# Q1. We Are Getting Close...

Mesut is trying to remotely control a car, which has gone out of his view. The unknown state of the car is represented by the random variable X. While Mesut can't see the car itself, his high-tech sensors on the car provides two useful readings: an estimate (E) of the distance to the car in front, and a detection model (D) that detects if the car is headed into a wall. Using these two readings, Mesut applies the controls (C), which determine the velocity of the car by changing the acceleration. The Dynamic Bayes Net below describes the setup.
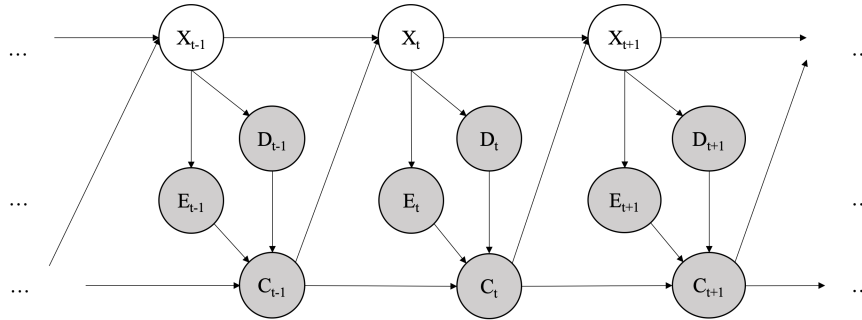
(a) For the above DBN, complete the equations for performing updates. (Hint: think about the prediction update and observation update equations in the forward algorithm for HMMs.)

Time elapse: $\underline{\quad \textbf{(i)} \quad} = \underline{\quad \textbf{(ii)} \quad}\ \underline{\quad \textbf{(iii)} \quad}\ \underline{\quad \textbf{(iv)} \quad}\ P\left(x_{t-1}|e_{0:t-1}, d_{0:t-1}, c_{0:t-1}\right)$

(i) ○ $P(x_t)$ ● $P\left(x_t|e_{0:t-1}, d_{0:t-1}, c_{0:t-1}\right)$ ○ $P\left(e_t, d_t, c_t|e_{0:t-1}, d_{0:t-1}, c_{0:t-1}\right)$

(ii) ○ $P(c_{0:t-1})$ ○ $P(x_{0:t-1}, c_{0:t-1})$ ○ $P(e_{0:t-1}, d_{0:t-1}, c_{0:t-1})$
     ○ $P(e_{0:t}, d_{0:t}, c_{0:t})$ ● 1

(iii) ● $\Sigma_{x_{t-1}}$ ○ Σ_{x_t} ○ $\max_{x_{t-1}}$ ○ $\max_{x_t}$ ○ 1

(iv) ○ $P(x_{t-1}|x_{t-2})$ ○ $P(x_{t-1}, x_{t-2})$ ○ $P(x_t|e_{0:t-1}, d_{0:t-1}, c_{0:t-1})$
     ○ $P(x_t|x_{t-1})$ ○ $P(x_t, x_{t-1})$ ○ $P(x_t, e_{0:t-1}, d_{0:t-1}, c_{0:t-1})$
     ● $P(x_t|x_{t-1}, c_{t-1})$ ○ $P(x_t, x_{t-1}, c_{t-1})$ ○ 1

Recall the prediction update of forward algorithm: $P(x_t|o_{0:t-1}) = \Sigma_{x_{t-1}} P(x_t|x_{t-1})P\left(x_{t-1}|o_{0:t-1}\right)$, where o is the observation. Here it is similar, despite that there are several observations at each time, which means o_t corresponds to e_t, d_t, c_t for each t, and that X is dependent on the C value of the previous time, so we need $P(x_t|x_{t-1}, c_{t-1})$ instead of $P(x_t|x_{t-1})$. Also note that X is independent of D_{t-1}, E_{t-1} given C_{t-1}, X_{t-1}.

Update to incorporate new evidence at time t:
$P\left(x_t|e_{0:t}, d_{0:t}, c_{0:t}\right) = \underline{\quad \textbf{(v)} \quad}\ \underline{\quad \textbf{(vi)} \quad}\ \underline{\quad \textbf{(vii)} \quad}\ \underline{\quad \textbf{Your choice for (i)} \quad}$

(v) ○ $\left(P\left(c_t|c_{0:t-1}\right)\right)^{-1}$ ○ $\left(P\left(e_t|e_{0:t-1}\right)P\left(d_t|d_{0:t-1}\right)P\left(c_t|c_{0:t-1}\right)\right)^{-1}$
     ● $\left(P\left(e_t, d_t, c_t|e_{0:t-1}, d_{0:t-1}, c_{0:t-1}\right)\right)^{-1}$ ○ $\left(P\left(e_{0:t-1}|e_t\right)P\left(d_{0:t-1}|d_t\right)P\left(c_{0:t-1}|c_t\right)\right)^{-1}$
     ○ $\left(P\left(e_{0:t-1}, d_{0:t-1}, c_{0:t-1}|e_t, d_t, c_t\right)\right)^{-1}$ ○ 1

(vi) ○ $\Sigma_{x_{t-1}}$ ○ Σ_{x_t} ○ Σ_{x_{t-1}, x_t} ○ $\max_{x_{t-1}}$ ○ $\max_{x_t}$ ● 1

(vii) ☐ $P(x_t|e_t, d_t, c_t)$ ☐ $P(x_t, e_t, d_t, c_t)$
     ☐ $P(x_t|e_t, d_t, c_t, c_{t-1})$ ☐ $P(x_t, e_t, d_t, c_t, c_{t-1})$
     ■ $P(e_t, d_t|x_t)P(c_t|e_t, d_t, c_{t-1})$ ☐ $P(e_t, d_t, c_t|x_t)$ ○ 1

Recall the observation update of forward algorithm: $P(x_t|o_{0:t}) \propto P(x_t, o_t|o_{0:t-1}) = P(o_t|x_t)P(x_t|o_{0:t-1})$.
Here the observations o_t corresponds to e_t, d_t, c_t for each t. Apply the Chain Rule, we are having

$P\left(x_t|e_{0:t}, d_{0:t}, c_{0:t}\right) \propto P\left(x_t, e_t, d_t, c_t|e_{0:t-1}, d_{0:t-1}, c_{0:t-1}\right) = P(e_t, d_t, c_t|x_t, c_{t-1})P(x_t|e_{0:t-1}, d_{0:t-1}, c_{0:t-1})$
$= P(e_t, d_t|x_t)P(c_t|e_t, d_t, c_{t-1})P(x_t|e_{0:t-1}, d_{0:t-1}, c_{0:t-1}).$

Note that in $P(e_t, d_t, c_t|x_t, c_{t-1})$, we cannot omit c_{t-1} due to the arrow between c_t and c_{t-1}.

To calculate the normalizing constant, use Bayes Rule: $P\left(x_t|e_{0:t}, d_{0:t}, c_{0:t}\right) = \frac{P(x_t, e_t, d_t, c_t|e_{0:t-1}, d_{0:t-1}, c_{0:t-1})}{P(e_t, d_t, c_t|e_{0:t-1}, d_{0:t-1}, c_{0:t-1})}.$

(viii) Suppose we want to do the above updates in one step and use normalization to reduce computation. Select all the terms that are <u>not explicitly calculated</u> in this implementation.
DO **NOT** include the choices if their values are 1.

☐ **(ii)**   ☐ **(iii)**   ☐ **(iv)**   ■ **(v)**   ☐ **(vi)**   ☐ **(vii)**   ○ None of the above

(v) is a constant, so we don't calculate it during implementation and simply do a normalization instead. Everything else is necessary.

Q2. HMMs: Help Your House Help You

Imagine you have a smart house that wants to track your location within itself so it can turn on the lights in the room you are in and make you food in your kitchen. Your house has 4 rooms (A, B, C, D) in the floorplan below (A is connected to B and D, B is connected to A and C, C is connected to B and D, and D is connected to A and C):
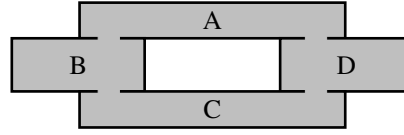
At the beginning of the day $(t = 0)$, your probabilities of being in each room are p_A, p_B, p_C, and p_D for rooms A, B, C, and D, respectively, and at each time t your position (following a Markovian process) is given by X_t. At each time, your probability of staying in the same room is q_0, your probability of moving clockwise to the next room is q_1, and your probability of moving counterclockwise to the next room is $q_{-1} = 1 - q_0 - q_1$.

(a) Initially, assume your house has no way of sensing where you are. What is the probability that you will be in room D at time $t = 1$?

- ○ $q_0 p_D$
- ○ $q_0 p_D + q_1 p_A + q_{-1} p_C + 2 q_1 p_B$
- ● $q_0 p_D + q_1 p_A + q_{-1} p_C$
- ○ $q_0 p_D + q_{-1} p_A + q_1 p_C$
- ○ $q_1 p_A + q_1 p_C + q_0 p_D$
- ○ None of these

This probability is given by the sum of three probabilities: 1) $q_0 p_D$: You are in room D to start (p_D) and stay there (q_0), 2) $q_1 p_A$: You are in room A to start (p_A) and move clockwise to room D (q_1), and 3) $q_{-1} p_C$: You are in room C to start (p_C) and move counterclockwise to room D (q_{-1}).

Now assume your house contains a sensor M^A that detects motion ($+m$) or no motion ($-m$) in room A. However, the sensor is a bit noisy and can be tricked by movement in adjacent rooms, resulting in the conditional distributions for the sensor given in the table below. The prior distribution for the sensor's output is also given.

M^A	$P(M^A \mid X = A)$	$P(M^A \mid X = B)$	$P(M^A \mid X = C)$	$P(M^A \mid X = D)$		M^A	$P(M^A)$
$+m^A$	$1 - 2\gamma$	γ	0.0	γ		$+m^A$	0.5
$-m^A$	2γ	$1 - \gamma$	1.0	$1 - \gamma$		$-m^A$	0.5

(b) You decide to help your house to track your movements using a particle filter with three particles. At time $t = T$, the particles are at $X^0 = A, X^1 = B, X^2 = D$. What is the probability that the particles will be resampled as $X^0 = X^1 = X^2 = A$ after time elapse? Select **all terms in the product.**

- ■ q_0
- □ q_0^2
- □ q_0^3
- ■ q_1
- □ q_1^2
- □ q_1^3
- ■ q_{-1}
- □ q_{-1}^2
- □ q_{-1}^3
- ○ None of these

The probability that all particles will be resampled as being in room A is $q_0 q_1 q_{-1}$ since particle X^0 stays in A with probability q_0, particle X^1 moves clockwise to A with probability q_1, and particle X^2 moves counterclockwise with probability q_{-1}.

(c) Assume that the particles are actually resampled after time elapse as $X^0 = D, X^1 = B, X^2 = C$, and the sensor observes $M^A = -m^A$. What are the particle weights given the observation?

Particle								Weight
$X^0 = D$	○ γ	● $1 - \gamma$	○ $1 - 2\gamma$	○ 0.0	○ 1.0	○ 2γ	○ None of these	
$X^1 = B$	○ γ	● $1 - \gamma$	○ $1 - 2\gamma$	○ 0.0	○ 1.0	○ 2γ	○ None of these	
$X^2 = C$	○ γ	○ $1 - \gamma$	○ $1 - 2\gamma$	○ 0.0	● 1.0	○ 2γ	○ None of these	

Now, assume your house also contains sensors M^B and M^D in rooms B and D, respectively, with the conditional distributions of the sensors given below and the prior equivalent to that of sensor M^A.

M^B	$P(M^B \mid X = A)$	$P(M^B \mid X = B)$	$P(M^B \mid X = C)$	$P(M^B \mid X = D)$
$+m^B$	γ	$1 - 2\gamma$	γ	0.0
$-m^B$	$1 - \gamma$	2γ	$1 - \gamma$	1.0

M^D	$P(M^D \mid X = A)$	$P(M^D \mid X = B)$	$P(M^D \mid X = C)$	$P(M^D \mid X = D)$
$+m^D$	γ	0.0	γ	$1 - 2\gamma$
$-m^D$	$1 - \gamma$	1.0	$1 - \gamma$	2γ

(d) Again, assume that the particles are actually resampled after time elapse as $X^0 = D, X^1 = B, X^2 = C$. The sensor readings are now $M^A = -m^A, M^B = -m^B, M^D = +m^D$. What are the particle weights given the observations?

Particle	Weight
$X^0 = D$	○ $\gamma^2 - 2\gamma^3$ ○ $3 - 2\gamma$ ○ 0.0 ○ $\gamma - \gamma^2 + \gamma^3$ ● $1 - 3\gamma + 2\gamma^2$ ○ $2 - \gamma$ ○ $1 - 2\gamma + \gamma^2$ ○ None of these
$X^1 = B$	○ $\gamma^2 - 2\gamma^3$ ○ $3 - 2\gamma$ ● 0.0 ○ $\gamma - \gamma^2 + \gamma^3$ ○ $1 - 3\gamma + 2\gamma^2$ ○ $2 - \gamma$ ○ $1 - 2\gamma + \gamma^2$ ○ None of these
$X^2 = C$	○ $\gamma^2 - 2\gamma^3$ ○ $3 - 2\gamma$ ○ 0.0 ○ $\gamma - \gamma^2 + \gamma^3$ ○ $1 - 3\gamma + 2\gamma^2$ ○ $2 - \gamma$ ○ $1 - 2\gamma + \gamma^2$ ● None of these

The sequence of observations from each sensor are expressed as the following: $m^A_{0:t}$ are all measurements $m^A_0, m^A_1, \ldots, m^A_t$ from sensor M^A, $m^B_{0:t}$ are all measurements $m^B_0, m^B_1, \ldots, m^B_t$ from sensor M^B, and $m^D_{0:t}$ are all measurements $m^D_0, m^D_1, \ldots, m^D_t$ from sensor M^D. Your house can get an accurate estimate of where you are at a given time t using the forward algorithm. The forward algorithm update step is shown here:

$$P(X_t \mid m^A_{0:t}, m^B_{0:t}, m^D_{0:t}) \propto P(X_t, m^A_{0:t}, m^B_{0:t}, m^D_{0:t}) \tag{1}$$

$$= \sum_{x_{t-1}} P(X_t, x_{t-1}, m^A_t, m^B_t, m^D_t, m^A_{0:t-1}, m^B_{0:t-1}, m^D_{0:t-1}) \tag{2}$$

$$= \sum_{x_{t-1}} \underline{\hspace{2cm}} P(X_t \mid x_{t-1})P(x_{t-1}, m^A_{0:t-1}, m^B_{0:t-1}, m^D_{0:t-1}) \tag{3}$$

(e) Which of the following expression(s) correctly complete the missing expression in line (3) above (regardless of whether they are available to the algorithm during execution)? Fill in **all** that apply.

■ $P(m_t^A, m_t^B, m_t^D \mid X_t, x_{t-1})$ □ $P(m_t^A, m_t^B, m_t^D \mid x_{t-1})$ □ $P(m_t^A \mid x_{t-1})P(m_t^B \mid x_{t-1})P(m_t^D \mid x_{t-1})$

□ $P(m_t^A \mid m_{0:t-1}^A)P(m_t^B \mid m_{0:t-1}^B)P(m_t^D \mid m_{0:t-1}^D)$ ■ $P(m_t^A, m_t^B, m_t^D \mid X_t, x_{t-1}, m_{0:t-1}^A, m_{0:t-1}^B, m_{0:t-1}^D)$

■ $P(m_t^A \mid X_t)P(m_t^B \mid X_t)P(m_t^D \mid X_t)$ ■ $P(m_t^A, m_t^B, m_t^D \mid X_t)$ ○ None of these

Using the chain rule from the previous step,

$$
\begin{aligned}
P(X_t, x_{t-1}, m_t^A, m_t^B, m_t^D, m_{0:t-1}^A, m_{0:t-1}^B, m_{0:t-1}^D) &= [P(m_t^A, m_t^B, m_t^D \mid X_t, x_{t-1}, m_{0:t-1}^A, m_{0:t-1}^B, m_{0:t-1}^D) \\
&\quad P(X_t \mid x_{t-1})P(x_{t-1}, m_{0:t-1}^A, m_{0:t-1}^B m_{0:t-1}^D)] \\
&= [P(m_t^A, m_t^B, m_t^D \mid X_t, x_{t-1}) \\
&\quad P(X_t \mid x_{t-1})P(x_{t-1}, m_{0:t-1}^A, m_{0:t-1}^B m_{0:t-1}^D)] \\
&\quad \text{(indep. of measurements from prev. measurements)} \\
&= [P(m_t^A, m_t^B, m_t^D \mid X_t) \\
&\quad P(X_t \mid x_{t-1})P(x_{t-1}, m_{0:t-1}^A, m_{0:t-1}^B m_{0:t-1}^D)] \\
&\quad \text{(indep. of measurements from prev. states)} \\
&= [P(m_t^A \mid X_t)P(m_t^B \mid X_t)P(m_t^D \mid X_t) \\
&\quad P(X_t \mid x_{t-1})P(x_{t-1}, m_{0:t-1}^A, m_{0:t-1}^B m_{0:t-1}^D)] \\
&\quad \text{(indep. of measurements from each other)}
\end{aligned}
$$

All of the expressions on the right side of the above equations should be selected.

Q3. Planning ahead with HMMs

Pacman is tired of using HMMs to estimate the location of ghosts. He wants to use HMMs to plan what actions to take in order to maximize his utility. Pacman uses the HMM (drawn to the right) of length T to model the planning problem. In the HMM, $X_{1:T}$ is the sequence of hidden states of Pacman's world, $A_{1:T}$ are actions Pacman can take, and U_t is the utility Pacman receives at the particular hidden state X_t. Notice that there are no evidence variables, and utilities are not discounted.
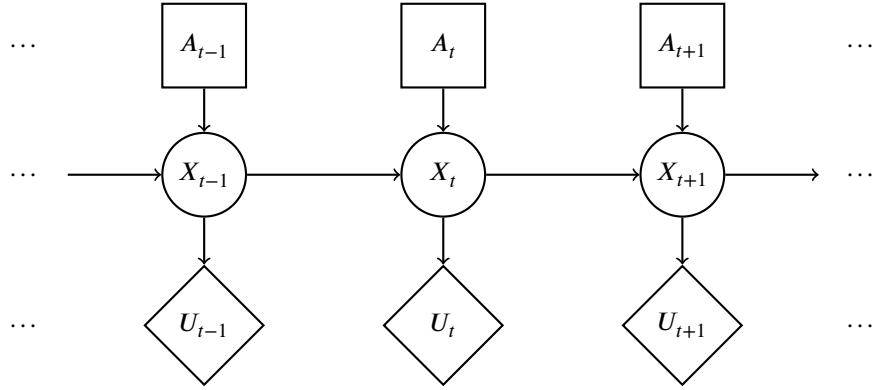
(a) The belief at time t is defined as $B_t(X_t) = p(X_t|a_{1:t})$. The forward algorithm update has the following form:

$$B_t(X_t) = \underline{\quad\text{(i)}\quad} \quad \underline{\quad\text{(ii)}\quad} \quad B_{t-1}(x_{t-1}).$$

Complete the expression by choosing the option that fills in each blank.

(i)	○ $\max_{x_{t-1}}$	● $\sum_{x_{t-1}}$	○ $\max_{x_t}$	○ $\sum_{x_t}$	○ 1
(ii)	○ $p(X_t|x_{t-1})$	○ $p(X_t|x_{t-1})p(X_t|a_t)$	○ $p(X_t)$	● $p(X_t|x_{t-1}, a_t)$	○ 1

○ None of the above combinations is correct

$$B_t(X_t) = p(X_t|a_{1:t})$$
$$= \sum_{x_{t-1}} p(X_t|x_{t-1}, a_t)p(x_{t-1}|a_{1:t-1})$$
$$= \sum_{x_{t-1}} p(X_t|x_{t-1}, a_t)B_{t-1}(x_{t-1})$$

(b) Pacman would like to take actions $A_{1:T}$ that maximizes the expected sum of utilities, which has the following form:

$$\text{MEU}_{1:T} = \underline{\quad\text{(i)}\quad} \quad \underline{\quad\text{(ii)}\quad} \quad \underline{\quad\text{(iii)}\quad} \quad \underline{\quad\text{(iv)}\quad} \quad \underline{\quad\text{(v)}\quad}$$

Complete the expression by choosing the option that fills in each blank.

(i)	● $\max_{a_{1:T}}$	○ $\max_{a_T}$	○ $\sum_{a_{1:T}}$	○ $\sum_{a_T}$	○ 1
(ii)	○ $\max_t$	○ $\prod_{t=1}^{T}$	● $\sum_{t=1}^{T}$	○ $\min_t$	○ 1
(iii)	○ $\sum_{x_t,a_t}$	● $\sum_{x_t}$	○ $\sum_{a_t}$	○ $\sum_{x_T}$	○ 1
(iv)	○ $p(x_t|x_{t-1}, a_t)$	○ $p(x_t)$	● $B_t(x_t)$	○ $B_T(x_T)$	○ 1
(v)	○ U_T	○ $\frac{1}{U_t}$	○ $\frac{1}{U_T}$	● U_t	○ 1

○ None of the above combinations is correct

$$\text{MEU}_{1:T} = \max_{a_{1:T}} \sum_{t=1}^{T} \sum_{x_t} B_t(x_t)U_t(x_t)$$

(c) A greedy ghost now offers to tell Pacman the values of some of the hidden states. Pacman needs your help to figure out if the ghost's information is useful. Assume that the transition function $p(x_t|x_{t-1}, a_t)$ is not deterministic. **With respect to the utility U_t, mark all that can be True:**

■ VPI$(X_{t-1}|X_{t-2}) > 0$ ☐ VPI$(X_{t-2}|X_{t-1}) > 0$ ■ VPI$(X_{t-1}|X_{t-2}) = 0$ ■ VPI$(X_{t-2}|X_{t-1}) = 0$ ☐
None of the above

It is always possible that VPI $= 0$. Can guarantee VPI$(E|e)$ is not greater than 0 if E is independent of parents(U) given e.

(d) Pacman notices that calculating the beliefs under this model is very slow using exact inference. He therefore decides to try out various particle filter methods to speed up inference. Order the following methods by how accurate their estimate of $B_T(X_T)$ is? If different methods give an equivalently accurate estimate, mark them as the same number.

	Most accurate			Least accurate
Exact inference	● 1	○ 2	○ 3	○ 4
Particle filtering with no resampling	○ 1	● 2	○ 3	○ 4
Particle filtering with resampling before every time elapse	○ 1	○ 2	○ 3	● 4
Particle filtering with resampling before every other time elapse	○ 1	○ 2	● 3	○ 4

Exact inference will always be more accurate than using a particle filter. When comparing the particle filter resampling approaches, notice that because there are no observations, each particle will have weight 1. Therefore resampling when particle weights are 1 could lead to particles being lost and hence prove bad.