Kalman's Mathematical Description of Linear Dynamical Systems

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0. Question and Scope

How much of the physical world can be determined from a given amount of experimental data?

- Real, finite dimensional, continuous-time, linear time-invariant (LTI) dynamical systems (LTV before Section 5).
- Impulse response.

1. Intro and Summary

- **1&2&3**: Intro; define a dynamical system
- **4**: Present the problem of system realization from impulse response
- **5**: Present the *canonical structure theorem*: if a system realization is controllable and observable, $\implies$ it is an irreducible realization of the impulse response.
- **6&7**: Define controllability and observability and compute the canonical structure of an LTI system.
- **8**: Define MIMO state variables for LTI systems from transfer functions.
- **9**: Present common errors made by equating transfer functions and state space system realizations.

2. Axioms of Dynamical Systems

A dynamical system consists of the following (developed for linear time-varying (LTV) systems by Kalman, but only presented for LTI systems here):

state, $\begin{bmatrix} \boldsymbol{x}(t) \\ \dot{\boldsymbol{x}}(t) \end{bmatrix} \in \Sigma = \mathbb{R}^n$

- Physical interpretation: instantaneous position and momentum.
- Definition: minimal information needed about the system's history which suffices to predict the effect of the past upon the future.
- Properties of state: $\begin{bmatrix} \boldsymbol{x}(t > \tau) \\ \dot{\boldsymbol{x}}(t > \tau) \end{bmatrix}$ is fully defined by $\begin{bmatrix} \boldsymbol{x}(\tau) \\ \dot{\boldsymbol{x}}(\tau) \end{bmatrix}$ and $\boldsymbol{u}(t \geq \tau)$.

time, $t \in \Theta = \mathbb{R}$

input, $u \in \Omega =$ **piecewise continuous functions**

- Physical interpretation: forcing function.

transition function, $\varphi : \Omega \times \Theta \times \Theta \times \Sigma \to \Sigma$

$$\boldsymbol{x}(t) = \varphi(\boldsymbol{u}, t; t_o, \boldsymbol{x}_o)$$

- Linear on $\Omega \times \Theta$.
- Continuous with respect to Σ, Θ, Ω and their induced products.

output function, $\psi : \Theta \times \Sigma \to \mathbb{R}$

- Physical interpretation: variables that are directly observed.
- Linear on Σ.
- Continuous with respect to Σ, Θ, Ω and their induced products.

Every linear time-invariant ($\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ linear in Σ and independent of time) dynamical system is thus governed by the equations[1]

$$\boxed{\begin{aligned} \frac{d}{dt}\boldsymbol{x}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) \end{aligned}}$$

The general solution to the differential equation is

$$\begin{aligned} \boldsymbol{x}(t) &= \Phi_{\boldsymbol{A}}(t, t_o)\boldsymbol{x}_o + \int_{t_o}^{t} \Phi_{\boldsymbol{A}}(t, \tau)\boldsymbol{B}\boldsymbol{u}(\tau) d\tau \\ \Phi_{\boldsymbol{A}}(t, \sigma) &= e^{\boldsymbol{A}(t-\sigma)} \end{aligned} \tag{1}$$

and it can be verified that $\Phi_{\boldsymbol{A}}(t, \sigma) = \Phi_{\boldsymbol{A}}(t, \tau)\Phi_{\boldsymbol{A}}(\tau, \sigma) \;\; \forall t, \tau, \sigma \in \mathbb{R}$.

3. Equivalent Dynamical Systems

The state vector $\boldsymbol{x}(t)$ is an abstract quantity in $\mathbb{R}^n$. Therefore it can be expressed in any $\mathbb{R}^n$ coordinates. The system with the state vector $\bar{\boldsymbol{x}}(t)$ is equivalent to the system with the state vector $\boldsymbol{x}(t)$ if there exists a nonsingular matrix $\boldsymbol{T} \in \mathbb{R}^{n \times n}$ such that

$$\bar{\boldsymbol{x}}(t) = \boldsymbol{T}\boldsymbol{x}(t) \; .$$

The equivalent governing equations for the system are as follows:

$$\begin{aligned} \frac{d}{dt}\bar{\boldsymbol{x}}(t) &= \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-\mathbf{1}}\bar{\boldsymbol{x}}(t) + \boldsymbol{T}\boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{T}^{-\mathbf{1}}\bar{\boldsymbol{x}}(t) \end{aligned}$$

4. System Realization from Impulse Response

The *impulse response matrix* of a system is a time-dependent array of the output at each $\boldsymbol{y}$ coordinate in response to an pulse input, $\delta(t - t_o)$, at each $\boldsymbol{u}$ coordinate.

That is, given the input $u_{ij}(t) = \delta_{ij}\delta(t - t_o)$, the output is $y_{ij} = S_{ij}(t, t_o)$, where (by plugging in (1),)

$$\boldsymbol{S}(t, \tau) = \boldsymbol{C}\Phi_{\boldsymbol{A}}(t, \tau)\boldsymbol{B} \; .$$

Given $\boldsymbol{S}(t, t_o)$ for a dynamical system, the output can be found for any input by the convolution integral:

$$\boldsymbol{y}(t) = \int_{t_o}^{t} \boldsymbol{S}(t, \tau)\boldsymbol{u}(\tau) d\tau \; .$$

The central question of this paper is,

When and how does the impulse-response matrix determine the dynamical equations of the system?

[1]Sometimes you'll see a system with a *feed-through* term, $\boldsymbol{D}$ in the output function, $\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$. This term can always be added or removed because inputs $\boldsymbol{u}(t)$ and outputs $\boldsymbol{y}(t)$ are deterministic quantities.

5. Kalman Canonical Staircase State Space Realization (K-CSSSR)

Kalman presents his canonical form for state space system realization of a dynamical system. He proves that any system realization can be transformed into one which isolates the state vector into four parts:

- controllable and unobservable ($\boldsymbol{x}_{cuo}$)
- controllable and observable ($\boldsymbol{x}_{co}$)
- uncontrollable and unobservable ($\boldsymbol{x}_{ucuo}$)
- ucontrollable and observable ($\boldsymbol{x}_{uco}$)

The governing equations are as follows:

$$\boxed{\begin{aligned} \frac{d}{dt}\begin{bmatrix} \boldsymbol{x}_{cuo}(t) \\ \boldsymbol{x}_{co}(t) \\ \boldsymbol{x}_{ucuo}(t) \\ \boldsymbol{x}_{uco}(t) \end{bmatrix} &= \begin{bmatrix} \boldsymbol{A}_{cuo} & \boldsymbol{A}_{12} & \boldsymbol{A}_{13} & \boldsymbol{A}_{14} \\ 0 & \boldsymbol{A}_{co} & 0 & \boldsymbol{A}_{24} \\ 0 & 0 & \boldsymbol{A}_{ucuo} & \boldsymbol{A}_{34} \\ 0 & 0 & 0 & \boldsymbol{A}_{uco} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{cuo}(t) \\ \boldsymbol{x}_{co}(t) \\ \boldsymbol{x}_{ucuo}(t) \\ \boldsymbol{x}_{uco}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B}_{cuo}(t) \\ \boldsymbol{B}_{co}(t) \\ 0 \\ 0 \end{bmatrix} \boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \begin{bmatrix} 0 & \boldsymbol{C}_{co} & 0 & \boldsymbol{C}_{uco} \end{bmatrix} \boldsymbol{x}(t) \end{aligned}} \tag{2}$$

The **impulse response matrix** of a linear dynamical system depends only on the dynamics of the controllable and observable modes $\boldsymbol{x}_{co}(t)$.

$$\boldsymbol{S}(t,\tau) = \boldsymbol{C}_{co}\Phi_{\boldsymbol{A}_{co}}(t,\tau)\boldsymbol{B}_{co}$$

Equivalently, any controllable and observable realization of an impulse response matrix is *irreducible* and equivalent to any other controllable and observable realization.

If a dynamical system is controllable and observable, it has a unique impulse response matrix.

The answer to the central question of this paper is,

The impulse-response matrix fully determines the dynamical equations of a controllable and observable system.

6. Definition of Controllability and Observability

The following statements are equivalent:

- The system is controllable.
- The pair, $\{\boldsymbol{A}, \boldsymbol{B}\}$, is controllable.
- The matrix $\boldsymbol{P} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A^{n-1}B} \end{bmatrix}$ has rank n.

The following statements are equivalent:

- The system is observable.
- The pair, $\{\boldsymbol{A}, \boldsymbol{C}\}$, is observable.
- The matrix $\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \cdots \\ \boldsymbol{CA^{n-1}} \end{bmatrix}$ has rank n.

7. Computing the Canonical Form (2)

The transformation matrix that separates us into controllable and uncontrollable modes is $\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_c & \boldsymbol{M}_{uc} \end{bmatrix}$, where the columns of $\boldsymbol{M}_c$ are the linearly independent columns of $\boldsymbol{P}$ and the columns of $\boldsymbol{M}_{uc}$ are chosen such that $\boldsymbol{M}$ is full rank.

The transformation matrix that separates us into observable and unobservable modes is $\boldsymbol{N} = \begin{bmatrix} \boldsymbol{N}_o \\ \boldsymbol{N}_{uo} \end{bmatrix}$, where the rows of $\boldsymbol{N}_o$ are the linearly independent rows of $\boldsymbol{Q}$ and the rows of $\boldsymbol{N}_{uo}$ are chosen such that $\boldsymbol{N}$ is full rank.