Lecture Twenty

Stationarity: $\mathbb{E}\, y_t$ does not change with time t

$\operatorname{var} y_t$ also does not change with t

$\operatorname{Cov}(y_t, y_{t+h})$ also does not change with t (for fixed h)

Eg: $\varepsilon_t \overset{iid}{\sim} N(0,\sigma^2)$

$$\boxed{y_t} = \boxed{\mu} + \boxed{\varepsilon_t} + \boxed{\theta\,\varepsilon_{t-1}}, \quad \text{all } t$$

μ, θ, σ

MA(1)

moving average

Slutzky

'Summation of Random Causes'

$$\cdot\ \varepsilon_0\ \ \varepsilon_1\ \ \varepsilon_2\ \ \varepsilon_3\ \cdots$$

$$y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

MA(q)

$$\mathbb{E}\, y_t = \mu$$

$$\operatorname{var}(y_t) = \sigma^2(1+\theta^2)$$

$$\operatorname{Cov}(y_t, y_{t+1}) = \operatorname{Cov}(\mu + \varepsilon_t + \theta\varepsilon_{t-1},\ \mu + \varepsilon_{t+1} + \theta\varepsilon_t)$$

$$= \operatorname{Cov}(\varepsilon_t, \theta\varepsilon_t)$$

$$= \theta\sigma^2$$

$$\mathrm{Cov}(y_t, y_{t+2}) = \mathrm{Cov}(\mu + \varepsilon_t + \theta\varepsilon_{t-1}, \mu + \varepsilon_{t+2} + \theta\varepsilon_{t+1})$$
$$= 0$$

$\mathrm{Cov}(y_t, y_{t+h}) = 0$ for $h \geq 2$
$\quad \overset{\shortparallel}{\gamma(h)}$

$$\gamma(h) = \begin{cases} \sigma^2(1+\theta^2) & \text{if } h = 0 \\ \theta\sigma^2 & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1 \end{cases}$$

ACF
$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \begin{cases} 1 & \text{if } h = 0 \\ \frac{\theta}{1+\theta^2} & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1 \end{cases}$$

AR models & Stationarity

AR(1): $y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t$
$\varepsilon_t \underset{\sim}{\text{iid}} N(0, \sigma^2)$

Is this stationary?

y_1 = fixed at the observed value.

$$\prod_{t=2}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[\frac{(y_t - \phi_0 - \phi_1 y_{t-1})^2}{2\sigma^2}\right]$$

y_1

$y_2 = \phi_0 + \phi_1 y_1 + \varepsilon_2$

$y_3 = \phi_0 + \phi_1 y_2 + \varepsilon_3 = \phi_0 + \phi_1(\phi_0 + \phi_1 y_1 + \varepsilon_2) + \varepsilon_3$

$$= \phi_0(1+\phi_1) + \phi_1^2 y_1 + \varepsilon_3 + \phi_1 \varepsilon_2$$

$$y_t = \phi_0\left(1+\phi_1+\cdots+\phi_1^{t-2}\right) + \phi_1^{t-1} y_1 + \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 \varepsilon_{t-2} + \cdots + \phi_1^{t-2} \varepsilon_2$$

Suppose $|\phi_1| < 1$, then y_t will be approximately equal to (at least for t which are not small)

$$y_t = \phi_0\left(1+\phi_1+\phi_1^2+\cdots\right) + \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 \varepsilon_{t-2} + \cdots$$

$$\boxed{y_t = \frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j \varepsilon_{t-j}}$$

$$y_t = \mu + \varepsilon_t + \underset{\phi_1}{\boxed{\theta_1}} \varepsilon_{t-1} + \underset{\phi_1^2}{\boxed{\theta_2}} \varepsilon_{t-2} + \cdots + \underset{\phi_1^q}{\boxed{\theta_q}} \varepsilon_{t-q}$$

$$\mathbb{E} y_t = \frac{\phi_0}{1-\phi_1}$$

$$\operatorname{Cov}(y_t, y_{t+h}) = \frac{\sigma^2 \phi_1^{|h|}}{1-\phi_1^2}$$

CHECK

AR(1) CAUSAL-stationary when $\boxed{|\phi_1| < 1.}$

$$y_t = \frac{\phi_0}{1-\phi_1} - \frac{\varepsilon_{t+1}}{\phi_1} - \frac{\varepsilon_{t+2}}{\phi_1^2} - \cdots$$

when $|\phi_1| > 1$ $\quad y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t$

$$y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t$$

$$\phi_1 y_{t-1} = -\phi_0 + y_t - \varepsilon_t$$

$$y_{t-1} = \frac{-\phi_0}{\phi_1} + \frac{y_t}{\phi_1} - \frac{\varepsilon_t}{\phi_1}$$

CAUSAL STATIONARY AR(1) $\Longleftrightarrow$ $|\phi_1| < 1$

AR(p) for $p \geq 1$

$$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t$$

y_1, y_2 $\quad y_3, y_4, \ldots$

BACKSHIFT TRICK

BACKSHIFT OPERATOR

$$B y_t = y_{t-1}$$

$$B^0 y_t = y_t$$

$$B^0 = I$$

$$B^2 y_t = B(B y_t) = B y_{t-1} = y_{t-2}$$

$$(B + 2B^2) y_t = y_{t-1} + 2 y_{t-2}$$

$$B^k y_t = y_{t-k}, \quad B^{-3} y_t = y_{t+3}$$

AR(p) model using Backshift Notation:

$$y_t = \phi_0 + \underbrace{\phi_1 y_{t-1} + \cdots + \phi_p y_{t-p}} + \varepsilon_t$$

$$y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \cdots - \phi_p y_{t-p} = \phi_0 + \varepsilon_t$$

$$y_t - \phi_1 B y_t - \phi_2 B^2 y_t - \cdots - \phi_p B^p y_t = \phi_0 + \varepsilon_t$$

$$(I - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p)\, y_t = \phi_0 + \varepsilon_t$$

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$$: AR polynomial

$$\boxed{\phi(B)\, y_t = \phi_0 + \varepsilon_t} \rightarrow \text{AR}(p) \text{ model}$$

AR(1): $\phi(B) = I - \phi_1 B$

$$\phi(z) = 1 - \phi_1 z$$

$$(I - \phi_1 B)\, y_t = \phi_0 + \varepsilon_t$$

$$y_t = \boxed{\frac{1}{I - \phi_1 B}} (\phi_0 + \varepsilon_t)$$

$$\boxed{\frac{1}{1 - \phi_1 z} = 1 + \phi_1 z + (\phi_1 z)^2 + (\phi_1 z)^3 + \cdots}$$

$$= (I + \phi_1 B + \phi_1^2 B^2 + \phi_1^3 B^3 + \cdots)(\phi_0 + \varepsilon_t)$$

$$= (I + \phi_1 B + \phi_1^2 B^2 + \cdots)\,\phi_0 + (I + \phi_1 B + \phi_1^2 B^2 + \cdots)\,\varepsilon_t$$

$$= (I + \phi_1 B + \phi_1^2 B^2 + \cdots$$

$$= \phi_0(1+\phi_1+\phi_1^2+\cdots) + \varepsilon_t + \phi_1\varepsilon_{t-1} + \phi_1^2\varepsilon_{t-2}+\cdots$$

$$\boxed{y_t = \frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty}\phi_1^j\varepsilon_{t-j}} \longleftrightarrow y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t$$

AR(2)

$$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t$$

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2$$

$$\boxed{\phi(B)\, y_t = \phi_0 + \varepsilon_t}$$

$$\Rightarrow y_t = \frac{1}{\phi(B)}(\phi_0 + \varepsilon_t)$$

$$= \left(\frac{1}{I - \phi_1 B - \phi_2 B^2}\right)(\phi_0 + \varepsilon_t)$$

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 = (1 - a_1 z)(1 - a_2 z)$$

$\frac{1}{a_1}, \frac{1}{a_2}$ denote roots of $\phi(z)$.

$$= \frac{1}{(I - a_1 B)(I - a_2 B)}(\phi_0 + \varepsilon_t)$$

$$= \underbrace{(I + a_1 B + a_1^2 B^2 + a_1^3 B^3 + \cdots)(I + a_2 B + a_2^2 B^2 + \cdots)}\ \underbrace{(\phi_0 + \varepsilon_t)}$$

$$= \boxed{\mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}}$$ for some $\{\psi_j\}$.

$$\mu = \phi_0 \sum_{j=0}^{\infty} \psi_j$$

ψ_1

$$\psi_1 = a_1 + a_2$$

$$\psi_2 = a_1^2 + a_2^2 + a_1 a_2$$

$$\psi_3 =$$

<u>Need</u>: a_1, a_2 to have moduli strictly smaller than 1.

<u>AR(1)</u>: $y_t = \phi_0 + \boxed{\phi_1} y_{t-1} + \varepsilon_t$

When does it have a causal stationary solution?

$|\phi_1| < 1$

<u>AR(p)</u>: $y_t = \phi_0 + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t$

When does it have a causal stationary solution?

AR polynomial: $1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$.

Calculate roots of the polynomial: $\frac{1}{a_1}, \ldots \frac{1}{a_p}$

Check: $\underline{|a_j| < 1 \text{ for every } j}$

<u>Note</u>: When $p=1$, AR polynomial $1 - \phi_1 z$

root: $\frac{1}{\phi_1}$ $a_1 = \phi_1$

$$\frac{1}{I - a_1 B} = I + a_1 B + (a_1 B)^2 + \cdots$$

$$\frac{1}{I - a_2 B} = I + a_2 B + (a_2 B)^2 + \cdots$$

$$\{I + a_1 B + (a_1 B)^2 + \cdots\}\{I + a_2 B + (a_2 B)^2 + \cdots\}$$

$$= I + (a_1 + a_2) B + (a_1^2 + a_1 a_2 + a_2^2) B^2 + (\quad) B^3 + \cdots$$

$$= (\psi_0 I + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots) \boxed{(\phi_0 + \varepsilon_t)}$$

$\psi_0 = 1$

$$\sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$$

$$\phi_0 \sum \psi_j$$