

Lecture Twenty Two

ARMA(p, q)

y_t is ARMA(p, q)

$$(y_t - \mu) - \phi_1(y_{t-1} - \mu) - \dots - \phi_p(y_{t-p} - \mu) = \varepsilon_t + \theta_1\varepsilon_{t-1} + \dots + \theta_q\varepsilon_{t-q}$$

$$y_t - \phi_0 - \phi_1 y_{t-1} - \dots - \phi_p y_{t-p} = \varepsilon_t + \theta_1\varepsilon_{t-1} + \dots + \theta_q\varepsilon_{t-q}$$

$$\phi_0 = \mu(1 - \phi_1 - \dots - \phi_p)$$

$$\mu = \frac{\phi_0}{1 - \phi_1 - \dots - \phi_p}$$

↑
ARMA(p, q)

$$y_t = \mu + \eta_t \text{ where } \eta_t \text{ is ARMA}(p, q) \text{ with no } \mu \text{ or } \phi_0$$

ARMA(p, q) = AR(p) with MA(q) errors.

$$\textcircled{1} \quad q=0 \rightarrow \text{ARMA}(p, 0) = \text{AR}(p)$$

$$\textcircled{2} \quad p=0 \rightarrow \text{ARMA}(0, q) = \text{MA}(q)$$

Backshift Notation:

$$\rightarrow \phi(B) (y_t - \mu) = \theta(B) \varepsilon_t \quad (\text{ARMA})$$

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$$

$$\Theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q$$

(Sometimes people
use ∇ instead of
B)
↑
Lag
operator

$$y_t - \mu = \frac{\Theta(B)}{\phi(B)} \varepsilon_t$$

$$\phi(z) = (1 - a_1 z) \dots (1 - a_p z)$$

where roots of ϕ are $\frac{1}{a_1} \dots \frac{1}{a_p}$

$$= \frac{\Theta(B)}{(1 - a_1 B) \dots (1 - a_p B)} \varepsilon_t$$

$$= \Theta(B) \left[1 + a_1 B + (a_1 B)^2 + \dots \right] \left[1 + a_2 B + (a_2 B)^2 + \dots \right] \dots \left[1 + (a_p B) + (a_p B)^2 + \dots \right] \varepsilon_t$$

If all $|a_j| < 1$, then the above is well-defined

$$= \psi_0 \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \dots$$

$$\sum |\psi_j| < \infty$$

$$\frac{1}{1 - x} = 1 + x + x^2 + \dots$$

$$\frac{\Theta(z)}{\phi(z)} = \psi(z) = \underbrace{\psi_0 + \psi_1 z + \psi_2 z^2 + \dots}$$

$$\boxed{\theta(z)} = (\psi_0 + \psi_1 z + \psi_2 z^2 + \dots) (1 - \phi_1 z - \dots - \phi_p z^p)$$

$$1 + \theta_1 z + \dots + \theta_q z^q$$

$$1 = \psi_0$$

$$\theta_1 = \psi_1 - \phi_1 \psi_0$$

$$\theta_2 = \psi_2 - \psi_1 \phi_1 - \phi_2 \psi_0$$

ARMA(ϕ, ψ): Causal - Stationary Regime
(All roots of ϕ have modulus > 1)

$$y_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$$

ACF & PACF

$\{y_t\}$ STATIONARY time series model.

ACF(h) = correlation between y_t & y_{t+h}

PACF(h) = partial correlation between y_t & y_{t+h} after removing the effects of $y_{t+1}, \dots, y_{t+h-1}$

Facts

- ① MA(q) \rightarrow ACF(h) = 0 for $|h| > q$
- ② AR(p) \rightarrow PACF(h) = 0 for $|h| > p$
- ③ Given data y_1, \dots, y_n , get Sample ACF(h) & Sample PACF(h)

④ ARMA(p, q): neither ACF(h) nor PACF(h) cut off after a finite lag.

$p \leq p_{\max}, q \leq q_{\max}$
 $\text{ARMA}(p, q)$ for all p, q use model selection
 e.g. AIC or BIC.

Parameter Estimation in ARMA(p, q)

ARIMA (data, order = (p, $\frac{d}{2}$, q))

statsmodels function

estimates

$\mu, \theta_1, \dots, \theta_q, \sigma^2$
 ϕ_1, \dots, ϕ_p

$p + q + 2$

Writing likelihood

Maximize log-likelihood

MA(1)

ARIMA uses
 Kalman filter
 to write the log-likelihood

AIC & BIC

$(-2) \times \text{Maximized log-likelihood} + 2 (\# \text{parameters})$

AIC
 ↑
 Akaike

Akaike Information Criterion

(-2) * Maximized log-likelihood + $(\log n)(\# \text{ parameters})$

↓

Bayesian Information Criterion (BIC)

Fit AR(2) to $\log y_t - \log y_{t-1}, t=2 \dots n$

$$\left. \begin{array}{l}
 \log y_{n+1} - \log y_n \\
 \log y_{n+2} - \log y_{n+1} \\
 \log y_{n+3} - \log y_{n+2} \\
 \vdots \\
 \log y_{n+100} - \log y_{n+99}
 \end{array} \right\} \quad \left. \begin{array}{l}
 \log y_{n+1} - \log y_n \\
 \log y_{n+2} - \log y_n \\
 \log y_{n+3} - \log y_n \\
 \vdots \\
 \log y_{n+100} - \log y_n
 \end{array} \right\}$$

ARIMA models

p, d, q

y_t is ARIMA(p, d, q)

Difference
 y_t :

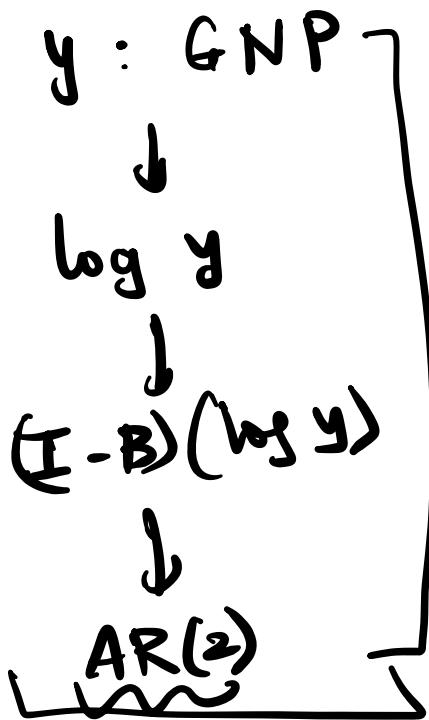
$$y_t - y_{t-1}$$

if $(I - B)^d y_t$ is ARMA(p, q)

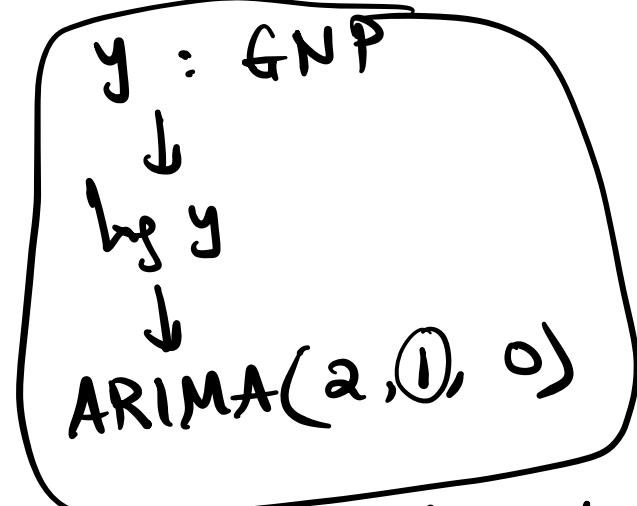
$$\begin{aligned}
 \nabla y_t &= y_t - y_{t-1} \\
 &= (I - B)y_t
 \end{aligned}$$

$$\begin{aligned}
 &\Phi(B) \left\{ (I - B)^d y_t - \mu \right\} \\
 &= \Theta(B) \varepsilon_t
 \end{aligned}$$

$$\begin{aligned}
 (I - B)y_t &= y_t - y_{t-1} \\
 (I - B)^2 y_t &= (I - 2B + B^2)y_t \\
 &= y_t - 2y_{t-1} + y_{t-2}
 \end{aligned}$$



$$\begin{aligned}
 & (I - B)(I - B)y_t \\
 &= (I - B)(y_t - y_{t-1}) \\
 &= (I - B)y_t - (I - B)y_{t-1} \\
 &= (y_t - y_{t-1}) - (y_{t-1} - y_{t-2})
 \end{aligned}$$



$\left\{ \text{arima}(\log y_t, \text{order} = (2, 1, 0), \text{trend} = 't') \right.$

$d=1$

$\text{ARIMA}(\phi, 1, \psi)$

$$\phi(B)(\nabla y_t - \mu) = \theta(B) \varepsilon_t$$

$$\nabla y_t = \mu + \sum_t \eta_t$$

$\eta_t \sim \text{ARMA}(\phi, \psi)$ with mean 0

$$y_t - y_0 = \mu + \eta_1$$

$$y_2 - y_1 = \mu + \eta_2$$

$$y_n - y_{n-1} = \mu + \eta_n$$

$$y_t = y_0 + t\mu + (\eta_1 + \dots + \eta_t)$$

$$\rightarrow y_t = y_0 + t\mu + \eta_t \quad \text{where } \nabla \eta_t = \text{ARMA}(p, q)$$

trend = 't' model ($d=1$)

$$y_t = y_0 + \eta_t \quad \text{where } \nabla \eta_t = \text{ARMA}(p, q)$$

trend = 'c' Default : $\mu = 0$ when $d=1$ or more

S ARIMA
↓
Seasonal