Journal Club Meeting

“Conserving energy and momentum in nonlinear dynamics: A simple implicit time integration scheme” by Klaus-Jürgen Bathe (2007)

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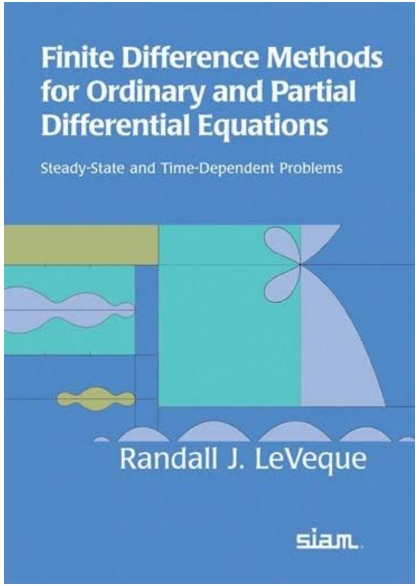
Fundamental Theorem of Finite Difference Methods

$$\underbrace{\text{consistency}}_{\substack{\text{local truncation}\\ \text{error (LTE)}}} + \text{stability} \Longrightarrow \underbrace{\text{convergence}}_{\text{global error}}$$



"The **global error** simply refers to the error that we are attempting to bound. The **LTE** refers to the error in our finite difference approximation of derivatives and hence is something that can be easily estimated using Taylor series expansions. **Stability** is the **magic ingredient** that allows us to go from these easily computed bounds on the local error to the estimates we really want for the global error."

"The **challenge** in analyzing finite difference methods for new classes of problems often is to **find an appropriate definition of "stability"** that allows one to prove convergence using the fundamental theorem. For nonlinear PDEs this frequently must be turned to each particular class of problems and relies on existing mathematical theory and techniques of analysis for this class of problems."

Linear Dynamic Problems

Newmark's method: a single-step, *second-order* accuracy scheme (a special trapezoidal rule method).

Stability

Newmark's method with constant average acceleration is *unconditionally stable procedures,* which leads to bounded solutions regardless of the time-step length.

Accuracy

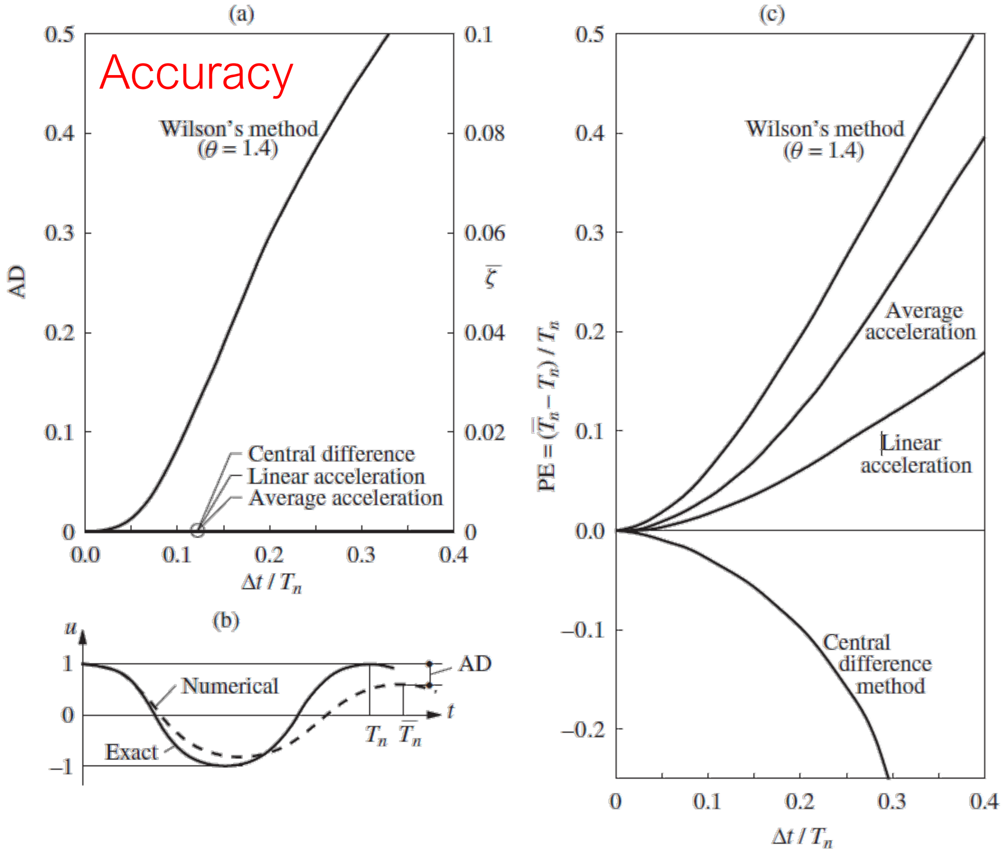


Figure 5.5.2 (a) Amplitude decay versus $\Delta t/T_n$; (b) definition of AD and PE; (c) period elongation versus $\Delta t/T_n$.

(Chopra, 2020)

Non-Linear Dynamic Problems

Stability

"In publications of time integration methods for linear dynamical systems, **the main topic is the order of accuracy, because the criterion for unconditionally stability is easy to satisfy for most of the algorithms.**

On the other hand, in the case of a non-linear dynamical system, the main interest in numerical dynamics is focused on the numerical stability of the algorithms. The reason for this change of interest is the problem of numerical stability in non-linear dynamics: **Algorithms which are unconditional stable for linear dynamics often lose this stability in the non-linear case.**

Instead, spectral stability is required, which is only a necessary condition for stable time integration schemes in non-linear dynamics. **A sufficient condition in the non-linear regime is the conservation or decay of the total energy within a time step.**" (Kuhl and Crisfield, 1999)

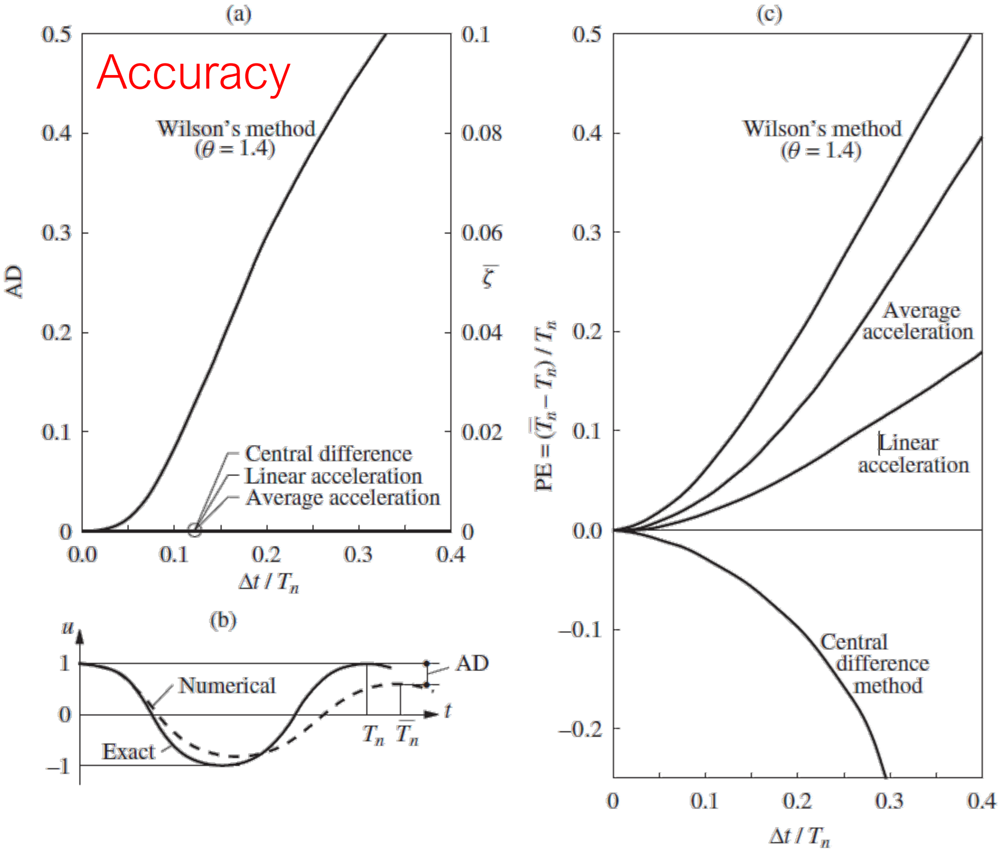


Figure 5.5.2 (a) Amplitude decay versus $\Delta t/T_n$; (b) definition of AD and PE; (c) period elongation versus $\Delta t/T_n$.

(Chopra, 2020)

Non-Linear Dynamic Problems

Once the stability is no long guarantee (i.e., size of time step matter), numerical damping or amplitude decay mentioned in Chopra's book would play an important role in both stability and accuracy of the numerical integration method.

In other words, we somehow need **Numerical Damping** to stabilizes the numerical integration scheme **by damping out the unwanted high frequency modes**, which contains frequency that is much smaller than the time step that we can practically use.

Numerical stability of Non-Linear Dynamic Problems

numerical stability $U_{n+1} - U_n + K_{n+1} - K_n \leq W_{ext}$

enforced conservation of energy

Constraint Energy Method

Hughes Caughey Liu [3]

numerical dissipation of energy

Newmark Algorithm

Generalized-α Method

e.g. Newmark [4],

Chung Hulbert [7]

algorithmic conservation of energy

Energy-Momentum Method

Lewis Simo [8],

Simo Tarnow [9]

enforced conservation of energy + numerical dissipation

Constraint Energy Momentum Algorithm

Kuhl Ramm [5, 6], Kuhl [10]

algorithmic conservation or decay of energy + numerical dissipation

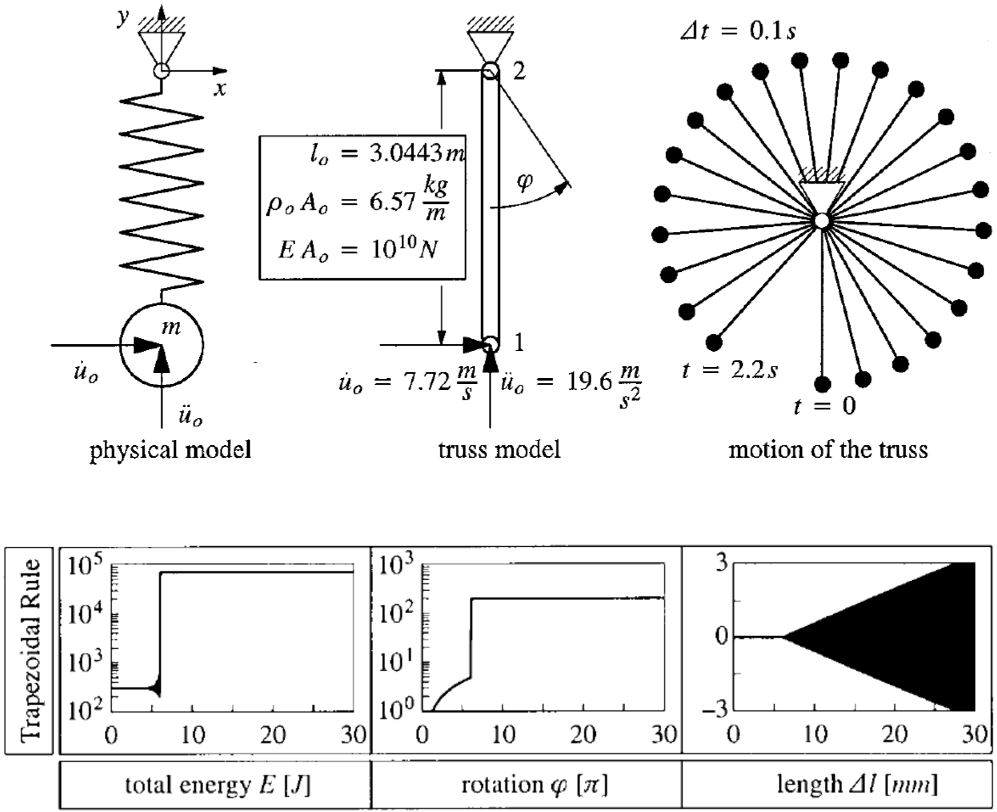
Modified Energy-Momentum Method;

Generalized Energy-Momentum Method

Armero Petőcz [11,12], Crisfield Galvanetto

Jelenić[13]; the present publication

controllable numerical dissipation of high frequencies



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- symmetric tangential stiffness matrix
- observable failure
- dissipation of high frequencies
- energy conserving/decaying - satisfy equation (1)
- order of accuracy

"High frequency modes which are inaccurately resolved with the time step used may then deteriorate the overall solution accuracy. Also, these methods may result in non-symmetric tangent stiffness matrices and the solution of a scalar variable either at the integration points or over each element in an averaged sense. Hence, these integration schemes are computationally costly." (Bathe, 2005)

Trapezoidal Rule	2	no	no	no	yes
Mid-Point Rule	2	no	no	no	yes
Newmark Method A	1	no	yes	no	yes
Newmark Method B	1	yes	yes	—	yes
α-methods A	2	no	yes	no	yes
α-methods B	2	yes	yes	—	yes
Energy-Momentum Method	2	yes	no	yes	no
Modified Energy-Momentum Method A	1	yes	yes	yes	no
Modified Energy-Momentum Method B	1	yes	yes	—	no
Generalized Energy-Momentum Method A	?	no	yes	?	no
Generalized Energy-Momentum Method B	?	yes	yes	—	no
Constraint Energy Method	2	yes	no	yes	yes
Constraint Energy Momentum Algorithm A	2	yes	yes	yes	yes
Constraint Energy Momentum Algorithm B	2	yes	yes	—	yes

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Stiff Differential Equation

- ❑ Equations where certain implicit methods perform better, usually tremendously better, than explicit ones (Curtiss and Hirschfelder, 1952).
- ❑ If the solution being sought is varying slowly, but there are nearby solutions that vary rapidly, so the numerical method must take small steps to obtain satisfactory results (Leveque, 2007).

trapezoidal rule + backward differentiation formula (TRBDF2):
A composite, single step, second-order accurate integration scheme with two sub-steps.

[Stiff Differential Equations - MATLAB & Simulink (mathworks.com)](https://www.mathworks.com/help/matlab/math/solve-stiff-odes.html)

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ode23tb

Solve stiff differential equations — trapezoidal rule + backward differentiation formula

Syntax

```
[t,y] = ode23tb(odefun,tspan,y0)
[t,y] = ode23tb(odefun,tspan,y0,options)
[t,y,te,ye,ie] = ode23tb(odefun,tspan,y0,options)
sol = ode23tb(___)
```

trapezoidal rule + backward differentiation formula (TRBDF2): 1st sub-step

Hence in the first sub-step, the equations solved are Eq. (1) applied at time $t+\Delta t/2$

$$\mathbf{M}^{t+\Delta t/2}\ddot{\mathbf{U}} + \mathbf{C}^{t+\Delta t/2}\dot{\mathbf{U}} = {}^{t+\Delta t/2}\mathbf{R} - {}^{t+\Delta t/2}\mathbf{F} \tag{2}$$

with the equations of the trapezoidal rule

$${}^{t+\Delta t/2}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + \left[\frac{\Delta t}{4}\right]\left({}^{t}\ddot{\mathbf{U}} + {}^{t+\Delta t/2}\ddot{\mathbf{U}}\right) \tag{3}$$

$${}^{t+\Delta t/2}\mathbf{U} = {}^{t}\mathbf{U} + \left[\frac{\Delta t}{4}\right]\left({}^{t}\dot{\mathbf{U}} + {}^{t+\Delta t/2}\dot{\mathbf{U}}\right) \tag{4}$$

Any iterative scheme can be used, but we employ the Newton–Raphson iteration, see Ref. [1], with the governing equations, for $i = 1, 2, 3, \ldots$

$$\begin{aligned}
&\left(\frac{16}{\Delta t^2}\mathbf{M} + \frac{4}{\Delta t}\mathbf{C} + {}^{t+\Delta t/2}\mathbf{K}^{(i-1)}\right)\Delta\mathbf{U}^{(i)} \\
&\quad = {}^{t+\Delta t/2}\mathbf{R} - {}^{t+\Delta t/2}\mathbf{F}^{(i-1)} \\
&\qquad - \mathbf{M}\left(\frac{16}{\Delta t^2}\left({}^{t+\Delta t/2}\mathbf{U}^{(i-1)} - {}^{t}\mathbf{U}\right) - \frac{8}{\Delta t}{}^{t}\dot{\mathbf{U}} - {}^{t}\ddot{\mathbf{U}}\right) \\
&\qquad - \mathbf{C}\left(\frac{4}{\Delta t}\left({}^{t+\Delta t/2}\mathbf{U}^{(i-1)} - {}^{t}\mathbf{U}\right) - {}^{t}\dot{\mathbf{U}}\right)
\end{aligned} \tag{5}$$

where

$${}^{t+\Delta t/2}\mathbf{U}^{(i)} = {}^{t+\Delta t/2}\mathbf{U}^{(i-1)} + \Delta\mathbf{U}^{(i)} \tag{6}$$

trapezoidal rule + backward differentiation formula (TRBDF2): 2nd sub-step

Once convergence has been reached, we use the calculated displacements ${}^{t+\Delta t/2}\mathbf{U}$ in Eqs. (3) and (4) to obtain the velocities, and accelerations if requested, at time $t+\Delta t/2$.

Then in the second sub-step, the equations solved are Eq. (1) applied at time $t+\Delta t$

$$\mathbf{M}^{t+\Delta t}\ddot{\mathbf{U}} + \mathbf{C}^{t+\Delta t}\dot{\mathbf{U}} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F} \tag{7}$$

with the equations of the three-point Euler backward method

$${}^{t+\Delta t}\dot{\mathbf{U}} = \frac{1}{\Delta t}{}^{t}\mathbf{U} - \frac{4}{\Delta t}{}^{t+\Delta t/2}\mathbf{U} + \frac{3}{\Delta t}{}^{t+\Delta t}\mathbf{U} \tag{8}$$

$${}^{t+\Delta t}\ddot{\mathbf{U}} = \frac{1}{\Delta t}{}^{t}\dot{\mathbf{U}} - \frac{4}{\Delta t}{}^{t+\Delta t/2}\dot{\mathbf{U}} + \frac{3}{\Delta t}{}^{t+\Delta t}\dot{\mathbf{U}} \tag{9}$$

Using Eqs. (7)–(9), and the earlier solutions obtained for time $t+\Delta t/2$, the governing equations for the Newton–Raphson iteration to obtain the solution at time $t+\Delta t$ are

$$\begin{aligned}
&\left(\frac{9}{\Delta t^2}\mathbf{M} + \frac{3}{\Delta t}\mathbf{C} + {}^{t+\Delta t}\mathbf{K}^{(i-1)}\right)\Delta\mathbf{U}^{(i)} \\
&\quad = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}\mathbf{F}^{(i-1)} \\
&\qquad - \mathbf{M}\left(\frac{9}{\Delta t^2}{}^{t+\Delta t}\mathbf{U}^{(i-1)} - \frac{12}{\Delta t^2}{}^{t+\Delta t/2}\mathbf{U} + \frac{3}{\Delta t^2}{}^{t}\mathbf{U} - \frac{4}{\Delta t}{}^{t+\Delta t/2}\dot{\mathbf{U}} + \frac{1}{\Delta t}{}^{t}\dot{\mathbf{U}}\right) \\
&\qquad - \mathbf{C}\left(\frac{3}{\Delta t}{}^{t+\Delta t}\mathbf{U}^{(i-1)} - \frac{4}{\Delta t}{}^{t+\Delta t/2}\mathbf{U} + \frac{1}{\Delta t}{}^{t}\mathbf{U}\right)
\end{aligned} \tag{10}$$

and

$${}^{t+\Delta t}\mathbf{U}^{(i)} = {}^{t+\Delta t}\mathbf{U}^{(i-1)} + \Delta\mathbf{U}^{(i)} \tag{11}$$

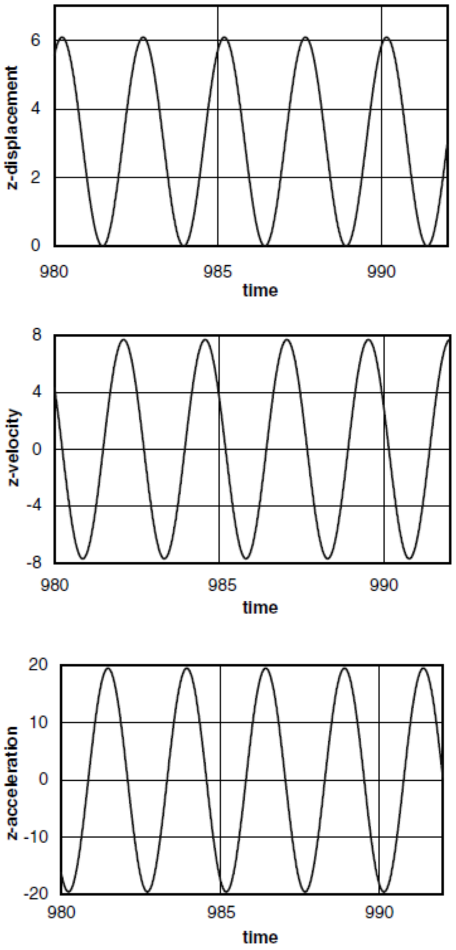


Fig. 4. Stiff pendulum in 400 cycles using the scheme with $\Delta t = 0.01$.

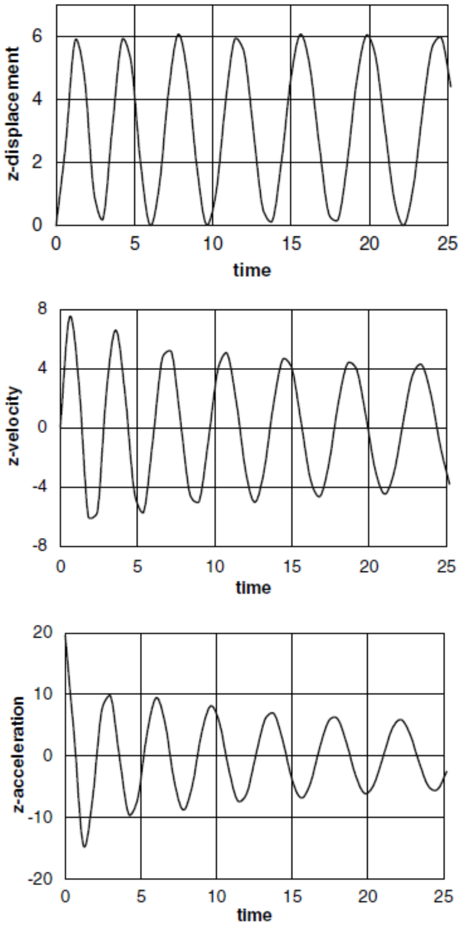
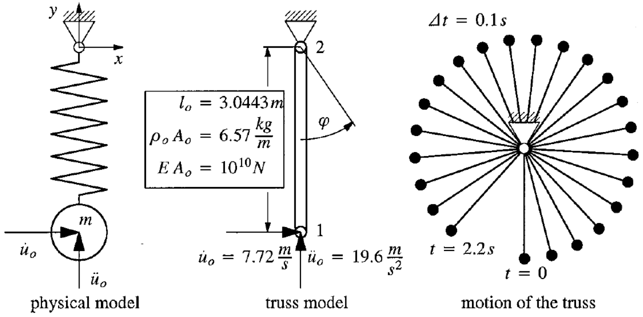


Fig. 5. Stiff pendulum using the scheme with $\Delta t = 0.6$.

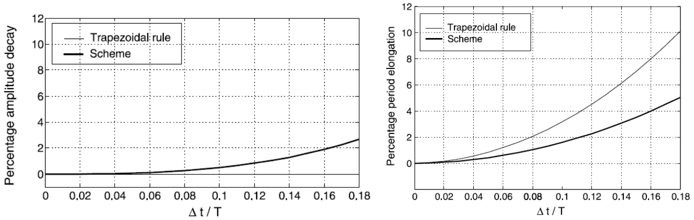


Stability

Conserving energy and linear/ angular momentum

Numerical damping

Slight damping from backward Euler



Discussion

- The so-called large deformation “nonlinear dynamic” problems in the previous studies usually refer to geometrically nonlinear problem, but what about the case with material nonlinearity/ contact or impact problem?
- In OpenSees, only the Generalized-alpha method and TRBDF2 can be found. I wonder why TRBDF2 doesn't receive more attention, at least in my experience? What about other schemes?

» 3. Command Manual » 3.2. Analysis Commands » 3.2.6. integrator Command » 3.2.6.9. TRBDF2 View page source

3.2.6.9. TRBDF2

`integrator TRBDF2`

Note

- As opposed to dividing the time-step in 2 as outlined in the papers, we just switch alternate between the 2 integration strategies,i.e. the time step in our implementation is double that described in the papers.

This command is used to construct a TRBDF2 integrator object. The TRBDF2 integrator is a composite scheme that alternates between the Trapezoidal scheme and a 3 point backward Euler scheme. It does this in an attempt to conserve energy and momentum, something newmark does not always do.

References

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