STAT 153 & 248 - Time Series
Lecture Eight

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1 Recap from last lecture

In the last lecture, we considered fitting the sinusoidal model

$$y_t = \beta_0 + \beta_1 \cos(2\pi f t) + \beta_2 \sin(2\pi f t) + \epsilon_t \qquad \text{with } \epsilon_t \overset{\text{i.i.d}}{\sim} N(0, \sigma^2) \tag{1}$$

to observed time series $y_1, \dots, y_n$. A key role for inferring the frequency parameter f in this model is played by:

$$RSS(f) := \min_{\beta_0, \beta_1, \beta_2} \sum_{t=1}^{n} (y_t - \beta_0 - \beta_1 \cos(2\pi f t) - \beta_2 \sin(2\pi f t))^2 = \|y - X_f \hat{\beta}_f\|^2$$

where

$$y = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix} \quad \text{and} \quad X_f = \begin{pmatrix} 1 & \cos(2\pi f(1)) & \sin(2\pi f(1)) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & \cos(2\pi f(n)) & \sin(2\pi f(n)) \end{pmatrix} \quad \text{and} \quad \hat{\beta}_f = (X_f^T X_f)^{-1} X_f^T y.$$

$RSS(f)$ is simply the residual sum of squares in the linear regression model obtained by fixing the frequency parameter f. It describes how well the sinusoid with frequency f fits the observed data $y_1, \dots, y_n$.

Calculating $RSS(f)$ separately for each value of f in a grid of values of f can be computationally quite inefficient (particularly when n is large). We have started discussing efficient computation of $RSS(f)$ in the last lecture. The key here is to recognize the following alternative formula for $RSS(f)$ (proved in last lecture) that holds when $f \in (0, 0.5)$ is a Fourier frequency i.e., nf is an integer:

$$RSS(f) = \sum_t (y_t - \bar{y})^2 - 2I(f) \qquad \text{when } f \in (0, 0.5) \text{ is a Fourier Frequency} \tag{2}$$

where $I(f)$ is defined by

$$I(f) := \frac{1}{n}\left(\sum_{t=1}^{n} y_t \cos(2\pi f t)\right)^2 + \frac{1}{n}\left(\sum_{t=1}^{n} y_t \sin(2\pi f t)\right)^2 \qquad \text{for } f \in (0, 0.5) \tag{3}$$

$I(f)$ is known as the *Periodogram* of the data $y_1, \dots, y_n$.

In this lecture, we study the Discrete Fourier Transform (DFT) of the observed time series data, and see how the DFT is related to the periodogram.

2 Discrete Fourier Transform (DFT) and Periodogram

The periodogram $I(f)$ clearly can be rewritten as:

$$I(f) = \frac{1}{n} \left| \sum_{t=1}^{n} y_t \exp(-2\pi i f t) \right|^2 .$$

The term inside the modulus sign above is almost the same as the Discrete Fourier Transform (DFT) of the data. There is only one difference which comes from changing the data indexing slightly. While discussing the DFT, it is a standard convention to write the data as $y_0, y_1, \dots, y_{n-1}$ (instead of $y_1, \dots, y_n$). In other words, we start the indexing with 0 (as in Python) as opposed to 1. With this indexing, the DFT is defined as follows.

The DFT of data $y_0, \dots, y_{n-1}$ is defined by:

$$b_j := \sum_{t=0}^{n-1} y_t \exp\left(-\frac{2\pi i j t}{n}\right) \qquad \text{for } j = 0, 1, \dots, n-1. \tag{4}$$

In words, b_j is the dot product between the data and the complex sinusoid with frequency $f = j/n$. Note that the complex sinusoid with frequency f is given by:

$$\exp(2\pi i f t) \qquad \text{for } t = 0, 1, 2, \dots, n-1.$$

The n (possibly) complex numbers $b_0, b_1, \dots, b_{n-1}$ are collectively called the DFT of $y_0, \dots, y_{n-1}$. Typically, $y_0, \dots, y_{n-1}$ will represent observed time series data. It is important to note that even though $y_0, \dots, y_{n-1}$ are real-valued, their DFT $b_0, \dots, b_{n-1}$ can be complex-valued.

More on indexing: As mentioned previously, while defining the DFT, we index the data starting from zero. In the definition (4), the first data point (which is y_0) is being multiplied by $\exp(-2\pi i j(0)/n) = 1$, the second data point is being multiplied by $\exp(-2i\pi j/n)$ and, in general, the t-th data point is being multiplied by $\exp(-2i\pi j(t-1)/n)$. If we instead define the DFT by:

$$\tilde{b}_j := \sum_{t=1}^{n} y_t \exp\left(-\frac{2\pi i j t}{n}\right),$$

then the t-th data point will be multiplied by $\exp(-2i\pi j t/n)$. b_j and $\tilde{b}_j$ will be different and they will be related by:

$$b_j = \tilde{b}_j \exp\left(\frac{2\pi i j}{n}\right).$$

The multiplier $\exp(2\pi i j/n)$ above has modulus one which means that

$$|b_j| = |\tilde{b}_j|.$$

So if we are only looking at the moduli of the DFT terms (note the periodogram only involves the moduli of DFT), then it does not matter whether we index data starting with 0 or 1. The standard convention while studying the DFT is to index starting from 0.

The connection between the periodogram and the DFT is given by:

$$I(j/n) = \frac{|b_j|^2}{n} \qquad \text{for } 0 < \frac{j}{n} < 1/2$$

and the connection between $RSS(j/n)$ and DFT is given by:

$$RSS(j/n) = \sum_t (y_t - \bar{y})^2 - 2I(j/n) \qquad \text{for } 0 > j/n < 0.5.$$

It is important to note that the DFT can be calculated very efficiently. The DFT of $y = (y_0, \dots, y_{n-1})^T$ can be obtained in `numpy` using the command `np.fft.fft(y)`. Here `fft` stands for Fast Fourier Transform which is a special efficient algorithm for computing the DFT. Naively, it would seem that to compute the DFT would need $O(n^2)$ computation (for each of n values of j, we have to compute b_j which requires a sum over the dataset of size n), but the FFT algorithm exploits symmetry and redundancy in the complex exponentials to compute the DFT in only $O(n \log n)$ time. We will not be going over the details of the FFT algorithm in class.

3 Basic Properties of the DFT

Here are some basic things to note about the DFT (defined in (4)).

1. b_0 is always equal to $y_0 + \cdots + y_{n-1}$. To see this, just plug in $j = 0$ in (4).
2. In general b_j is a complex number with real and imaginary parts given by:

$$\text{real part of } b_j = \sum_{t=0}^{n-1} y_t \cos\left(\frac{2\pi jt}{n}\right) \text{ and imaginary part of } b_j = -\sum_{t=0}^{n-1} y_t \sin\left(\frac{2\pi jt}{n}\right)$$

   Sometimes the imaginary part will be zero (for example, when n is even and $j = n/2$) but, in general, b_j will be complex-valued.

3. For each $j = 1, \dots, n-1$, the DFT term b_{n-j} equals the complex conjugate of b_j:

$$b_{n-j} = \bar{b}_j. \tag{5}$$

   The reason for the above is

$$b_{n-j} = \sum_t y_t \exp\left(-\frac{2\pi i(n-j)t}{n}\right) = \sum_t y_t \exp\left(\frac{2\pi ijt}{n}\right) \exp(-2\pi it) = \bar{b}_j,$$

   where, in the above, we used that $\exp(-2\pi it) = 1$ (because t is an integer) and that $\exp\left(\frac{2\pi ijt}{n}\right)$ is the complex conjugate of $\exp\left(-\frac{2\pi ijt}{n}\right)$. Note that, for the above argument, it is crucial that $y_0, \dots, y_{n-1}$ are real. If some of $y_0, \dots, y_{n-1}$ are complex, the relation (5) is no longer true.

Because of (5), the DFT terms for later indices j can be determined as complex conjugates for the DFT terms for earlier indices. For example, when $n = 11$, the DFT can be written as:

$$b_0, b_1, b_2, b_3, b_4, b_5, \bar{b}_5, \bar{b}_4, \bar{b}_3, \bar{b}_2, \bar{b}_1,$$

and, for $n = 12$, it is

$$b_0, b_1, b_2, b_3, b_4, b_5, b_6 = \bar{b}_6, \bar{b}_5, \bar{b}_4, \bar{b}_3, \bar{b}_2, \bar{b}_1.$$

Note that when $n = 12$, the term b_6 is necessarily real because $b_6 = \bar{b}_6$.

Thus when $n = 11$, the data consists of 11 real numbers while the DFT consists of one real number (b_0) and 5 complex numbers. On the other hand, when $n = 12$, the data consists of 12 real numbers while the DFT consists of two real numbers (b_0 and b_6) and 5 complex numbers.

It turns out that the data $y_0, y_1, \ldots, y_{n-1}$ can be recovered from the DFT $b_0, \ldots, b_{n-1}$ using a simple formula (this formula is sometimes known as the inverse DFT formula). Before seeing this, it is necessary to understand orthogonality properties of complex sinusoids with Fourier frequencies.

4 Complex Sinuoids and Orthogonality

4.1 Complex Sinusoids

As we saw in the last lecture, sinusoids are linear combinations of $\cos(2\pi ft)$ and $\sin(2\pi ft)$. While doing algebra with sinusoids, it is very useful to represent them in terms of complex exponentials as:

$$\cos(2\pi ft) = \frac{1}{2}\exp(2\pi ift) + \frac{1}{2}\exp(-2\pi ift) \text{ and } \sin(2\pi ft) = \frac{1}{2i}\exp(2\pi ift) - \frac{1}{2i}\exp(-2\pi ift)$$

In the last lecture, we saw that while dealing with sinusoids at integer valued time points t, we can restrict the frequency f to $[0, 1/2]$. However, if we use the formulae above, note that we have to deal with $-f$ as well (because of the second term $e^{-2\pi ift} = e^{2\pi i(-f)t}$) and $-f$ lies between $-1/2$ and 0. Thus when discussing sinusoids in terms of complex exponentials $e^{2\pi ift}, t = 0, 1, \ldots, n-1$, one takes $f \in [-0.5, 0.5)$ (note that $f = -0.5$ leads to the same $e^{2\pi ift}$ as $f = 0.5$ so we drop $f = 0.5$ from consideration). For example, the function `np.fft.fftfreq(n)` gives all Fourier frequencies in $[-0.5, 0.5)$.

If one does not want to deal with negative frequencies, then we can use

$$e^{-2\pi ift} = \cos(2\pi ft) - i\sin(2\pi ft) = \cos(2\pi(1-f)t) + i\sin(2\pi(1-f)t) = e^{2\pi i(1-f)t}$$

because $\cos(2\pi(1-f)t) = \cos(2\pi t - 2\pi ft) = \cos(2\pi ft)$ (note t is an integer) and $\sin(2\pi(1-f)t) = \sin(2\pi t - 2\pi ft) = -\sin(2\pi ft)$.

Therefore, if we want to use complex exponentials $e^{2\pi ift}$ but we do not want to deal with negative frequencies, then we can restrict f to $[0, 1)$. From here on, whenever we consider the complex sinusoid $x_t = e^{2\pi ift}$ for $t = 0, 1, \ldots, n-1$, we restrict $f \in [0, 1)$.

4.2 Complex Sinusoidal Vectors

For every $0 \le j \le (n-1)$, let us define the $n \times 1$ vector

$$u^j = (1, \exp(2\pi ij/n), \exp(2\pi i2j/n), \ldots, \exp(2\pi i(n-1)j/n))^T.$$

This vector can be interpreted as the complex sinusoid $e^{2\pi ift}$ with Fourier frequency $f = j/n$ evaluated at the time points $t = 0, 1, \ldots, (n-1)$. It is easy to see that

1. When $j = 0$, we have $u^0 = (1, 1, \ldots, 1)$.
2. When $1 \le j \le n-1$, we have $u^j = \overline{u^{n-j}}$. Here $\bar{u}$ denotes complex conjugate of u (the complex conjugate $\bar{u}$ of a vector u is defined as the vector obtained by taking the complex conjugates of each entry of u).

The most important property of these complex valued vectors $u^0, u^1, \ldots, u^{n-1}$ is **orthogonality**. Specifically, for $0 \leq j \neq k \leq n-1$, we have

$$\left\langle u^j, u^k \right\rangle = 0. \tag{6}$$

Recall that the inner product between two complex valued vectors $a = (a_1, \ldots, a_n)^T$ and $b = (b_1, \ldots, b_n)^T$ is given by

$$\langle a, b \rangle = \sum_{j=1}^{n} a_j \bar{b}_j.$$

Note specially the complex conjugate of b_j above.

Here is the proof of (6). Fix $0 \leq j \neq k \leq n-1$ and write

$$\begin{aligned}
\left\langle u^j, u^k \right\rangle &= \sum_{t=0}^{n-1} \exp\left(2\pi i \frac{j}{n} t\right) \overline{\exp\left(2\pi i \frac{k}{n} t\right)} \\
&= \sum_{t=0}^{n-1} \exp\left(2\pi i \frac{j}{n} t\right) \exp\left(-2\pi i \frac{k}{n} t\right) \\
&= \sum_{t=0}^{n-1} \exp\left(2\pi i \frac{j-k}{n} t\right) \\
&= \sum_{t=0}^{n-1} \left[\exp\left(2\pi i \frac{j-k}{n}\right)\right]^t \\
&= \frac{1 - \left(\exp\left(2\pi i \frac{j-k}{n}\right)\right)^n}{1 - \exp\left(2\pi i \frac{j-k}{n}\right)} \\
&= \frac{1 - \exp\left(2\pi i (j-k)\right)}{1 - \exp\left(2\pi i \frac{j-k}{n}\right)} \\
&= \frac{1 - \cos\left(2\pi (j-k)\right) - i \sin\left(2\pi (j-k)\right)}{1 - \exp\left(2\pi i \frac{j-k}{n}\right)} = \frac{1 - 1 - 0}{1 - \exp\left(2\pi i \frac{j-k}{n}\right)} = 0
\end{aligned}$$

This proves (6). It is also easy to see that (just take $j = k$ in the above calculation and the answer can be found in the third line)

$$\left\langle u^j, u^j \right\rangle = \|u^j\|^2 = n.$$

Therefore the n complex-valued vectors $u^0, u^1, \ldots, u^{n-1}$ are orthogonal and they all have the same squared length equal to n. This immediately implies that they form a **basis** for the space $\mathbb{C}^n$ consisting of all complex-valued vectors of length n. In other words, every complex-valued vector of length n can be written as a linear combination of $u^0, u^1, \ldots, u^n$.

4.3 The Inverse DFT formula

The inverse DFT formula recovers the data $y_0, \ldots, y_{n-1}$ from their DFT $b_0, \ldots, b_{n-1}$. We derive this formula below.

The main observation is the following: Because $u^0, \ldots, u^{n-1}$ form a basis, we can write any $n \times 1$ vector of complex entries:

$$y = (y_0, \ldots, y_{n-1})^T$$

as a linear combination of $u^0, \ldots, u^{n-1}$. More specifically, we can write

$$y = a_0 u^0 + a_1 u^1 + \cdots + a_{n-1} u^{n-1} \tag{7}$$

Take the inner product of both sides of the above equation with u^j for a fixed j and use orthogonality so that $\langle u^j . u^k \rangle = 0$ for $k \neq j$ and the fact that $\langle u^j, u^j \rangle = n$ to obtain

$$a_j = \frac{1}{n} \langle y, u^j \rangle = \frac{1}{n} \sum_{t=0}^{n-1} y_t \exp\left(-\frac{2\pi i j t}{n}\right). \tag{8}$$

By the formula (4) for the DFT b_j, it is easy to see that $a_j = b_j/n$. As a consequence (7) becomes:

$$y = \frac{1}{n} \left(b_0 u^0 + b_1 u^1 + \cdots + b_{n-1} u^{n-1}\right).$$

Writing the t-th entry on both sides, we get

$$y_t = \frac{1}{n} \sum_{j=0}^{n-1} b_j \exp\left(\frac{2\pi i j t}{n}\right) \qquad \text{for each } t = 0, 1, \ldots, n-1. \tag{9}$$

This is the inverse DFT formula. Note that the inverse DFT formula (9) as well as the DFT definition (4) look similar; the differences being in the sign of the exponent in the complex exponential and the presence of the factor $1/n$ in (9).