Time Series Lab 1 (Practice Problems)

September 8, 2025

Exercise 1 - Change of Variables

Let $n > 0$. Consider the unnormalized function on $(0, \infty)$

$$g_n(s) = s^{-n-1} \exp\left(-\frac{1}{2s^2}\right).$$

Show that

$$\int_0^\infty s^{-n-1} \exp\left(-\frac{1}{2s^2}\right) ds = 2^{\frac{n}{2}-1} \Gamma\left(\frac{n}{2}\right).$$

Hint: Use change of variables $u = \dfrac{1}{2s^2}$

Proof. Let $n > 0$ and consider

$$I = \int_0^\infty s^{-n-1} \exp\left(-\frac{1}{2s^2}\right) ds.$$

Use the change of variables $u = \dfrac{1}{2s^2}$ so that

$$s = (2u)^{-1/2}, \qquad ds = -(2u)^{-3/2}\, du.$$

Then

$$s^{-(n+1)}\, ds = \left((2u)^{-1/2}\right)^{-(n+1)} \left(-(2u)^{-3/2}\, du\right) = -\,(2u)^{(n-2)/2}\, du.$$

As $s : 0 \to \infty$, we have $u : \infty \to 0$, hence

$$\begin{aligned} I &= \int_\infty^0 e^{-u} \left(-(2u)^{(n-2)/2}\right) du = \int_0^\infty e^{-u}\, (2u)^{\frac{n}{2}-1}\, du \\ &= 2^{\frac{n}{2}-1} \int_0^\infty u^{\frac{n}{2}-1} e^{-u}\, du = 2^{\frac{n}{2}-1}\, \Gamma\left(\frac{n}{2}\right), \end{aligned}$$

Recall the Gamma(α, β) density (shape–rate parameterization)

$$f(u) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, u^{\alpha-1} e^{-\beta u}, \qquad u > 0,\ \alpha > 0,\ \beta > 0.$$

Thus the integrand $u^{\frac{n}{2}-1}e^{-u}$ is the (unnormalized) Gamma kernel with $\alpha = \frac{n}{2}$ and $\beta = 1$, so

$$\int_0^\infty u^{\frac{n}{2}-1}e^{-u}\,du = \Gamma\left(\frac{n}{2}\right).$$

Therefore,

$$\boxed{\int_0^\infty s^{-n-1}\exp\left(-\frac{1}{2s^2}\right)\,ds = 2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right).}$$

$\square$

Exercise 2 - MLE

Given data $y_1, \dots, y_n$, consider the model $y_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$ for two unknown parameters μ and $\sigma > 0$.

Find the maximum likelihood estimators (MLEs) of μ and σ by maximizing the log-likelihood (use first-order derivatives).

Requirement. Show that the solution is the *unique interior maximizer* by arguing concavity of the log-likelihood OR by showing a sign change of the derivative and checking boundary behavior.

Proof. **1) Likelihood and log-likelihood.** The likelihood is

$$L(\mu, \sigma) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{(y_i - \mu)^2}{2\sigma^2}\right\} = (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2\right\}.$$

Hence the log-likelihood is

$$\ell(\mu, \sigma) = -\frac{n}{2} \log(2\pi) \;-\; n \log \sigma \;-\; \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2.$$

2) Maximize in μ for fixed σ. Differentiate w.r.t. μ:

$$\frac{\partial \ell}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \mu) = \frac{n}{\sigma^2} (\bar{y} - \mu), \qquad \frac{\partial^2 \ell}{\partial \mu^2} = -\frac{n}{\sigma^2} \;<\; 0.$$

Setting the first derivative to zero yields

$$\widehat{\mu} = \bar{y} \;=\; \frac{1}{n} \sum_{i=1}^{n} y_i.$$

Since $\partial^2 \ell / \partial \mu^2 < 0$, $\ell(\mu, \sigma)$ is strictly concave down in μ (for fixed σ), so $\widehat{\mu}$ is the unique maximizer in μ. Equivalently, maximizing ℓ in μ is the same as minimizing $\sum_{i=1}^{n} (y_i - \mu)^2$, a strictly convex (concave up) quadratic in μ with unique minimizer $\bar{y}$.

3) Maximize in σ for $\mu = \bar{y}$. Let $S = \sum_{i=1}^{n} (y_i - \bar{y})^2$. The profile log-likelihood is

$$\ell(\bar{y}, \sigma) = -\frac{n}{2} \log(2\pi) \;-\; n \log \sigma \;-\; \frac{S}{2\sigma^2}, \qquad \sigma > 0.$$

Differentiate w.r.t. σ:

$$\frac{d\ell(\bar{y}, \sigma)}{d\sigma} \;=\; -\frac{n}{\sigma} \;+\; \frac{S}{\sigma^3} \;=\; \frac{S - n\sigma^2}{\sigma^3}.$$

Set to zero:

$$S - n\sigma^2 = 0 \implies \widehat{\sigma}^2 = \frac{S}{n}, \qquad \widehat{\sigma} = \sqrt{\frac{S}{n}}.$$

Uniqueness and interiority.
As $\sigma \to 0^+$, $\ell(\bar{y}, \sigma) \to -\infty$ and as $\sigma \to \infty$, $\ell(\bar{y}, \sigma) \to -\infty$. Moreover,

$$\frac{d\ell}{d\sigma} = \frac{S - n\sigma^2}{\sigma^3} \begin{cases} > 0, & \sigma < \sqrt{S/n} = \widehat{\sigma} \\ < 0, & \sigma > \sqrt{S/n} = \widehat{\sigma} \end{cases}$$

so the log-likelihood increases on $(0, \widehat{\sigma})$ and decreases on $(\widehat{\sigma}, \infty)$. Therefore $\widehat{\sigma}$ is the *unique interior maximizer* over $\sigma > 0$.

Conclusion. The MLEs are

$$\boxed{\widehat{\mu} = \bar{y}, \qquad \widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2, \qquad \widehat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2}}.$$

These form the unique interior maximizer of the log-likelihood over $\mu \in \mathbb{R},\ \sigma > 0$.

$\square$