Lecture Twenty-Three

ARMA(p, q): $\phi(B)(y_t - \mu) = \theta(B)\varepsilon_t$

Causal-stationary regime: ϕ having roots all with modulus > 1

ARIMA(data, order = (p, 0, q))

Preprocessing: $y \longrightarrow \underbrace{(I-B)y}_{y_t - y_{t-1}} \longrightarrow (I-B)^2 y$

$y_t - 2y_{t-1} + y_{t-2}$

ARIMA(p, d, q):

$$\phi(B)\left((I-B)^d y - \mu\right) = \theta(B)\varepsilon_t$$

$\rightarrow$ $d \leq 2$, $p \leq 5$, $q \leq 5$

d: $(I-B)y$, $(I-B)^2 y$

Seasonal ARIMA

Quite often, we deal with monthly data.

p, q

Sample ACF

① Seasonal ARMA models
② Multiplicative Seasonal ARMA models
③ Seasonal ARIMA models (SARIMA models)

Seasonal ARMA / MA

MA(1): $y_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}$

$$ACF(h) = \begin{cases} 1 & \text{if } h=0 \\ \frac{\theta}{1+\theta^2} & \text{if } h=1 \\ 0 & \text{if } |h| \geq 2 \end{cases}$$

Suppose we want:

$$ACF(h) = \begin{cases} 1 & \text{if } h=0 \\ \frac{\theta}{1+\theta^2} & \text{if } h=12 \\ 0 & \text{for all other } h \end{cases}$$

$$\boxed{y_t = \mu + \varepsilon_t + \theta \varepsilon_{t-12}}$$

$Cov(y_t, y_{t+h})$

$= Cov(\varepsilon_t + \theta \varepsilon_{t-12}, \varepsilon_{t+h} + \theta \varepsilon_{t+h-12})$

$\neq 0$ only when $h = 0, 12, -12$

$$ACF(h) = \begin{cases} 1 & h=0 \\ \neq 0 & |h| = 12 \\ 0 & \text{for all other } h \end{cases}$$

$\boxed{\text{Seasonal } MA(1)_{12}}$

$MA(1)$: $\quad y_t - \mu = \theta(B)\varepsilon_t$

$\theta(z) = 1 + \theta z$

Seasonal $MA(1)$: $\quad y_t - \mu = \theta(B^{12})\varepsilon_t$

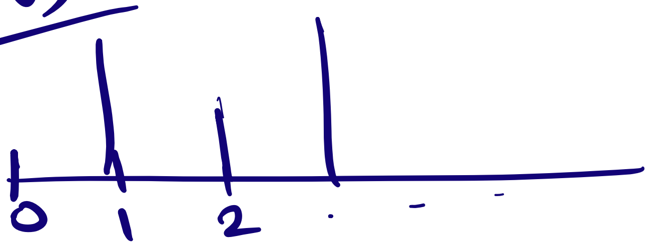
$B^{12} y_t = y_{t-12}$

$ARMA(p,q)$: $\quad \phi(B)(y_t - \mu) = \theta(B)\varepsilon_t$

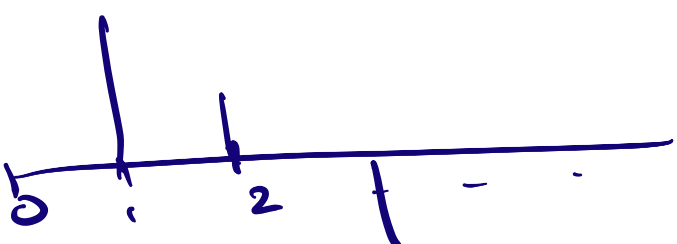
Seasonal $ARMA(p,q)$ with period s

$$\phi(B^s)(y_t - \mu) = \theta(B^s)\varepsilon_t$$

ARMA(p,q)



ACF

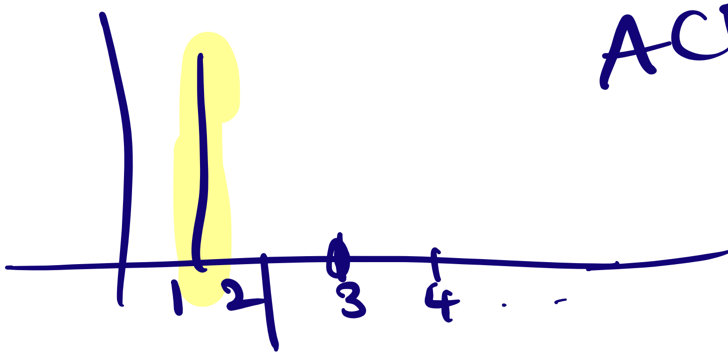


PACF

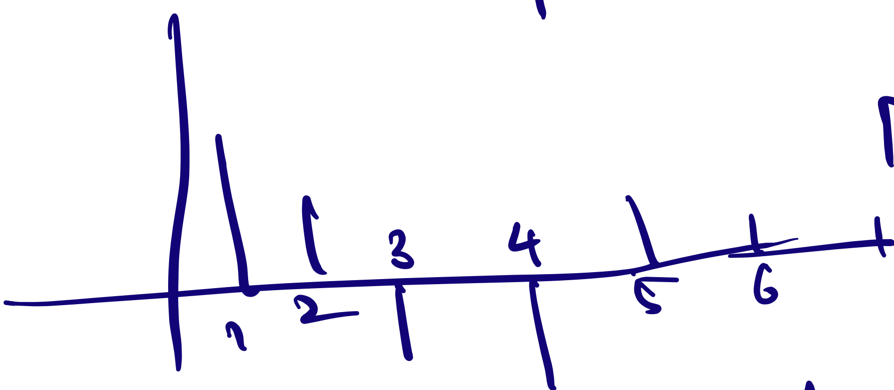
Seasonal ARMA(p, q) with period s

will the same structure of ACF, PACF as regular ARMA(p, q) but at lags that are multiples of s.

MA(2):

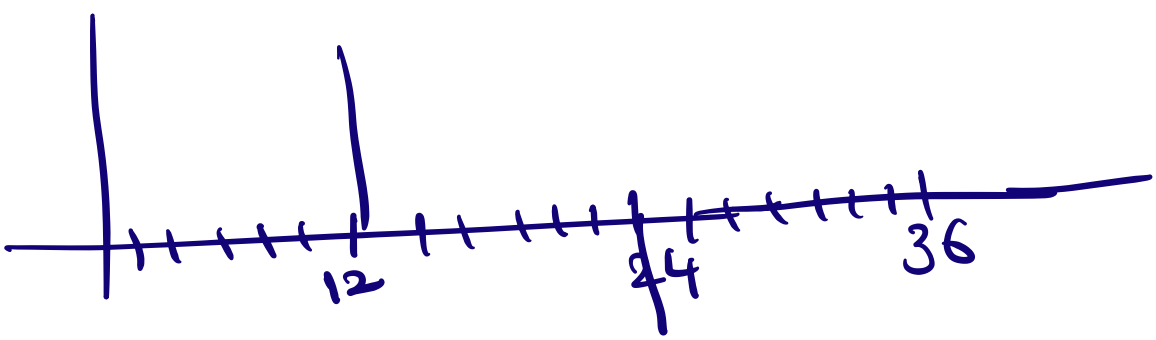


ACF



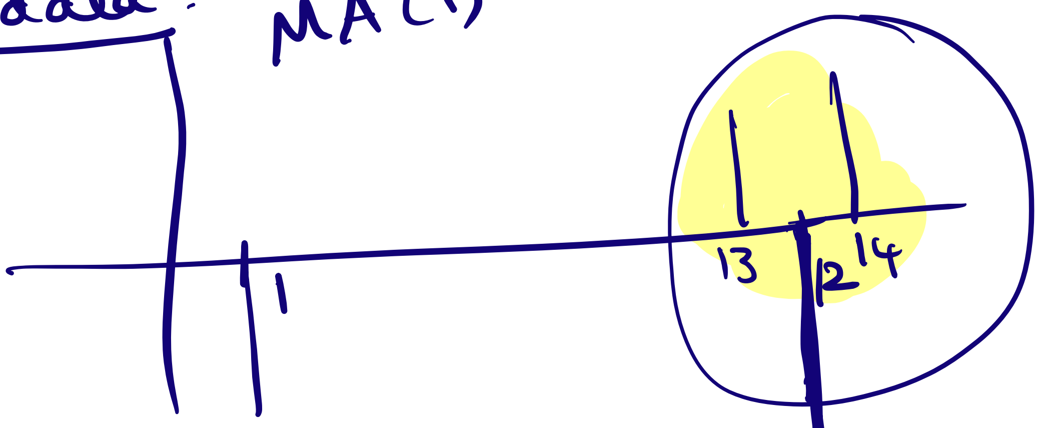
PACF

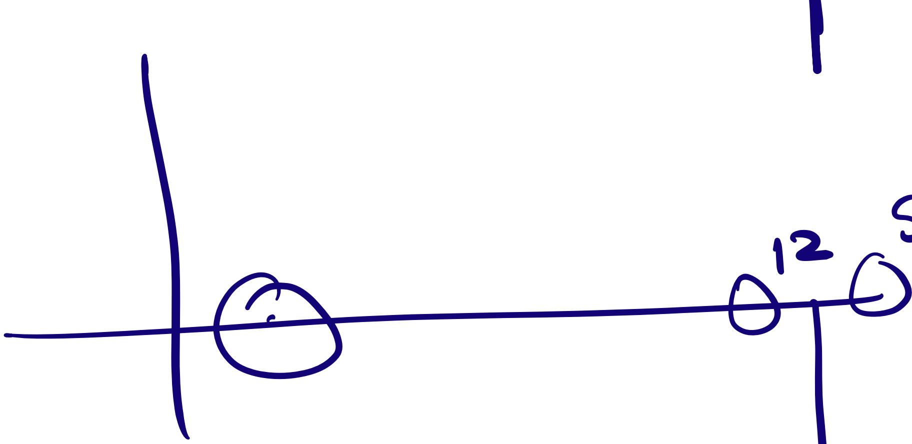
Seasonal MA(2) with period s



CO_2 data: MA(1)

ACF





Seasonal MA(1) with period $s=12$

$\boxed{\text{Multiplicative}}$ Seasonal ARMA models

$y_t - \mu = \theta(B)\varepsilon_t$ → regular MA(1)

$\theta(z) = 1 + \theta z$

$y_t - \mu = \theta(B^s)\varepsilon_t$ → seasonal MA(1) with period 12

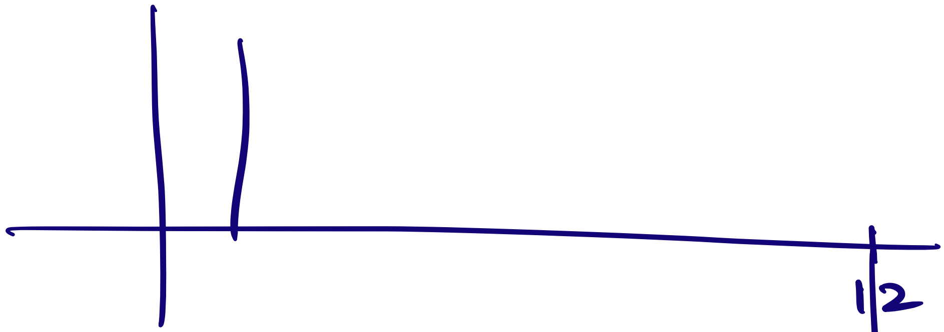
$$\boxed{y_t - \mu = \theta(B)\Theta(B^s)\varepsilon_t} \rightarrow$$

$$\boxed{MA(1) \times MA(1)_{12}}$$

$$y_t - \mu = (1 + \theta B)(1 + \Theta B^s)\varepsilon_t$$

$$\rightarrow \boxed{y_t - \mu = \varepsilon_t + \theta\varepsilon_{t-1} + \Theta\varepsilon_{t-s} + \theta\Theta\varepsilon_{t-s-1}}$$

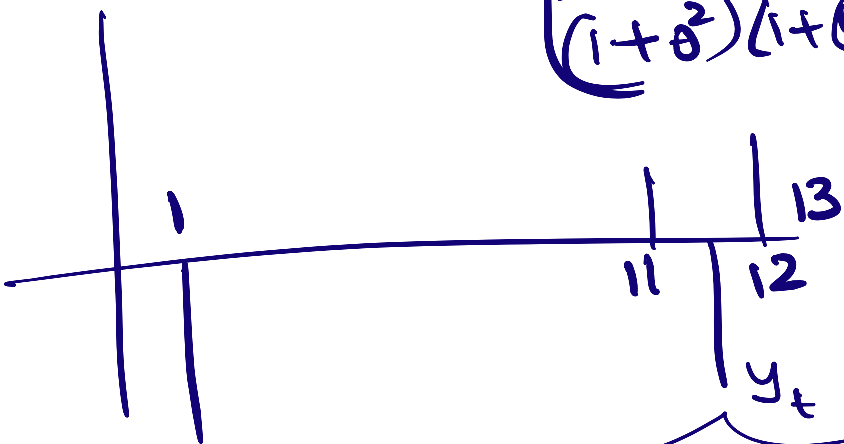
→ special case of MA(13).



ACF of $MA(1) \times MA(1)_{12}$:

$\boxed{\text{Check:}}$

$$ACF(h) = \begin{cases} 1 & \text{if } h=0 \\ \dfrac{\theta}{1+\theta^2} & \text{if } h=1 \\ \dfrac{\Theta}{1+\Theta^2} & h=12 \\ \dfrac{\theta\Theta}{(1+\theta^2)(1+\Theta^2)} & h=11,13 \end{cases}$$



$$\mathrm{Cov}\Big(\overbrace{\varepsilon_t+\theta\varepsilon_{t-1}+\Theta\varepsilon_{t-12}+\theta\Theta\varepsilon_{t-13}}^{y_t},\ \underbrace{\varepsilon_{t+11}+\theta\varepsilon_{t+10}+\Theta\varepsilon_{t-1}+\theta\Theta\varepsilon_{t-2}}_{y_{t+11}}\Big)$$

$$\varepsilon_{t+13}+\theta\varepsilon_{t+12}+\Theta\varepsilon_{t+1}+\theta\Theta\varepsilon_t$$

Multiplicative Seasonal ARMA

ARMA(p,q) $\rightarrow$ $\phi(B)(y_t-\mu) = \theta(B)\varepsilon_t$

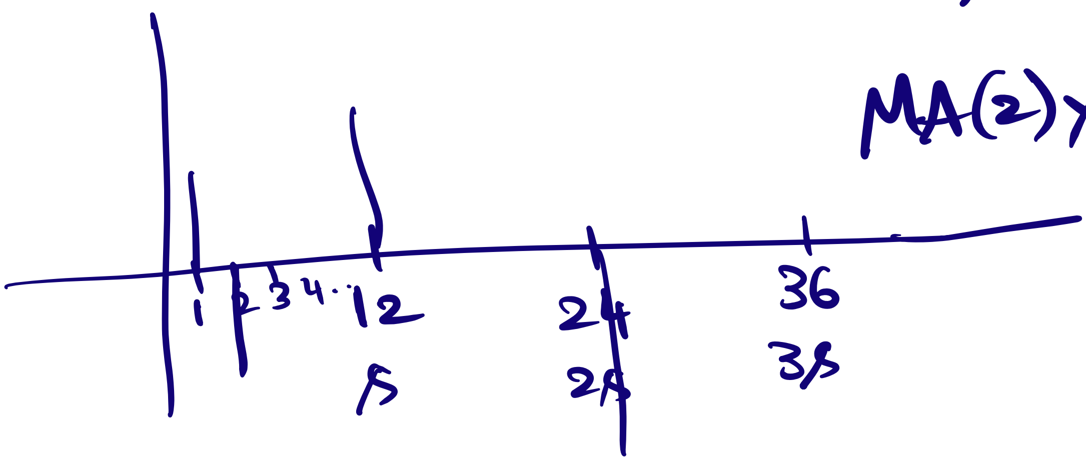
Seasonal ARMA(p,q) with period s $\rightarrow$ $\Phi(B^s)(y_t-\mu) = \Theta(B^s)\varepsilon_t$

$$\boxed{\phi(B)\Phi(B^s)(y_t-\mu) = \theta(B)\Theta(B^s)\varepsilon_t}$$

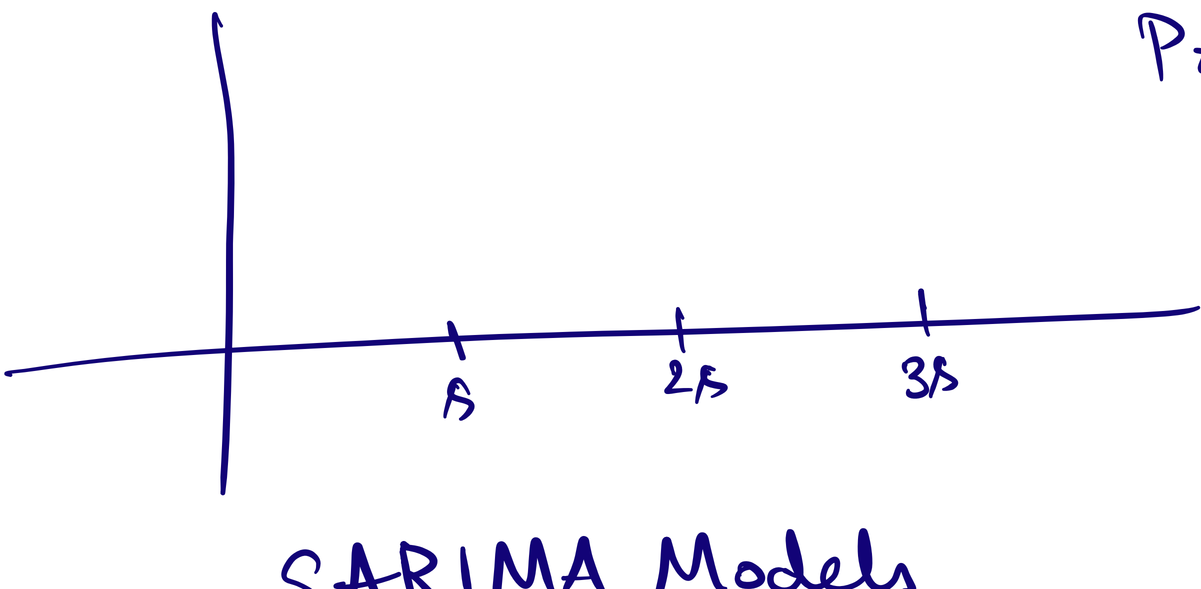
Heuristic for understanding ACF & PACF

ACF

$MA(2)\times MA(2)_{12}$



PACF



SARIMA Models

ARIMA(data, order = (p, d, q),
seasonal order
= (P, D, Q, s))

$$(I-B^s)^D (I-B)^d y_t = x_t$$ → preprocessing

$$\Phi(B^s)\phi(B)(x_t-\mu) = \Theta(B^s)\theta(B)\varepsilon_t$$

$$\Phi(B^s)\phi(B)\left((I-B^s)^D (I-B)^d y_t - \mu\right) = \Theta(B^s)\theta(B)\varepsilon_t$$

SARIMA