_t + \varepsilon_t}$$



$$\boxed{s(t) = \beta_0 + \beta_1 \cos 2\pi f t + \beta_2 \sin 2\pi f t}$$

$$s''(t) = -(2\pi f)^2 \left[\beta_1 \cos 2\pi f t + \beta_2 \sin 2\pi f t\right]$$

$$\boxed{s''(t) = -(2\pi f)^2 \left(s(t) - \beta_0\right)}$$

$$(s_t - s_{t-1}) - (s_{t-1} - s_{t-2}) = 2[\cos(2\pi f) - 1](s_{t-1} - \beta_0) \iff s_t = \beta_0 + \beta_1 \cos 2\pi f t + \beta_2 \sin 2\pi f t$$

$\boxed{y_t}$

$$(y_t - y_{t-1}) - (y_{t-1} - y_{t-2}) = 2(\cos 2\pi f - 1)(y_{t-1} - \beta_0) + \eta_t$$

$$\Updownarrow$$

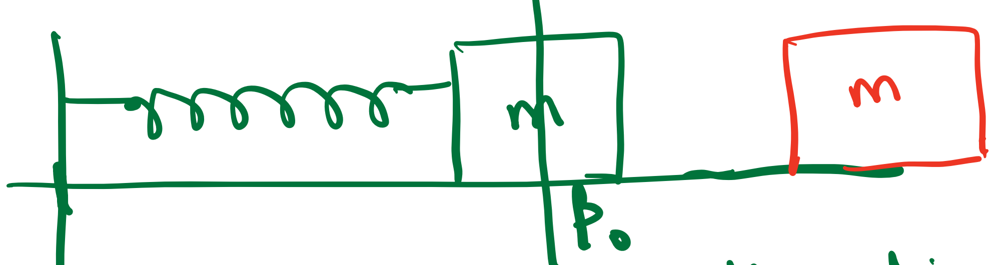
$$y_t = \phi_0 + \phi_1 y_{t-1} \boxed{-} y_{t-2} + \eta_t$$

$+ \phi_2$

special case of AR(2)

① $y_t = \beta_0 + \beta_1 \cos 2\pi f t + \beta_2 \sin 2\pi f t + \varepsilon_t$

② $y_t = \phi_0 + \phi_1 y_{t-1} - y_{t-2} + \eta_t$



$s(t)$: position of the object at time t.

$$s''(t) = -k\,(s(t) - \beta_0) \rightarrow \text{random noise due to stone throwing}$$

spring

constant

$$s''(t) = -\frac{K}{m}(s(t) - \beta_0)$$

<u>Scenario 1</u> : $y_t = s(t) + \varepsilon_t, \quad \varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$

<u>Scenario 2</u> : $\boxed{y''(t) = -K(y(t) - \beta_0) + \eta_t,}$

$y(t)$ = will be smooth.

<u>YULE MODEL</u>:

$$\boxed{y_t = \phi_0 + \phi_1 y_{t-1} - y_{t-2} + \varepsilon_t}$$

$$\boxed{\phi_1 = 2\cos 2\pi f}$$

$$y_t + y_{t-2} = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t$$