EECS 16A   Designing Information Devices and Systems I
Spring 2021
Discussion 2B

1. Matrix Multiplication

Consider the following matrices:

$$A = \begin{bmatrix} 1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \quad F = \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix} \quad G = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \quad H = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix}$$

For each matrix multiplication problem, *if the product exists*, find the product by hand.
Otherwise, explain why the product does not exist.

(a) **A B Answer:** A is a 1×2 vector and B_1 is a 2×1 vector, so the product exists!
   $\mathbf{AB} = 1 \cdot 3 + 4 \cdot 2 = 11$

(b) **C D Answer:**   Since both **C** and **D** are 2×2 matrices, the product exists and is a 2×2 matrix.
   $$\mathbf{CD} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 4 \cdot 2 & 1 \cdot 2 + 4 \cdot 1 \\ 2 \cdot 3 + 3 \cdot 2 & 2 \cdot 2 + 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ 12 & 7 \end{bmatrix}.$$

(c) **D C Answer:**   Since both **C** and **D** are 2×2 matrices, the product exists and is a 2×2 matrix.
   $$\mathbf{DC} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + 2 \cdot 2 & 3 \cdot 4 + 2 \cdot 3 \\ 2 \cdot 1 + 1 \cdot 2 & 2 \cdot 4 + 1 \cdot 3 \end{bmatrix} = \begin{bmatrix} 7 & 18 \\ 4 & 11 \end{bmatrix}.$$

(d) **C E Answer:**   Since **C** is a 2×2 matrix and **E** is a 2×4 matrix, the product exists and is a 2×4 matrix.
   $$\mathbf{CE} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 4 \cdot 4 & 1 \cdot 9 + 4 \cdot 3 & 1 \cdot 5 + 4 \cdot 2 & 1 \cdot 7 + 4 \cdot 2 \\ 2 \cdot 1 + 3 \cdot 4 & 2 \cdot 9 + 3 \cdot 3 & 2 \cdot 5 + 3 \cdot 2 & 2 \cdot 7 + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 17 & 21 & 13 & 15 \\ 14 & 27 & 16 & 20 \end{bmatrix}.$$

(e) **F E** (only note whether or not the product exists) **Answer:**
   Since **E** is a 2×4 matrix and **F** is a 4×3 matrix, the product does not exist.
   This is because the number of columns in the first matrix (**F**) should match the number of rows in the
   second matrix (**E**) for this product to be defined.

(f) **E F** (only note whether or not the product exists) **Answer:**
   Since **E** is a 2×4 matrix and **F** is a 4×3 matrix, the product exists and is a 2×3 matrix.

$$\mathbf{EF} = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot5+9\cdot6+5\cdot4+7\cdot3 & 1\cdot5+9\cdot1+5\cdot1+7\cdot2 & 1\cdot8+9\cdot2+5\cdot7+7\cdot2 \\ 4\cdot5+3\cdot6+2\cdot4+2\cdot3 & 4\cdot5+3\cdot1+2\cdot1+2\cdot2 & 4\cdot8+3\cdot2+2\cdot7+2\cdot2 \end{bmatrix}$$

$$= \begin{bmatrix} 100 & 33 & 75 \\ 52 & 29 & 56 \end{bmatrix}$$

(g) **G H** (Practice on your own) **Answer:** Since **G** and **H** are both 3×3 matrices, the product exists and is another 3×3 matrix.

$$\mathbf{GH} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 8\cdot5+1\cdot1+6\cdot2 & 8\cdot3+1\cdot8+6\cdot3 & 8\cdot4+1\cdot2+6\cdot5 \\ 3\cdot5+5\cdot1+7\cdot2 & 3\cdot3+5\cdot8+7\cdot3 & 3\cdot4+5\cdot2+7\cdot5 \\ 4\cdot5+9\cdot1+2\cdot2 & 4\cdot3+9\cdot8+2\cdot3 & 4\cdot4+9\cdot2+2\cdot5 \end{bmatrix} = \begin{bmatrix} 53 & 50 & 64 \\ 34 & 70 & 57 \\ 33 & 90 & 44 \end{bmatrix}.$$

(h) **H G** (Practice on your own) **Answer:** Since **H** and **G** are both 3×3 matrices, the product exists and is another 3×3 matrix.

$$\mathbf{GH} = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} = \begin{bmatrix} 5\cdot8+3\cdot3+4\cdot4 & 5\cdot1+3\cdot5+4\cdot9 & 5\cdot6+3\cdot7+4\cdot2 \\ 1\cdot8+8\cdot3+2\cdot4 & 1\cdot1+8\cdot5+2\cdot9 & 1\cdot6+8\cdot7+2\cdot2 \\ 2\cdot8+3\cdot3+5\cdot4 & 2\cdot1+3\cdot5+5\cdot9 & 2\cdot6+3\cdot7+5\cdot2 \end{bmatrix} = \begin{bmatrix} 65 & 56 & 59 \\ 40 & 59 & 66 \\ 45 & 62 & 43 \end{bmatrix}.$$

2. Visualizing Matrices as Operations

This problem is going to help you visualize matrices as operations. For example, when we multiply a vector by a "rotation matrix," we will see it "rotate" in the true sense here. Similarly, when we multiply a vector by a "reflection matrix," we will see it be "reflected." The way we will see this is by applying the operation to all the vertices of a polygon and seeing how the polygon changes.

Your TA will now show you how a unit square can be rotated, scaled, or reflected using matrices!

Part 1: Rotation Matrices as Rotations

(a) We are given matrices $\mathbf{T}_1$ and $\mathbf{T}_2$, and we are told that they will rotate the unit square by $15°$ and $30°$, respectively. Suggest some methods to rotate the unit square by $45°$ using only $\mathbf{T}_1$ and $\mathbf{T}_2$. How would you rotate the square by $60°$? Your TA will show you the result in the iPython notebook.

Answer:

Apply $\mathbf{T}_1$ and $\mathbf{T}_2$ in succession to rotate the unit square by $45°$. To rotate the square by $60°$, you can either apply $\mathbf{T}_2$ twice, or if you prefer variety, apply $\mathbf{T}_1$ twice and $\mathbf{T}_2$ once.

(b) Find a single matrix $\mathbf{T}_3$ to rotate the unit square by $60°$. Your TA will show you the result in the iPython notebook.

Answer: This matrix will look like the rotation matrix that rotates a vector by $60°$. This matrix can be composed by multiplying $\mathbf{T}_1$ by $\mathbf{T}_1$ by $\mathbf{T}_2$ (or equivalently, $\mathbf{T}_2$ by $\mathbf{T}_2$).

(c) $\mathbf{T}_1$, $\mathbf{T}_2$, and the matrix you used in part (b) are called "rotation matrices." They rotate any vector by an angle θ. Show that a rotation matrix has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ is the angle of rotation. To do this consider rotating the unit vector $\begin{bmatrix} cos(\alpha) \\ sin(\alpha) \end{bmatrix}$ by θ degrees using the matrix $\mathbf{R}$.

(**Definition:** A vector, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix}$, is a unit vector if $\sqrt{v_1^2 + v_2^2 + \ldots} = 1$.)

(*Hint: Use your trigonometric identities!*)

Answer:

The reason the matrix is called a rotation matrix is because it transforms the unit vector $\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ to give $\begin{bmatrix} \cos(\alpha+\theta) \\ \sin(\alpha+\theta) \end{bmatrix}$.

Proof:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \cos\alpha \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \sin\alpha \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\theta - \sin\alpha\sin\theta \\ \cos\alpha\sin\theta + \sin\alpha\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\theta) \\ \sin(\alpha+\theta) \end{bmatrix}$$

Alternative solution:

Let's try to derive this matrix using trigonometry. Suppose we want to rotate the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by θ.
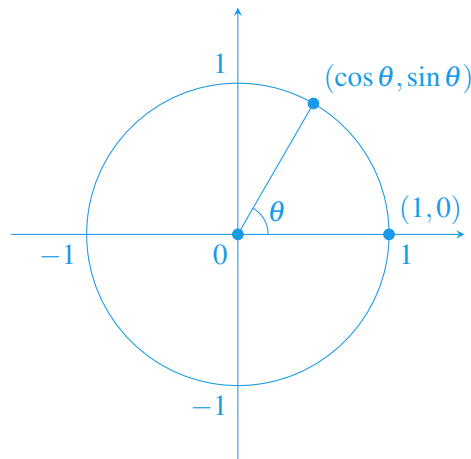
We can use basic trigonometric relationships to see that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotated by θ becomes $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$. Similarly, rotating the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by θ becomes $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$:
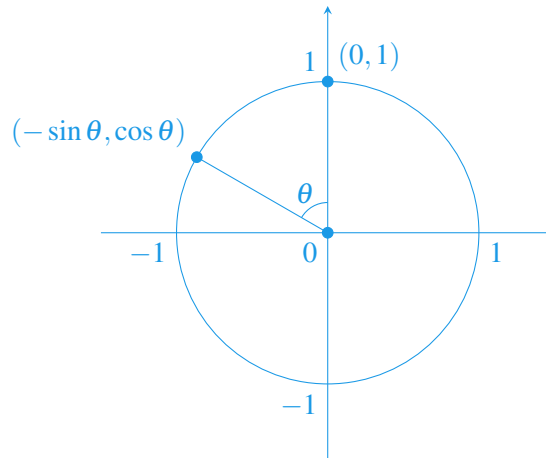
We can also scale these pre-rotated vectors to any length we want, $\begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ y \end{bmatrix}$, and we can observe graphically that they rotate to $\begin{bmatrix} x\cos\theta \\ x\sin\theta \end{bmatrix}$ and $\begin{bmatrix} -y\sin\theta \\ y\cos\theta \end{bmatrix}$, respectively. Rotating a vector solely in the x-direction produces a vector with both x and y components, and, likewise, rotating a vector solely in the y-direction produces a vector with both x and y components.

Finally, if we want to rotate an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix}$, we can combine what we derived above. Let x' and y' be the x and y components after rotation. x' has contributions from both x and y: $x' = x\cos\theta - y\sin\theta$. Similarly, y' has contributions from both components as well: $y' = x\sin\theta + y\cos\theta$. Expressing this in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, we've derived the 2-dimensional rotation matrix.

(d) Now, we want to get back the original unit square from the rotated square in part (b). What matrix should we use to do this? (**Note:** Don't use inverses! Answer this question using your intuition, we will visit inverses very soon in lecture!)

Answer:

Use a rotation matrix that rotates by $-60°$.

$$\begin{bmatrix} \cos(-60°) & -\sin(-60°) \\ \sin(-60°) & \cos(-60°) \end{bmatrix}$$

(e) Use part (d) to obtain the "inverse" rotation matrix for a matrix that rotates a vector by θ. Multiply the inverse rotation matrix with the rotation matrix and vice-versa. What do you get?

Answer:

The inverse matrix is as follows:

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

We can see that for any $\vec{v} \in \mathbb{R}^2$ that the product of the rotation matrix with $\vec{v}$ followed by the product of the inverse results in the original $\vec{v}$.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \vec{v} \right) = \vec{v}$$

(f) What are the matrices that reflect a vector about the (i) x-axis, (ii) y-axis, and (iii) $x = y$

Answer:

The matrix that reflects about the x-axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The matrix that reflects about the y-axis:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the matrix that reflects about $x = y$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Part 2: Commutativity of Operations

A natural question to ask is the following: Does the *order* in which you apply these operations matter? Your TA will demonstrate parts (a) and (b) in the iPython notebook.

(a) Let's see what happens to the unit square when we rotate the square by $60°$ and then reflect it along the y-axis.

(b) Now, let's see what happens to the unit square when we first reflect the square along the y-axis and then rotate it by $60°$. Is this the same as in part (a)?

Answer: (For parts (a) and (b)): The two operations are not the same.

(c) Try to do steps (a) and (b) by multiplying the reflection and rotation matrices together (in the correct order for each case). What does this tell you?

Answer:

The resulting matrices that are obtained (by multiplying the two matrices) are different depending on the order of multiplication.

(d) If you reflected the unit square twice (along any pair of axes), do you think the order in which you applied the reflections would matter? Why/why not?

Answer:

It turns out that reflections are not commutative unless the two reflection axes are perpendicular to each other. For example, if you reflect about the x-axis and the y-axis, it is commutative. But if you reflect about the x-axis and $x = y$, it is not commutative.

Part 3: Distributivity of Operations

(a) The distributivity property of matrix-vector multiplication holds for any vectors and matrices. Show for general $\mathbf{A} \in \mathbb{R}^{2 \times 2}$ and $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ that $\mathbf{A}(\vec{v}_1 + \vec{v}_2) = \mathbf{A}\vec{v}_1 + \mathbf{A}\vec{v}_2$.

Answer: Matrix-vector multiplication distributes because scalar multiplication distributes.

$$
\begin{align}
\mathbf{A}(\vec{v}_1 + \vec{v}_2) &= \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} (\vec{v}_1 + \vec{v}_2) \tag{1} \\
&= (v_{11} + v_{21})\vec{a}_1 + (v_{12} + v_{22})\vec{a}_2 \tag{2} \\
&= \begin{bmatrix} a_{11}(v_{11} + v_{21}) + a_{12}(v_{12} + v_{22}) \\ a_{21}(v_{11} + v_{21}) + a_{22}(v_{12} + v_{22}) \end{bmatrix} \tag{3} \\
&= \begin{bmatrix} a_{11}v_{11} + a_{12}v_{12} \\ a_{21}v_{11} + a_{22}v_{12} \end{bmatrix} + \begin{bmatrix} a_{11}v_{21} + a_{12}v_{22} \\ a_{21}v_{21} + a_{22}v_{22} \end{bmatrix} \tag{4} \\
&= \mathbf{A}\vec{v}_1 + \mathbf{A}\vec{v}_2 \tag{5}
\end{align}
$$