VI Some matrix algebra

① Spectral decomposition

A : (real) symmetric $n \times n$ matrix

$\Rightarrow \exists$ $n \times n$ orthogonal matrix P ($PP^T = I = PP^T$)

and $n \times n$ diagonal matrix D s.t.

$$A = PDP^T$$

If we write $P = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$,

(i) $\{v_1, \cdots, v_n\}$: orthonormal vectors

(ii) v_i is an eigen vector of A

and λ_i is a corresponding eigen value of A.

$$\left(\begin{aligned} & A = PDP^T \Rightarrow AP = PD \\ & \Rightarrow A \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \\ & \Rightarrow \begin{pmatrix} | & & | \\ Av_1 & \cdots & Av_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \lambda_1 v_1 & \cdots & \lambda_n v_n \\ | & & | \end{pmatrix} \end{aligned} \right)$$

(iii) $A = PDP^T = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} - v_1^T - \\ \vdots \\ - v_n^T - \end{pmatrix}$

$$= \sum_{i=1}^{n} \lambda_i v_i v_i^T$$

② Positive semidefinite matrices

and positive definite matrices

A : symmetric $n\times n$ matrix

$$\left(A = PDP^T \quad \begin{cases} P = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} : \text{orthogonal matrix} \\ D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} : \text{diagonal matrix} \end{cases} \right)$$

(i) We say A is positive semidefinite if

$x^T A x \geqq 0$ for all $x \in \mathbb{R}^n$

* All eigenvalues are nonnegative

$\left(0 \leqq v_i^T A v_i = \lambda_i \text{ for } i = 1, \cdots, n \right)$

* $\exists B$: $n\times n$ matrix s.t. $A = BB^T$

$\left(B = PD^{\frac{1}{2}} \text{, where } D^{\frac{1}{2}} = \begin{pmatrix} \lambda_1^{\frac{1}{2}} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{\frac{1}{2}} \end{pmatrix} \right)$

* $\exists C$: $n\times n$ symmetric matrix s.t. $A = C^2$

$\left(C = PD^{\frac{1}{2}}P^T \right)$

We sometime write C as $A^{\frac{1}{2}}$

(ii) We say A is positive definite if

$x^T A x > 0$ for all $x \in \mathbb{R}^n$

* All eigenvalues are positive

$(0 < v_i^T A v_i = \lambda_i$ for $i = 1, \cdots, n)$

* A is invertible

  $(A^{-1} = PD^{-1}P^T)$

* $\exists B$: $n \times n$ invertible matrix s.t. $A = BB^T$

  $\left(B = PD^{\frac{1}{2}}, \text{ where } D^{\frac{1}{2}} = \begin{pmatrix} \lambda_1^{\frac{1}{2}} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{\frac{1}{2}} \end{pmatrix}\right)$

* $\exists C$: $n \times n$ invertible symmetric matrix s.t. $A = C^2$

  $(C = PD^{\frac{1}{2}}P^T)$

$\boxed{\text{II}}$ Multivariate normal distributions

• Equivalent definitions

① Z : n-dimensional random variable

We say $Z \sim N_n(\underset{(0,\cdots,0)^T}{0}, \underset{n \times n \text{ identity matrix}}{I_n})$ if

$Z = (Z_1, \cdots, Z_n)^T$, where $Z_i \overset{iid}{\sim} N(0,1)$

② X : n-dimensional random variable

We say $Z \sim N_n(\underset{\mathbb{R}^n}{\mu}, \underset{n \times n \text{ positive semi-definite matrix}}{\Sigma})$ if

(i) $X = AZ + \mu$, where $Z \sim N(0, I_n)$

and $AA^T = \Sigma$

(ii) $X = \Sigma^{\frac{1}{2}} Z + \mu$, where $Z \sim N(0, I_n)$

(iii) $M_X(u) = \exp\{\mu^T u + \frac{1}{2} u^T \Sigma u\}$ for $u \in \mathbb{R}^n$

moment generating function of X

if Σ is invertible

(iv) $p_X(x) = |2\pi\Sigma|^{-\frac{1}{2}} \exp\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\}$

for $x \in \mathbb{R}^n$

· Properties

* Suppose $X \sim N(\mu, \Sigma)$.

For a $m \times n$ matrix A and $b \in \mathbb{R}^m$,

$$AX + b \sim N(A\mu + b, A\Sigma A^T)$$

$$\left(\begin{aligned} & X = \Sigma^{\frac{1}{2}} Z + \mu, \text{ where } Z \sim N(0, I_n) \\ & AX + b = A\Sigma^{\frac{1}{2}} Z + (A\mu + b) \\ & (A\Sigma^{\frac{1}{2}})(A\Sigma^{\frac{1}{2}})^T = A\Sigma A^T \end{aligned} \right)$$

* If $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right)$

and $\mathrm{cov}(X_1, X_2) = \Sigma_{12} = 0$,

then X_1 and X_2 are independent.

In other words, if X_1 and X_2 are jointly normal

and $\mathrm{Cov}(X_1, X_2) = 0$, then they are independent.

$$
\left(
\begin{aligned}
&M_{X_1,X_2}(u_1,u_2) \\
&\quad = \exp\left[(\mu_1^T\ \mu_2^T)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \frac{1}{2}(u_1^T\ u_2^T)\begin{pmatrix} \Sigma_{11} & \overset{=0}{\Sigma_{12}} \\ \underset{=0}{\Sigma_{21}} & \Sigma_{22} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right] \\
&\quad = \exp\left\{ \mu_1^T u_1 + \frac{1}{2} u_1^T \Sigma_{11} u_1 \right\} \\
&\qquad\quad \times \exp\left\{ \mu_2^T u_2 + \frac{1}{2} u_2^T \Sigma_{22} u_2 \right\}
\end{aligned}
\right)
$$

$*$ Suppose $X \sim N(\mu, \Sigma)$

For matrices A and B, if $\mathrm{Cov}(AX, BX) = 0$,

then AX and BX are independent.

$$\left(\begin{pmatrix} AX \\ BX \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} X \Rightarrow \text{jointly normal} \right)$$

$*$ If $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right)$,

then $X_1 \sim N(\mu_1, \Sigma_{11})$

$$\left(\begin{aligned} & X_1 = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\ & \Rightarrow X_1 \sim N\left(\begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} \right) \end{aligned} \right)$$

* If $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right)$

and $\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$ is invertible,

then $X_2 \mid X_1 = x_1 \sim N\left(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1), \right.$
$\left. \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \right)$

$$\left(\begin{aligned} & \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} : \text{Positive definite} \\ & \Rightarrow \Sigma_{11} : \text{Positive definite} \\ & \Rightarrow \Sigma_{11} : \text{invertible} \\ & \begin{pmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}(X_1 - \mu_1) \end{pmatrix} = \begin{pmatrix} I & 0 \\ -\Sigma_{21}\Sigma_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \end{pmatrix} \\ & \begin{pmatrix} I & 0 \\ -\Sigma_{21}\Sigma_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\Sigma_{21}\Sigma_{11}^{-1} & I \end{pmatrix}^T \\ & \quad = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \end{pmatrix} \end{aligned} \right.$$

$\Rightarrow X_1 - \mu_1$ and $X_2 - \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}(X_1 - \mu_1)$

are independent

$\Rightarrow X_2 - \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}(X_1-\mu_1) \mid X_1 = x_1$

$$\sim N(0, \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$$

* If $X \sim N_k(\mu, \Sigma)$ and Σ is invertible,

then

$$(X-\mu)^T \Sigma^{-1} (X-\mu) \sim \chi^2(k)$$

$X = \Sigma^{\frac{1}{2}} Z + \mu$, where $Z \sim N_k(0, I)$

$(X-\mu)^T \Sigma^{-1}(X-\mu) = Z^T Z$

$= Z_1^2 + \cdots + Z_k^2 \sim \chi^2(k)$