Lecture Eighteen

Last lecture: Estimation in AR(1)

$$y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t, \quad \varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$$

$$t = 2, \ldots, n$$

① Likelihood: $\displaystyle\prod_{t=2}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(y_t - \phi_0 - \phi_1 y_{t-1})^2}{2\sigma^2}\right)$

$f_{y_2 \ldots y_n \mid y_1}^{\theta}$ Conditional MLE or least squares

AR(p): $y_t = \phi_0 + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t$

$$t = p+1, \ldots, n$$

Conditional likelihood:

$$\prod_{t=p+1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(y_t - \phi_0 - \cdots - \phi_p y_{t-p})^2}{2\sigma^2}\right)$$

Two implementation methods:

ⓐ Create $\underbrace{y}_{(n-p)\times 1}$ & $\underbrace{X}_{(n-p)\times(p+1)}$ & use OLS

sm.OLS(y, X).fit()

ⓑ AutoReg(y, p = 1 or 2 etc)

→ Identical parameter estimates of $\phi_0, \phi_1, \ldots, \phi_p$

→ For σ, method (a): $\hat{\sigma} = \sqrt{\dfrac{RSS}{(n-p)-(p+1)}}$

method (b): $\hat{\sigma} = \sqrt{\frac{RSS}{n-p}}$

→ t vs z

② AR(1): $y_t = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t$,

$t = 2, \dots, n$

$t = 1, 0, -1, \dots$

y_1 y_0

$$y_1 = \frac{\phi_0}{1-\phi_1} + \sum_{j=0}^{\infty} \phi_1^j \varepsilon_{1-j}$$

Assume $|\phi_1| < 1$

$$\sim N\left(\frac{\phi_0}{1-\phi_1}, \frac{\sigma^2}{1-\phi_1^2}\right)$$

Likelihood:

$$\frac{\sqrt{1-\phi_1^2}}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{\left(y_1 - \frac{\phi_0}{1-\phi_1}\right)^2 (1-\phi_1^2)}{2\sigma^2}\right]$$

$$\prod_{t=2}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(y_t - \phi_0 - \phi_1 y_{t-1})^2}{2\sigma^2}\right)$$

→ Full likelihood $|\phi_1| < 1$

Unconditional

AutoReg does not have an option to use this complicated likelihood.

→ arima (function in statsmodels)

$y_1 \cdots y_p$

How are predictions & standard errors calculated?

$y_1 \cdots y_n$

$$\hat{y}_{n+1}(\theta) = \phi_0 + \phi_1 y_n + \cdots + \phi_p y_{n+1-p}$$

$$\theta = (\phi_0, \cdots, \phi_p) \qquad \hat{y}_{n+1}(\hat{\theta})$$

$$= \mathbb{E}(y_{n+1} \mid y_1 \cdots y_n, \theta)$$

$$\hat{y}_{n+2}(\theta) = \mathbb{E}(y_{n+2} \mid y_1 \cdots y_n, \theta)$$

$$= \mathbb{E}\left[\phi_0 + \phi_1 y_{n+1} + \phi_2 y_n + \cdots + \phi_p y_{n+2-p} \mid y_1 \cdots y_n, \theta\right]$$

$$= \phi_0 + \phi_1 \underbrace{\mathbb{E}(y_{n+1} \mid y_1 \cdots y_n, \theta)}_{\hat{y}_{n+1}(\theta)} + \phi_2 y_n + \cdots + \phi_p y_{n+2-p}$$

$$\hat{y}_{n+k}(\theta) = \phi_0 + \phi_1 \hat{y}_{n+k-1}(\theta) + \phi_2 \hat{y}_{n+k-2}(\theta) + \cdots + \phi_p \hat{y}_{n+k-p}(\theta)$$

Recursion

$k = 1, 2, \cdots$

$$\mathbb{E}(y_{n+k} \mid y_1 \cdots y_n, \theta)$$

$$\hat{y}_j(\theta) = y_j \quad \text{for } j \le n$$

$y_1 \quad \cdots \quad y_n$

Standard Errors corresponding to predictions

$$\boxed{\operatorname{var}\left(y_{n+k} \,\middle|\, \begin{smallmatrix} y_1 \cdots y_n \\ \theta \end{smallmatrix}\right)} \quad k = 1, 2, \dots$$

$$\operatorname{var}\left(y_{n+1} \,\middle|\, \begin{smallmatrix} y_1 \cdots y_n \\ \theta \end{smallmatrix}\right)$$

$$= \operatorname{var}\left(\phi_0 + \phi_1 y_n + \phi_2 y_{n-1} + \dots + \phi_p y_{n+1-p} + \varepsilon_{n+1} \,\middle|\, \begin{smallmatrix} y_1 \cdots y_n \\ \theta \end{smallmatrix}\right)$$

$$= \operatorname{var}\left(\varepsilon_{n+1} \,\middle|\, \begin{smallmatrix} y_1 \cdots y_n \\ \theta \end{smallmatrix}\right) = \sigma^2$$

$$\operatorname{var}\left(y_{n+2} \,\middle|\, \begin{smallmatrix} y_1 \cdots y_n \\ \theta \end{smallmatrix}\right)$$

$$= \operatorname{var}\left(\phi_0 + \phi_1 y_{n+1} + \phi_2 y_n + \dots + \phi_p y_{n+2-p} + \varepsilon_{n+2} \,\middle|\, \begin{smallmatrix} y_1 \cdots y_n \\ \theta \end{smallmatrix}\right)$$

$$= \operatorname{var}\left(\phi_1 y_{n+1} + \varepsilon_{n+2} \,\middle|\, \begin{smallmatrix} y_1 \cdots y_n \\ \theta \end{smallmatrix}\right)$$

$$\operatorname{var}\left(y_{n+k} \,\middle|\, \begin{smallmatrix} y_1 \cdots y_n \\ \theta \end{smallmatrix}\right)$$

$$\operatorname{Cov}\left(\begin{pmatrix} y_{n+1} \\ \vdots \\ y_{n+k} \end{pmatrix} \,\middle|\, \begin{smallmatrix} y_1 \cdots y_n \\ \theta \end{smallmatrix}\right) = \Gamma_k(\theta)$$

Covariance Matrices

Y : random variable $\mathbb{E}Y$, $\operatorname{var} Y$

$Y_1 \dots Y_p$: p- random variables

$\underset{p\times 1}{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_p \end{pmatrix}$: Random vector

$\mathbb{E} Y = \begin{pmatrix} \mathbb{E} Y_1 \\ \vdots \\ \mathbb{E} Y_p \end{pmatrix}$: $p \times 1$

$\underbrace{Cov(Y)}_{p\times p \text{ matrix}} = \begin{bmatrix} & & \\ & & \cdot \\ & & \end{bmatrix}$ $\to Cov(Y_i, Y_j)$

① $\mathbb{E}(AY + b) = A(\mathbb{E} Y) + b$

② $\underbrace{Cov(AY + b)}$ $= \underbrace{A (Cov\, Y) A^T}$

$$\boxed{③\ Cov(AY + b, BW + d) = A\, Cov(Y, W)\, B^T}$$

$\underbrace{Y}_{p\times 1}$: random vector $\quad W_{q\times 1}$: another random vector

$Cov(Y, W)_{p\times q}$ with $(i,j)^{th}$ entry $Cov(Y_i, W_j)$

$var(y_{n+k} \mid y_1 \dots y_n, \theta)$

$$\mathrm{Cov}\left(\begin{pmatrix} y_{n+1} \\ \vdots \\ y_{n+k} \end{pmatrix} \,\middle|\, \begin{matrix} y_1 \dots y_n \\ \theta \end{matrix}\right) = \underbrace{\Gamma_k(\theta)}_{k \times k \text{ covariance matrix}}$$

$$\Gamma_1(\theta) = \mathrm{Cov}\left(y_{n+1} \,\middle|\, \begin{matrix} y_1 \dots y_n \\ \theta \end{matrix}\right) = \mathrm{var}\left(y_{n+1} \,\middle|\, \begin{matrix} y_1 \dots y_n \\ \theta \end{matrix}\right) = \sigma^2$$

$$[\sigma^2] \qquad [\quad]_{2\times 2} \qquad [\quad]_{3 \times 3}$$

$$\underbrace{\Gamma_k(\theta)}_{} = \mathrm{Cov}\left(\begin{pmatrix} y_{n+1} \\ \vdots \\ y_{n+k} \end{pmatrix} \,\middle|\, \begin{matrix} y_1 \dots y_n \\ \theta \end{matrix}\right)$$

$(i,j)^{th}$ entry in

$$\mathrm{Cov}\left(y_{n+i}, y_{n+j} \,\middle|\, \begin{matrix} y_1 \dots y_n \\ \theta \end{matrix}\right)$$

$$\mathrm{Cov}\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{pmatrix}$$

$$= \left[\begin{array}{c|c} \Gamma_{k-1}(\theta) & \boxed{\gamma_k(\theta)} \\ \hline \boxed{\gamma_k^T(\theta)} & \bigcirc \end{array}\right]_{k \times k} \qquad = \left[\begin{array}{c|c} \mathrm{Cov}\begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix} & \\ \hline & \end{array}\right]$$

$$\gamma_k(\theta) = \mathrm{Cov}\left(\begin{pmatrix} y_{n+1} \\ \vdots \\ y_{n+k-1} \end{pmatrix}, y_{n+k} \,\middle|\, \begin{matrix} y_1 \dots y_n \\ \theta \end{matrix}\right)$$

$$= \mathrm{Cov}\left(\underbrace{\begin{pmatrix} y_{n+1} \\ \vdots \\ y_{n+k-1} \end{pmatrix}}, \ \overbrace{\underbrace{\phi_0 + \phi_1 y_{n+k-1} + \phi_2 y_{n+k-2} + \cdots + \phi_p y_{n+k-p}} + \underbrace{\varepsilon_{n+k}}} \,\middle|\, \underset{\theta}{y_1 \ldots y_n} \right)$$

$$a_1 y_{n+1} + a_2 y_{n+2} + \cdots + a_{k-1} y_{n+k-1}$$

$$\mathrm{var}\left(y_{n+1} \,\middle|\, \underset{\theta}{y_1 \ldots y_n} \right) = \sigma^2$$

$$\mathrm{var}\left(y_{n+2} \,\middle|\, \underset{\theta}{y_1 \ldots y_n} \right)$$

$$= \mathrm{var}\left(\phi_0 + \phi_1 y_{n+1} + \phi_2 y_n + \cdots + \varepsilon_{n+2} \,\middle|\, \underset{\theta}{y_1 \ldots y_n} \right)$$

$$= \mathrm{var}\left(\phi_1 y_{n+1} + \varepsilon_{n+2} \,\middle|\, \underset{\theta}{y_1 \ldots y_n} \right)$$

$$= \phi_1^2 \sigma^2 + \sigma^2$$

$$\mathrm{var}\left(y_{n+3} \,\middle|\, \underset{\theta}{y_1 \ldots y_n} \right)$$

$$= \mathrm{var}\left(\phi_1 y_{n+2} + \phi_2 y_{n+1} + \varepsilon_{n+3} \,\middle|\, \underset{\theta}{y_1 \ldots y_n} \right)$$

$$\mathrm{Cov}\left(\underbrace{\begin{pmatrix} y_{n+1} \\ \vdots \\ y_{n+k-1} \end{pmatrix}}, \ \underbrace{a_1 y_{n+1} + \cdots + a_{k-1} y_{n+k-1}}_{\underbrace{a^T \begin{pmatrix} y_{n+1} \\ \vdots \\ y_{n+k-1} \end{pmatrix}}} \,\middle|\, \underset{\theta}{y_1 \ldots y_n} \right)$$

$$= \Gamma_{k-1}(\theta)\, a$$

$$\mathrm{Cov}(AY+b, BW+c) = A\,\mathrm{Cov}(Y,W)\,B^T$$

$$\Gamma_k(\theta) = \left[\begin{array}{c|c} \Gamma_{k-1}(\theta) & \xrightarrow{} \Gamma_{k-1}(\theta)a \\ \hline a^T\Gamma_{k-1}(\theta) & \square \end{array}\right]$$

$$\mathrm{var}(y_{n+k} \mid y_1 \dots y_n, \theta)$$

$$= \mathrm{var}(\underbrace{\phi_1 y_{n+k-1} + \dots + \phi_p y_{n+k-p}}_{a^T \begin{pmatrix} y_{n+1} \\ \vdots \\ y_{n+k-1} \end{pmatrix}} + \underbrace{\varepsilon_{n+k}} \mid y_1 \dots y_n, \theta)$$

(a)

$$= \sigma^2 + a^T \Gamma_{k-1}(\theta)\, a$$

$$\Gamma_k(\theta) = \left[\begin{array}{c|c} \Gamma_{k-1}(\theta) & \Gamma_{k-1}(\theta)a \\ \hline a^T\Gamma_{k-1}(\theta) & a^T\Gamma_{k-1}(\theta)a \end{array}\right]$$

① $\Gamma_1(\theta) = \sigma^2$

② For each $k = 2, 3, \dots$

Calculate a dependng on k, p & $\phi_1 \dots \phi_p$

$$\Gamma_k(\theta) = \begin{bmatrix} \Gamma_{k-1}(\theta) & \Gamma_{k-1}(\theta)a \\ a^T\Gamma_{k-1}(\theta) & a^T\Gamma_{k-1}(\theta)a \end{bmatrix}$$