Lecture Nineteen

AR models

$$y_t = \phi_0 + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t$$

AR(p) $\qquad \varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$

$y_1, y_2, \ldots, y_p$ $\quad \boxed{\text{OLS AutoReg}} \quad \hat{\phi}_j, \hat{\sigma}$

$$\hat{y}_{n+1} = \hat{\phi}_0 + \hat{\phi}_1 y_n + \hat{\phi}_2 y_{n-1} + \cdots + \hat{\phi}_p y_{n+1-p}$$

$$\hat{y}_{n+2} = \hat{\phi}_0 + \hat{\phi}_1 \hat{y}_{n+1} + \hat{\phi}_2 y_n + \cdots + \hat{\phi}_p y_{n+2-p}$$

$\hat{y}_{n+3}, \ldots$ $\quad$ get_prediction

$$\boxed{\operatorname{Cov}\left(\begin{matrix} y_{n+1} \\ \vdots \\ y_{n+k} \end{matrix} \,\middle|\, \begin{matrix} y_1 \ldots y_n \\ \theta \end{matrix}\right), \quad k = 1, 2, \ldots}$$

1. How to select p? Sample PACF
2. Different types of AR models

e.g: (a) Suppose we fit AR(1) to $y_1 \ldots y_n$.

$$\hat{\phi}_0 = 5, \quad \hat{\phi}_1 = \frac{1}{2}$$

SHORT-RANGE PREDICTIONS

$$\hat{y}_{n+1} = 5 + \frac{1}{2} y_n$$

$$\hat{y}_{n+2} = 5 + \frac{1}{2}\hat{y}_{n+1} = 5 + \frac{1}{2}\left(5 + \frac{1}{2} y_n\right)$$

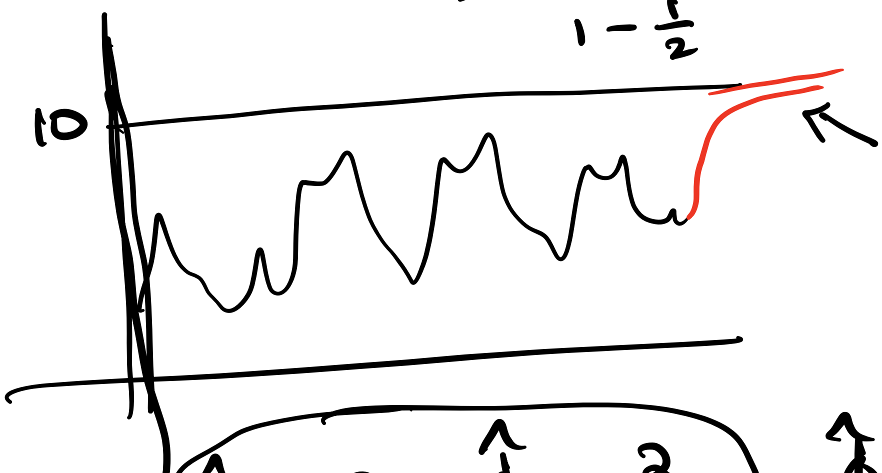
$$= 5\left(1 + \frac{1}{2}\right) + \frac{1}{4} y_n$$

$$\hat{y}_{n+3} = 5 + \frac{1}{2}\hat{y}_{n+2}$$

$$= 5\left(1+\frac{1}{2}+\frac{1}{2^2}\right) + \frac{y_n}{2^3}$$

$$\hat{y}_{n+k} = 5\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{k-1}}\right) + \frac{y_n}{2^k}$$

$$\xrightarrow{k\to\infty} \frac{5}{1-\frac{1}{2}} = 10$$



(b) $\hat{\phi}_0 = 5, \quad \hat{\phi}_1 = 2$ $\qquad \hat{\phi}_1 = 1.09$ or 1.1

$y_k \to \infty$ $\qquad \hat{\phi}_1 = 1.03$

(c) $\hat{\phi}_0 = 5, \quad \hat{\phi}_1 = 1$

$$\hat{y}_{n+1} = 5 + y_n$$

$$\hat{y}_{n+2} = 5 + y_{n+1} = 10 + y_n$$

$$\hat{y}_{n+3} = 15 + y_n$$

$\hat{y}_{n+k} = 5k + y_n$ → linear growth in k

AR(1) : exploding prediction
ϕ_1 $\quad$ settling to a constant
$\quad$ linear predictions

Next Topics

1. Mathematics of AR(p)
2. MA (Moving Average)
3. ARMA & ARIMA

Stationary time series models
Non-stationary models

Mean models
Variance models

Definition of Stationary Time Series Models

$$y_t, \quad t = \dots, -3, -2, -1, 0, 1, 2, \dots$$

$$(y_1 \dots y_n)$$

Our models in this part of the course are jointly Gaussian.
As a result, the models can be described using means & covariances.

Defn (Stationarity) y_t, $t = \dots, -3, -2, -1, 0, 1, 2 \dots$ is stationary if

(a) $\mathbb{E}\, y_t$ does not change with t

(b) $\operatorname{var} y_t$ does not change with t

(c) $\operatorname{cov}(y_{t_1}, y_{t_2})$ only depends on the gap between t_1 & t_2

i.e. $\operatorname{cov}(y_{t_1}, y_{t_2})$ only depends on $|t_1 - t_2|$

$\mathrm{cov}(y_{99}, y_{100}) = \mathrm{cov}(y_{35}, y_{36})$

Suppose $\{y_t\}$ is a stationary time series model.

$$\gamma(h) = \underbrace{\mathrm{cov}(y_t, y_{t+h})}, \quad h = \ldots, -2, -1, 0, 1, 2, \ldots$$

ACVF of $\{y_t\}$

Auto CoVariance Function

$$\gamma(0) = \mathrm{cov}(y_t, y_t) = \mathrm{var}(y_t)$$

$$\rho(h) = \mathrm{corr}(y_t, y_{t+h})$$

AutoCorrelation Function ACF

$$= \frac{\mathrm{cov}(y_t, y_{t+h})}{\sqrt{\mathrm{var}(y_t)\,\mathrm{var}(y_{t+h})}} = \frac{\gamma(h)}{\sqrt{\gamma(0) \times \gamma(0)}}$$

$$\boxed{\rho(h) = \frac{\gamma(h)}{\gamma(0)}} \qquad \rho(0) = 1$$

Note: ① Stationarity applies to a time series model but not to actual data.

② ACVF & ACF apply only to stationary models

③ Many time series models are <u>not</u> stationary

Examples

① $y_t \overset{iid}{\sim} N(\mu, \sigma^2)$

$$y_t = \mu + \varepsilon_t, \quad \varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$$

STATIONARY

$$\rho(h) = \begin{cases} 1 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0 \end{cases}$$

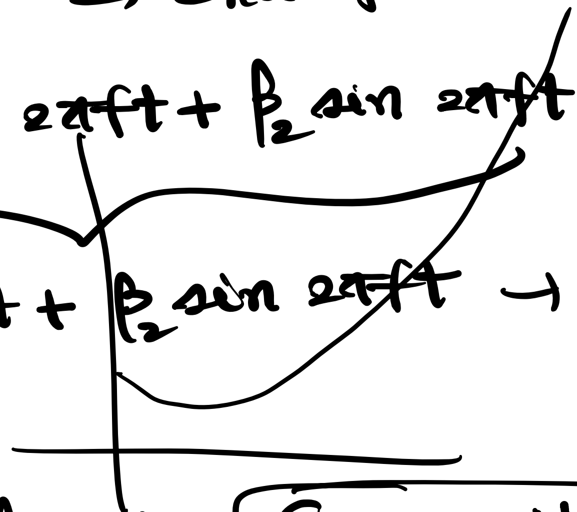
② $y_t = \beta_0 + \beta_1 t + \varepsilon_t, \quad \varepsilon_t \underset{\sim}{\text{iid}} N(0, \sigma^2)$

$\mathbb{E} y_t = \beta_0 + \beta_1 t \rightarrow$ changes with t.

③ $y_t = \underbrace{\beta_0 + \beta_1 \cos 2\pi f t + \beta_2 \sin 2\pi f t}$ $+ \varepsilon_t$,

$\varepsilon_t \underset{\sim}{\text{iid}} N(0, \sigma^2)$

$\mathbb{E} y_t = \beta_0 + \beta_1 \cos 2\pi f t + \beta_2 \sin 2\pi f t \rightarrow$ changes with t



④ SPECTRUM Model : $\boxed{\text{STATIONARY}}$

$$y_t = \beta_0 + \sum_{j=1}^{m} \left[\beta_{j1} \cos \frac{2\pi j t}{n} + \beta_{j2} \sin \frac{2\pi j t}{n} \right]$$

n : sample size

$\beta_{j1}, \beta_{j2} \underset{\sim}{\text{iid}} N(0, \tau_j^2)$

m = largest integer strictly smaller than n.

parameters : $\beta_0, \tau_1^2, \ldots, \tau_m^2$

$$\mathbb{E} y_t = \beta_0$$

$\mathrm{Cov}(y_{t_1}, y_{t_2})$

$$= \mathrm{Cov}\left(\sum_{j=1}^{m} \left(\beta_{1j} \cos \frac{2\pi j t_1}{n} + \beta_{2j} \sin \frac{2\pi j t_1}{n} \right), \right.$$

$$\sum_{j=1}^{m}\left(\beta_{1j}\cos\frac{2\pi j t_2}{n} + \beta_{2j}\sin\frac{2\pi j t_2}{n}\right)\Bigg)$$

β_{1j}, β_{2j}, $j=1,\ldots,m$ are all independent

& $\beta_{1j}, \beta_{2j} \underset{\sim}{iid} N(0, \tau_j^2)$

$$= \sum_{j=1}^{m} \operatorname{Cov}\left(\beta_{1j}\cos\frac{2\pi j t_1}{n} + \beta_{2j}\sin\frac{2\pi j t_1}{n},\ \beta_{1j}\cos\frac{2\pi j t_2}{n} + \beta_{2j}\sin\frac{2\pi j t_2}{n}\right)$$

$$= \sum_{j=1}^{m} \tau_j^2 \left\{\cos\frac{2\pi j t_1}{n}\cos\frac{2\pi j t_2}{n} + \sin\frac{2\pi j t_1}{n}\sin\frac{2\pi j t_2}{n}\right\}$$

$$= \sum_{j=1}^{m} \tau_j^2 \cos\left(\frac{2\pi j}{n}(t_1 - t_2)\right)$$

$$= \sum_{j=1}^{m} \tau_j^2 \cos\left(\frac{2\pi j}{n}|t_1 - t_2|\right)$$

$$y_t = \beta_0 + \sum_{j=1}^{m}\left(\beta_{1j}\cos 2\pi \frac{j}{n} t + \beta_{2j}\sin 2\pi\frac{j}{n} t\right)$$

$$\boxed{y_t = \beta_0 + \int_0^{1/2}\left[\beta_1(\omega)\cos 2\pi\omega t + \beta_2(\omega)\sin 2\pi \omega t\right] d\omega}$$

$\beta_1(\omega), \beta_2(\omega) \underset{\sim}{iid} N(0, \sigma^2)$

<u>Qn</u>: Is AR(1) or AR(p), $p \geq 1$ stationary?

AR(1): $\boxed{y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t,}$

$\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$

Implicit definition

$x^2 + 5x + 3 = 0$

$$y_1 = \phi_0 + \phi_1 y_0 + \varepsilon_1$$

$$y_2 = \phi_0 + \phi_1 y_1 + \varepsilon_2$$
$$= \phi_0 + \phi_1(\phi_0 + \phi_1 y_0 + \varepsilon_1) + \varepsilon_2$$

For every $t = 1, 2, \ldots$
can write y_t in terms of $y_0, \varepsilon_1, \ldots, \varepsilon_t$

y_{-1}

$$y_0 = \phi_0 + \phi_1 y_{-1} + \varepsilon_0$$

$$y_{-1} = \frac{y_0 - \phi_0}{\phi_1} - \frac{\varepsilon_0}{\phi_1}$$

$$y_{-2} = \ldots$$

Examples: Fix $|\phi_1| < 1$.

$$\boxed{y_t = \frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j \varepsilon_{t-j}}$$ → STATIONARY MODEL

$$y_t = \frac{\phi_0}{1-\phi_1} + \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 \varepsilon_{t-2} + \phi_1^3 \varepsilon_{t-3} + \ldots$$

$\ldots\ \varepsilon_0,\ \varepsilon_1,\ \varepsilon_2\ \varepsilon_{t-2}\ \varepsilon_{t-1}\ \varepsilon_t$

y_t

$$\mathbb{E}\, y_t = \frac{\phi_0}{1-\phi_1}$$

Check: $\boxed{\mathrm{Cov}(y_t, y_{t+h}) = \frac{\sigma^2}{1-\phi_1^2}\,\phi_1^{|h|}}$

Check: $y_t = \frac{\phi_0}{1-\phi_1} + \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 \varepsilon_{t-2} + \cdots$

satisfies

$\varepsilon_t \overset{iid}{\sim} N(0,\sigma^2)$

$$y_t - \phi_1 y_{t-1}$$

$$= \left(\frac{\phi_0}{1-\phi_1} + \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 \varepsilon_{t-2} + \cdots\right)$$

$$- \phi_1\left(\frac{\phi_0}{1-\phi_1} + \varepsilon_{t-1} + \phi_1 \varepsilon_{t-2} + \phi_1^2 \varepsilon_{t-3} + \cdots\right)$$

$$= \phi_0 + \varepsilon_t$$

$$\boxed{y_t = \frac{\phi_0}{1-\phi_1} - \frac{\varepsilon_{t+1}}{\phi_1} - \frac{\varepsilon_{t+2}}{\phi_1^2} - \frac{\varepsilon_{t+3}}{\phi_1^3} - \cdots}$$

Assume $|\phi_1| > 1$

Check: $y_t - \phi_1 y_{t-1} = \phi_0 + \varepsilon_t$

& $\{y_t\}$ is STATIONARY.

<u>Qn</u>: Is AR(1) stationary?

<u>Ans</u>: $y_t = \frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j \varepsilon_{t-j}, \; |\phi_1|<1$

$\longrightarrow$ STATIONARY

$y_t = \frac{\phi_0}{1-\phi_1} - \sum_{j=1}^{\infty} \frac{\varepsilon_{t+j}}{\phi_1^j}, \; |\phi_1|>1$

$\longrightarrow$ STATIONARY

If $|\phi_1|=1$, then there CANNOT be any stationary AR(1).

<u>e.g</u>: $\phi_1=1$ $\quad y_t = \phi_0 + y_{t-1} + \varepsilon_t$

$y_1 = \phi_0 + y_0 + \varepsilon_1 \rightarrow$

$y_k = k\phi_0 + y_0 + \underbrace{\varepsilon_1 + \cdots + \varepsilon_k}$

<u>SUMMARY</u>

AR(1) $\rightarrow \underbrace{y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t}$

AR(1) DIFFERENCE EQUATION

$x^2-4=0$

<u>$|\phi_1|<1$</u>: UNIQUE STATIONARY SOLUTION

$y_t = \underbrace{\frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j \varepsilon_{t-j}}$

CAUSAL STATIONARY SOLUTION

$\ldots \varepsilon_{t+1}, \varepsilon_t$
$\downarrow$
y_t

$|\phi_1|>1$: UNIQUE STATIONARY SOLUTION

$$y_t = \frac{\phi_0}{1-\phi_1} - \sum_{j=1}^{\infty} \frac{\varepsilon_{t+j}}{\phi_1^j} \quad X$$

$\rightarrow$ NON CAUSAL STATIONARY SOLUTION

$\boxed{|\phi_1|=1}$: NO STATIONARY SOLUTION

$\phi_1=1$

Practice: $\phi_1=1$ & $|\phi_1|<1$