Lecture Twenty Two

ARMA(p, q)

y_t is ARMA(p, q)

$$\boxed{(y_t - \mu) - \phi_1 (y_{t-1} - \mu) - \cdots - \phi_p (y_{t-p} - \mu) = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}}$$

$$y_t - \phi_0 - \phi_1 y_{t-1} - \cdots - \phi_p y_{t-p} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$$

$$\phi_0 = \mu (1 - \phi_1 - \cdots - \phi_p)$$

ARMA(p, q)

$$\boxed{\mu = \frac{\phi_0}{1 - \phi_1 - \cdots - \phi_p}}$$

$$\boxed{y_t = \mu + \eta_t \text{ where } \eta_t \text{ is ARMA}(p, q) \text{ with no } \mu \text{ or } \phi_0}$$

ARMA(p, q) = AR(p) with MA(q) errors.

1. $q = 0 \rightarrow$ ARMA(p, 0) = AR(p)
2. $p = 0 \rightarrow$ ARMA(0, q) = MA(q)

Backshift Notation:

$$\rightarrow \phi(B)(y_t - \mu) = \theta(B) \varepsilon_t \text{ — (ARMA)}$$

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$$

$\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$

(Sometimes people use L instead of B)

Lag Operator

$$y_t - \mu = \boxed{\frac{\theta(B)}{\phi(B)}} \varepsilon_t$$

$\phi(z) = (1 - a_1 z) \cdots (1 - a_p z)$

where roots of ϕ are $\frac{1}{a_1} \cdots \frac{1}{a_p}$

$$= \frac{\theta(B)}{(1 - a_1 B) \cdots (1 - a_p B)} \varepsilon_t$$

$$= \theta(B)\left[1 + a_1 B + (a_1 B)^2 + \cdots\right]\left[1 + a_2 B + (a_2 B)^2 + \cdots\right] \cdots \left[1 + (a_p B) + (a_p B)^2 + \cdots\right] \varepsilon_t$$

If all $|a_j| < 1$, then the above is well-defined

$$= \psi_0 \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \cdots,$$

$$\sum |\psi_j| < \infty$$

$$\boxed{\frac{1}{1 - x} = 1 + x + x^2 + \cdots}$$

$$\frac{\theta(z)}{\phi(z)} = \psi(z) = \underbrace{\psi_0 + \psi_1 z + \psi_2 z^2 + \cdots}$$

$$\boxed{\underset{\substack{\| \\ 1+\theta_1 z+\cdots+\theta_q z^q}}{\theta(z)}} = (\psi_0+\psi_1 z+\psi_2 z^2+\cdots)(1-\phi_1 z-\cdots-\phi_p z^p)$$

$$\left.\begin{aligned} 1 &= \psi_0 \\ \theta_1 &= \psi_1-\phi_1\psi_0 \\ \theta_2 &= \psi_2-\psi_1\phi_1-\phi_2\psi_0 \end{aligned}\right\} \text{ARMA} \to \text{MA}$$

ARMA(p,q): Causal-Stationary Regime (All roots of ϕ have modulus >1)

$$y_t = \mu + \sum_{j=0}^{\infty} \boxed{\psi_j}\,\varepsilon_{t-j}$$

ACF & PACF

$h \geq 0$ $\{y_t\}$ STATIONARY time series model.

ACF(h) = correlation between y_t & y_{t+h}

PACF(h) = partial correlation between y_t & y_{t+h} after removing the effects of $y_{t+1}, \ldots, y_{t+h-1}$

Facts

1. MA(q) $\to$ ACF$(h) = 0$ for $|h|>q$
2. AR(p) $\to$ PACF$(h) = 0$ for $|h|>p$
3. Given data $y_1, \ldots, y_n$, get Sample ACF(h) & Sample PACF(h)

④ ARMA(p, q): neither ACF(h) nor PACF(h) cut off after a finite lag.

$p \le p_{max}$, $q \le q_{max}$

ARMA(p, q) for all & use Automatic model selection e.g AIC or BIC.

Parameter Estimation in ARMA(p, q)

ARIMA (data, order = (p, $\boxed{d}$, q))

statsmodels function

estimates
μ, $\theta_1 \dots \theta_q$, σ^2
$\phi_1 \dots \phi_p$

$\boxed{p + q + 2}$

Writing likelihood
Maximize log-likelihood

MA(1)

ARIMA uses Kalman filter to write the log-likeliho

AIC & BIC

$(-2) \times$ Maximized log-likelihood $+ \underbrace{2 \text{ (\# parameters)}}$

AIC ← Akaike

Akaike Information Criterion

$(-2) \times$ Maximized log-likelihood $+ (\log n)$ (# parameters)

$\downarrow$

Bayesian Information Criterion (BIC)

Fit AR(2) to $\underbrace{\log y_t - \log y_{t-1}}$, $t = 2 \ldots n$

$$\begin{bmatrix} \log y_{n+1} - \log y_n \\ \log y_{n+2} - \log y_{n+1} \\ \log y_{n+3} - \log y_{n+2} \\ \vdots \\ \log y_{n+100} - \log y_{n+99} \end{bmatrix} \Bigg\} \qquad \left\{ \begin{matrix} \log y_{n+1} - \log y_n \\ \log y_{n+2} - \log y_n \\ \log y_{n+3} - \log y_n \\ \vdots \\ \log y_{n+100} - \log y_n \end{matrix} \right\}$$

ARIMA models

p, d, q

y_t is ARIMA(p, d, q)

if

$\underbrace{(I-B)^d y_t \text{ is ARMA}(p, q)}$

$\phi(B)\{(I-B)^d y_t - \mu\}$

$= \theta(B) \varepsilon_t$

Difference y_t:

$y_t - y_{t-1}$

$\nabla y_t = y_t - y_{t-1}$

$= (I-B) y_t$

$(I-B) y_t = y_t - y_{t-1}$

$\underbrace{(I-B)^2} y_t = (I - 2B + B^2) y_t$

$= \underline{y_t - 2y_{t-1} + y_{t-2}}$

$\downarrow$

y : GNP

$\downarrow$

$\log y$

$\downarrow$

$(I-B)(\log y)$

$\downarrow$

$\underbrace{AR(2)}$

$$(I-B)((I-B)y_t)$$
$$= (I-B)(y_t - y_{t-1})$$
$$= (I-B)y_t - (I-B)y_{t-1}$$
$$= \underbrace{(y_t - y_{t-1}) - (y_{t-1} - y_{t-2})}$$

y : GNP

$\downarrow$

$\log y$

$\downarrow$

ARIMA(2, 1, 0)

arima($\log y_t$, order = (2,1,0),
trend = 't')

d=1 ARIMA(p, 1, q)

$$\phi(B)(\nabla y_t - \mu) = \theta(B)\varepsilon_t$$

$$\nabla y_t = \mu + \underbrace{\eta_t}$$

$\eta_t \sim$ ARMA(p, q)
with mean 0

$$y_1 - y_0 = \mu + \eta_1$$
$$y_2 - y_1 = \mu + \eta_2$$
$$\vdots$$

$$y_n - y_{n-1} = \mu + \eta_n$$

$$y_t = y_0 + t\mu + \underbrace{(\eta_1 + \cdots + \eta_t)}$$

$$\boxed{y_t = y_0 + \boxed{t\mu} + \gamma_t}$$ where $\nabla \gamma_t = ARMA(p, q)$ with zero mean

trend = 't' model (d=1)

$$\boxed{y_t = y_0 + \gamma_t \text{ where } \nabla \gamma_t = ARMA(p,q) \text{ with zero mean}}$$

trend = 'c'

Default : $\mu = 0$ when $d = 1$ or more

$$\boxed{\underset{\text{Seasonal}}{\underset{\downarrow}{S}}\ ARIMA}$$