Lecture Seventeen

Topic for today: Estimation in AR models
AR(p)

Revisit: Linear Regression

$(x_1, y_1), \dots, (x_n, y_n)$

(A1) $\rightarrow$ $\boxed{y_i = \beta_0 + \beta_1 x_i + \varepsilon_i,}$ $\quad \varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$

$\theta = (\beta_0, \beta_1, \sigma)$

MLE likelihood: $\underbrace{\prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2} \right) \right)}$

Derivation of the likelihood

density of observations (assuming parameters are fixed).

$$f_{x_1, y_1, x_2, y_2, \dots, x_n, y_n \mid \theta}(x_1, y_1, x_2, y_2, \dots, x_n, y_n)$$

$$= \underbrace{f_{y_1 \dots y_n \mid x_1 \dots x_n, \theta}(y_1 \dots y_n)} \; \underbrace{f_{x_1 \dots x_n \mid \theta}(x_1 \dots x_n)}$$

$$= f_{\beta_0 + \beta_1 x_1 + \varepsilon_1, \beta_0 + \beta_1 x_2 + \varepsilon_2, \dots, \beta_0 + \beta_1 x_n + \varepsilon_n \mid x_1 \dots x_n, \theta}(y_1 \dots y_n) \; f_{x_1 \dots x_n \mid \theta}(x_1 \dots x_n)$$

$$= f_{\varepsilon_1, \dots, \varepsilon_n \mid x_1 \dots x_n, \theta}(y_1 - \beta_0 - \beta_1 x_1, y_2 - \beta_0 - \beta_1 x_2, \dots, y_n - \beta_0 - \beta_1 x_n) \times f_{x_1 \dots x_n \mid \theta}(x_1 \dots x_n)$$

$$f_{\beta_0 + \beta_1 x + \varepsilon}(y) = f_{\varepsilon}(y - \beta_0 - \beta_1 x)$$

Assume: $\varepsilon_1 \dots \varepsilon_n$ are independent of $x_1 \dots x_n$ given θ.

A2

$$= f_{\varepsilon_1 \dots \varepsilon_n | \theta}(y_1 - \beta_0 - \beta_1 x_1, \dots, y_n - \beta_0 - \beta_1 x_n)\, f_{x_1 \dots x_n | \theta}(x_1 \dots x_n)$$

A3: $\varepsilon_1 \dots \varepsilon_n \overset{iid}{\sim} N(0, \sigma^2)$

$$= \left[\prod_{i=1}^{n} f_{\varepsilon_i | \theta}(y_i - \beta_0 - \beta_1 x_i)\right] \times f_{x_1 \dots x_n | \theta}(x_1 \dots x_n)$$

$$= \left[\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}\right)\right] f_{x_1 \dots x_n | \theta}(x_1 \dots x_n)$$

A4: $f_{x_1 \dots x_n | \theta}(x_1 \dots x_n)$ does not depend on θ.

$$\propto \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}\right)$$

1. $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, $i = 1 \dots n$
2. $(\varepsilon_1, \dots, \varepsilon_n)$ is independent of $(x_1 \dots x_n)$
3. $\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$
4. density of $x_1 \dots x_n$ does not depend on $(\beta_0, \beta_1, \sigma)$

likelihood: $\prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}\right)\right]$

Maximize likelihood to get MLEs.

$\hat{\beta}_0, \hat{\beta}_1$: least squares estimates $\left[\underbrace{(X^TX)^{-1}X^Ty}_{\hat{\beta} \sim N(\beta,\ \sigma^2 (X^TX)^{-1})} \right]$

$$\hat{\sigma}_{MLE} = \sqrt{\frac{RSS}{n}}$$

Bayesian: $\beta_0, \beta_1, \log\sigma \overset{iid}{\sim} \text{Unif}(-C, C)$, $C = \infty$

$$\beta \mid \text{data} \sim t_{n-2,2}\left(\hat{\beta}, \hat{\sigma}^2 (X^TX)^{-1}\right)$$

$$\hat{\sigma} = \sqrt{\frac{RSS}{n-2}}$$

Auto Regression AR(1)

(A) → $y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t, \quad t = 2, \ldots, n$

$\boxed{y_t = \phi_0 + \phi_1 x_t + \varepsilon_t, \quad t = 2 \ldots n \quad \text{with } x_t = y_{t-1}}$ $\quad \theta = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \sigma \end{pmatrix}$

How to write the likelihood now?

$$f_{y_1 \ldots y_n \mid \theta}(y_1 \ldots y_n)$$

$$= f_{y_1 \mid \theta}(y_1)\, f_{y_2 \mid y_1, \theta}(y_2)\, f_{y_3 \mid y_1, y_2, \theta}(y_3) \cdots f_{y_n \mid y_1 \cdots y_{n-1}, \theta}(y_n)$$

$$= f_{y_1 \mid \theta}(y_1) \prod_{t=2}^{n} f_{y_t \mid y_1 \ldots y_{t-1}, \theta}(y_t)$$

$$= f_{y_1|\theta}(y_1) \prod_{t=2}^{n} f_{\phi_0+\phi_1 y_{t-1}+\underbrace{\varepsilon_t}|y_1\ldots y_{t-1},\theta}(y_t)$$

$$= f_{y_1|\theta}(y_1) \prod_{t=2}^{n} f_{\varepsilon_t|y_1\ldots y_{t-1},\theta}(y_t - \phi_0 - \phi_1 y_{t-1})$$

A2 ε_t is independent of $y_1 \ldots y_{t-1}$ (given θ) for each $t = 2,\ldots,n$

$$= f_{y_1|\theta}(y_1) \prod_{t=2}^{n} f_{\varepsilon_t}(y_t - \phi_0 - \phi_1 y_{t-1})$$

A3 $\varepsilon_t \overset{iid}{\sim} N(0,\sigma^2)$

$$= f_{y_1|\theta}(y_1) \underbrace{\left[\prod_{t=2}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2\sigma^2}(y_t - \phi_0 - \phi_1 y_{t-1})^2\right)\right]}_{x_t = y_{t-1}}$$

$$\underbrace{\left[\prod_{t=2}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2\sigma^2}(y_t - \phi_0 - \phi_1 x_t)^2\right)\right]}_{\text{Same likelihood as in linear regression.}}$$

A1 $y_t = \phi_0 + \phi_1 x_t + \varepsilon_t$ with $x_t = y_{t-1}$

A2 ε_t is independent of $y_1 \ldots y_{t-1}$, $t = 2,\ldots,n$

A3 $\varepsilon_t \overset{iid}{\sim} N(0,\sigma^2)$

(A4) The density of y_1 does not depend on θ.

Under A1, A2, A3, A4,

likelihood $\propto \underbrace{\prod_{t=2}^{n-1} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2\sigma^2}\left(y_t - \phi_0 - \phi_1 y_{t-1}\right)^2\right)}_{\text{identical to linear regression likelihood.}}$

MLE: same as in linear regression

$$\hat{\phi} = (X^T X)^{-1} X^T y \qquad X = \begin{pmatrix} 1 & y_1 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{pmatrix}$$

$$y = \begin{pmatrix} y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$\hat{\sigma}_{MLE} = \sqrt{\frac{RSS}{n-1}}, \quad \hat{\sigma} = \sqrt{\frac{RSS}{(n-1)-2}}$$

$\phi_0, \phi_1, \log \sigma \overset{iid}{\sim} \text{Unif}(-C, C)$

$$\boxed{\phi \mid \text{data} \sim t_{n-3,2}\left(\underset{\text{least squares}}{\hat{\phi}}, \hat{\sigma}^2 (X^T X)^{-1}\right)}$$

$\hat{\phi} = (X^T X)^{-1} X^T y$, can longer say:

$$\hat{\phi} \sim N(\phi, \sigma^2 (X^T X)^{-1})$$

standard errors corresponding to ϕ_0, ϕ_1:

$\rightarrow \sqrt{\text{diag}\left[\hat{\sigma}^2 (X^T X)^{-1}\right]}$

$$\hat{\sigma} = \sqrt{\frac{RSS}{\text{residual degrees of freedom}}}$$

$\boxed{\text{AutoReg:}}$ $\sqrt{\operatorname{diag}\left(\hat{\sigma}^2_{MLE}(X^TX)^{-1}\right)}$ $\hat{\sigma}_{MLE} = \sqrt{\frac{RSS}{n-1}}$

AR(1)

How to write a model for $y_1 | \theta$?

$$y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t, \quad t = 2, \dots, n$$

$$y_1 = \phi_0 + \phi_1 y_0 + \varepsilon_1$$

$$= \phi_0 + \phi_1(\phi_0 + \phi_1 y_{-1} + \varepsilon_0) + \varepsilon_1$$

$$= \phi_0 + \phi_1\phi_0 + \varepsilon_1 + \phi_1\varepsilon_0 + \phi_1^2 y_{-1}$$

$$= \phi_0(1+\phi_1) + \varepsilon_1 + \phi_1\varepsilon_0 + \phi_1^2 \underbrace{y_{-1}}$$

$$= \phi_0(1+\phi_1+\phi_1^2) + \varepsilon_1 + \phi_1\varepsilon_0 + \phi_1^2\varepsilon_{-1} + \phi_1^3 y_{-2}$$

$$= \phi_0(1+\phi_1+\phi_1^2+\dots+\phi_1^M) + \boxed{\varepsilon_1 + \phi_1\varepsilon_0 + \dots + \phi_1^M\varepsilon_{1-M}} + \underbrace{\phi_1^{M+1} y_{-M}}$$

$\boxed{\text{Suppose } |\phi_1| < 1.}$

let $M \to \infty$ and assume $|\phi_1| < 1$:

$$y_1 = \phi_0\underbrace{(1+\phi_1+\phi_1^2+\dots)} + \sum_{j=0}^{\infty}\phi_1^j\varepsilon_{1-j}$$

$$\boxed{y_1 = \frac{\phi_0}{1-\phi_1} + \underbrace{\sum_{j=0}^{\infty}\phi_1^j\varepsilon_{1-j}}}$$

$$y_1 \sim N\left(\frac{\phi_0}{1-\phi_1}, \sum_{j=0}^{\infty} \phi_1^{2j} \sigma^2\right)$$

$$= N\left(\frac{\phi_0}{1-\phi_1}, \frac{\sigma^2}{1-\phi_1^2}\right), \quad |\phi_1|<1$$

Likelihood

$$= \left[\frac{\sqrt{1-\phi_1^2}}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{\left(y_1 - \frac{\phi_0}{1-\phi_1}\right)^2 (1-\phi_1^2)}{2\sigma^2}\right\}\right] \left[\prod_{t=2}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{(y_t - \phi_0 - \phi_1 y_{t-1})^2}{2\sigma^2}\right\}\right]$$

$|\phi_1|<1$

Two AR(1) Models

1. y_1 has a density not depending on parameters (y_1 treated as constant)
2. $y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t, \quad t=2,\dots n$
3. ε_t independent of $y_1, \dots y_{t-1}$
4. $\varepsilon_t \overset{iid}{\sim} N(0,\sigma^2)$

FIRST AR(1)

Second AR(1)

① $y_1 = \frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j \varepsilon_{1-j}$

② $y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t$, $t = 2, \ldots, n$.

$$y_2 = \phi_0 + \phi_1 y_1 + \varepsilon_2$$

$$= \phi_0 + \phi_1 \left[\frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j \varepsilon_{1-j} \right] + \varepsilon_2$$

$$= \phi_0 + \frac{\phi_1 \phi_0}{1-\phi_1} + \left[\underbrace{\sum_{j=0}^{\infty} \phi_1^{j+1} \varepsilon_{1-j} + \varepsilon_2}_{\varepsilon_2 + \phi_1 \varepsilon_1 + \phi_2 \varepsilon_0 + \cdots} \right]$$

$$= \frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j \varepsilon_{2-j}$$

Second AR(1)

① $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$, $|\phi_1| < 1$

② $y_t = \frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j \varepsilon_{t-j}$ for every t

y_t is calculated from $\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots$

for every $t = \ldots, -3, -2, -1, 0, 1, 2, \ldots$

STATIONARY

$\varepsilon_t \underset{\sim}{iid} N(0, \sigma^2)$

$$\varepsilon_1 + \frac{1}{2}\varepsilon_0 + \frac{1}{4}\varepsilon_{-1} + \frac{1}{8}\varepsilon_{-2} + \cdots$$