Finite Rotations

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Rotations

Key	$\boldsymbol{\Lambda}_{n+1}^{(i+1)}$	Reference
Iterative	$\exp\left(\Delta\boldsymbol{\omega}_{n+1}^{(i)}\right)\boldsymbol{\Lambda}_{n+1}^{(i)}$	Simo et al. [7]
Incremen-tal	$\exp\left(\boldsymbol{\theta}_{n+1}^{(i)}+\Delta\boldsymbol{\theta}_{n+1}^{(i)}\right)\boldsymbol{\Lambda}_{n}$	Ibrahimbegovic [4]
Total	$\exp\left(\boldsymbol{\theta}_{n+1}^{(i)}+\Delta\boldsymbol{\theta}_{n+1}^{(i)}\right)$	Ibrahimbegović et al. [5]

- Just about every element formulated with finite rotations (corotational, exact, etc.) begins as a nonlinear problem with unknowns in SO(3), and a residual that looks like:

$$\begin{aligned}\mathbf{0} &= \left(\boldsymbol{P}_r(\boldsymbol{\Lambda}) - \boldsymbol{P}_f\right)\cdot\delta\boldsymbol{\Lambda}\\ &= \boldsymbol{P}_u(\boldsymbol{\Lambda})\cdot\delta\boldsymbol{\Lambda}\end{aligned}$$

- The tangent space for such elements takes the natural multiplicative form, so that variations look like $\delta\boldsymbol{\Lambda} = \boldsymbol{\Omega}\boldsymbol{\Lambda}$ (i.e., the product of a skew-symmetric and othogonal matrix). The first successful finite rotation elements stopped here (Simo and Vu-Quoc, Rankin and Nour-Omid, etc). This corresponds to the iterative update option.

  Later, with the work of Cardona et al. [2], Ibrahimbegović et al. [5] and Battini [1], several re-parameterized formulations were introduced. All of these were derived by applying a change of variable of the form $\boldsymbol{\Lambda} = g(\boldsymbol{U})$ to one of the aforementioned multiplicative formulations to produce the total and incremental family of options.

- The "tangent space" is where we solve for iterative corrections in the Newton-Raphson method. In typical problems, even those with finite deformations, this space is often structured exactly like the configurations space. However, for rotations, the natural tangent space is $\mathfrak{so}(3)$ (i.e., the space of 3×3 skew-symmetric matrices). *This is not affected by the selection of the rotation's representation.*

- When treating problems which involve rotations as unknowns, the distinction between *representing rotations*, and *parameterizing the problem* is sometimes unclear.

  - The **representation** of a rotation concerns how a single rotation is stored in computer memory, and how it's operations are computed. There are several possibilities ("rotation vector", quaternions, etc). Regardless of this choice, the domain of the problem (i.e., the configuration space) is still the set of maps to SO(3), and updates are *still* multiplicative in nature.

– A **(re)parameterization** is a re-mapping of the actual problem domain, and corresponding tangent space. This is equivalent to applying a change of variable, $\boldsymbol{\Lambda} = g(\boldsymbol{U})$ and $\delta\boldsymbol{\Lambda} = \nabla g(\boldsymbol{U})\delta\boldsymbol{U}$ in the unbalance equation. This produces a new problem:

$$\boldsymbol{P}_u(\boldsymbol{\Lambda}) \cdot \delta\boldsymbol{\Lambda} \to \boldsymbol{P}_u(g(\boldsymbol{U})) \cdot \nabla g(\boldsymbol{U})\delta\boldsymbol{U}$$

The solution to this new problem, $\boldsymbol{U}$, is not necessarily an element of SO(3). In order to alleviate the multiplicative burdens of SO(3), all one needs is to find some function g that associates some vectorial parameter $\boldsymbol{U}$ predictably with the rotations of SO(3).

A change of *representation* is not observable by the global solution routine, and does not necessitate any changes to the fundamental element response quantities (e.g., unbalanced forces or stiffness). For example, most formulations can use quaternions internally for their representation, and this has been suggested from the beginning (Simo et al. [7], Nour-Omid et al. [6], Crisfield [3], De Souza, etc.). However, Ibrahimbegović et al. [5] and later Battini [1] go a step further and actually *reparameterize the underlying problem.* This does require a transformation in force and stiffness formulations.

References

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