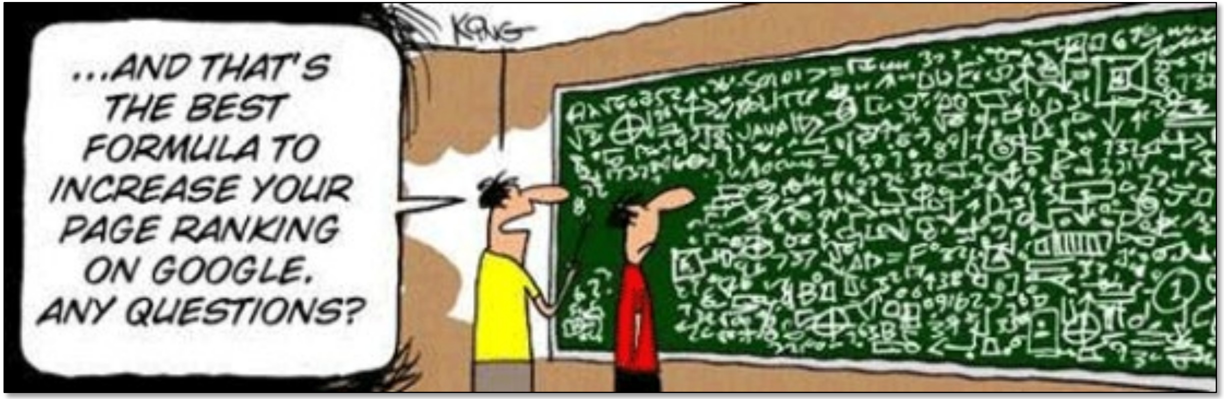
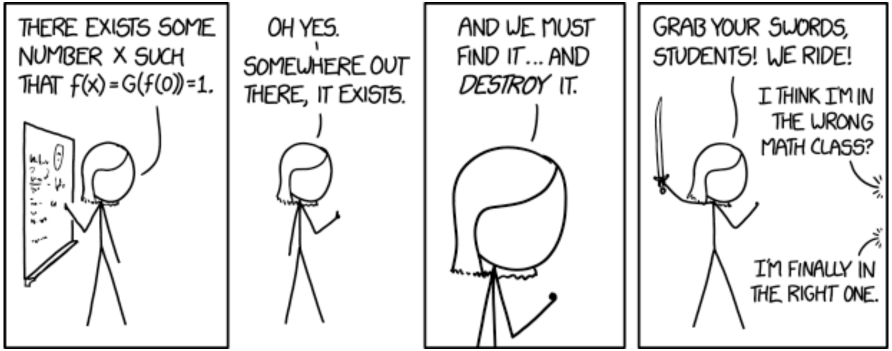


EE16A

Page Rank, Eigenvalues and Eigenspaces

Recall: Equivalent Statements:

- Matrix A is **invertible**
- Ax=b has a **unique solution**
- A has linearly independent columns (A is **full rank**)
- A has a **trivial nullspace**
- The **determinant** of A is not zero



Jargon Roundup

- **range/span** of matrix A is the set of all possible linear combinations of the column vectors (all the outputs it can get to)

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- **rank** is the dimension of the span of the columns of matrix A

$$\dim(\text{colspan}(A)) = \#\text{ ind. cols.}$$
$$= \#\text{ of cols. w/ } \underline{\text{pivots}}$$
$$= \dim(\text{rowspan}(A))$$

$$\boxed{\#\text{ lin. ind. cols } A = \#\text{ lin. indep. rows } A}$$

$$A = \begin{bmatrix} x & x \\ x & x \\ x & \cancel{x} \end{bmatrix}$$

3×2

Jargon Roundup

- **range/span** of matrix A is the set of all possible linear combinations of the column vectors (all the outputs it can get to)
- **rank** is the dimension of the span of the columns of matrix A
- **nullspace** of matrix A is the set of solutions to $A\vec{x}=\vec{0}$ (all the places it can't get to)

Def'n:

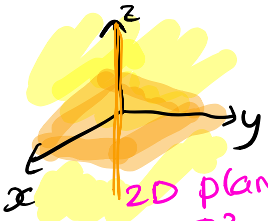
$$\underbrace{N(A)}_{\text{Null Space}} \equiv \{ \vec{x} \mid \underbrace{A\vec{x}=\vec{0}}_{\text{vectors that solve}},\ \underbrace{\vec{x}\in\mathbb{R}^N}_{\text{size of input}} \}$$

$$\underset{M\times N}{\begin{bmatrix} A \end{bmatrix}} \underset{N\times 1}{\begin{bmatrix} \vec{x} \end{bmatrix}} = \underset{M\times 1}{\begin{bmatrix} \vec{b} \end{bmatrix}}$$

$$A = \begin{array}{ccc} x & y & z \\ \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] \end{array}$$

lin. ind.

can't reach



2D plane in $\mathbb{R}^3$ pass through $\vec{0}$

$\text{rank}(A) = 2$

$\dim(\text{Null}(A)) = 1$

$\mathbb{R}^3$ 3 dim total

$2+1=3$

"Rank-Nullity Theorem"

$\text{rank}(A) + \boxed{\dim(\text{Null}(A))} = N$

# of cols of A

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- **rank** is the dimension of the span of the columns of matrix A
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- **vector space** is a set of vectors connected by two operators (+,x) that obeys the 10 axioms

A **vector space** $\mathbb{V}$ is a set of vectors and two operators that satisfy the following properties:

Note 7

- Vector Addition
  - Associative: $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$ for any $\vec{v},\vec{u},\vec{w}\in\mathbb{V}$.
  - Commutative: $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ for any $\vec{v},\vec{u}\in\mathbb{V}$.
  - Additive Identity: There exists an additive identity $\vec{0}\in\mathbb{V}$ such that $\vec{v}+\vec{0}=\vec{v}$ for any $\vec{v}\in\mathbb{V}$.
  - Additive Inverse: For any $\vec{v}\in\mathbb{V}$, there exists $-\vec{v}\in\mathbb{V}$ such that $\vec{v}+(-\vec{v})=\vec{0}$. We call $-\vec{v}$ the additive inverse of $\vec{v}$.
  - Closure under vector addition: For any two vectors $\vec{v},\vec{u}\in\mathbb{V}$, their sum $\vec{v}+\vec{u}$ must also be in $\mathbb{V}$.
- Scalar Multiplication
  - Associative: $\alpha(\beta\vec{v})=(\alpha\beta)\vec{v}$ for any $\vec{v}\in\mathbb{V}$, $\alpha,\beta\in\mathbb{R}$.
  - Multiplicative Identity: There exists $1\in\mathbb{R}$ where $1\cdot\vec{v}=\vec{v}$ for any $\vec{v}\in\mathbb{V}$. We call 1 the multiplicative identity.
  - Distributive in vector addition: $\alpha(\vec{u}+\vec{v})=\alpha\vec{u}+\alpha\vec{v}$ for any $\alpha\in\mathbb{R}$ and $\vec{u},\vec{v}\in\mathbb{V}$.
  - Distributive in scalar addition: $(\alpha+\beta)\vec{v}=\alpha\vec{v}+\beta\vec{v}$ for any $\alpha,\beta\in\mathbb{R}$ and $\vec{v}\in\mathbb{V}$.
  - Closure under scalar multiplication: For any vector $\vec{v}\in\mathbb{V}$ and scalar $\alpha\in\mathbb{R}$, the product $\alpha\vec{v}$ must also be in $\mathbb{V}$.

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- vector **subspace** is a subset of vectors from a vector space that obey 3 properties

Definition 8.1 (Subspace): A subspace $\mathbb{U}$ consists of a subset of the vector space $\mathbb{V}$ that satisfies the following three properties:

Note 8

- Contains the zero vector: $\vec{0} \in \mathbb{U}$.
- Closed under vector addition: For any two vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{U}$, their sum $\vec{v}_1 + \vec{v}_2$ must also be in $\mathbb{U}$.
- Closed under scalar multiplication: For any vector $\vec{v} \in \mathbb{U}$ and scalar $\alpha \in \mathbb{R}$, the product $\alpha\vec{v}$ must also be in $\mathbb{U}$.

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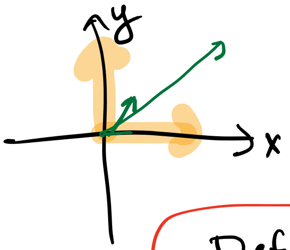
- A **basis** for a vector space is a minimum set of vectors needed to represent all vectors in the space

Jargon Roundup

Most "efficient representation" of a vector space.

$$\left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0.5\\0.7\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}\pi\\e\end{bmatrix} \right\} \Rightarrow \text{vector space } \mathbb{R}^2$$



Basis set: $\left\{ \begin{bmatrix}0.5\\0.7\end{bmatrix}, \begin{bmatrix}\pi\\e\end{bmatrix} \right\}$

Basis set of vectors is not unique

but dim is.

dim = 2

Def'n: Given a vector set $(\mathbb{V}, \mathbb{F})$

→ a set of vectors $\{\vec{v}_1, \vec{v}_2 \dots \vec{v}_n\}$, is a BASIS SET

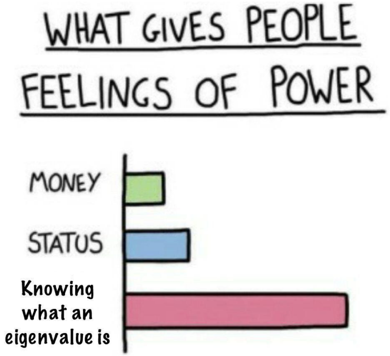
if ① $\vec{v}_1, \vec{v}_2 \dots \vec{v}_n$ are lin. indep.

② $\vec{v} \in \mathbb{V}$ (any vect) $\longrightarrow \vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots \alpha_n \vec{v}_n$

- A **basis** for a vector space is a minimum set of vectors needed to represent all vectors in the space

Today's Jargon: determinant, eigenvalue, eigenvector

- We will use Google's PageRank algorithm as a new application example to learn about these things!

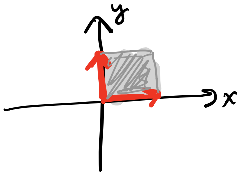


$\boxed{\text{Determinant}}$

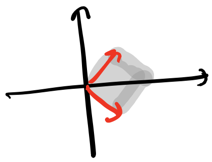
$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

$\det(A) = |A|$

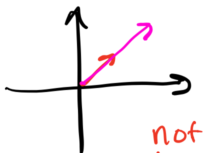
$\det(A) \neq 0$ when A invertible



$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Area $\neq 0$

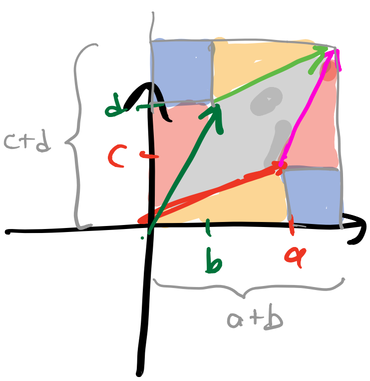


$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\neq 0$



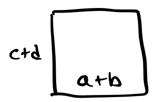
$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ $= 0$

not invertible
$\det(A) = 0$



$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

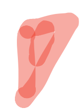
Compute area of parallelogram as:



$(c+d)(a+b)$

$-$  $\times 2$ $\quad bc \times 2$

$-$  $\times 2$ $\quad \frac{1}{2} ac \times 2$

$-$  $\times 2$ $\quad \frac{1}{2} bd \times 2$

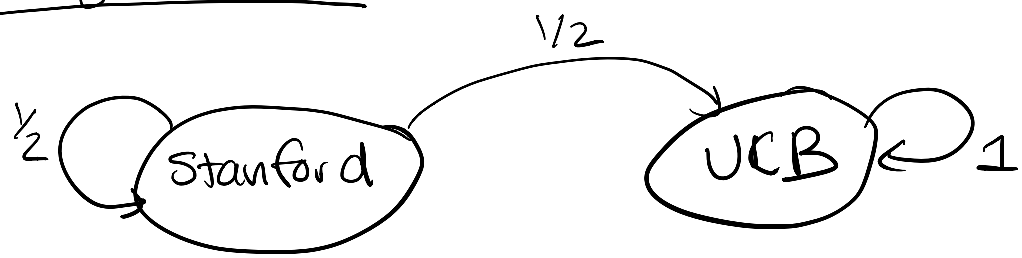
$$\square \text{area} = (c+d)(a+b) - 2bc - ac - bd$$
$$= \cancel{ca} + \cancel{cb} + da + \cancel{db} - \cancel{2}bc - \cancel{ac} - \cancel{bd}$$
$$= \underline{ad - bc}$$

yay!

Higher dimensional?

$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

Pagerank



$$\vec{x} = \begin{bmatrix} x_{\text{Stanford}} \\ x_{\text{UCB}} \end{bmatrix} \qquad Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

State transition mtx

Start with everyone @ Stanford $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

After one timestep: $\vec{x}(1) = Q \cdot \vec{x}(0)$

$$= \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

After two timesteps: $\vec{x}(2) = Q \cdot \vec{x}(1)$

$$= \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$$

After 3 timesteps: $\vec{x}(3) = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix} = \begin{bmatrix} 1/8 \\ 7/8 \end{bmatrix}$

$$\vec{x}(t) = \begin{bmatrix} (1/2)^t \\ 1 - (1/2)^t \end{bmatrix}$$

$$\vec{x}(\infty) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Everyone's at UCB!

What if instead $\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\vec{x}(1) = Q \cdot \vec{x}(0) = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

steady state sol'n

In general,

$$\boxed{\vec{x}_{\text{steady state}} = Q \cdot \vec{x}_{\text{steady state}}}$$

$$I \vec{x}_{ss} = Q \cdot \vec{x}_{ss}$$

$$Q \cdot \vec{x}_{ss} - I \vec{x}_{ss} = \vec{0}$$

$$\boxed{(Q - I)\, \vec{x}_{ss} = \vec{0}}$$

trans. mtx   identity   steady state sol'n

Want to find $\text{Null}(Q - I) \rightarrow$ steady state sol'ns.

$$\begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} -1/2 & 0 \\ 1/2 & 0 \end{bmatrix}}_{Q - I}$$

$$\text{Null}(Q-I): \left[\begin{array}{cc|c} -1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1/2 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

basis x_s

x_{free}

$$\text{Null}(Q-I) = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \underbrace{\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}}$$

Eigen space of Q corresponding to eigenvalues 1 $\rightarrow$ steady state

---

Def'n: Let Q be square matrix, $\lambda \in \mathbb{R}$

if $\vec{x} \neq 0$ such that $\boxed{Q \cdot \vec{x} = \lambda \cdot \vec{x}}$ "lambda" eig val.

then λ is an $\boxed{\text{eigenvalue}}$ of Q and $\text{Null}(Q-\lambda I)$ is

it's $\boxed{\text{eigenspace}}$ corresponding to λ

Find eigenvalues & Eigspaces: $Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$

Want to find $\lambda, \vec{x}$ such that $\underbrace{Q \cdot \vec{x}}_{\text{mtrix}} = \underbrace{\lambda \cdot \vec{x}}_{\text{scalar}}$

$$Q \cdot \vec{x} - \lambda \cdot I \cdot \vec{x} = \vec{0}$$

$$(Q - \lambda I)\,\vec{x} = \vec{0}$$

Find $\vec{x} \in \text{Null}(Q - \lambda I)$

$$Q - \lambda I = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1/2-\lambda & 0 \\ 1/2 & 1-\lambda \end{bmatrix}$$

① find λ

② find $\vec{x}$

If we have non-trivial nullspace

$$\det(Q - \lambda I) = 0$$

$$\boxed{(\tfrac{1}{2} - \lambda)(1 - \lambda) = 0}$$

characteristic polynomial

$\lambda_1 = \frac{1}{2}$ $\quad$ $\lambda_2 = 1$