Lab 7: Scale-Mixture of N

$x \mid \tau^2 \sim N(0, \tau^2)$

$\tau^2 \sim \text{Inv-Gamma}(\alpha, \beta) \Rightarrow \underbrace{[\tau^2]^{-1}}_{\equiv s} \sim \text{Gamma}(\overset{\text{shape}}{\alpha}, \underset{\text{scale}}{\beta})$

$$\cdot\, p(x; \alpha, \beta) = \int_{s=0}^{\infty} \underbrace{\underset{\uparrow \frac{1}{\sqrt{2\pi\tau^2}}}{\left(\frac{\sqrt{s}}{\sqrt{2\pi}}\right)} e^{-\frac{1}{2} \underset{\uparrow (\frac{x}{\tau})^2}{s x^2}}}_{\text{normal density}} \underbrace{\frac{1}{\Gamma(\alpha)\beta^{\alpha}} s^{(\alpha-1)} e^{-s/\beta}}_{\text{gamma density}} \, ds$$

$$\propto \int_{s=0}^{\infty} \underbrace{s^{(\alpha - 1/2)} e^{-\left[\frac{1}{2}x^2 + \frac{1}{\beta}\right] s}} \, ds$$

this is the form of a gamma density

we can find the integral in closed form by referencing the normalization factor of a gamma density

let: $\alpha' = (\alpha + 1/2)$, $\beta'(x) = \left[\frac{1}{2}x^2 + \frac{1}{\beta}\right]^{-1}$

$$\int_0^{\infty} \underbrace{s^{(\alpha'-1)} e^{-s/\beta'(x)}}_{\text{gamma w/ parameters } \alpha',\, \beta'(x)} \, ds = \underbrace{\Gamma(\alpha')\, \beta(x)^{\alpha'}}_{\text{norm. factor of gamma}} \propto \beta(x)^{\alpha'} = \left[\frac{1}{\beta} + \frac{1}{2}x^2\right]^{-(\alpha + 1/2)}$$

$$\propto \underbrace{\left[1 + \frac{1}{2}\beta x^2\right]^{-\left(\frac{2\alpha+1}{2}\right)}}$$

form of a student's-t distribution w/ scale parameter $\beta^{-1/2}$ and shape parameter 2α...

$$\boxed{X \sim \text{Student's-}t(\text{scale} = \beta^{-1/2},\ \text{shape} = 2\alpha)}$$

so, the marginal prior on x_i is a student's-t

(a scale mix of N w/ inv-gamma var ⇒ student's t)