LECTURE FIFTEEN

Spectrum Model

Periodogram: $y_0, y_1, \dots, y_{n-1}$

DFT : $b_0, b_1, \dots, b_{n-1}$

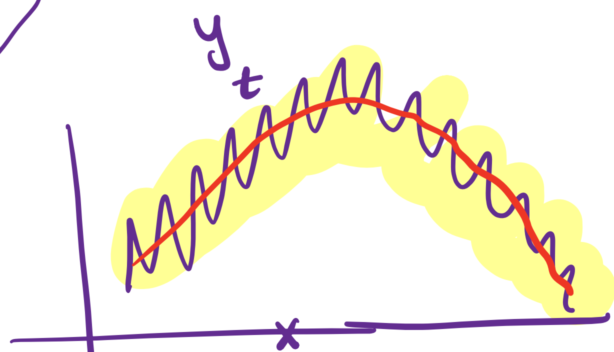
$$I\left(\frac{j}{n}\right) = \frac{|b_j|^2}{n}, \quad 0 < \frac{j}{n} < \frac{1}{2}$$

n : odd, $m = \frac{n-1}{2}$

$\boxed{I\left(\frac{j}{n}\right),\ j = 1, \dots, m}$: tells us sinusoids at which frequencies are contributing to the data

Most often in practice, $I\left(\frac{j}{n}\right)$ will have many peaks, usually very noisy. especially looking at the log-scale.

Qn: How to SMOOTH PERIODOGRAM?



$$y_t = \underbrace{\boxed{\mu_t}}_{\text{smooth trend we want to estimate}} + \underbrace{\varepsilon_t}_{N(0,\sigma^2)}$$

$$\sum (y_t - \mu_t)^2 + \lambda \sum_t \left| (\mu_t - \mu_{t-1}) - (\mu_{t-1} - \mu_{t-2}) \right|^2$$

To smooth the periodogram, we shall write a model.

$$\rightarrow I\left(\frac{j}{n}\right) = \underbrace{f\left(\frac{j}{n}\right)}_{\mathbb{E}\left(I\left(\frac{j}{n}\right)\right)} + \boxed{\varepsilon_j}, \quad \varepsilon_j \overset{iid}{\sim} N(0, \sigma^2)$$

$\mathbb{E}\left(I\left(\frac{j}{n}\right)\right)$: mean of the periodogram

WE WILL NOT USE THIS MODEL BECAUSE

① $I\left(\frac{j}{n}\right)$ is always positive.

② Additive model is better suited for log-periodogram (not $I(j/n)$ directly)

Multiplicative Model:

$$I\left(\frac{j}{n}\right) = \underbrace{f\left(\frac{j}{n}\right)}_{\text{represents smooth pattern in the periodogram}} \boxed{\eta_j} \longrightarrow \text{multiplicative noise}$$

We want ⓐ $\mathbb{E}\, \eta_j = 1$ so that $f\left(\frac{j}{n}\right) = \mathbb{E}\, I\left(\frac{j}{n}\right)$

ⓑ η_j to be positive valued

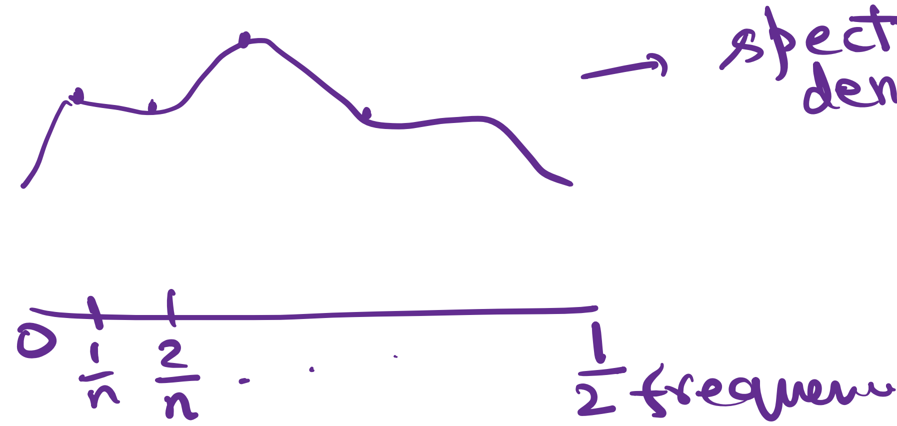
$$\eta_j \overset{iid}{\sim} \text{Exp}(1)$$

$$\boxed{I\left(\frac{j}{n}\right) = \underbrace{f\left(\frac{j}{n}\right)}_{\text{smooth}} \eta_j \qquad \eta_j \overset{iid}{\sim} \text{Exp}(1)}$$

$$\text{Exp}(1) = \text{Gamma}(1, 1) = \frac{\chi^2_2}{2}$$

$\boxed{2\eta_j \overset{iid}{\sim} \chi^2_2}$

$\boxed{f\left(\frac{j}{n}\right)}$ = mean of $I\left(\frac{j}{n}\right)$ → $\boxed{\text{power of frequency } \frac{j}{n}}$



→ spectral density (f)

$I\left(\frac{j}{n}\right) = f\left(\frac{j}{n}\right)\eta_j, \quad \eta_j \overset{iid}{\sim} \text{Exp}(1)$

$\log I\left(\frac{j}{n}\right) = \underbrace{\log f\left(\frac{j}{n}\right)} + \log \eta_j$

$\mathbb{E}(\log \eta_j) \approx -0.527$

$y_t = \mu_t + \varepsilon_t$
$\varepsilon_t \sim N(0, \sigma^2)$

Connection to DFT

$y_0, \cdots, y_{n-1}$

$b_0, \cdots b_{n-1}$

$\boxed{\text{Re}(b_j), \text{Im}(b_j) \overset{iid}{\sim} N(0, \gamma_j^2)}$

$$I\left(\frac{j}{n}\right) = \frac{|b_j|^2}{n} = \frac{Re(b_j)^2 + Im(b_j)^2}{n}$$

$$= \frac{\gamma_j^2 (N(0,1))^2 + \gamma_j^2 (N(0,1))^2}{n}$$

Independent

$$= \frac{\gamma_j^2}{n}\left(N(0,1)^2 + N(0,1)^2\right)$$

independent

$$= \frac{\gamma_j^2}{n} \chi_2^2$$

$$= \frac{2\gamma_j^2}{n} \; \boxed{\frac{\chi_2^2}{2}} \; \boxed{Exp(1)}$$

$$\boxed{f\left(\frac{j}{n}\right) = \frac{2\gamma_j^2}{n}}$$

Spectrum Model

① $I\left(\frac{j}{n}\right) = f\left(\frac{j}{n}\right)\eta_j$, $\quad \eta_j \overset{iid}{\sim} Exp(1)$

② $Re(b_j), Im(b_j) \overset{iid}{\sim} N(0, \gamma_j^2)$

$$\boxed{\frac{2\gamma_j^2}{n} = f\left(\frac{j}{n}\right)}$$

Data: y_t

Representation of the Model in terms of the data $\{y_t\}$

Inverse DFT formula:

$$y_t = \frac{1}{n}\sum_{j=0}^{n-1} b_j \exp\left(\frac{2\pi i j t}{n}\right) \rightarrow \text{IDFT}$$

$$b_j = \sum_{t=0}^{n-1} y_t \exp\left(-\frac{2\pi i j t}{n}\right) \rightarrow \text{DFT}$$

$$y_t = \underbrace{\frac{1}{n}\sum_{j=0}^{n-1} b_j \exp\left(\frac{2\pi i j t}{n}\right)}$$

$\left.\begin{matrix} Re(b_j) \\ Im(b_j) \end{matrix}\right\} N(0, \gamma_j^2)$

$$= \frac{1}{n}\sum_{j=0}^{n-1}\left(Re(b_j) + i\, Im(b_j)\right)\left(\cos\frac{2\pi j t}{n} + i \sin\frac{2\pi j t}{n}\right)$$

$$= \frac{1}{n}\sum_{j=0}^{n-1}\left(Re(b_j)\cos\frac{2\pi j t}{n} - Im(b_j)\sin\frac{2\pi j t}{n}\right)$$

$$+ \frac{i}{n}\sum_{j=0}^{n-1}\left(Re(b_j)\sin\frac{2\pi j t}{n} + Im(b_j)\cos\frac{2\pi j t}{n}\right)$$

$\| \quad b_{n-j} = \bar{b}_j$

0

because data is real.

$$y_t = \frac{1}{n}\sum_{j=0}^{n-1}\left(Re(b_j)\cos\frac{2\pi j t}{n} - Im(b_j)\sin\frac{2\pi j t}{n}\right)$$

$Re(b_j), Im(b_j) \overset{iid}{\sim} N(0, \gamma_j^2)$,

$j = 1, \ldots, m$

$$= \frac{b_0}{n} + \frac{1}{n}\sum_{j=1}^{m}\left[Re(b_j)\cos\frac{2\pi j t}{n} - Im(b_j)\sin\frac{2\pi j t}{n}\right]$$

$$+ \frac{1}{n}\sum_{j=m+1}^{n-1}\left(\mathrm{Re}(b_j)\cos\frac{2\pi jt}{n} - \mathrm{Im}(b_j)\sin\frac{2\pi jt}{n}\right)$$

Check: last two terms coincide $\quad b_{n-j} = \bar{b}_j$

$$y_t = \frac{b_0}{n} + \sum_{j=1}^{m}\left[\boxed{\frac{2\,\mathrm{Re}(b_j)}{n}}\cos\frac{2\pi jt}{n} + \boxed{\frac{-2\,\mathrm{Im}(b_j)}{n}}\sin\frac{2\pi jt}{n}\right]$$

$\rightarrow$ Real IDFT formula

$$y_t = \beta_0 + \sum_{j=1}^{m}\left(\beta_{1j}\cos\frac{2\pi jt}{n} + \beta_{2j}\sin\frac{2\pi jt}{n}\right) \quad (A)$$

where β_0 is a constant

β_{1j}, β_{2j} independent across j

$\beta_{1j}, \beta_{2j} \overset{iid}{\sim} N(0, \tau_j^2)$

$$y_t = \beta_0 + \beta_1\cos 2\pi ft + \beta_2\sin 2\pi ft + \varepsilon_t$$

(A) has no additive error term.

Spectrum Model

$y_0, \ldots, y_{n-1}$: Data

$b_0 \ldots b_{n-1}$: DFT

$I(\frac{j}{n}), j = 1 \ldots, m$: Periodogram

Model can be described in terms of either of these

$$\boxed{① \text{ Periodogram: } I\left(\tfrac{j}{n}\right) = \underbrace{f\left(\tfrac{j}{n}\right)}_{\text{parameters}} \eta_j \qquad \eta_j \overset{iid}{\sim} \mathrm{Exp}(1)}$$

② DFT: $b_1 \ldots b_m$ independent

$Re(b_j), Im(b_j) \overset{iid}{\sim} N(0, \gamma_j^2)$

$$\boxed{f\left(\tfrac{j}{n}\right) = \frac{2\gamma_j^2}{n}}$$

③ $y_t = \beta_0 + \sum_{j=1}^{m}\left(\beta_{1j} \cos \frac{2\pi j t}{n} + \beta_{2j} \sin \frac{2\pi j t}{n}\right)$ → all Fourier frequency sinusoids

$\beta_{1j}, \beta_{2j} \overset{iid}{\sim} N(0, \tau_j^2)$

Parameters: $\tau_1^2, \ldots, \tau_m^2$

$$\beta_{1j} = \frac{2 Re(b_j)}{n} \sim N\left(0, \frac{4\gamma_j^2}{n^2}\right)$$

$$\boxed{\tau_j = \frac{2\gamma_j}{n}}$$

Estimation of parameters

$$\prod_{j=1}^{m} f\left(\underbrace{I(j/n)}_{I(j/n)}\right) \qquad I\left(\tfrac{j}{n}\right) = f\left(\tfrac{j}{n}\right)\eta_j$$

$$= \prod_{j=1}^{m} f\left(\underbrace{I(j/n)}_{f(\frac{j}{n})\eta_j}\right)$$

$$= \prod_{j=1}^{m} \frac{1}{f(j/n)} \exp\left(-\frac{I(j/n)}{f(j/n)}\right)$$

WHITTLE LIKELIHOOD

$$\sum_{j=1}^{m} -\frac{I(j/n)}{f(j/n)} - \log f(j/n)$$

neg log: $$\sum_{j=1}^{m} \left[\log f\left(\tfrac{j}{n}\right) + \frac{I(j/n)}{f(j/n)} \right]$$

$\log f\left(\tfrac{j}{n}\right) = g_j$

$$\sum_{j} \left[g_j + e^{-g_j} I(j/n) \right]$$

$$+ \lambda \sum_{j=3}^{m} \left((g_j - g_{j-1}) - (g_{j-1} - g_{j-2}) \right)^2$$