

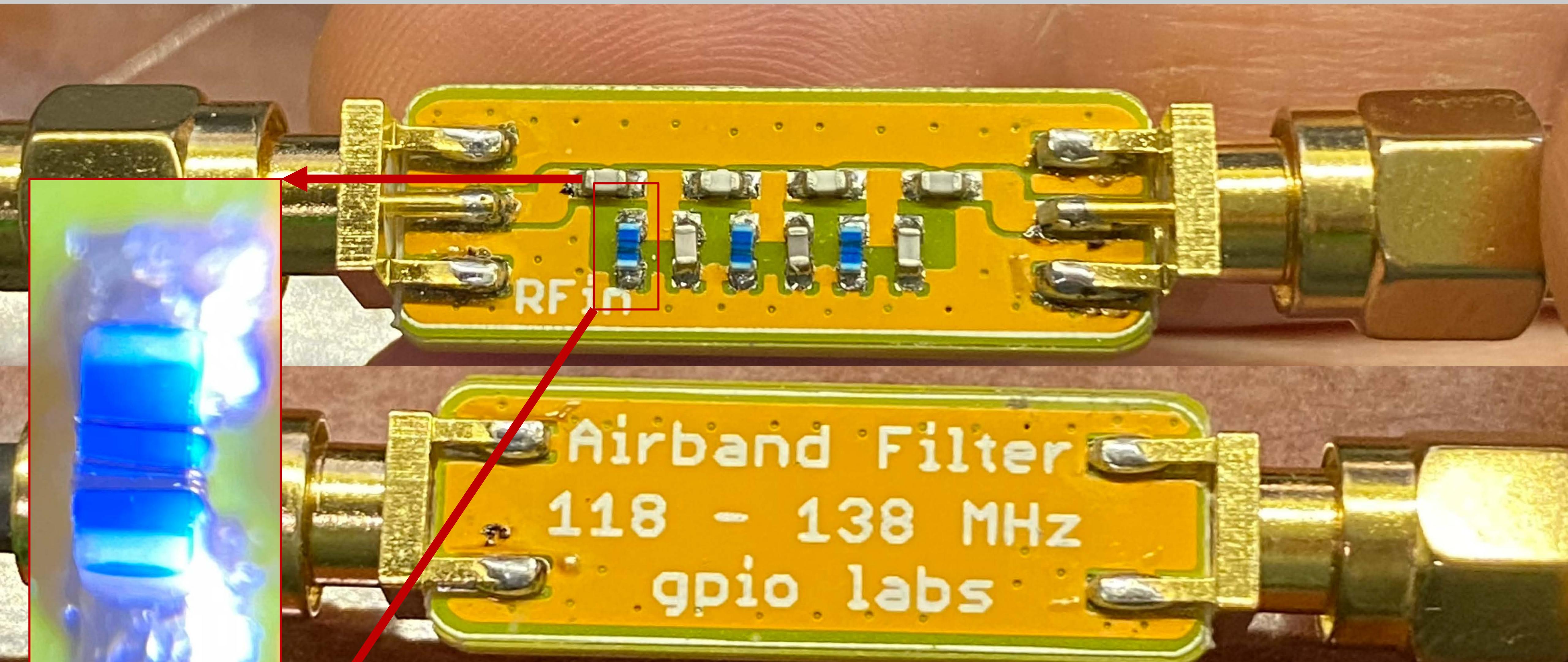
EE16B
Designing Information
Devices and Systems II

Lecture 6A
Control- state space representation

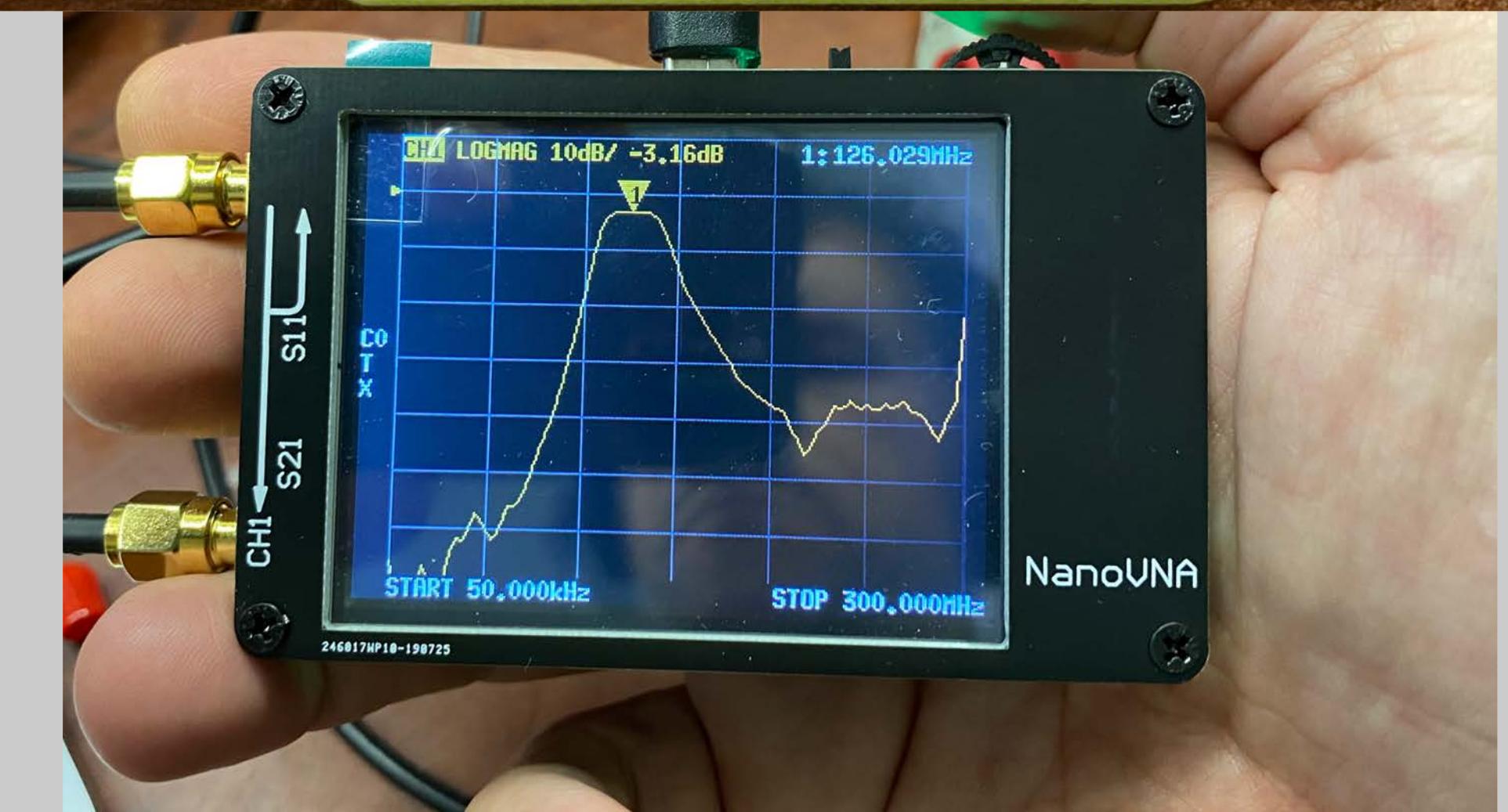
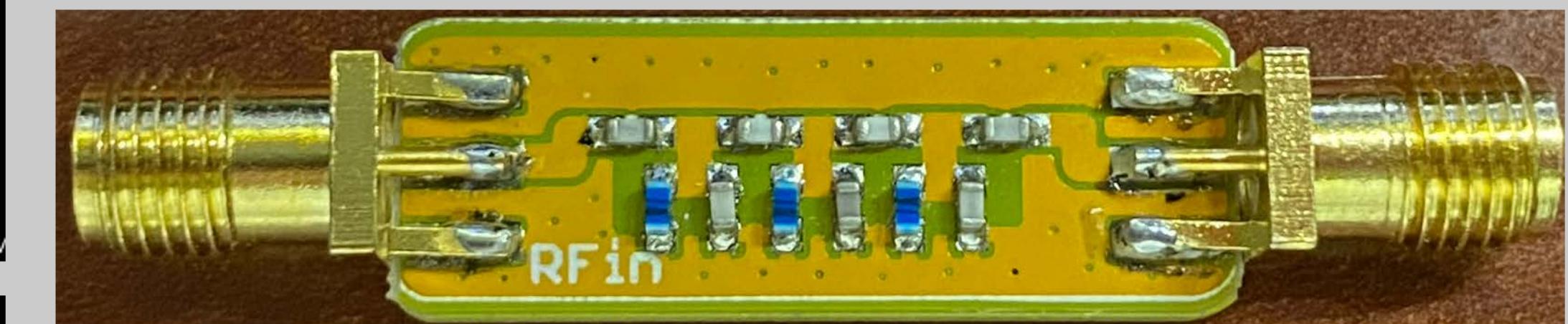
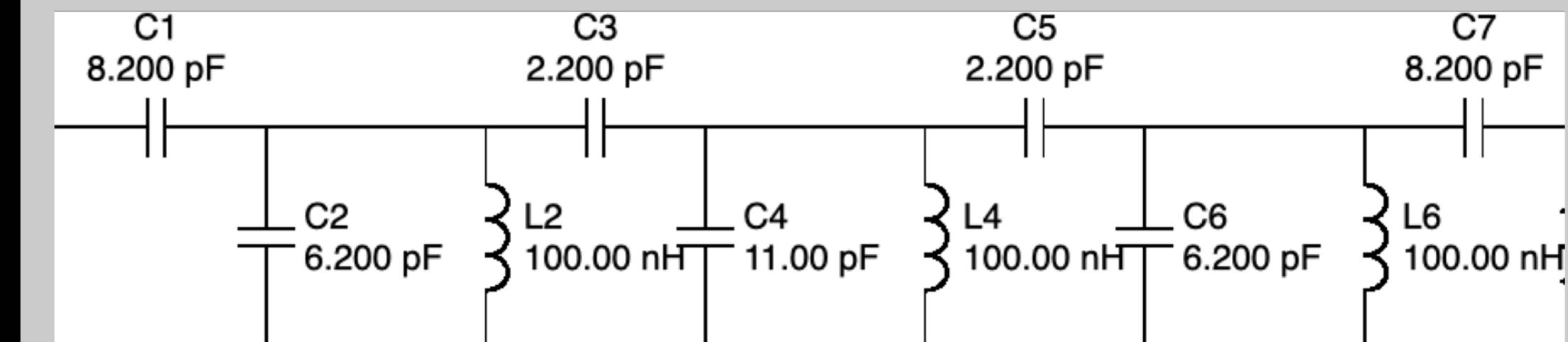
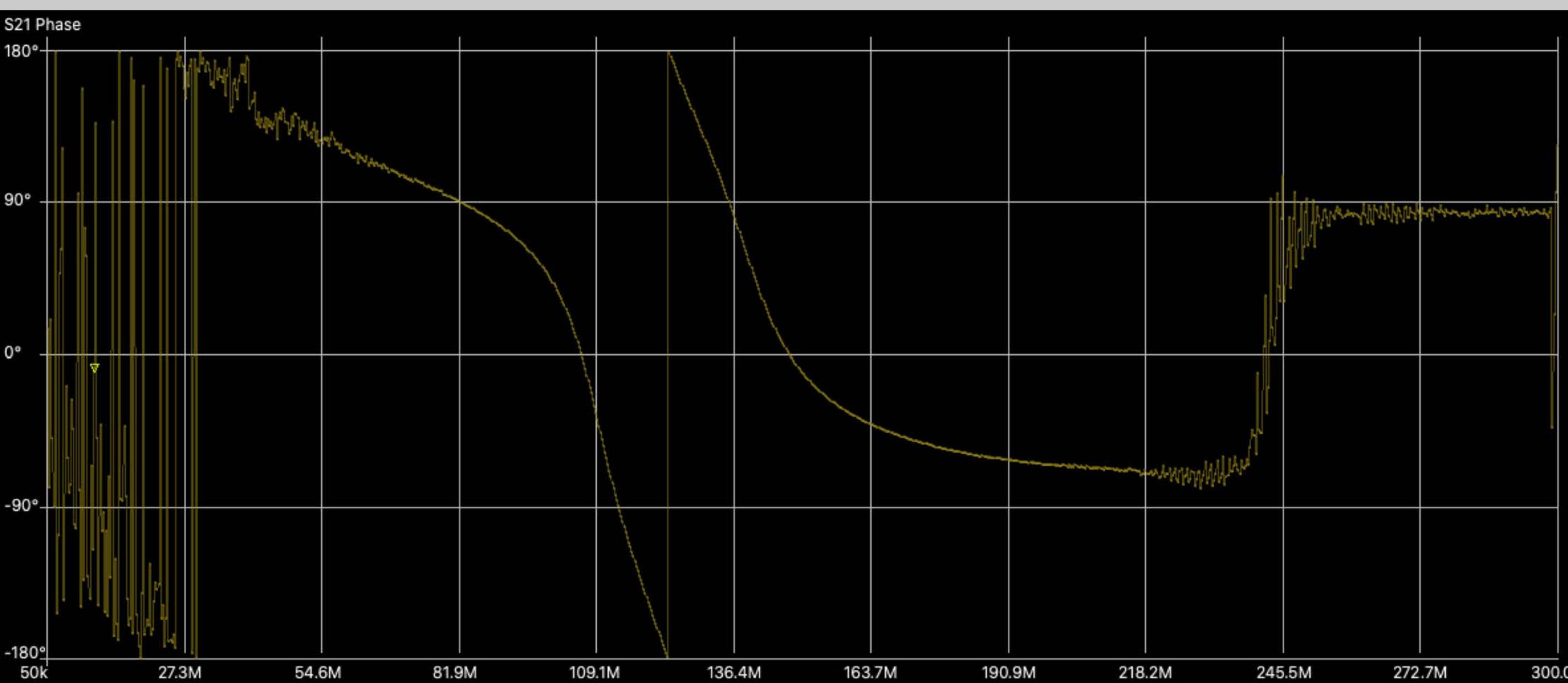
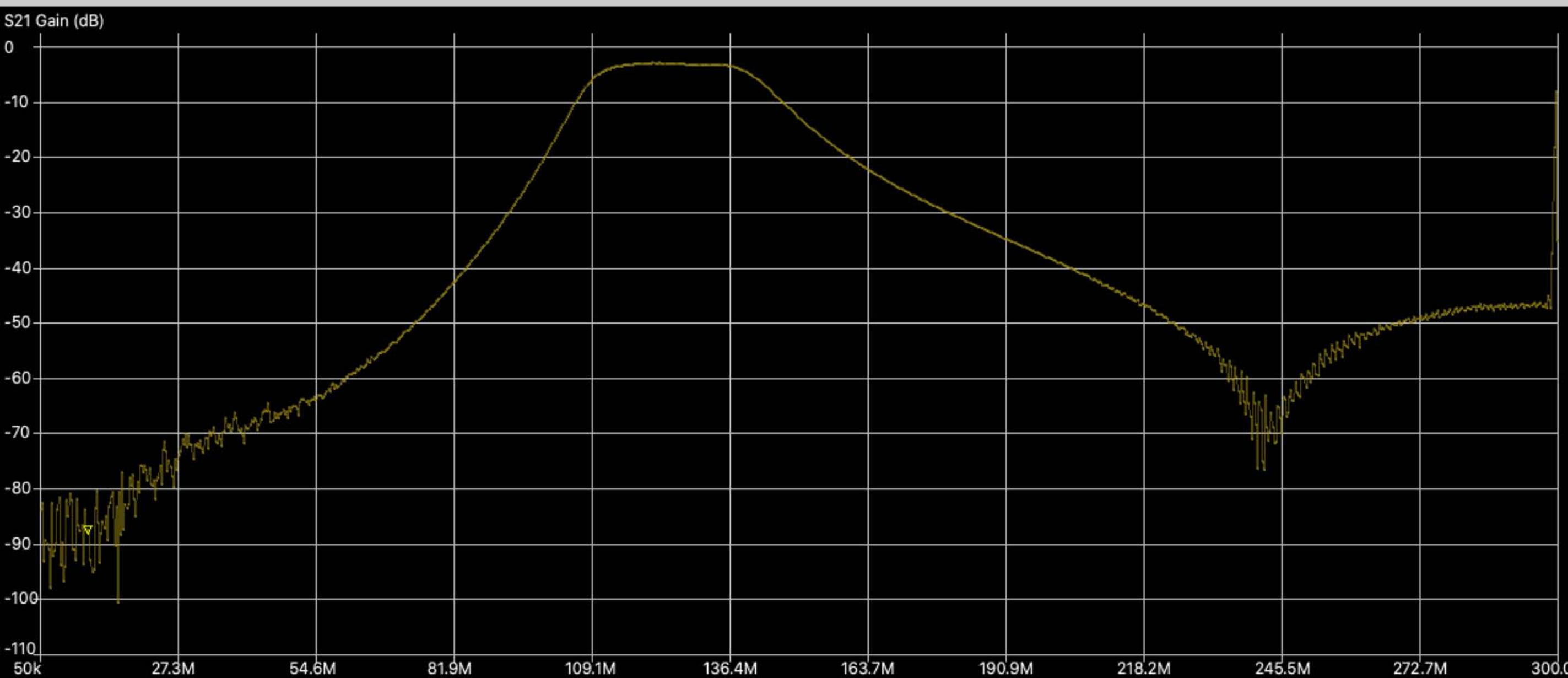
Announcements

- Last time:
 - Bode plots
 - Resonance systems and Q

Airband Bandpass Filter



Frequency Response



Today

- Start a new module: Control
- Describe dynamic systems as a state-space model
 - Extremely powerful model
- Show some concrete examples of how to construct state space models

SELF-DRIVING CARS



Google car
2016

The idea has existed for a long time. Recent progress was enabled by sensors, radar, machine learning, and mapping.

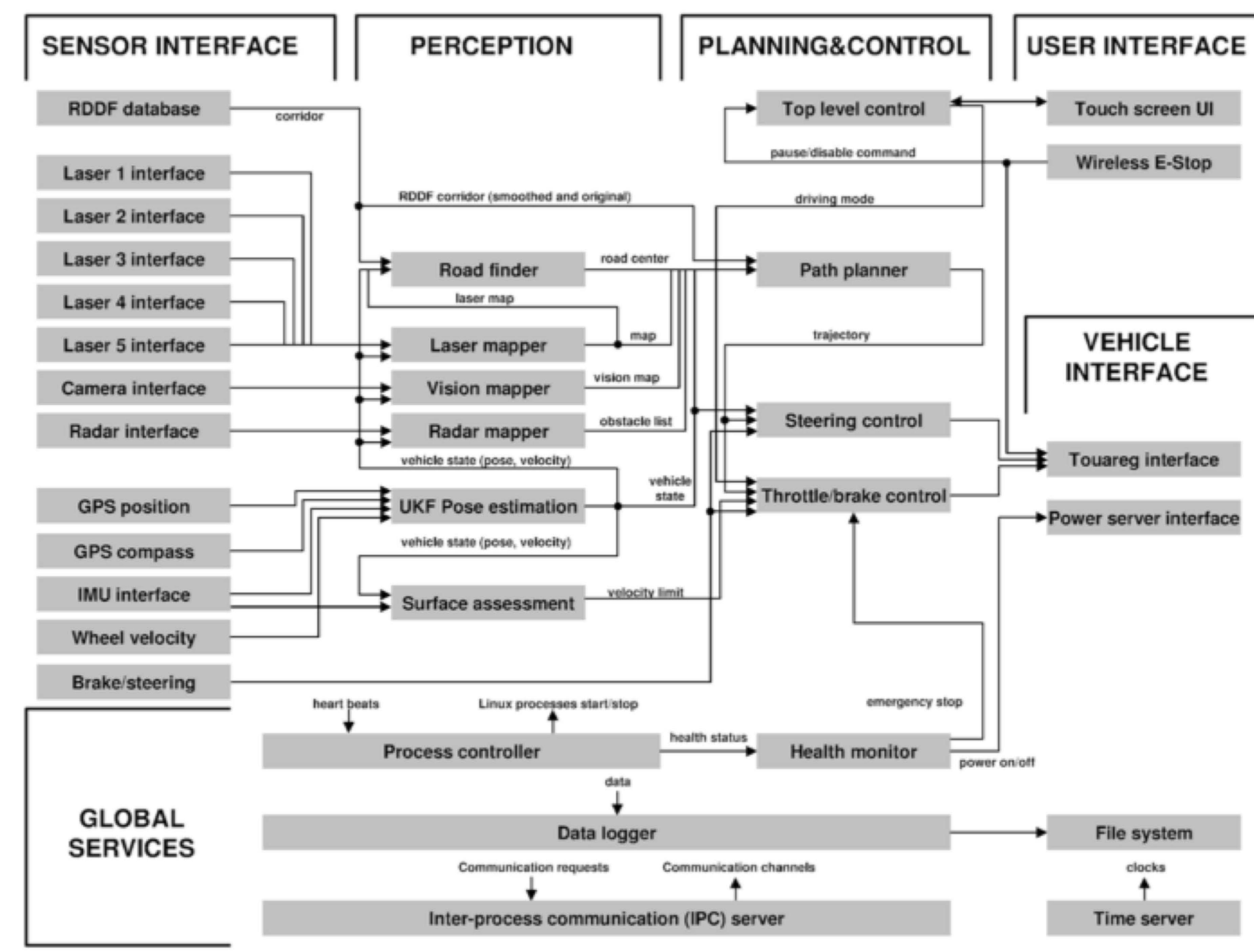


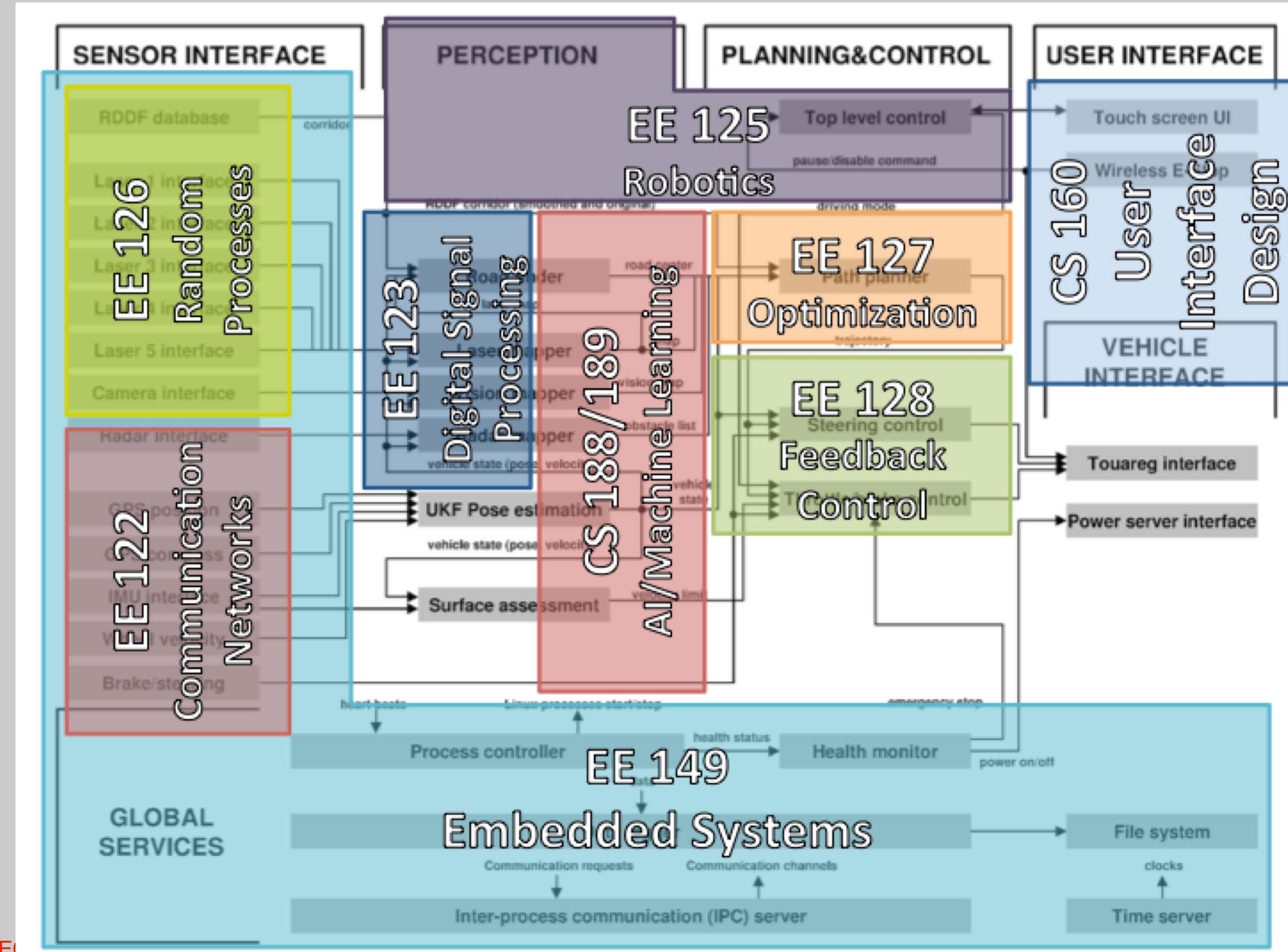
ca. 1957

Stanley: winner of the DARPA Grand Challenge



Credit: Thrun, Journal of Field Robotics,
2006. DOI: 10.1002/rob.20147

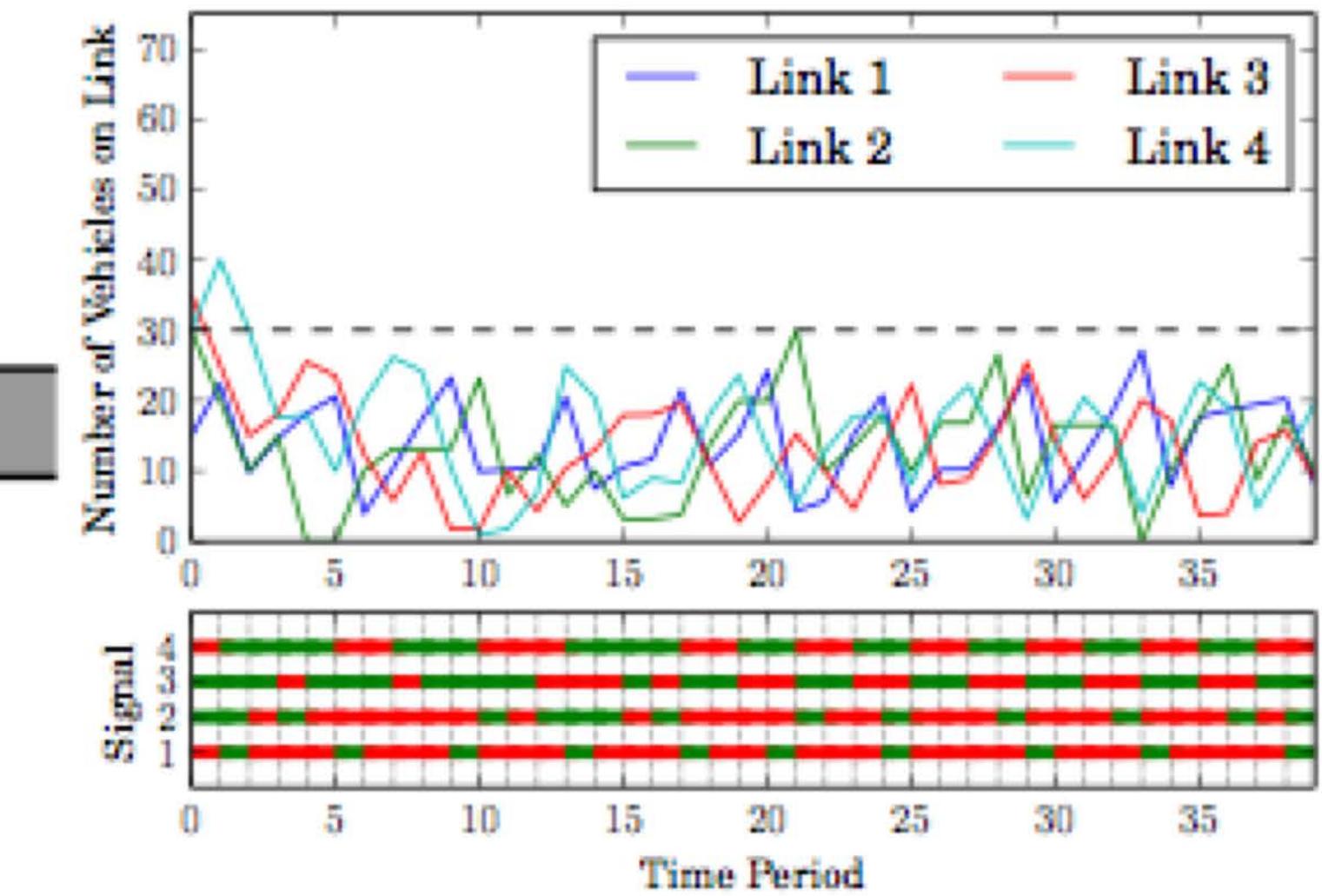
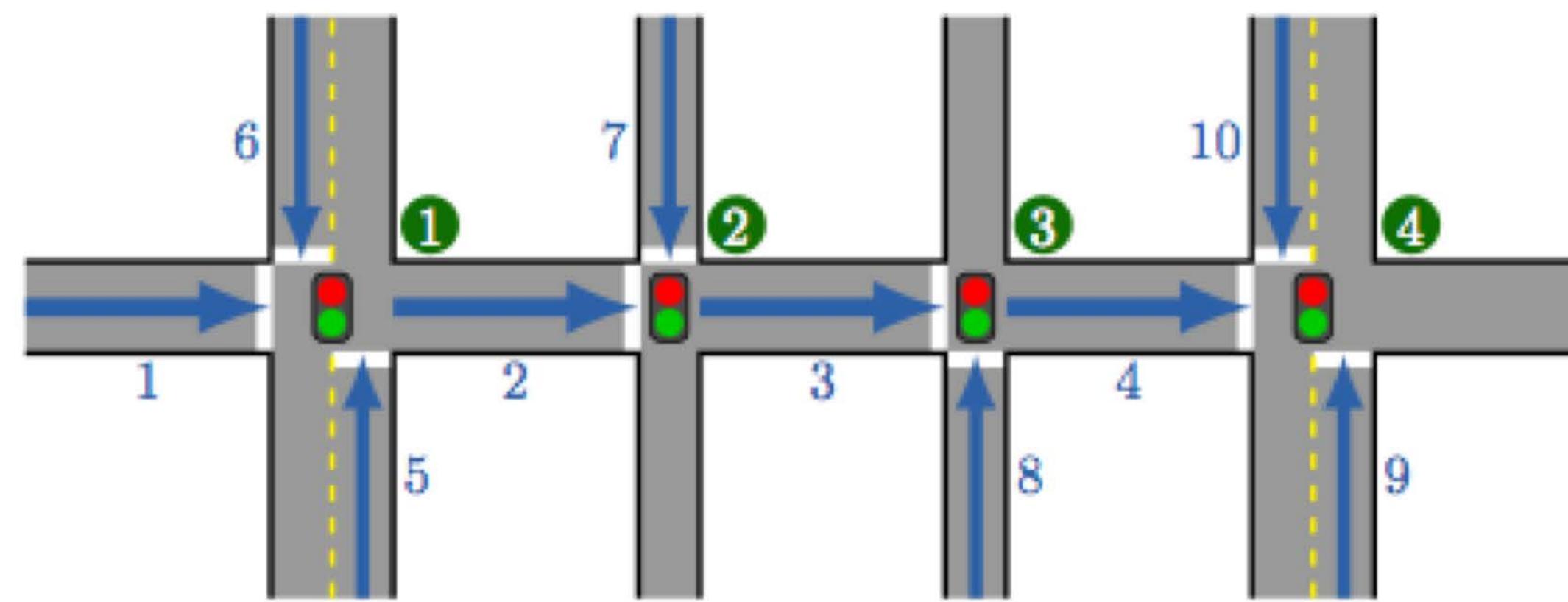




TRAFFIC CONTROL

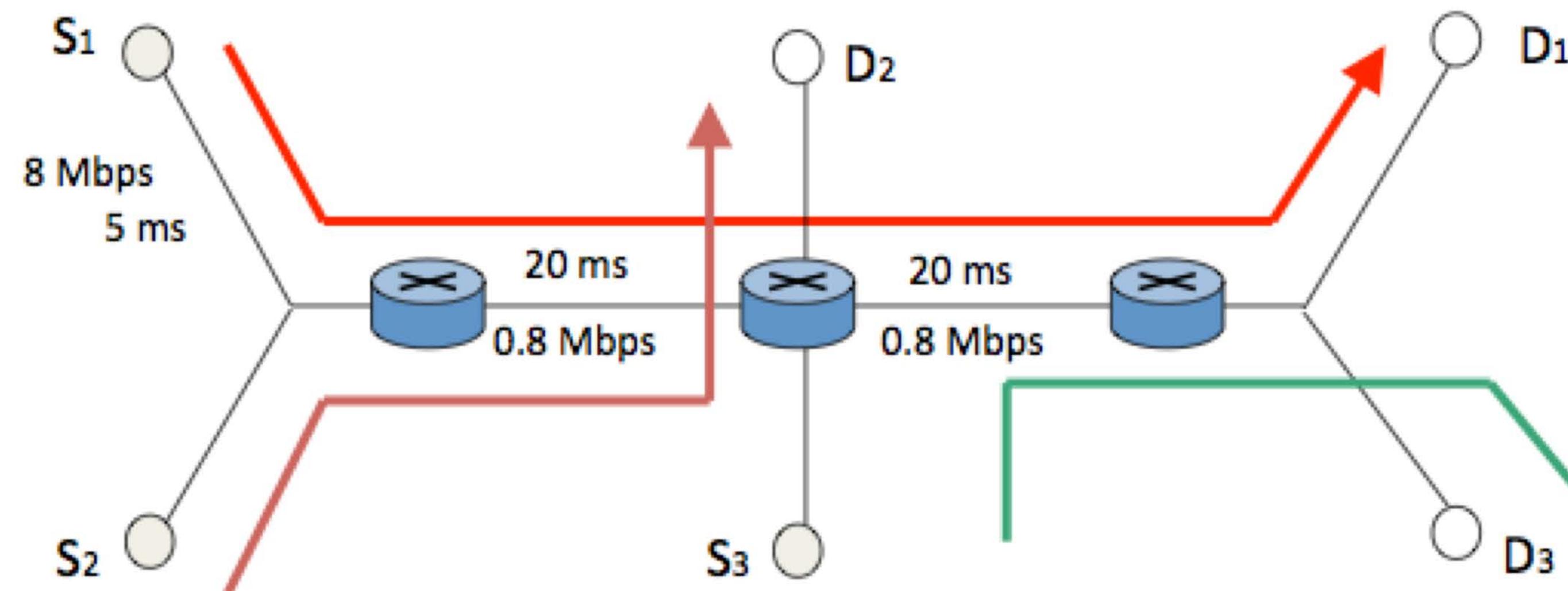
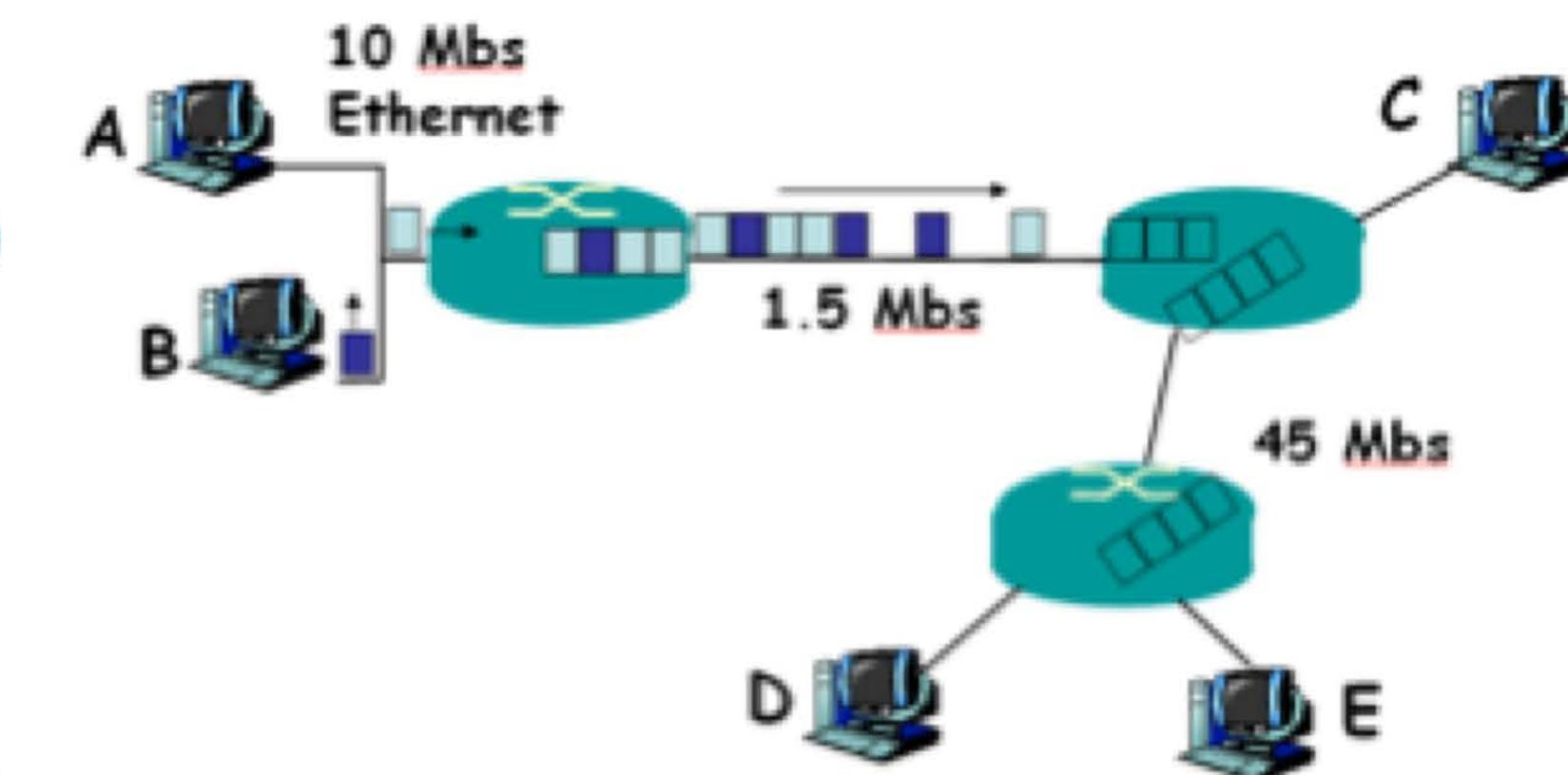
Goal: increase throughput, reduce travel time, vehicle miles traveled, emissions.

Control mechanisms: signal timing at intersections and onramps, speed advisory, lane management, etc.



INTERNET CONGESTION CONTROL

TCP and variants increase sending rate when there is no congestion, and decrease when there is congestion, inferred from “ack” messages.

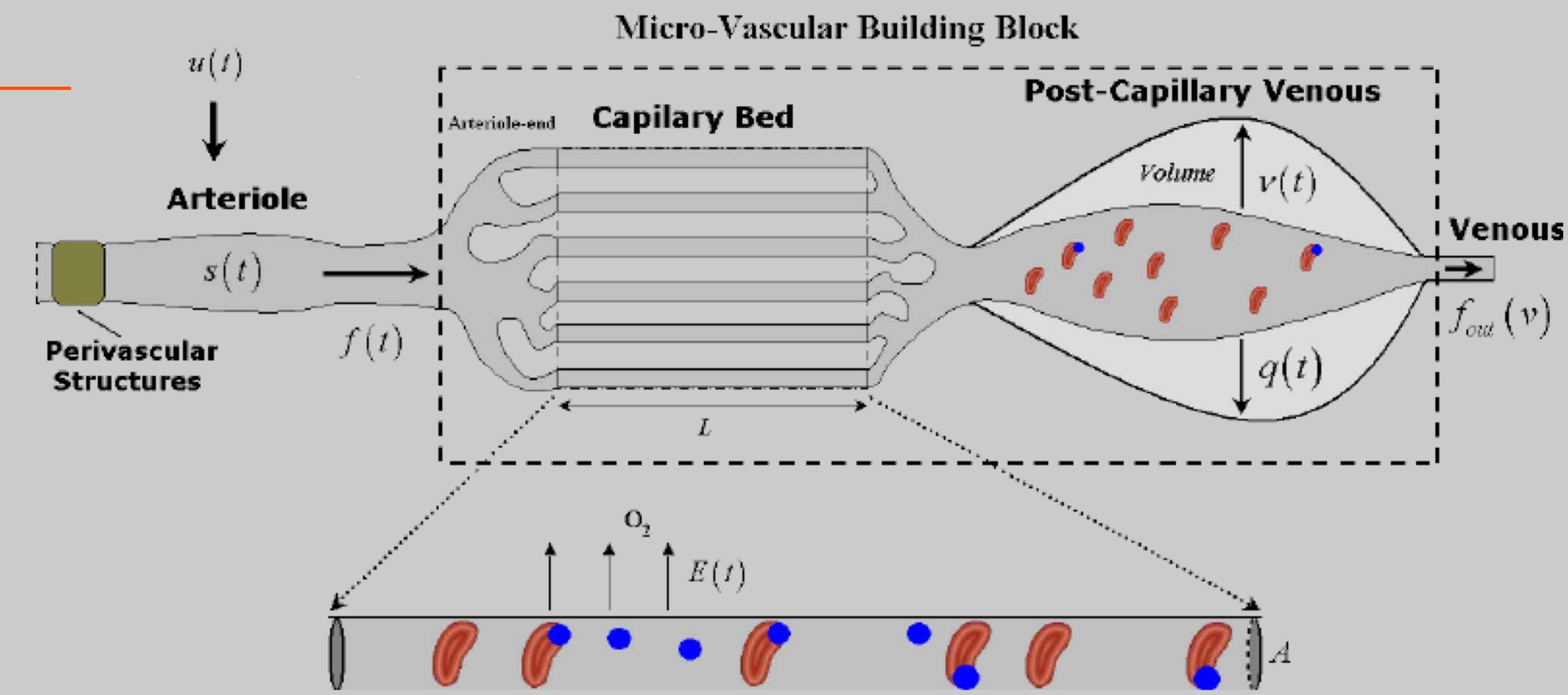
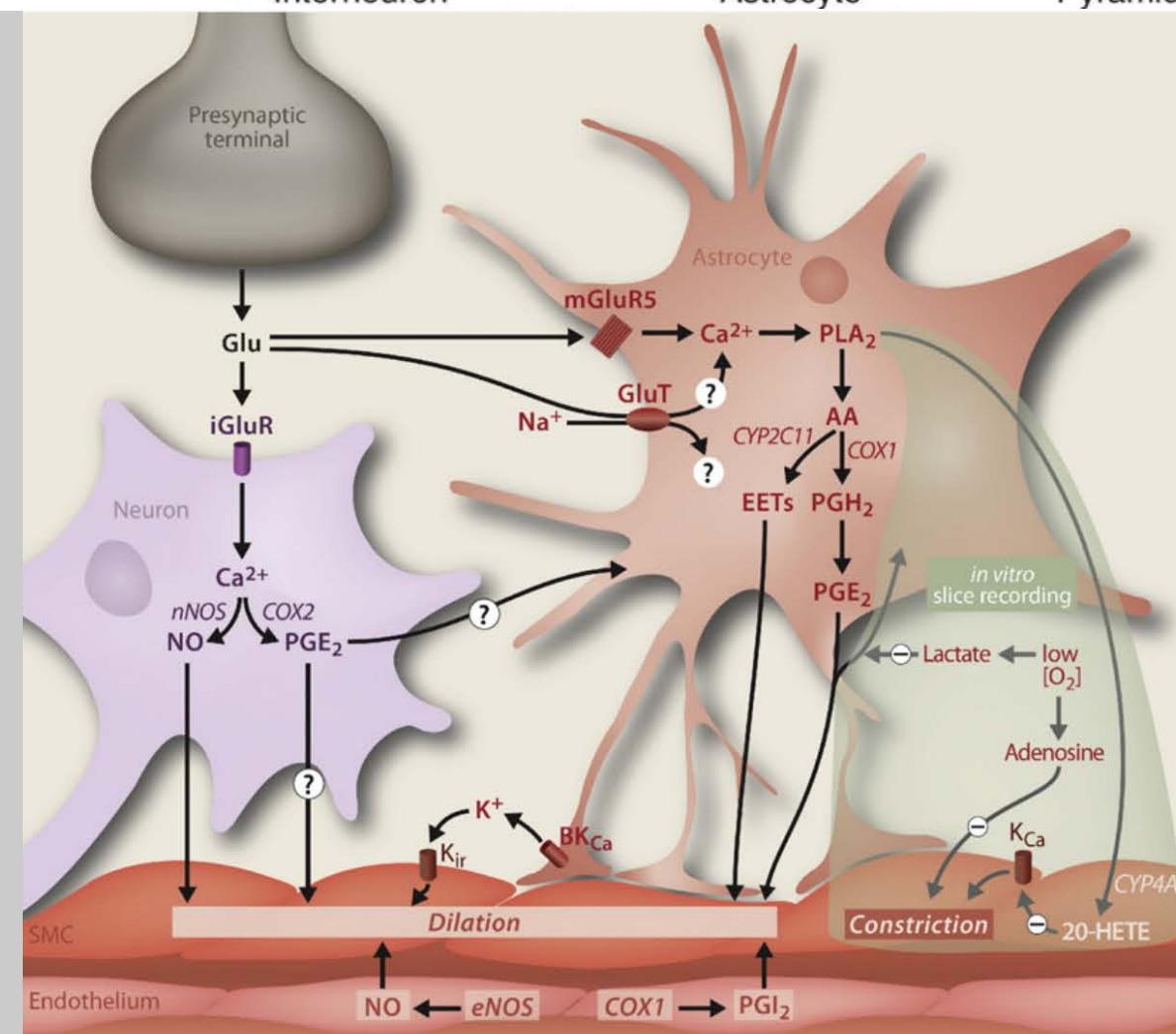
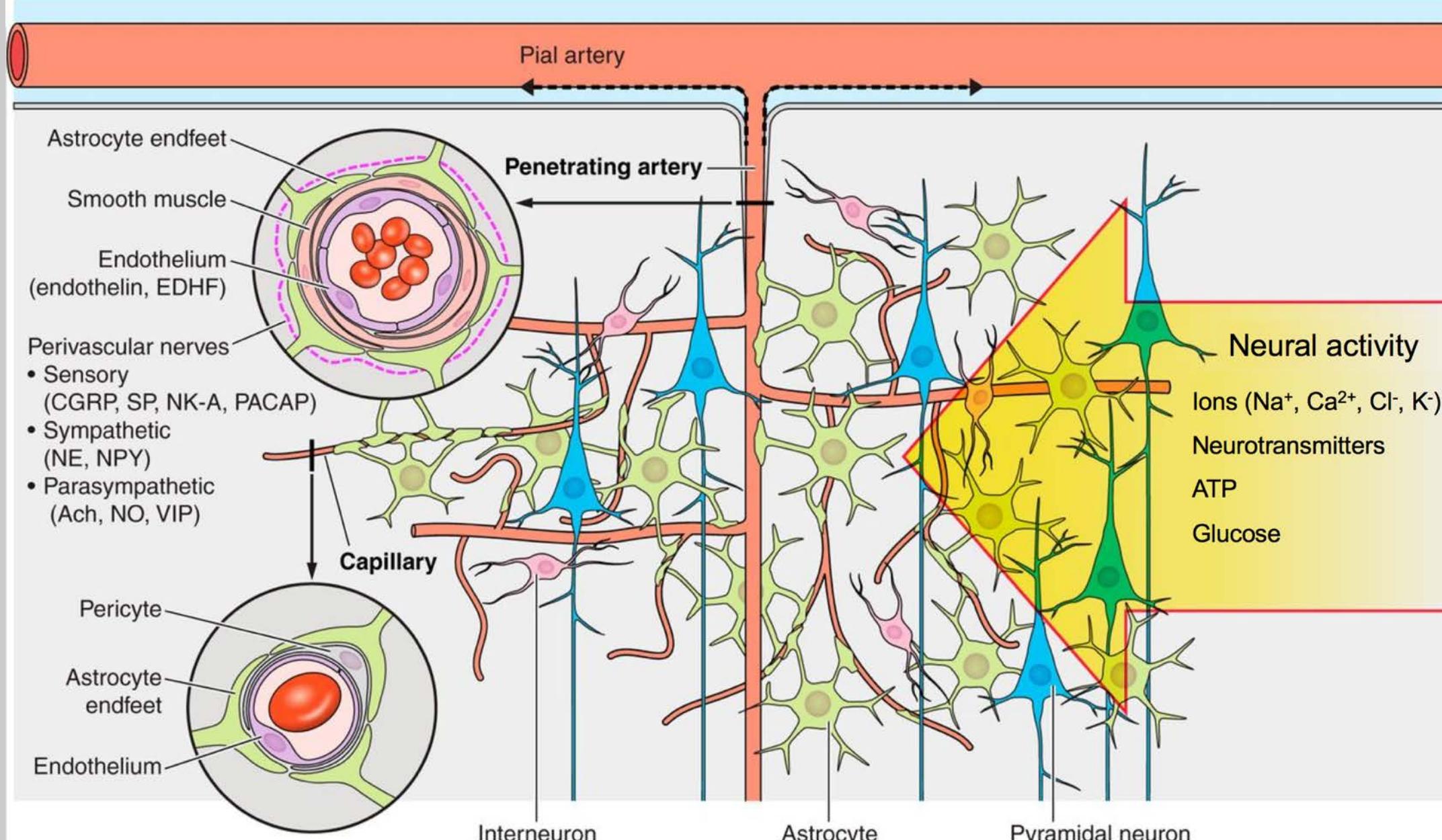


Brain Machine Interface



<http://news.sky.com/story/woman-uses-her-mind-to-control-robotic-arm-10460512>

Blood Oxygenation in the Brain



$Q(t)$: Total deoxyhemoglobin

$V(t)$: Venous volume

$F_{in}(t)$: Volume flow rate into tissue (ml/s)

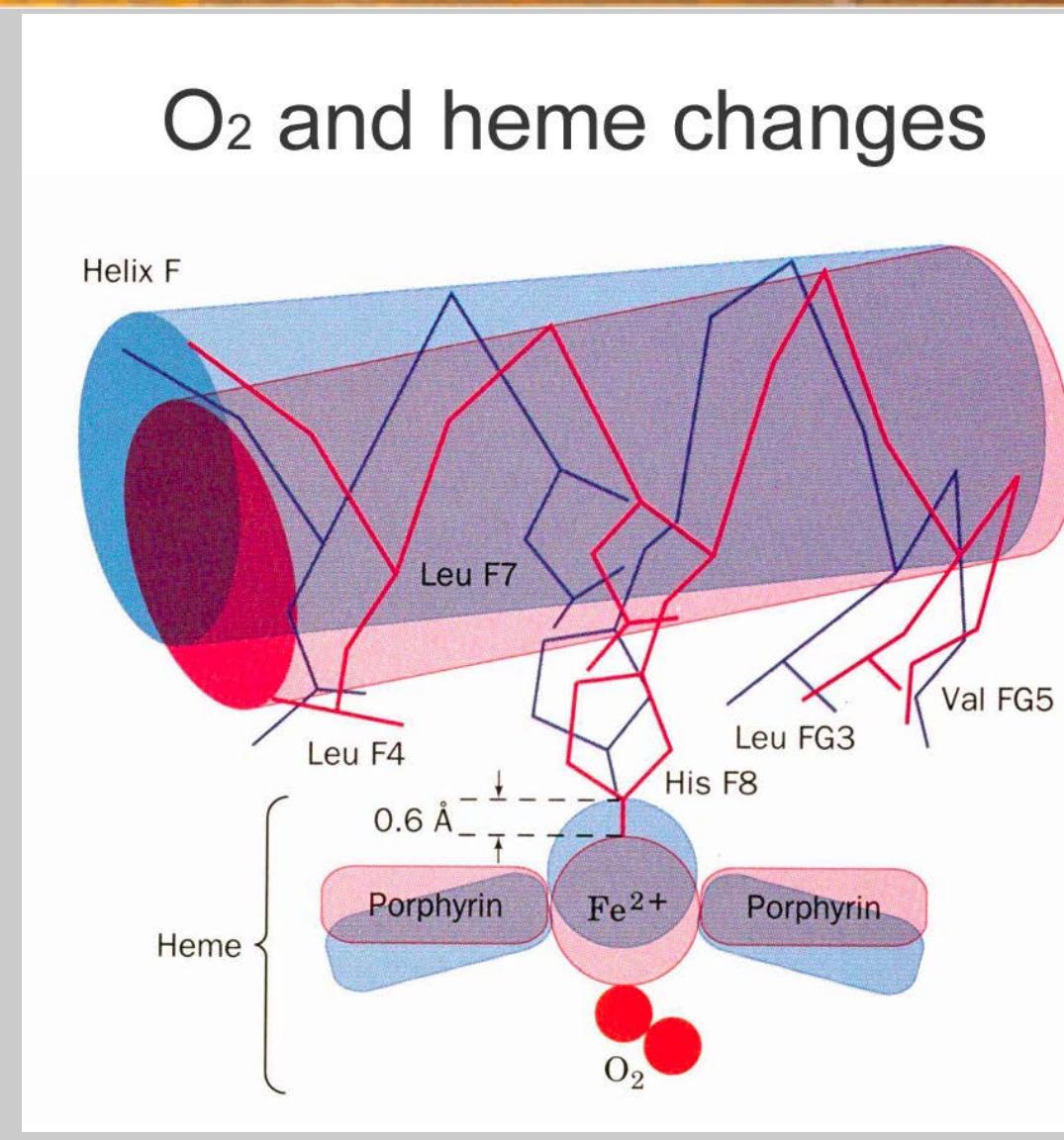
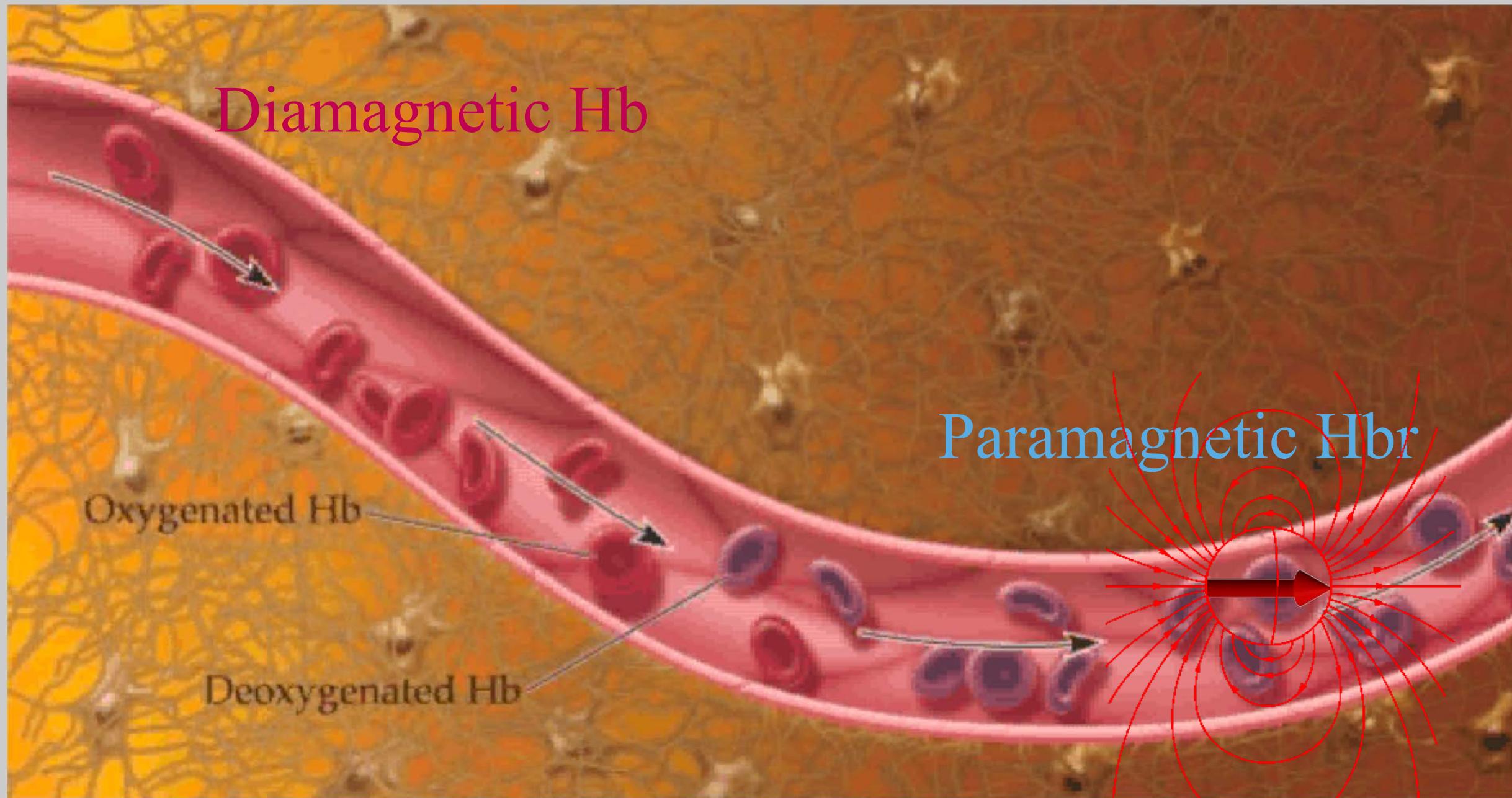
$F_{out}(t)$: Volume flow rate out of tissue (ml/s)

C_a : Arterial O₂ concentration

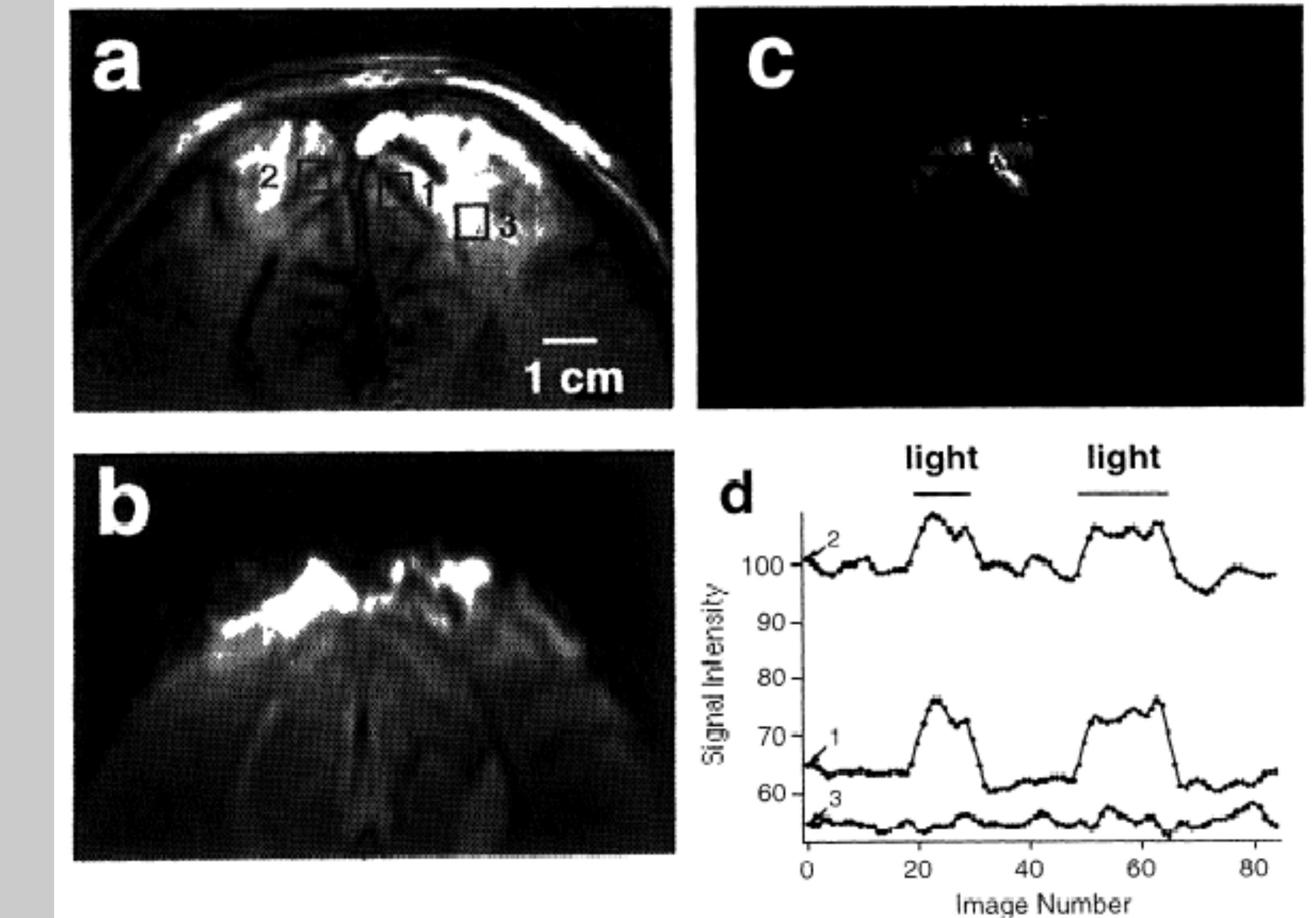
E : Net extract of O₂ from blood

Buxton, Richard B., Eric C. Wong, and Lawrence R. Frank. "Dynamics of blood flow and oxygenation changes during brain activation: the balloon model." Magnetic resonance in medicine 39.6 (1998): 855-864.

Blood-Oxygen-Level-Dependent Signal

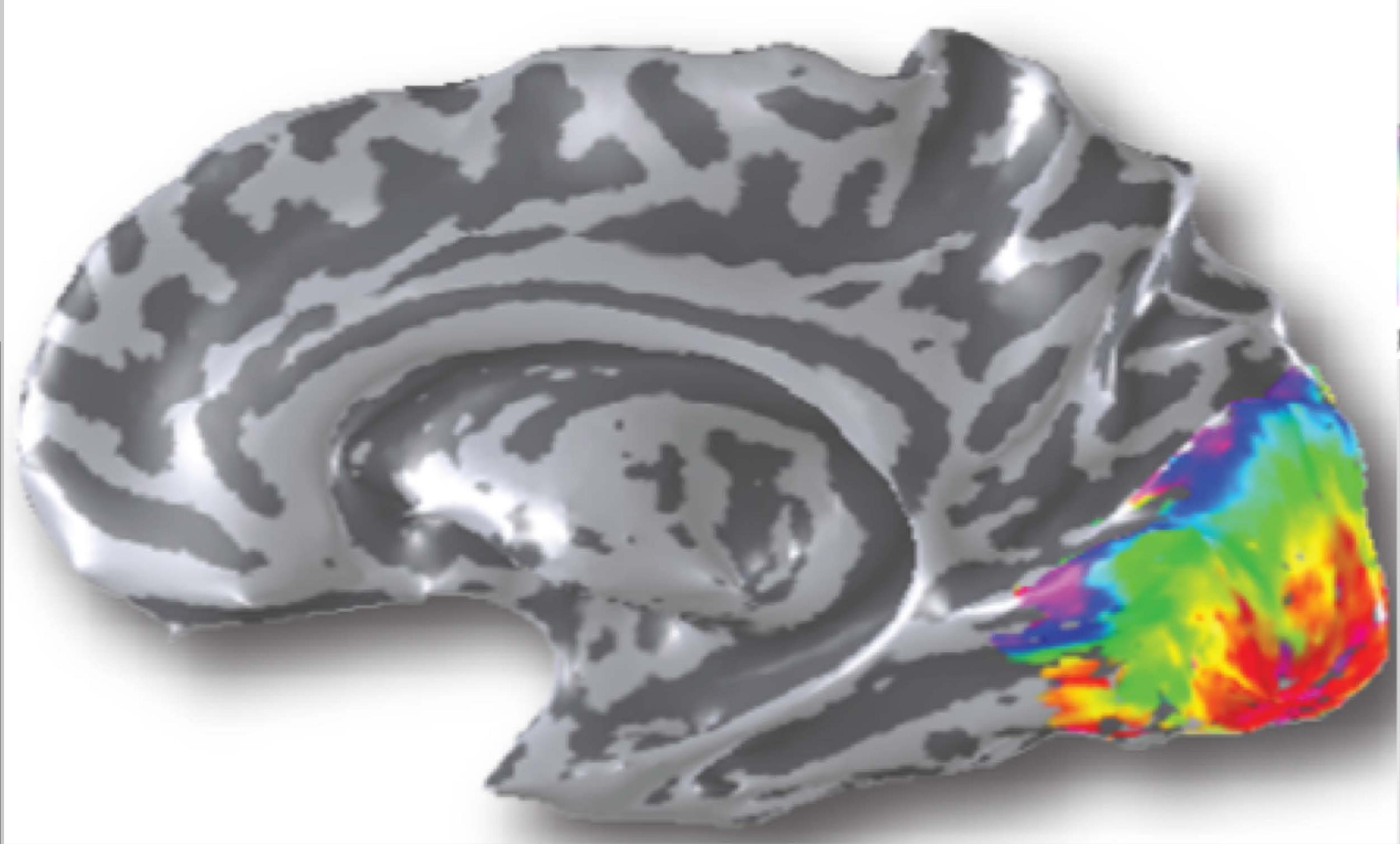
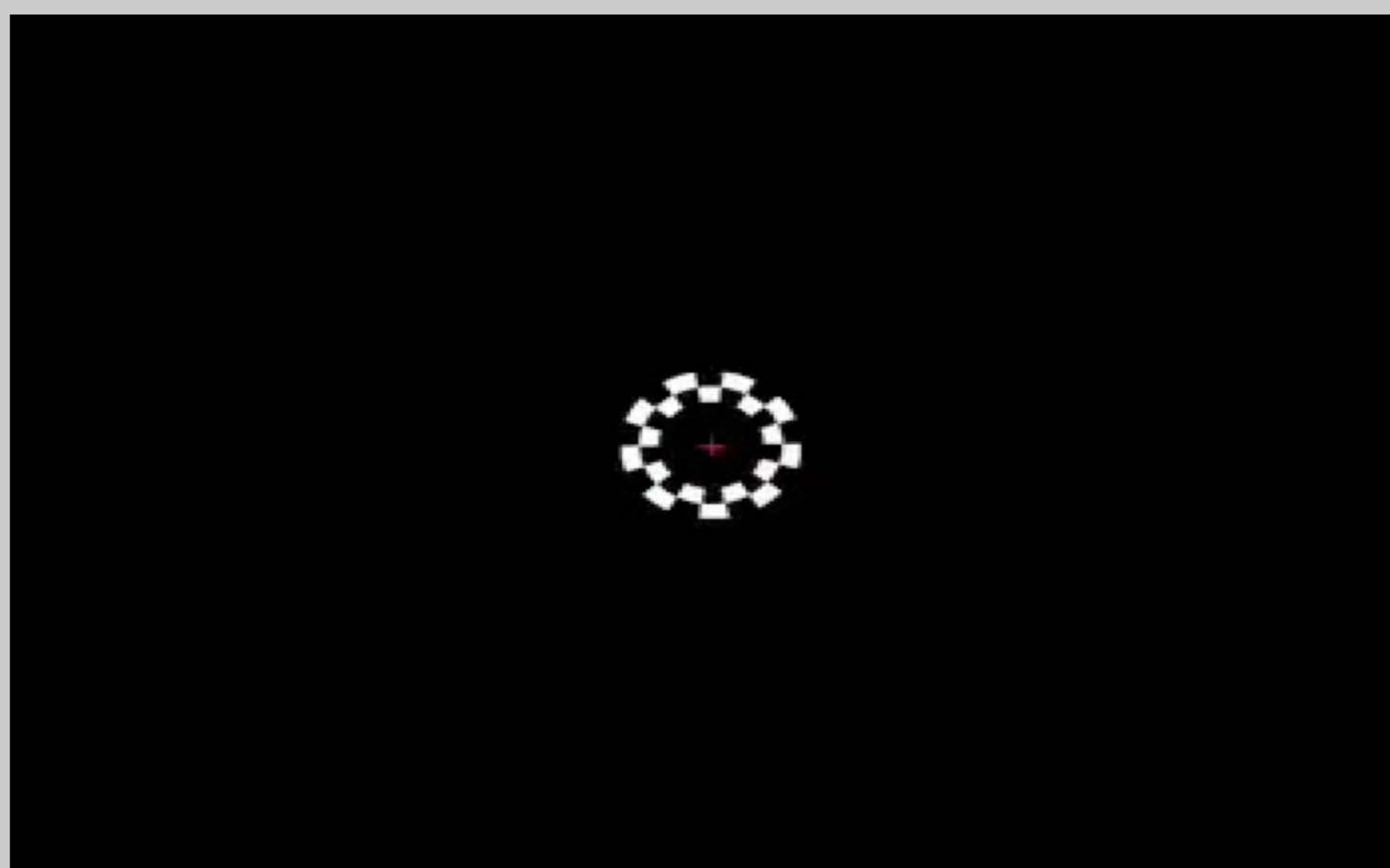
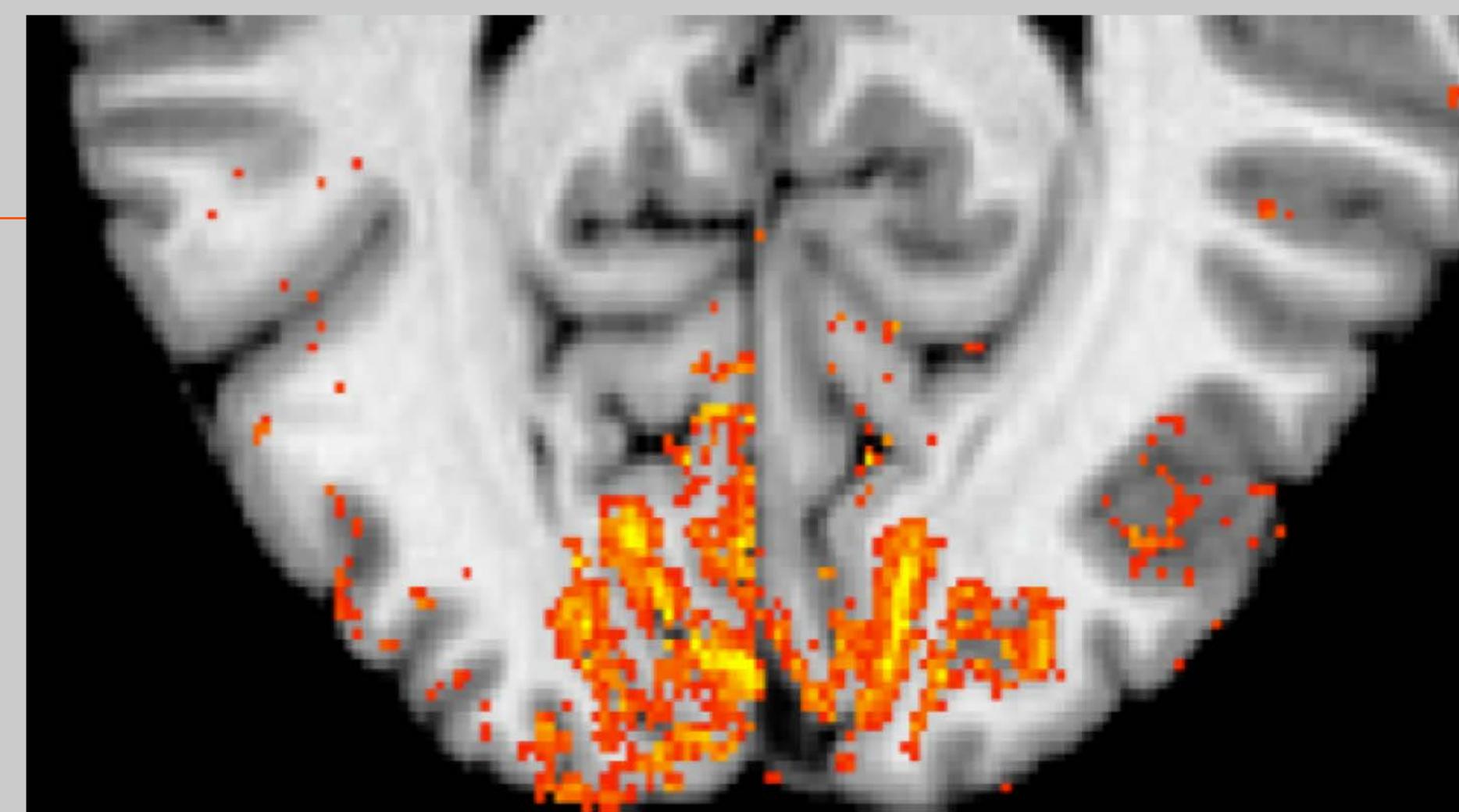


Brain tissue
(diamagnetic)
Oxyhemoglobin
(diamagnetic)
Deoxyhemoglobin
(paramagnetic)



Ogawa S. et al, 1992

Brain Mapping with MRI



Control Design Steps

1. Describe physics with a differential or
difference equations

⇒ Continuous-time

⇒ Discrete-time

2. Design control algorithms that manipulate
these equations for desired behavior

Often where the “art” is

Resources

- Lecture Slides – will be posted before class
- Murat Arcak Reader (linked in resources on class website)

State Space Models

n first order (but typically coupled)
differential equations instead of a single nth order

$x_1, x_2, \dots, x_n :$

“state variables”

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

“state vector”

State eqn:

$$\frac{d}{dt} \vec{x}(t) = f(\vec{x}(t))$$

$$\frac{d}{dt} \vec{x}(t) = f(\vec{x}(t), \vec{u}(t))$$

control variables

State Space Models

Free running

$$\frac{d}{dt} \vec{x}(t) = f(\vec{x}(t))$$

Controlled input

$$\frac{d}{dt} \vec{x}(t) = f(\vec{x}(t), \vec{u}(t))$$

control variables

with disturbance

$$\frac{d}{dt} \vec{x}(t) = f(\vec{x}(t), \vec{u}(t), \vec{w}(t))$$

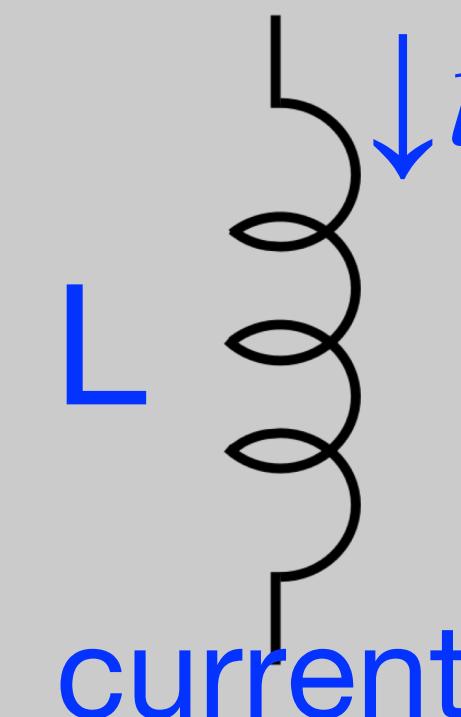
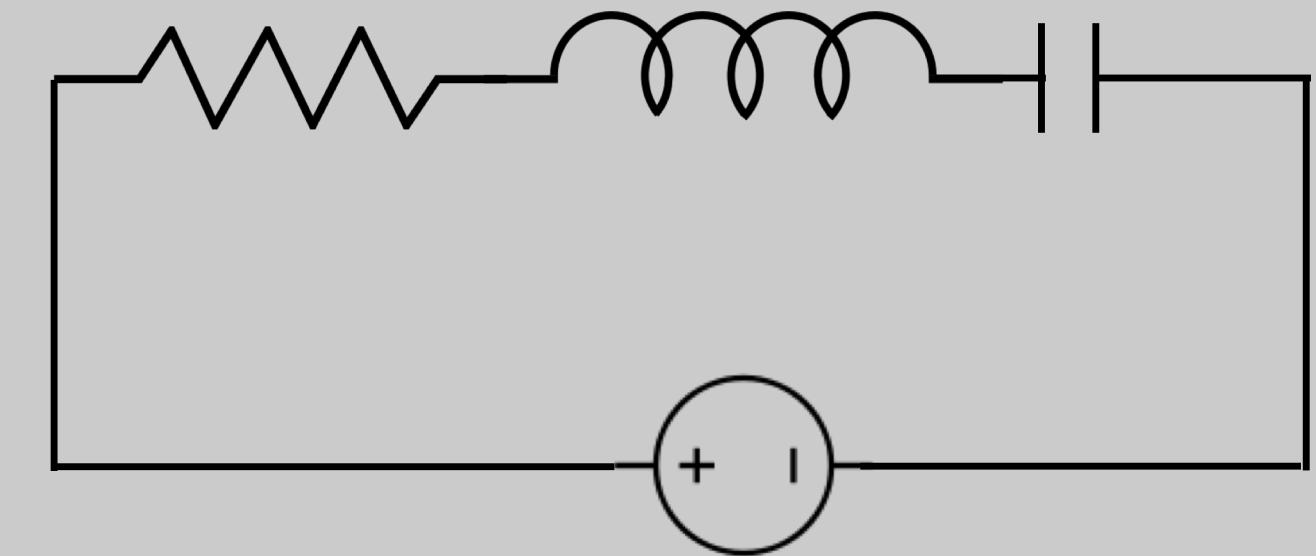
disturbance

Example 1 RLC Circuits

General state-space formulation:

$$\frac{d}{dt} \vec{x}(t) = f(\vec{x}(t), \vec{u}(t))$$

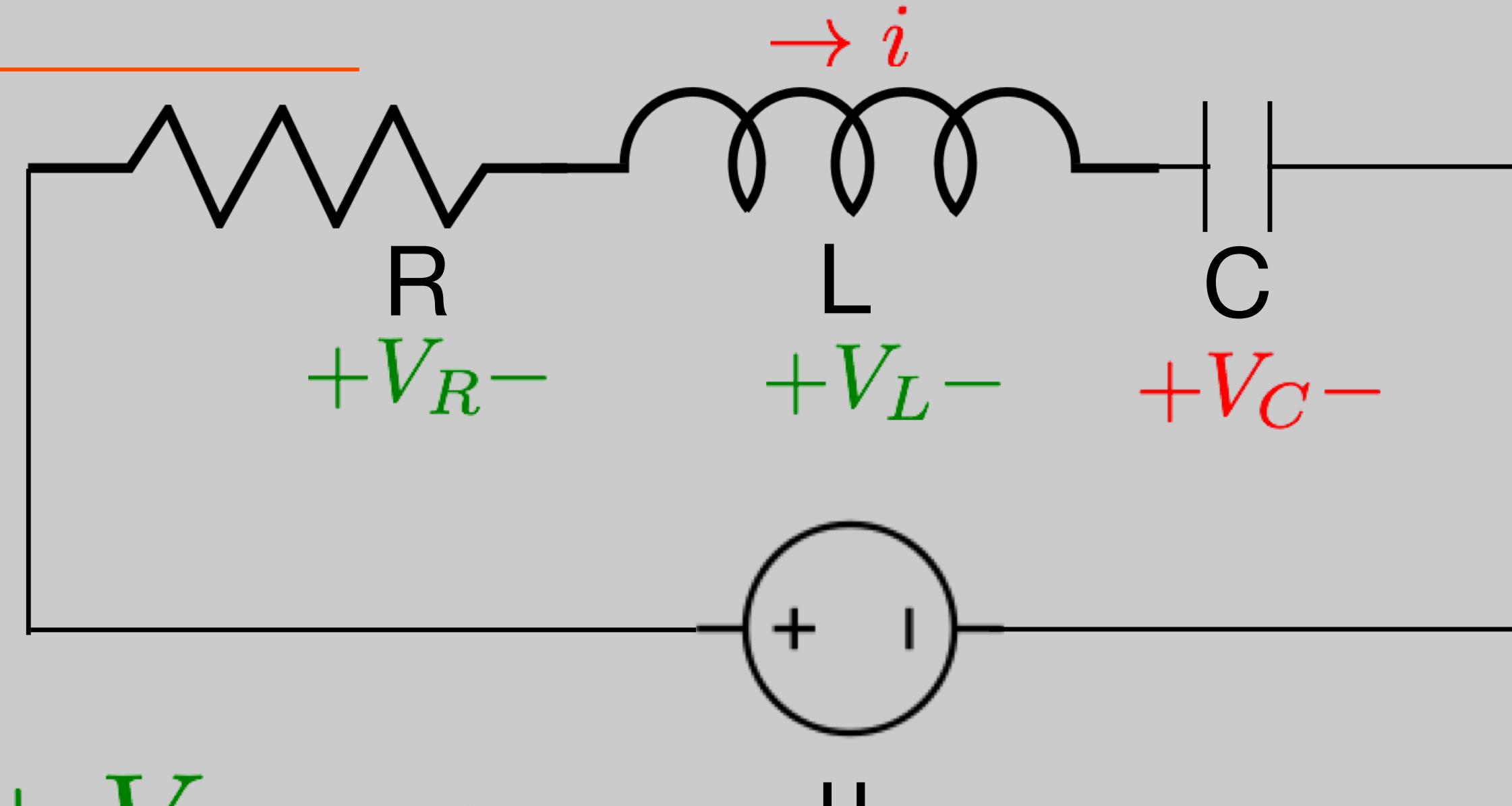
Which state variables to choose?



$$L \frac{di}{dt} = \text{voltage}$$

$$C \frac{dv}{dt} = \text{current}$$

Example 1: RLC Circuits



$$x_1 = V_c$$
$$x_2 = i$$

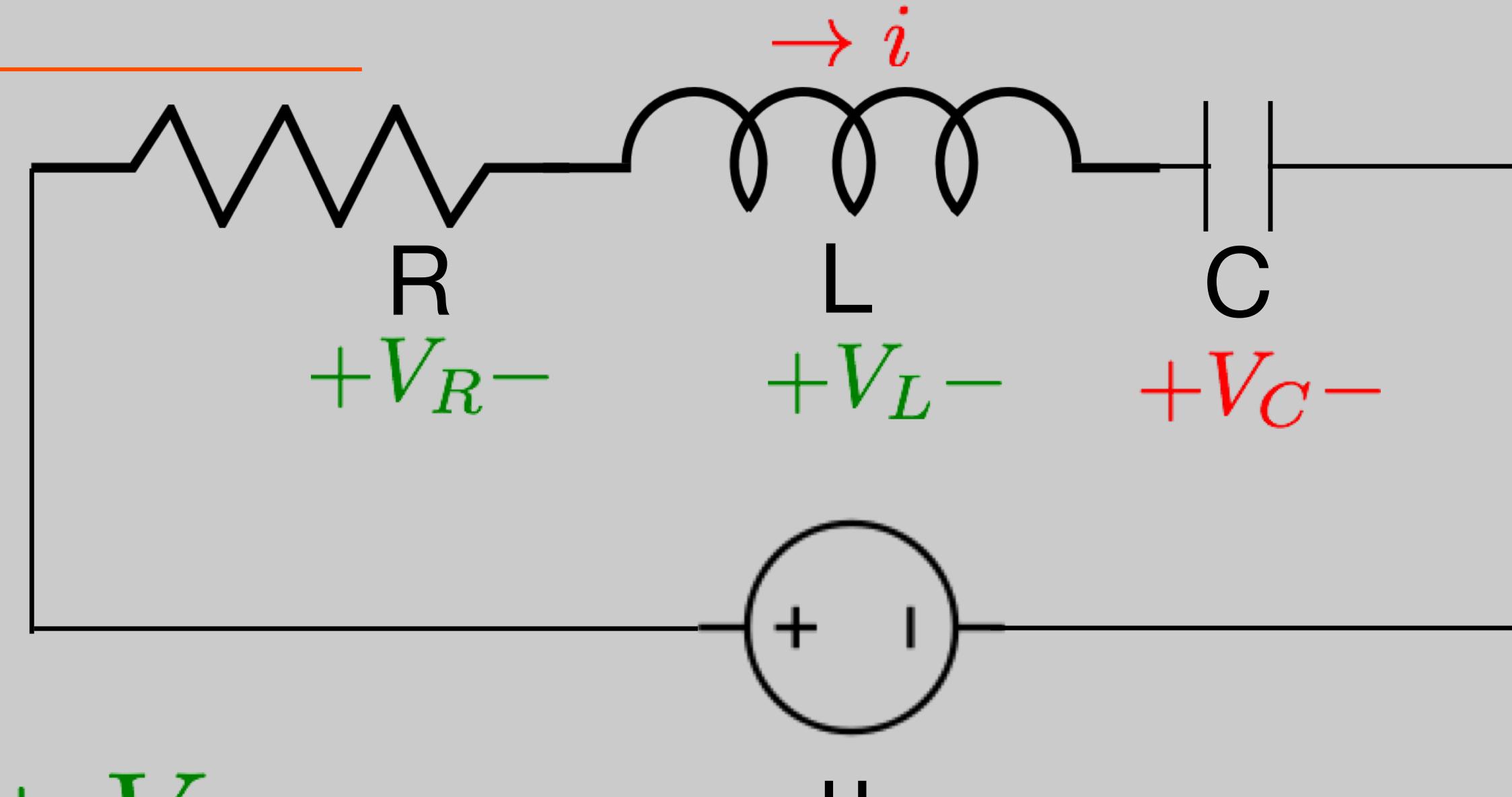
From KVL: $V_R + V_L + V_C = u$

$$L \frac{di(t)}{dt} = V_L = u - V_c - V_R = u - V_c - Ri$$

$$C \frac{dV_C(t)}{dt} = i$$

$$\boxed{\frac{dx_1(t)}{dt} =}$$
$$\boxed{\frac{dx_2(t)}{dt} =}$$

Example 1: RLC Circuits



$$x_1 = V_c$$
$$x_2 = i$$

From KVL: $V_R + V_L + V_C = u$

$$L \frac{di(t)}{dt} = V_L = u - V_c - V_R = u - V_c - Ri$$

$$C \frac{dV_C(t)}{dt} = i$$

$$\boxed{\begin{aligned}\frac{dx_1(t)}{dt} &= \frac{1}{C}x_2(t) \\ \frac{dx_2(t)}{dt} &= \frac{1}{L}u(t) - \frac{1}{L}x_1(t) - \frac{R}{L}x_2(t)\end{aligned}}$$

Reminder:

2 coupled 1st order diff. eq.

$$\frac{d}{dt} \vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \vec{x}(t)$$

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -a_0 x_1(t) - a_1 x_2(t)$$

2nd order diff. eq.

$$\Rightarrow \ddot{x}_1(t) = -a_0 x_1(t) - a_1 \dot{x}_1(t)$$

Reminder:

$$\frac{d^2y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = 0 \quad \text{2nd order diff. eq.}$$

$$\vec{x}(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \Rightarrow \frac{d}{dt} \vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \vec{x}(t)$$

2 coupled 1st order diff. eq.

Example 2: Pendulum

$$x_1(t) = \theta(t)$$

$$x_2(t) = \dot{\theta}(t)$$

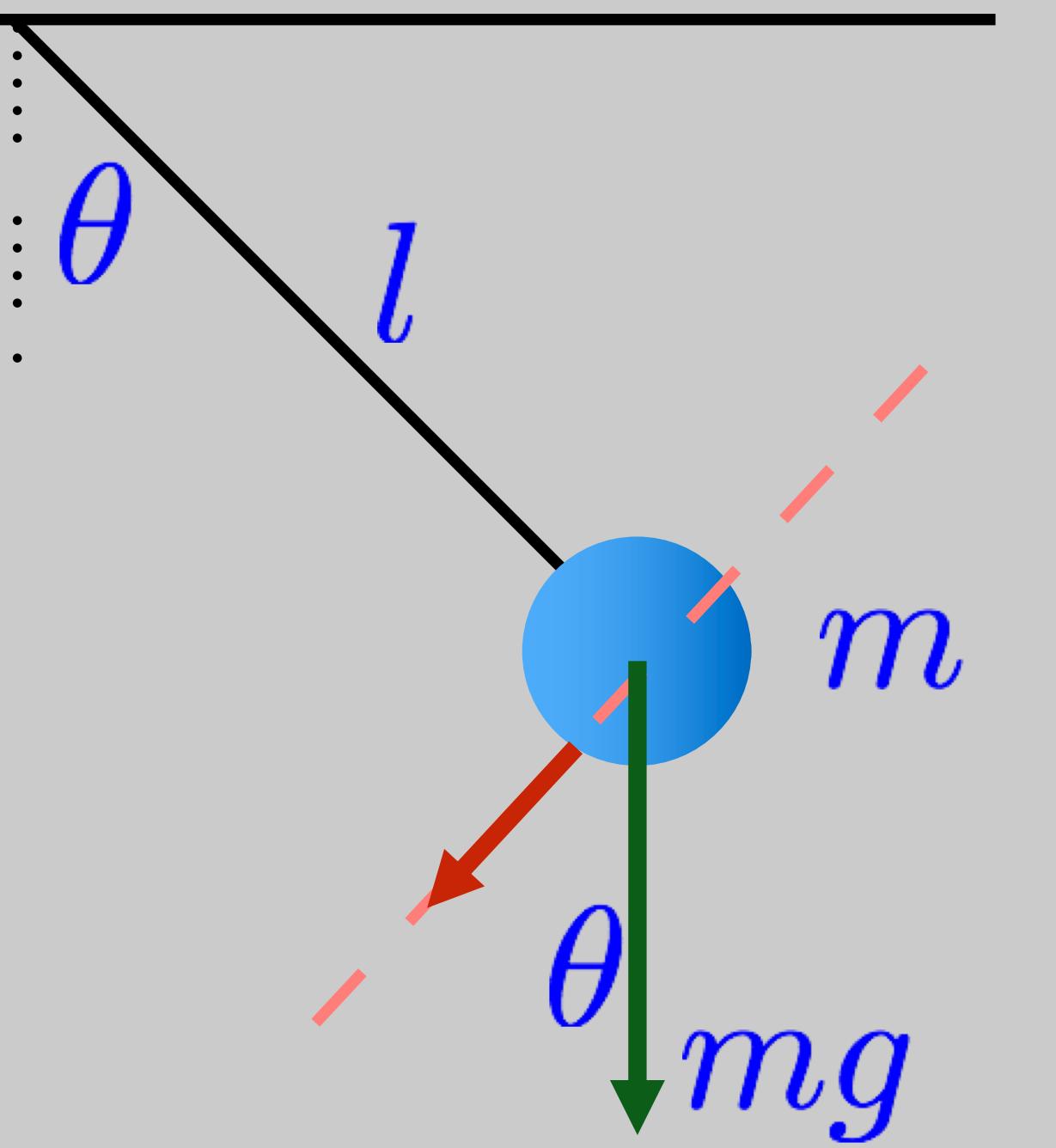
$$ma = F$$

$$\frac{dx_1(t)}{dt} = x_2(t)$$

$$\frac{dx_2(t)}{dt} = \ddot{\theta}(t) = -\frac{g}{l} \sin(\theta(t)) - \frac{k}{m} \dot{\theta}(t)$$

$$m \left(l \frac{d^2\theta}{dt^2} \right) = -mg \sin(\theta) - k(l\dot{\theta}(t))$$

\Rightarrow tangent velocity



Example 2 Cont.

Two 1st order diff. Eq. instead of 1 2nd order

$$\frac{dx_1(t)}{dt} = x_2(t)$$

$$\frac{dx_2(t)}{dt} = -\frac{g}{l} \sin(x_1(t)) - \frac{k}{m}x_2(t)$$

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$f(\vec{x}(t))$$

$$\Rightarrow \frac{d\vec{x}(t)}{dt} = f(\vec{x}(t)) = \begin{bmatrix} x_2(t) \\ -\frac{g}{l} \sin(x_1(t)) - \frac{k}{m}x_2(t) \end{bmatrix}$$

Discrete Time Systems

In discrete-time systems, $\vec{x}(t)$,evolves according to a *difference* equation

$$\vec{x}(t + 1) = f(\vec{x}(t), \vec{u}(t), \vec{w}(t)) \quad t = 0, 1, 2, \dots$$

Example 3: Manufacturing

State { $s(t)$: inventory at the start of day t
 $g(t)$: goods manufactured on day t (tomorrow inventory)
 $r(t)$: raw material available in the morning (becomes goods)

control { $u(t)$: raw materials ordered today (arrives next AM)

Disturbance { $w(t)$: amount sold

$$s(t + 1) =$$

$$g(t + 1) = r(t)$$

$$r(t + 1) = u(t)$$

Example 3: Manufacturing

State {

- $s(t)$: inventory at the start of day t
- $g(t)$: goods manufactured on day t (tomorrow inventory)
- $r(t)$: raw material available in the morning (becomes goods)

control {

- $u(t)$: raw materials ordered today (arrives next AM)

Disturbance {

- $w(t)$: amount sold

$$s(t + 1) = s(t) + g(t) - w(t)$$

$$g(t + 1) = r(t)$$

$$r(t + 1) = u(t)$$

Example 4: EECS Professors

$p(t)$: EECS professors in year t

$r(t)$: # of industry researchers

$\delta < 1$: fraction that leave the profession

$$\begin{aligned} p(t+1) &= p(t) - \delta p(t) \\ r(t+1) &= r(t) - \delta r(t) \end{aligned} \quad \text{without input will diminish to 0}$$

$u(t)$: average # of PhD/prof/year

γ : fraction of new PhD that become professors

Example 4: EECS Professors

$u(t)$: average # of PhD/prof/year

γ : fraction of new PhD that become professors

of new PhDs = $p(t)u(t)$

$$p(t+1) = p(t) - \delta p(t) + \gamma p(t)u(t)$$

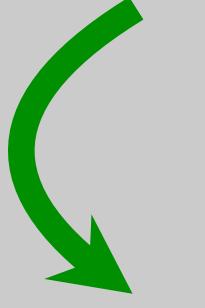
$$r(t+1) = r(t) - \delta r(t) + (1 - \gamma)p(t)u(t)$$

Linear Systems

Arguably...the most important class of systems

When the state equation is linear

$$\frac{d\vec{x}(t)}{dt} = f(\vec{x}(t), \vec{u}(t), \vec{w}(t)) = A\vec{x}(t) + B_u\vec{u}(t) + B_w\vec{w}(t)$$

 $n \times n$  $n \times 1$ for scalar u
 $n \times m$ for m inputs

$$\vec{x}(t+1) = A\vec{x}(t) + B_u\vec{u}(t) + B_w\vec{w}(t)$$

Revisiting Example 1:

Q: example 1 linear?

A: Linear

$$\frac{dx_1(t)}{dt} = \frac{1}{C}x_2(t)$$

$$\frac{dx_2(t)}{dt} = \frac{1}{L}u(t) - \frac{1}{L}x_1(t) - \frac{R}{L}x_2(t)$$

Linear relationship:

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} + \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix} u(t)$$

Revisiting Example 1:

Q: example 1 linear?

A: Linear

$$\frac{dx_1(t)}{dt} = \frac{1}{C}x_2(t)$$

$$\frac{dx_2(t)}{dt} = \frac{1}{L}u(t) - \frac{1}{L}x_1(t) - \frac{R}{L}x_2(t)$$

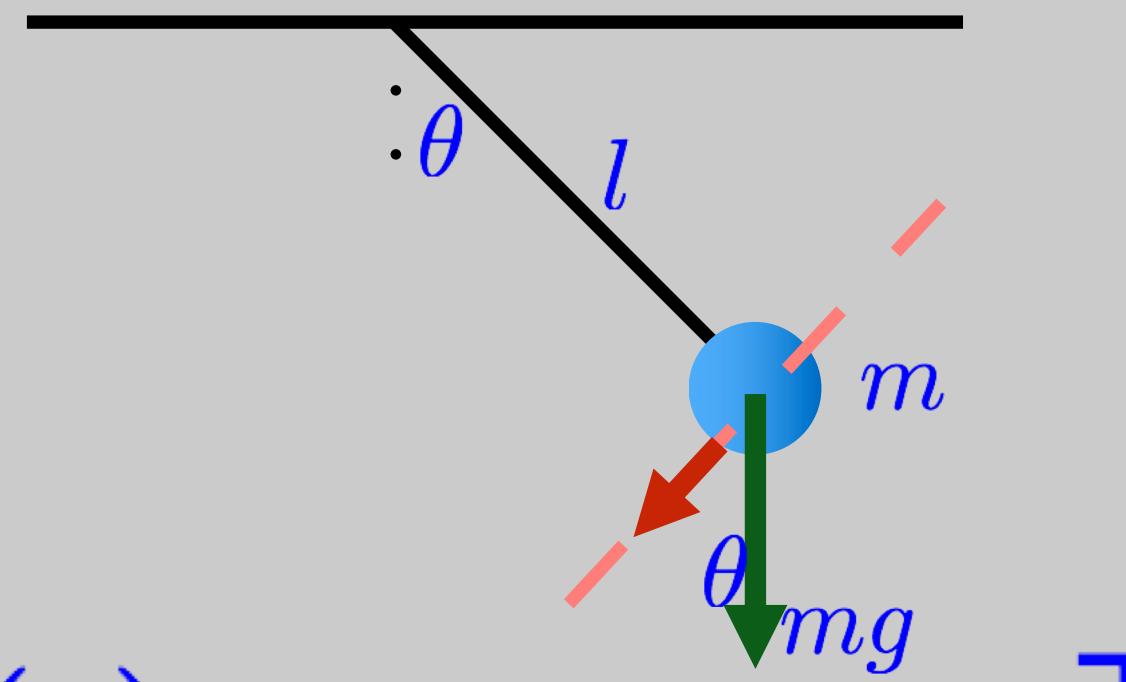
Linear relationship:

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t)$$

Revisiting Examples

Q: Is example 2 linear?

$$\frac{d\vec{x}(t)}{dt} = f(\vec{x}(t)) = \begin{bmatrix} x_2(t) \\ -\frac{g}{l} \sin(x_1(t)) - \frac{k}{m}x_2(t) \end{bmatrix}$$



A: Non Linear

Revisiting Examples

Q: Is example 3 linear?

$$s(t+1) = s(t) + g(t) - w(t)$$

$$g(t+1) = r(t)$$

$$r(t+1) = u(t)$$

A: Linear

$$\begin{bmatrix} \vec{x}(t+1) \\ \vec{x}(t) \end{bmatrix} = \begin{bmatrix} A & B_u \\ C & D \end{bmatrix} \begin{bmatrix} \vec{x}(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} \vec{w}(t) \\ 0 \end{bmatrix}$$

Revisiting Examples

Q: Is example 3 linear?

$$s(t+1) = s(t) + g(t) - w(t)$$

$$g(t+1) = r(t)$$

$$r(t+1) = u(t)$$

A: Linear

$$\begin{bmatrix} s(t+1) \\ g(t+1) \\ r(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s(t) \\ g(t) \\ r(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} w(t)$$

$\overrightarrow{x}(t+1)$ A $\overrightarrow{x}(t)$ B_u W_u

Revisiting Examples

Q: Is example 4 linear?

$$p(t+1) = p(t) - \delta p(t) + \gamma p(t)u(t)$$

$$r(t+1) = r(t) - \delta r(t) + (1 - \gamma)p(t)u(t)$$

A: non Linear

Changing State Variable

State variables are not unique!

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t)$$

Let T be an invertible matrix:

$$\vec{z} = T\vec{x}$$

Then,

$$\begin{aligned}\vec{z}(t+1) &= T\vec{x}(t+1) = TA\vec{x}(t) + TB\vec{u}(t) \\ &= TAT^{-1}\vec{z}(t) + TB\vec{u}(t)\end{aligned}$$

Changing State Variables

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t)$$

Define:

$$\vec{z} = T\vec{x} \quad A_{\text{new}} = TAT^{-1} \quad B_{\text{new}} = TB$$

Can be written as,

$$\vec{z}(t+1) = A_{\text{new}}\vec{z}(t) + B_{\text{new}}\vec{u}(t)$$

Similarly for continuous systems!

Next: We will see how a special choice of T will make it easy to analyze system properties like *stability*, and *controllability*

Summary

- Learned to describe systems in a state-space model
 - Extremely powerful model!!!
- State space model leads to coupled
 - 1st order (coupled) differential equations (Cont. time)
 - 1st order (coupled) difference equations (Disc. time)
- Talked about linear systems
 - Described state evolution in matrix form
- Showed how to change state variables
- Next: Linearization of non-linear systems & Stability