Outline

1) Testing with nuisance parameters

2) UMPU multivariate tests

3) Conditioning on null sufficient stat

Nuisance Parameters

Common setup: Extra unknown parameters which are not of direct interest

$\mathcal{P} = \{ P_{\theta,\lambda} : (\theta, \lambda) \in \Omega \}$, $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$

θ parameter of interest

λ nuisance parameter

Issue: λ unknown but might affect type I error or power of a given test

Ex $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ $Y_1, \dots, Y_m \overset{iid}{\sim} N(\nu, \sigma^2)$

μ, ν, σ^2 unknown

$H_0 : \mu = \nu$ vs $H_1 : \mu \neq \nu$

$\theta = \mu - \nu$ $\lambda = (\mu + \nu, \sigma^2)$ or (μ, σ^2)

Ex $X_1 \sim \text{Binom}(n_1, \pi_1)$ $X_2 \sim \text{Binom}(n_2, \pi_2)$

n_1, n_2 known $\Rightarrow$ not nuisance parameters

$H_0 : \pi_1 \leq \pi_2$ vs $H_1 : \pi_1 > \pi_2$

Multiparameter Exp. Families

Assume $X \sim p_{\theta,\lambda}(x) = e^{\theta' T(x) + \lambda' U(x) - A(\theta,\lambda)} h(x)$

$\theta \in \mathbb{R}^s$, $\lambda \in \mathbb{R}^r$, both unknown.

How to test $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$?

Idea: Condition on $U(X)$ to eliminate dep. on λ

1) Sufficiency reduction:

$g\,dt\,du$ = push-forward of $h\,du$

$$(T(X), U(X)) \sim q_{\theta,\lambda}(t,u)$$

$$= e^{\theta' t + \lambda' u - A(\theta,\lambda)} g(t,u)$$

(density wrt e.g. Lebesgue on $\mathbb{R}^{s+r}$)

2) Condition on $U(X)$:

$$q_\theta(t \mid u) = \frac{q_{\theta,\lambda}(t,u)}{\int q_{\theta,\lambda}(z,u)\,dz}$$

$$= \frac{e^{\theta' t + \cancel{\lambda' u} - \cancel{A(\theta,\lambda)}} g(t,u)}{e^{B_u(\theta)} = \int e^{\theta' z + \cancel{\lambda' u} - \cancel{A(\theta,\lambda)}} g(z,u)\,dz}$$

$$= e^{\theta' t - B_u(\theta)} g(t,u)$$

3) Conditional test:

Test $H_0: \theta \in \Theta_0$ vs. $H_1: \theta \in \Theta_1$ in

s-parameter model $\mathcal{Q}_u = \{q_\theta(t|u) : \theta \in \Theta\}$

Note if $s=1$, this family has MLR in T

Even if $s>1$, we still have gotten rid of λ

Theorem Let $\mathcal{P}$ be full rank exp. fam. with densities $$p_{\theta,\lambda}(x) = e^{\theta T(x) + \lambda' U(x) - A(\theta,\lambda)} h(x)$$

$\theta \in \mathbb{R}$, $\lambda \in \mathbb{R}^r$, $(\theta,\lambda) \in \Omega$ open, θ_0 possible

a) To test $H_0: \theta \le \theta_0$ vs. $H_1: \theta > \theta_0$, there is a UMPU test $\phi^*(x) = \psi(T(x); U(x))$ where

$$\psi(t;u) = \begin{cases} 1 & t > c(u) \\ \gamma(u) & t = c(u) \\ 0 & t < c(u) \end{cases}$$

With $c(u)$, $\gamma(u)$ chosen to make

$$\mathbb{E}_{\theta_0}[\phi^*(x) \mid U(x) = u] = \alpha$$

b) To test $H_0: \theta = \theta_0$ vs. $H_1: \theta \ne \theta_0$ there is a UMPU test $\phi^*(x) = \psi(T(x); U(x))$ where

$$\psi(t;u) = \begin{cases} 1 & t < c_1(u) \text{ or } t > c_2(u) \\ \gamma_i(u) & t = c_i(u) \\ 0 & t \in (c_1(u), c_2(u)) \end{cases}$$

with $c_i(u)$, $\gamma_i(u)$ chosen to make

$$\mathbb{E}_{\theta_0}[\phi^*(x) \mid U(x) = u] = \alpha$$

$$\mathbb{E}_{\theta_0}[T(X)(\phi^*(x) - \alpha) \mid U(X) = u] = 0$$

[Note λ has disappeared from the problem.]

Ex : $X_i \overset{ind.}{\sim} \text{Pois}(\mu_i)$ $\quad i = 1, 2$

$H_0 : \mu_1 \le \mu_2$ vs. $H_1 : \mu_1 > \mu_2$

$$p_\mu(x) = \prod_{i=1}^{2} \frac{\mu_i^{x_i} e^{-\mu_i}}{x_i!}$$

$$= e^{x_1\eta_1 + x_2\eta_2 - (e^{\eta_1} + e^{\eta_2})} \frac{1}{x_1!x_2!}$$

(Where $\eta_i = \log \mu_i$. $\quad H_0 : \eta_1 \le \eta_2 \quad H_1 : \eta_1 > \eta_2$)

$$= e^{\overset{T(x)}{x_1} \cdot \overbrace{(\eta_1 - \eta_2)}^{\theta} + \overbrace{(x_1 + x_2)}^{u(x)} \cdot \overset{\lambda}{\eta_2} - A(\eta)} \frac{1}{x_1!x_2!}$$

$H_0 : \theta \le 0$ vs $H_1 : \theta > 1$

Reject for conditionally large values of

X_1, given $X_1 + X_2 = u$

$$\mathbb{P}_\theta(X_1 = x_1 \mid U = u) = e^{x_1\theta + \cancel{u\lambda - A(\cdot)}} \cdot \frac{1}{x_1!(u-x_1)!} \Big/ \sum_{x_1=0}^{u} (\cdot)$$

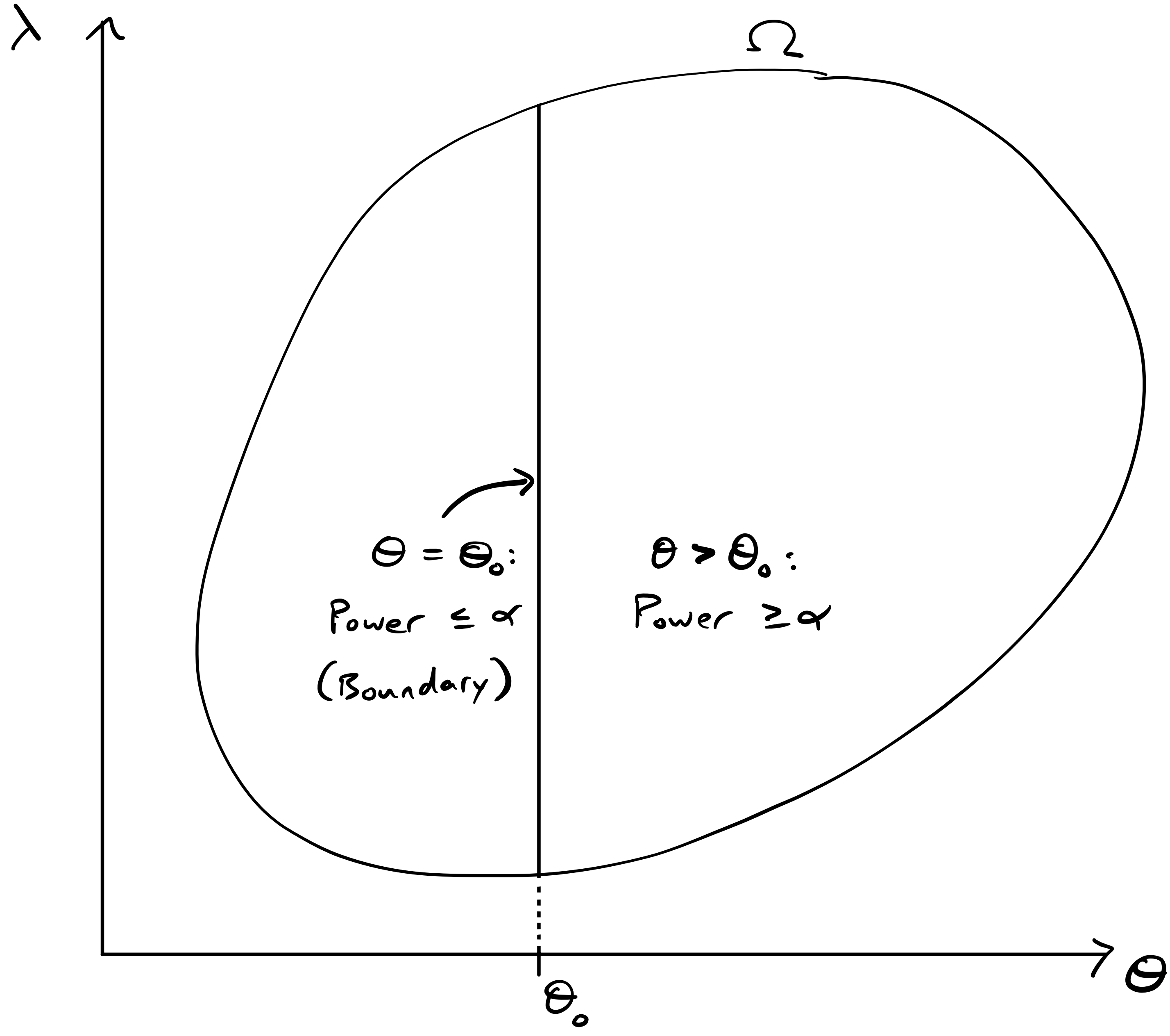
$$\propto_{x_1} e^{x_1\theta} \cdot \frac{u!}{x!(u-x_1)!}$$

$$= \text{Binom}\left(u, \frac{e^\theta}{1+e^\theta}\right) \qquad e^\theta = \mu_1/\mu_2$$

$$= \text{Binom}\left(u, \frac{\mu_1}{\mu_1 + \mu_2}\right)$$

So in the end we do a Binomial test.

Proof Sketch



1) Any unbiased test has $\beta(\theta_0, \lambda) = \alpha \quad \forall \lambda$

   (continuity of $\beta(\theta, \lambda)$)

2) Power $\equiv \alpha$ on boundary $\Rightarrow$ $\mathbb{E}_{\theta_0}[\phi \mid U] \overset{a.s.}{=} \alpha$

   ($U(X)$ complete sufficient on boundary submodel)

3) ϕ^* optimal among all tests with conditional level α

   (by reduction to univariate model)

Proof Assume ϕ any unbiased test

Step 1: $\mathbb{E}_{\theta,\lambda}|\phi(X)| \le 1 < \infty \quad \forall(\theta,\lambda)\in\Omega$

(Keener Thm 2.4)

$\Rightarrow \mathbb{E}_{\theta,\lambda}\phi(X)$ infinitely diff. on Ω, can diff. under $\int$

ϕ unbiased $\Rightarrow \mathbb{E}_{\theta_0,\lambda}[\phi(X)] = \alpha \quad \forall\ (\theta_0,\lambda)\in\Omega$

Step 2: Boundary submodel: $\mathcal{P}_{\theta_0} = \{P_{\theta_0,\lambda} : (\theta_0,\lambda)\in\Omega\}$

$$p_{\theta_0,\lambda}(x) = e^{\lambda' U(x) - A(\theta_0,\lambda)} \cdot \frac{e^{\theta_0 T(x)}}{h(x)}$$

$\mathcal{P}_{\theta_0}$ is full-rank, s-param exp. fam, $U(X)$ comp. suff.

Let $f(u) = \mathbb{E}_{\theta_0}[\phi(X) \mid U(X)=u] - \alpha$

$$\mathbb{E}_{\theta_0,\lambda}[f(U(X))] = \mathbb{E}_{\theta_0,\lambda}[\phi(X)] - \alpha = 0 \quad \forall\lambda$$

$$\Rightarrow f(u) \overset{a.s.}{=} 0$$

$$\Rightarrow \mathbb{E}_{\theta_0}[\phi(X) \mid U(x)=u] = 0 \quad \forall u$$

Two-sided case: $g(u) = \frac{d}{d\theta}\mathbb{E}_{\theta_0}[\phi \mid U=u]$

$$= \mathbb{E}_{\theta_0}\left[(T - \mathbb{E}_{\theta_0}[T \mid u])\phi \mid U\right]$$

$$= \mathbb{E}_{\theta_0}[T(\phi-\alpha) \mid U]$$

$$\mathbb{E}_{\theta_0,\lambda}\, g(u) = \mathbb{E}_{\theta_0,\lambda}[T(\phi-\alpha)] = \frac{\partial}{\partial\theta}\beta_\phi(\theta_0) = 0 \ \forall\lambda$$

$\Rightarrow \frac{d}{d\theta}\mathbb{E}_{\theta_0}[\phi \mid U] \overset{a.s.}{=} 0$ (Cond'l power has derivative 0 at θ_0)

Step 3: For any value u, the conditional model is

$$q_\theta(t \mid u) = e^{\theta t - B_u(\theta)} g(t,u), \quad \text{1-param. exp. fam}$$

In one- / two-sided case, we have shown $\psi(t;u)$ is UMP / UMPU in Q_u

Let $\bar{\phi}(t;u) = \mathbb{E}\left[\phi(X) \mid T(X)=t, U(X)=u\right]$

$$\mathbb{E}_\theta\left[\bar{\phi}(T;u) \mid U=u\right] = \mathbb{E}_\theta\left[\phi(X) \mid U(X)=u\right]$$
$$= \alpha \text{ if } \theta = \theta_0$$

$\Rightarrow \bar{\phi}(\cdot\,;u)$ is a (cond.'l) test of H_0 vs. H_1 in Q_u with power $= \alpha$ at boundary

One-sided case: (or $\theta \leq \theta_0$)

$\psi(t;u)$ is the UMP test of $\theta=\theta_0$ vs $\theta > \theta_0$ in Q_u, which is a 1-param. exp. fam.

Two-sided case:

$\psi(t;u)$ is the UMP test of $\theta=\theta_0$ vs. $\theta \neq \theta_0$ among tests with power $= \alpha$, $\frac{d}{d\theta}$power $= 0$ @ θ_0

In either case ψ has higher <u>cond</u>. power than $\bar{\phi}$, a.s.

For $(\theta, \lambda) \in \Omega_1$:

$$\begin{aligned} \mathbb{E}_{\theta,\lambda}[\phi(X)] &= \mathbb{E}_{\theta,\lambda}\left[\mathbb{E}_{\theta}\left[\bar{\phi}(T; u) \mid u\right] \right] \\ &\leq \mathbb{E}_{\theta,\lambda}\left[\mathbb{E}_{\theta}\left[\psi(T; u) \mid u\right] \right] \\ &= \mathbb{E}_{\theta,\lambda}\left[\phi^{*}(X)\right] \end{aligned}$$

Ex $\quad X_1,\dots,X_n \overset{iid}{\sim} N(\mu,\sigma^2) \qquad \sigma^2>0$ unknown

$$H_0: \mu=0 \quad \text{vs.} \quad H_1: \mu\neq 0$$

$$p_{\mu,\sigma^2}(x) = e^{\overbrace{\frac{\mu}{\sigma^2}}^{\theta}\overbrace{\sum X_i}^{T=\bar{X}} \overbrace{-\frac{1}{2\sigma^2}}^{\lambda}\overbrace{\sum X_i^2}^{u=\|X\|^2} - \frac{n\mu}{2\sigma^2}}\cdot\left(\frac{1}{2\pi\sigma^2}\right)^{n/2}$$

Optimal test rejects when $\bar{X}$ is extreme given $\|X\|^2$

If $\mu=0$, p is rotationally symmetric

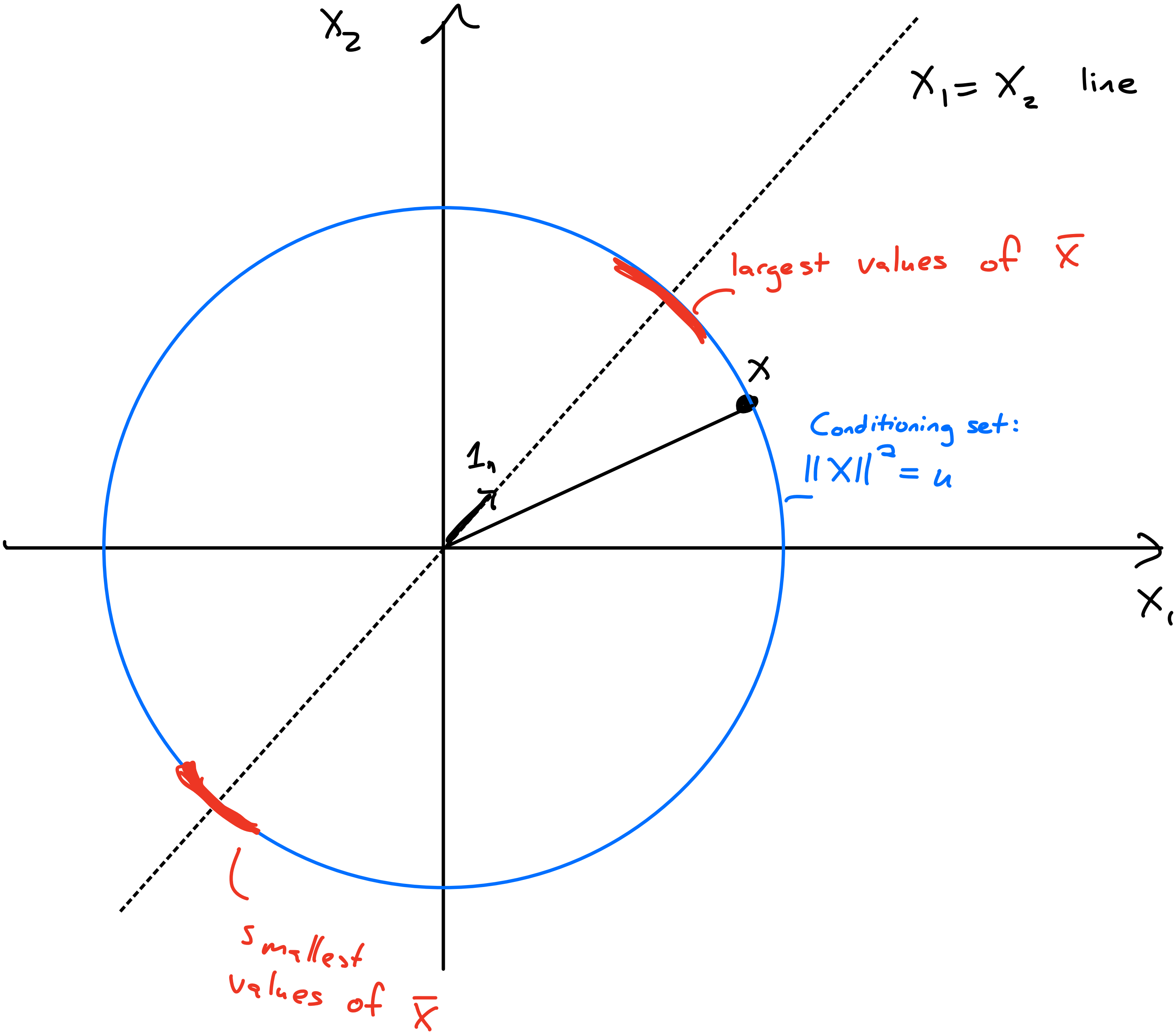
$$\Rightarrow X \,/\, \|X\|^2=u \overset{H_0}{\sim} \text{Unif}\left(\sqrt{u}\cdot S^{n-1}\right)$$

$$\left(\Leftrightarrow \frac{X}{\|X\|} \overset{H_0}{\sim} \text{Unif}\left(S^{n-1}\right), \text{ indep. of } \|X\|\right)$$

Optimal test rejects when $\frac{\bar{X}}{\|X\|}$ extreme (marginally)

Could stop here & simulate

Geometric Picture $(n=2)$



t - statistic

Above test rejects for

- conditionally extreme $\bar{X}$ given $\|X\|^2$

OR (equiv.)

- marginally extreme $\frac{\bar{X}}{\|X\|}$ $\left(\perp\!\!\!\perp \|X\|^2\right)$

Equivalent: reject for marginally extreme

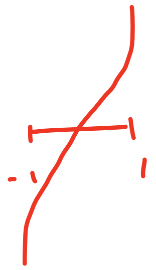
$$T = \frac{\sqrt{n}\,\bar{X}}{\sqrt{S^2}} \quad , \text{ where}$$

$$S^2 = \frac{1}{n-1}\sum (X_i - \bar{X})^2$$ (sample variance)

$$= \frac{1}{n-1}\left(\sum X_i^2 - 2\bar{X}\sum X_i + n\bar{X}^2\right)$$

$$= \frac{1}{n-1}\left(\|X\|^2 - n\bar{X}^2\right)$$

$r \to \frac{r}{\sqrt{1-r^2}}$:

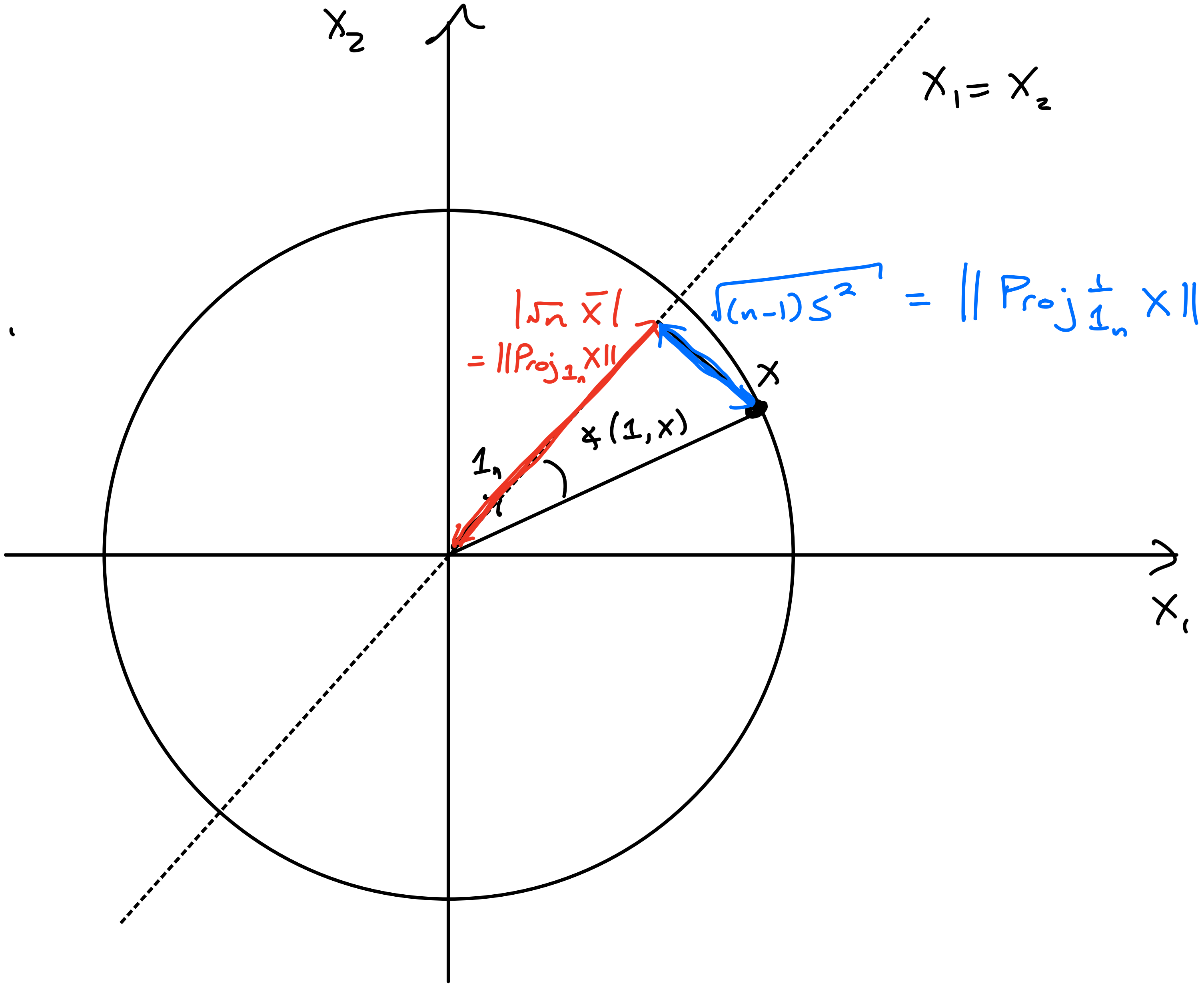


$$\Rightarrow \quad T = \sqrt{n-1}\cdot\frac{\sqrt{n}\,\bar{X}}{\sqrt{\|X\|^2 - n\bar{X}^2}} = \sqrt{n-1}\cdot\frac{R}{\sqrt{1-R^2}}$$

for $$R = \frac{\sqrt{n}\,\bar{X}}{\|X\|} = \underbrace{\overbrace{\frac{1}{\sqrt{n}}1_n}^{\text{unit vectors}}{}'\;\overbrace{\frac{X}{\|X\|}}}_{f\left(\frac{X}{\|X\|}\right) \Rightarrow \perp\!\!\!\perp \|X\|} = \cos\measuredangle(1_n, X)$$

Geometric Picture

$$T = \frac{\sqrt{n}\,\bar{X}}{\sqrt{S^2}} = \frac{\| \mathrm{Proj}_{1_n} X \|}{\| \mathrm{Proj}^{\perp}_{1_n} X \|} \cdot \sqrt{n-1}\ \mathrm{sgn}(\bar{X})$$



Next major theme: ratios of projections

Permutation Tests

Even if we don't get a UMPU test at the end, conditioning on null suff. stat. still helps.

Ex. $X_1, \dots, X_n \overset{iid}{\sim} P$ $\quad Y_1, \dots, Y_m \overset{iid}{\sim} Q$ $\quad H_0 : P = Q \quad H_1 : P \neq Q$

Under H_0, $P = Q$, $X_1, \dots, X_n, Y_1, \dots, Y_m \overset{iid}{\sim} P$

Let $(Z_1, \dots, Z_{n+m}) = (X_1, \dots, X_n, Y_1, \dots, Y_m)$

Under H_0, $U(Z) = (Z_{(1)}, \dots, Z_{(n+m)})$ compl. suff.

Let $S_{n+m} = \{\text{Permutations on } n+m \text{ elements}\}$

$$(X, Y) \mid U \overset{H_0}{\sim} \text{Unif}\left(\{ \pi U : \pi \in S_{n+m} \} \right)$$

Thus, for <u>any</u> test stat T, if $P = Q$,

$$\mathbb{P}_{P,Q}\left(T(Z) \geq t \mid U \right) = \frac{1}{(n+m)!} \sum_{\pi \in S_{n+m}} 1\{ T(\pi Z) \geq t \}$$

<u>Monte Carlo test</u>: In practice, we sample

$\pi_1, \dots, \pi_B \overset{iid}{\sim} S_{n+m}$, $\quad$ e.g. $B = 1000$

Then $Z, \pi_1 Z, \dots, \pi_B Z \overset{iid}{\sim} \text{Unif}(S_{n+m} U)$ under H_0

MC p-value $\quad p = \frac{1}{1+B} \sum_{b=1}^{B} 1\{ T(Z) \leq T(\pi_b Z) \}$

$$\overset{H_0}{\sim} \text{Unif}\left(\left\{ \tfrac{1}{1+B}, \dots, \tfrac{B-1}{1+B}, 1 \right\} \right) \quad \text{(if no ties)}$$

$(p \geq \text{Unif}(\cdot)$ if there are ties $)$