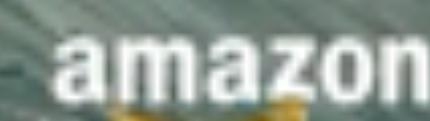
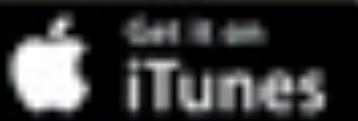


Ayreon
01011001

CONNECT THE DOTS



EE16B
Designing Information
Devices and Systems II

Lecture 11B
Sampling and Interpolation

Polynomial Interpolation

- Given n distinct points, then there's exist a unique $(n-1)$ order polynomial passing through them

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$\rightarrow \left[\begin{array}{c} \\ \\ \\ \end{array} \right] \left[\begin{array}{c} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{array} \right] = \left[\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array} \right]$$

Polynomial Interpolation

- Given n distinct points, then there's exist a unique $(n-1)$ order polynomial passing through them

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$$

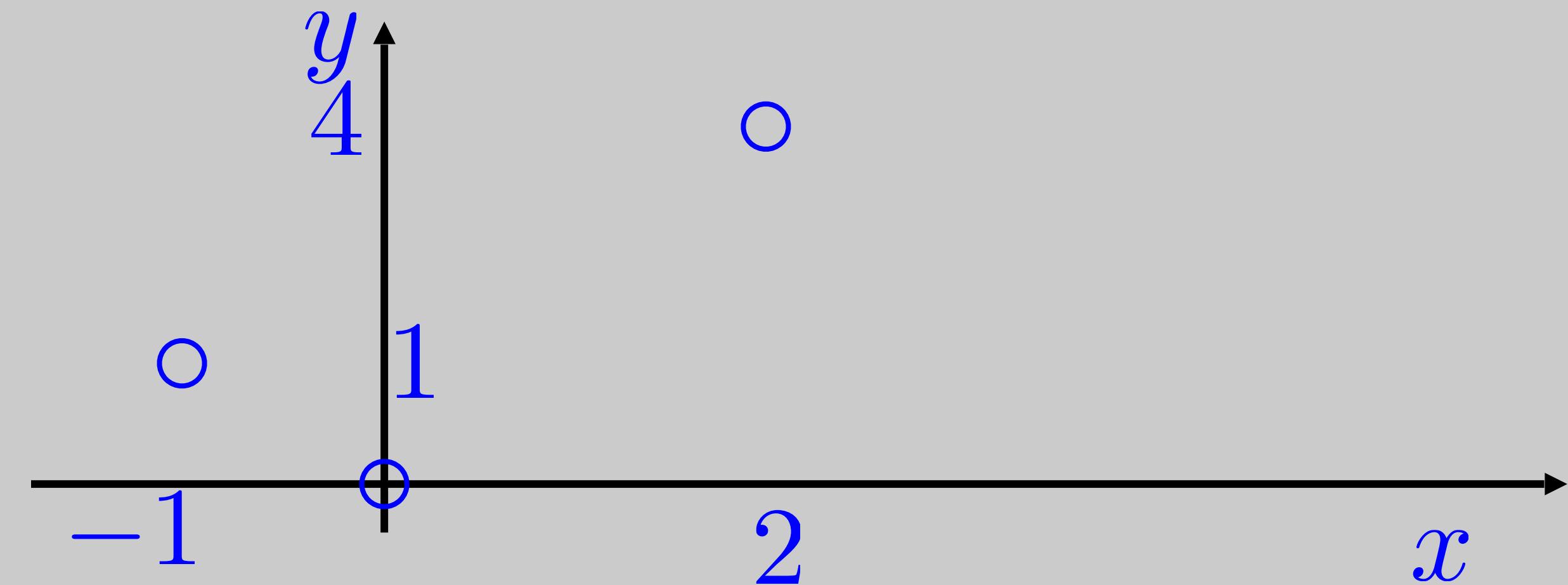
$$\rightarrow \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

“Vandermonde” Matrix $\det(v) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$

Quiz

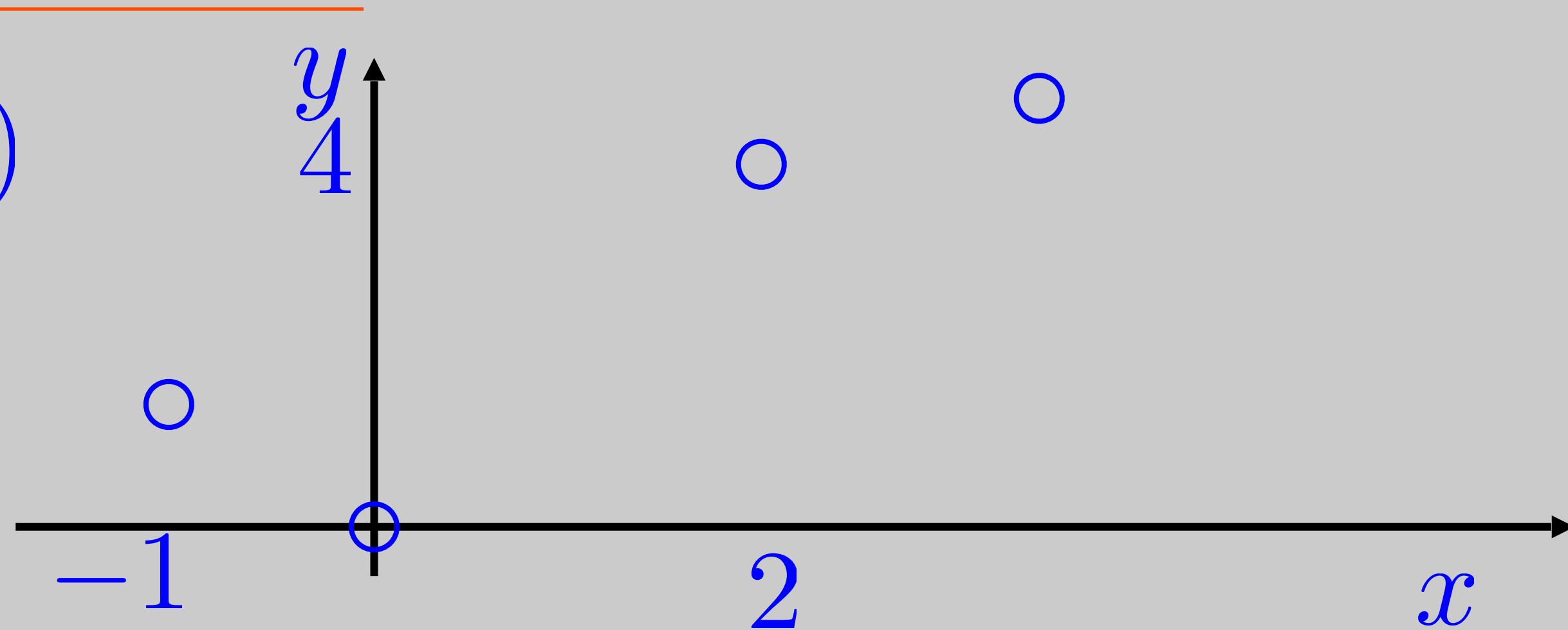
- What's the polynomial that passes through these points:

$$(-1, 1), (0, 0), (2, 4)$$



Polynomial Regression

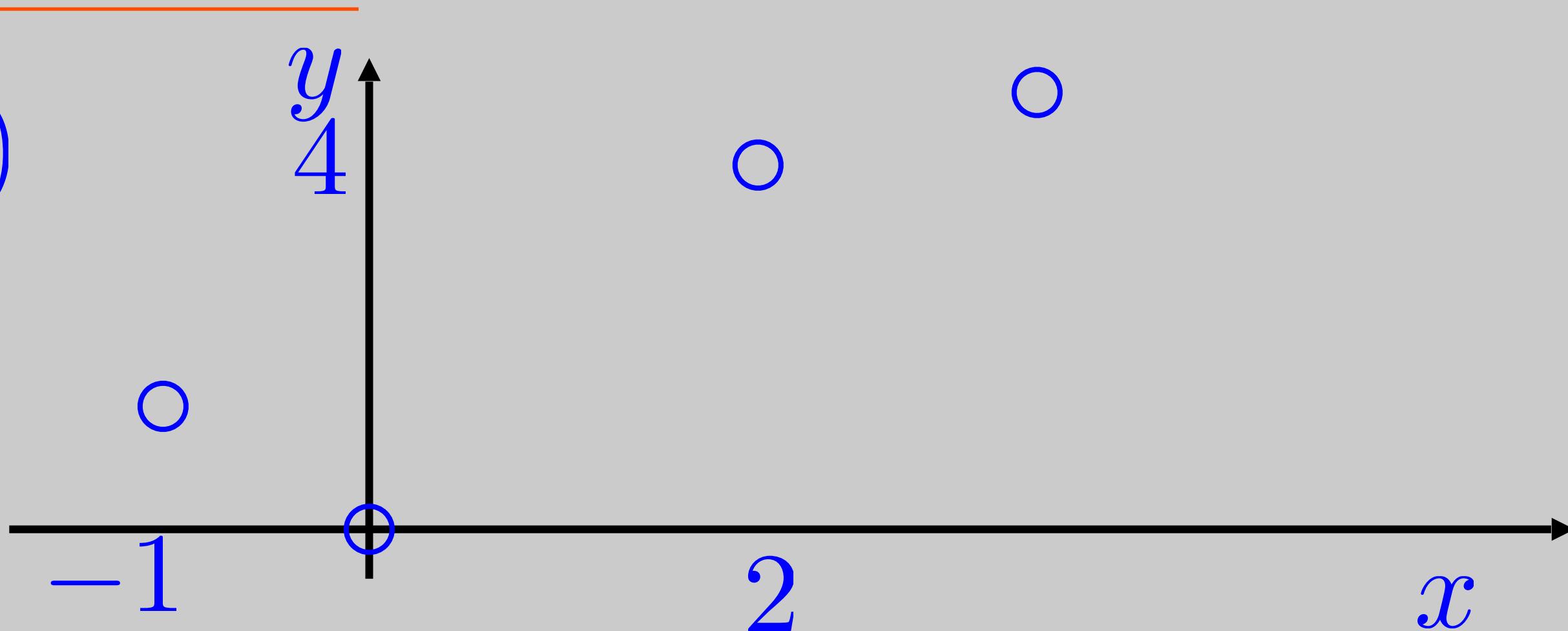
$(-1, 1), (0, 0), (2, 4), (3, 5)$



- What is the “best” quadratic polynomial that passes through the points?

Polynomial Regression

$(-1, 1), (0, 0), (2, 4), (3, 5)$



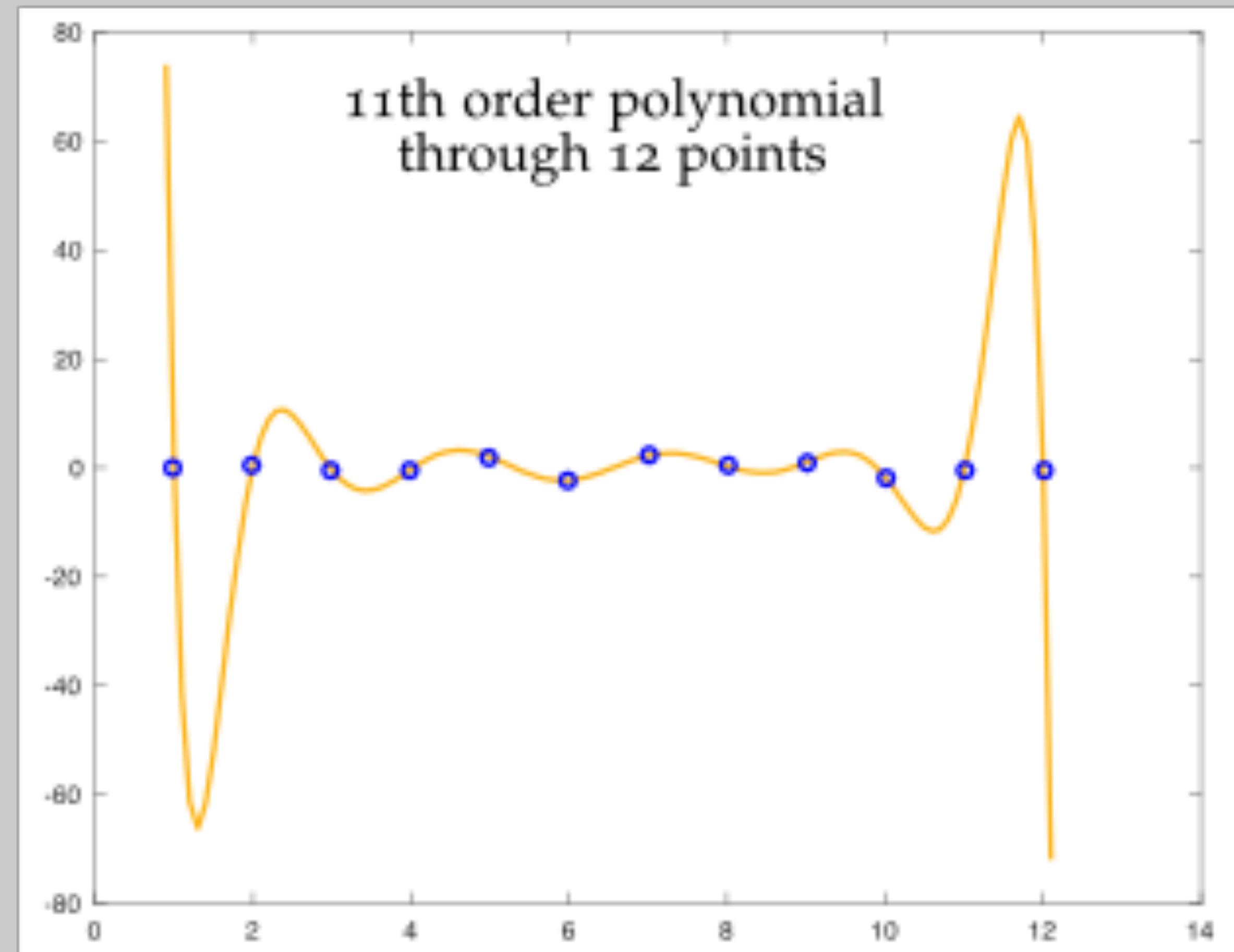
What is the “best” quadratic polynomial that passes through the points?

$$y = a_0 + a_1 x + a_2^2 + \cdots a_{n-1} x^{n-1}$$

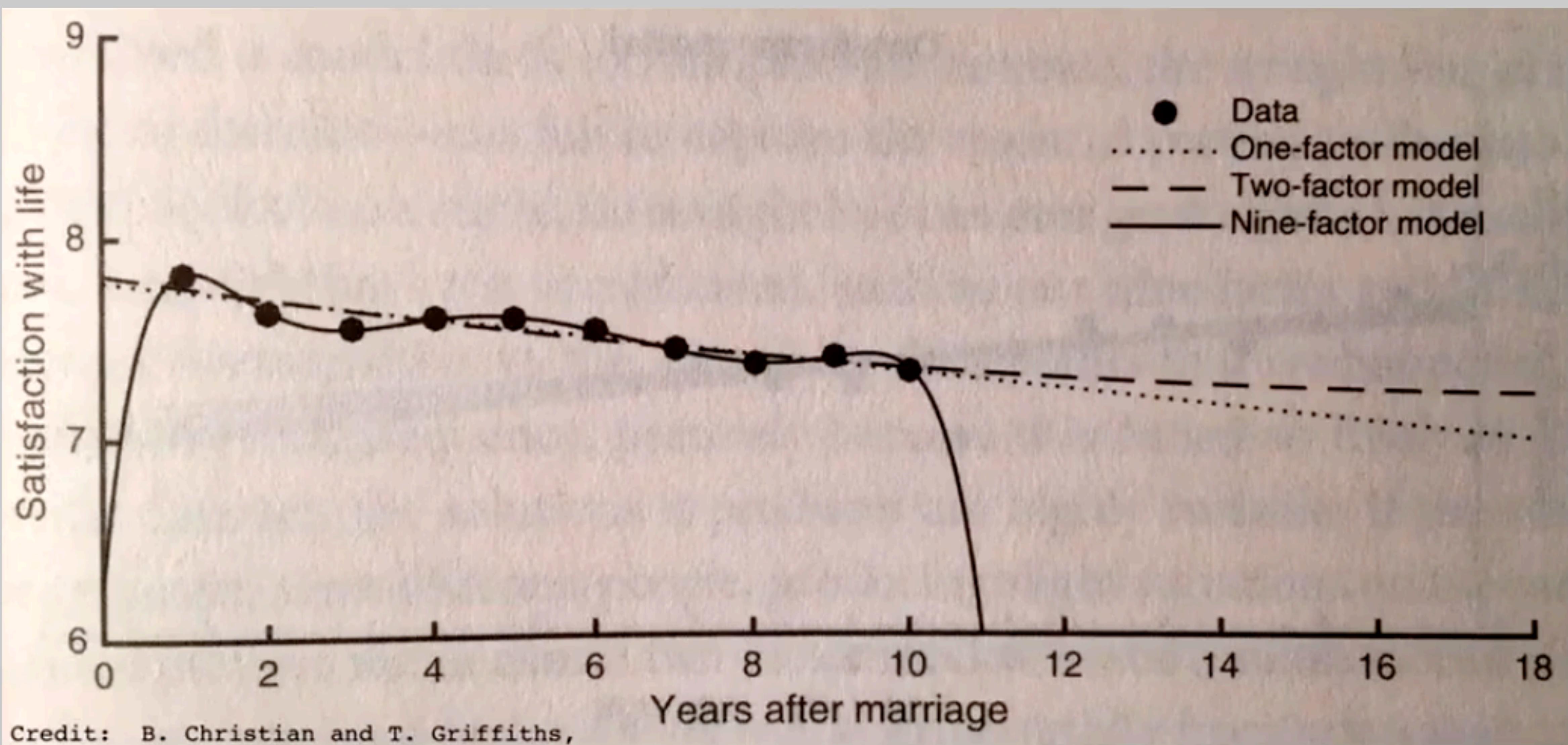
$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

Issue with Polynomial Interpolation

- Tend to be oscillatory in high order interp/regression
- Not numerically stable! x^{n-1}



Example



Commercial Break

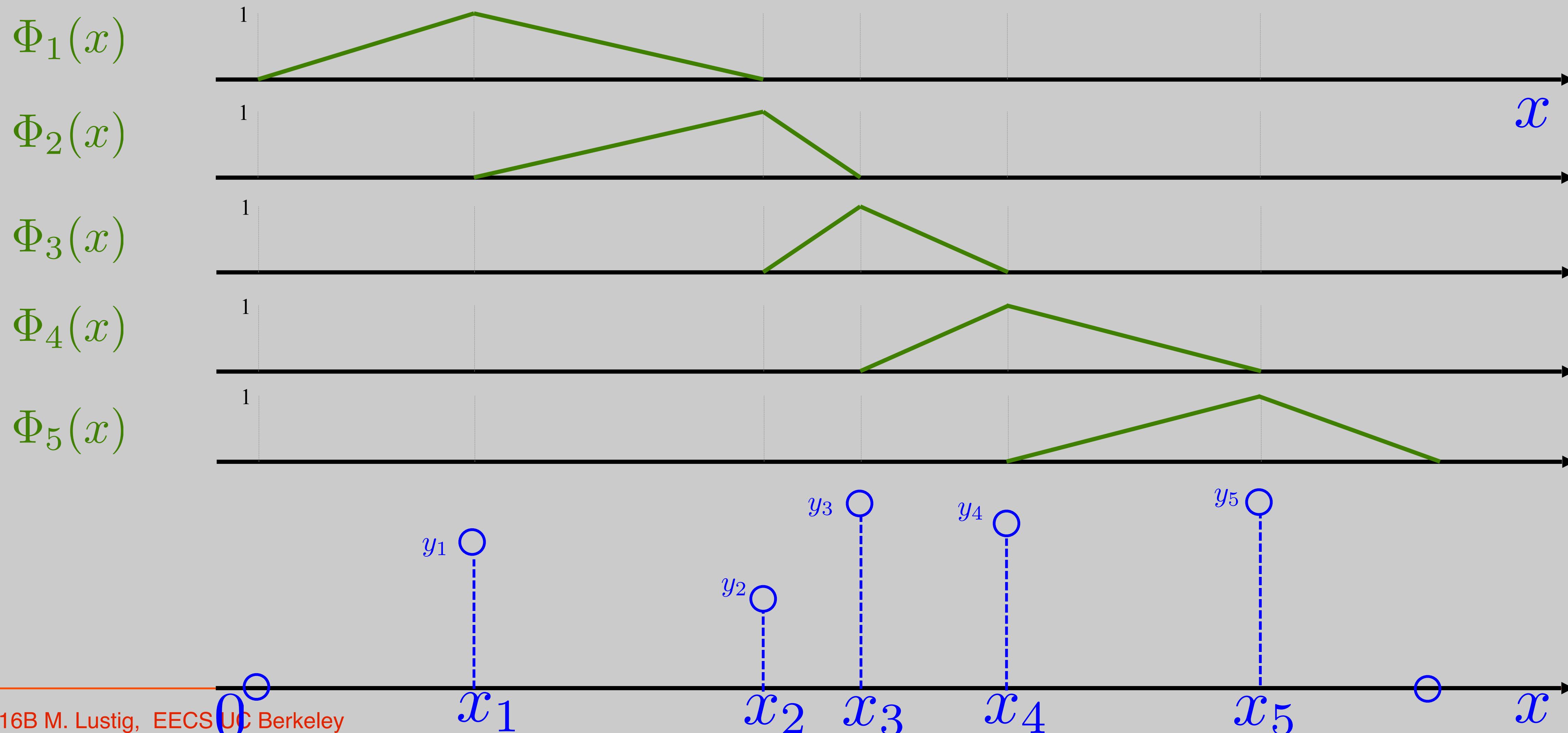


Interpolation with Basis Functions

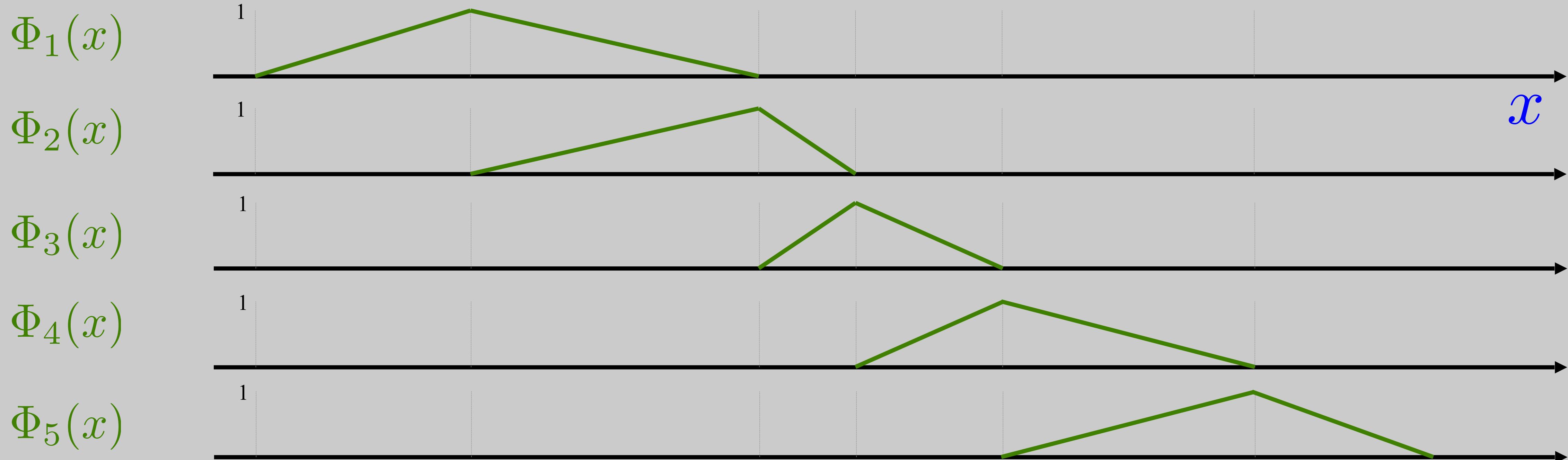
Given, $(x_i, y_i) \quad i=1, \dots, n$

Define a set of Cont. functions $\Phi_i(x)$

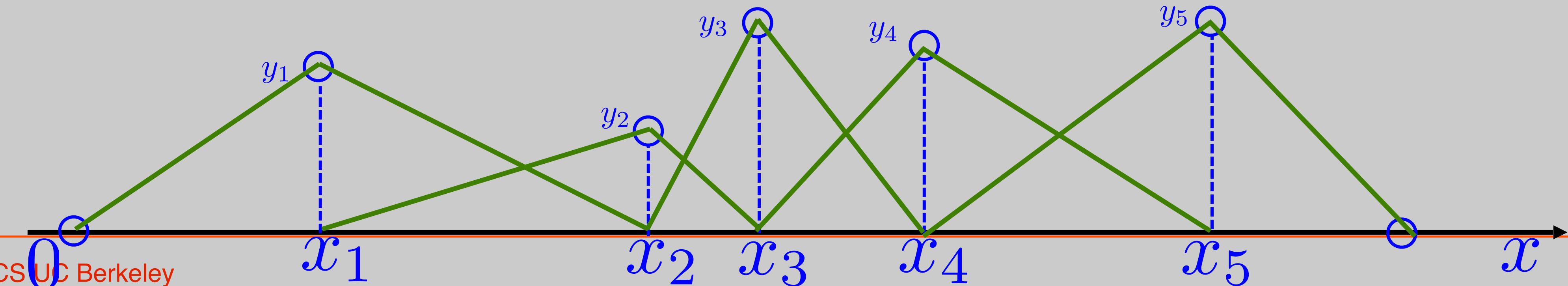
$$\begin{aligned}\Phi_i(x_i) &= 1 \\ \Phi_i(x_j) &= 0\end{aligned}$$



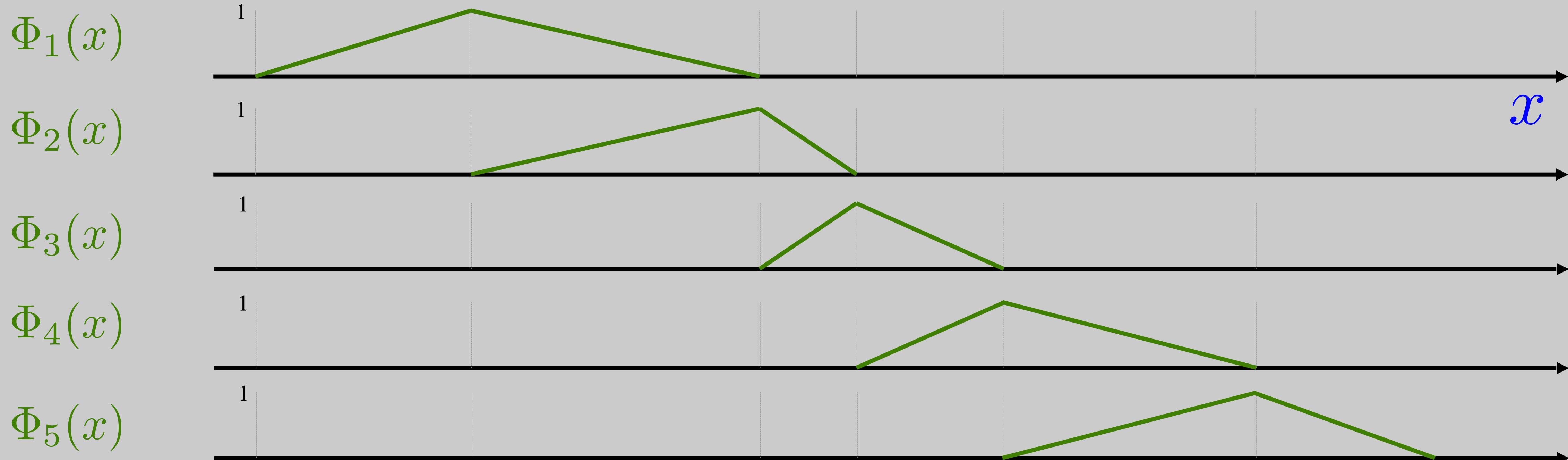
Interpolation with Basis Function



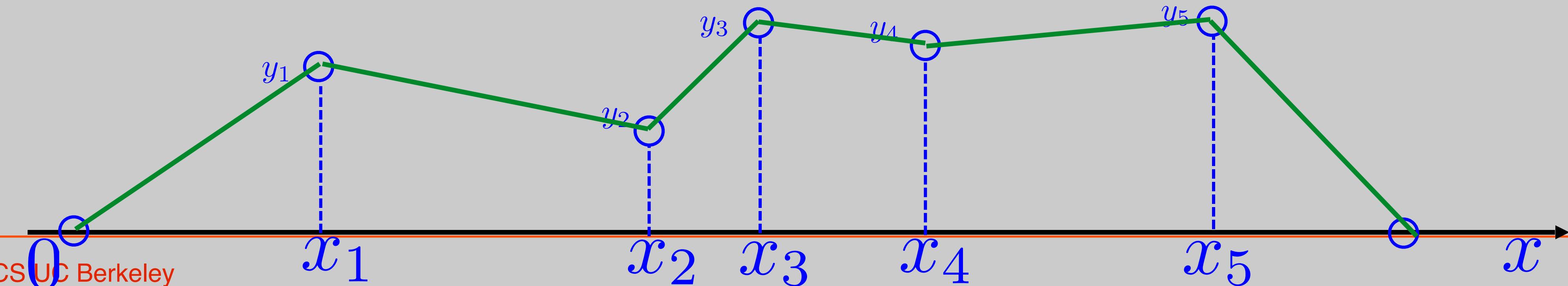
$$y(x) = y_1 \Phi_1(x) + y_2 \Phi_2(x) + y_3 \Phi_3(x) + y_4 \Phi_4(x) + y_5 \Phi_5(x)$$



Interpolation with Basis Function



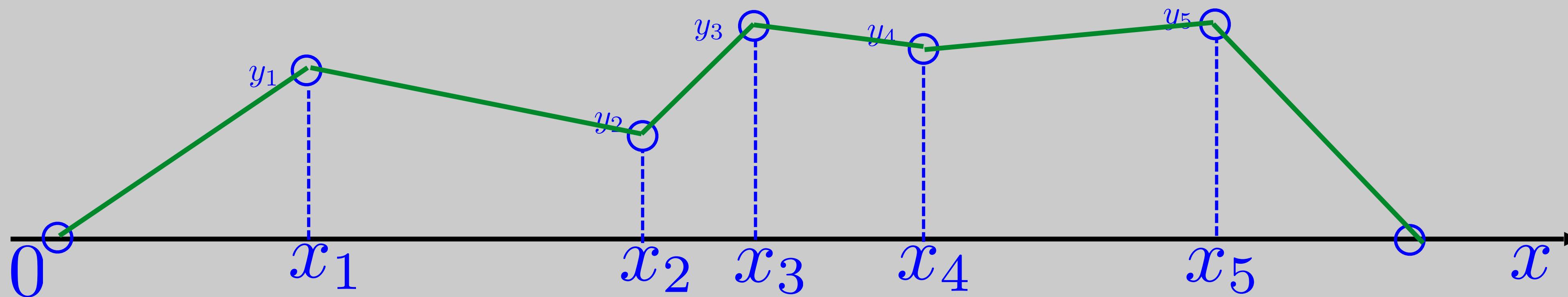
$$y(x) = y_1 \Phi_1(x) + y_2 \Phi_2(x) + y_3 \Phi_3(x) + y_4 \Phi_4(x) + y_5 \Phi_5(x)$$



Interpolation with Basis Functions

- Interpolation:

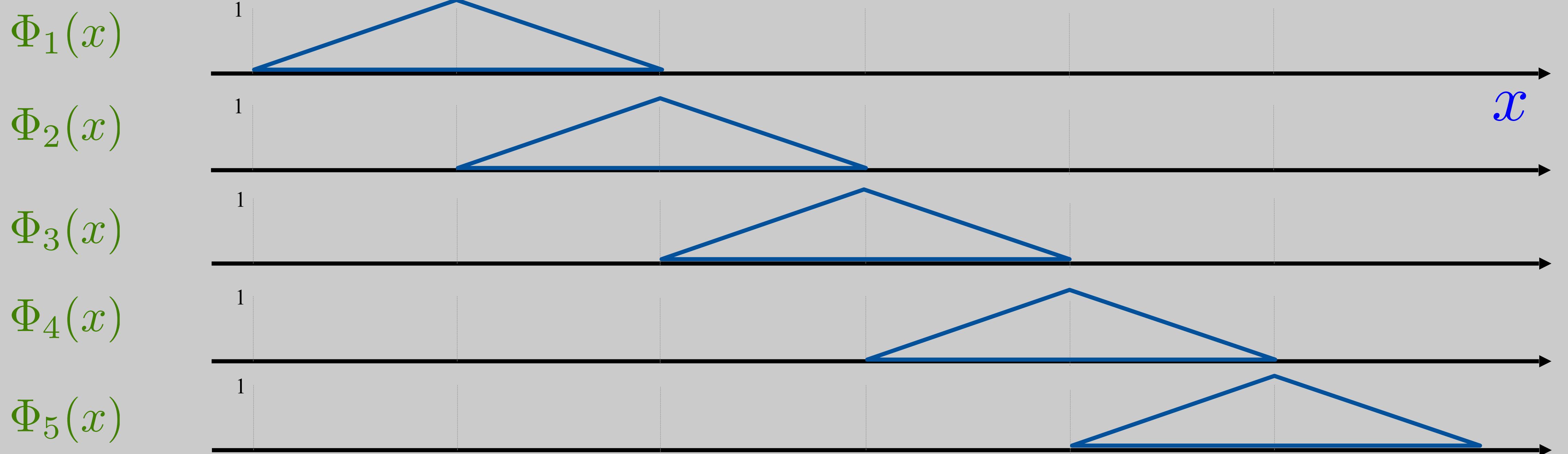
$$y(x) = \sum_{i=1}^n y_i \Phi_i(x) \Rightarrow y(x_i) = \sum_{i=1}^n y_i \Phi_i(x_i) = y_i$$



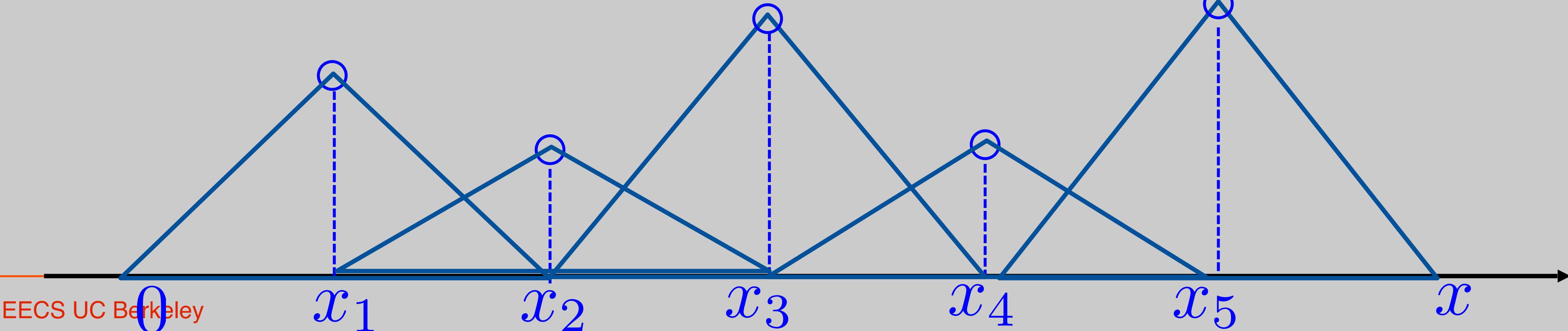
Interpolation with Basis Functions

- Guaranteed value of known points
- Control of continuous function behaviour between known points
- Often used for equispaced points
 - Can use a single “kernel” and shift it to get a basis

Equispaced Interpolation



$$y(x) = y_1\Phi_1(x) + y_2\Phi_2(x) + y_3\Phi_3(x) + y_4\Phi_4(x) + y_5\Phi_5(x)$$



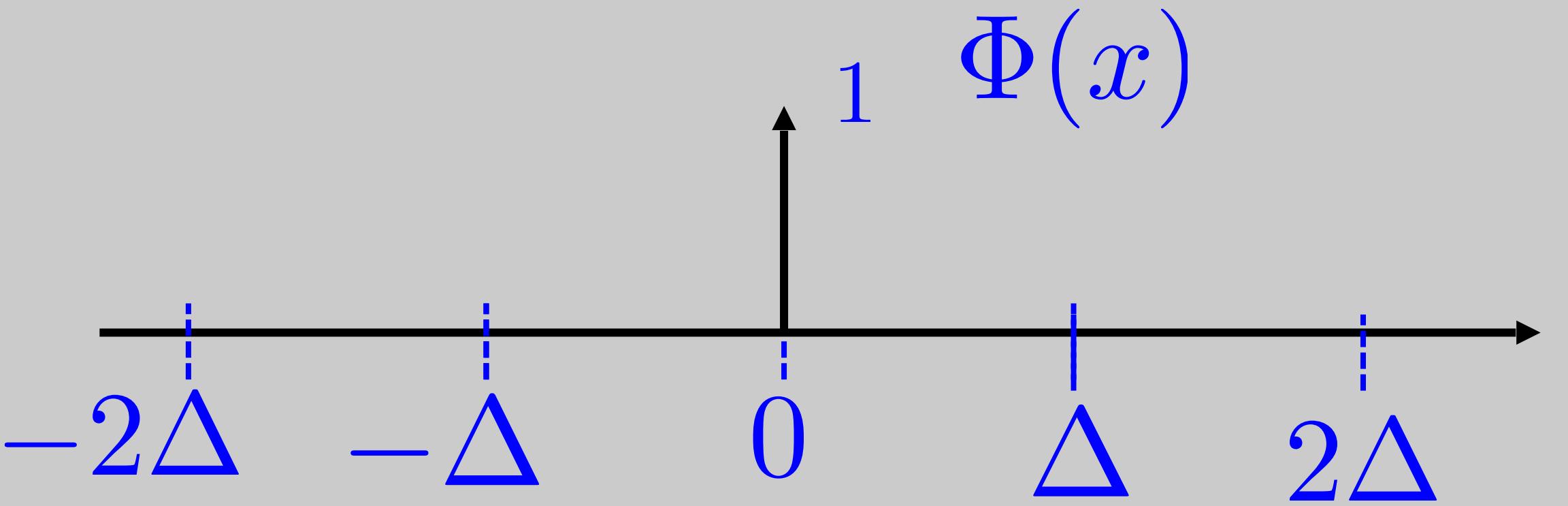
Kernel Interpolation, equispaced sampling

$$(x_i, y_i) \quad i = 1, 2, 3, \dots, N \quad x_{i+1} - x_i = \Delta \quad \forall i$$

Define: $\Phi(x)$

$$\Phi(0) = 1$$

$$\Phi(k\Delta) = 0 \quad k = \text{integer} \neq 0$$



$$y(x) = \sum_{k=1}^N y_k \Phi(x - k\Delta)$$

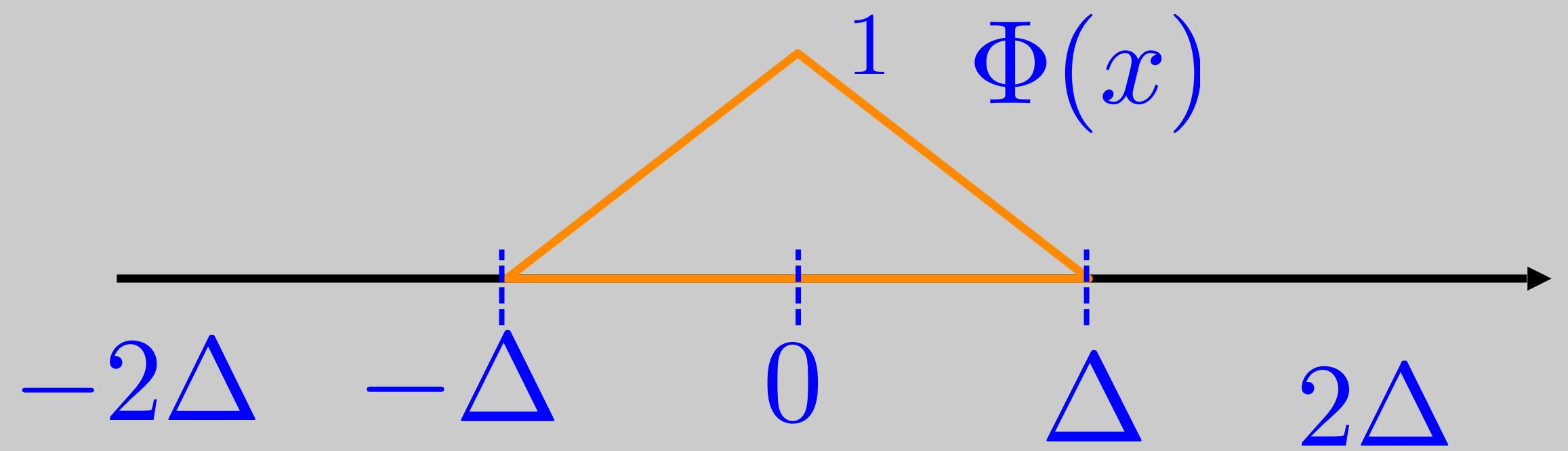
Kernel Interpolation, equispaced sampling

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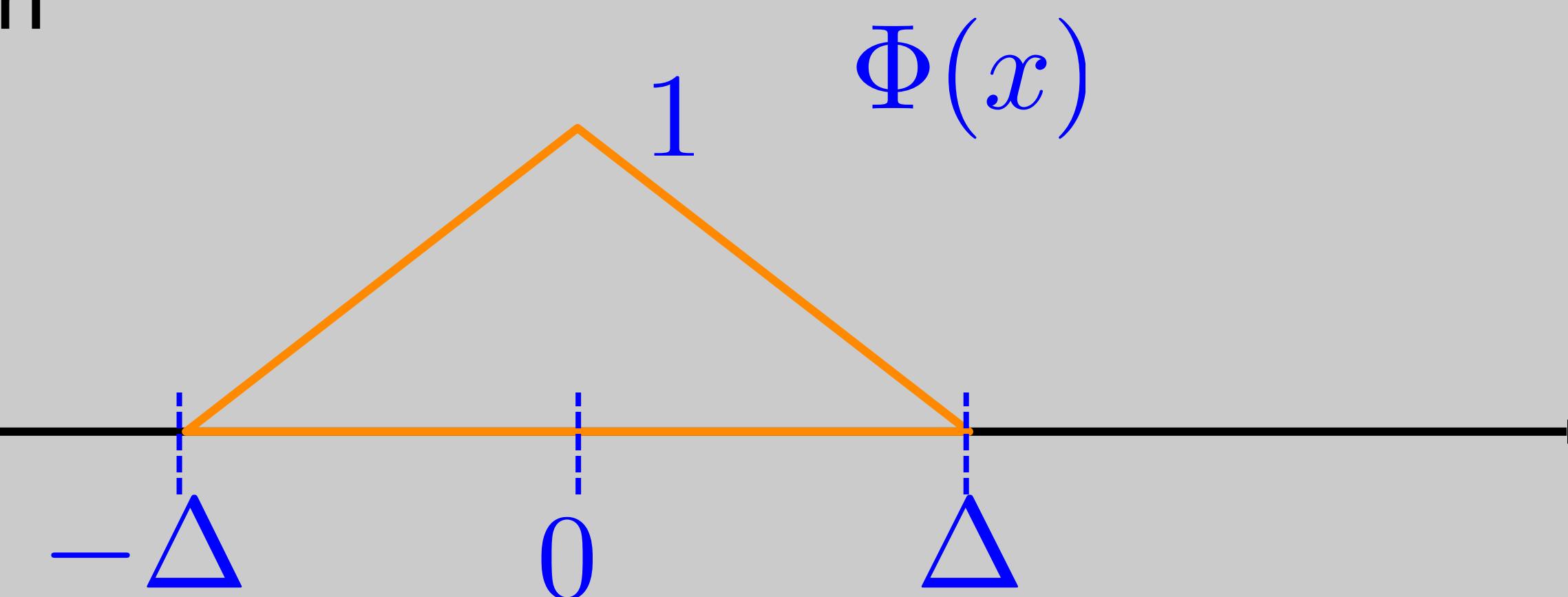
$$\Phi(k\Delta) = 0 \quad k = \text{integer} \neq 0$$



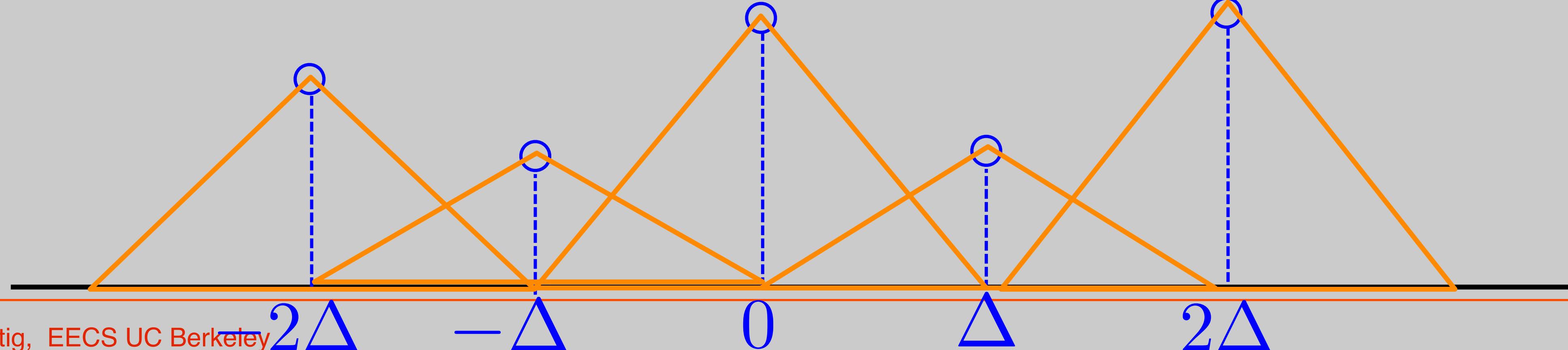
$$y(x) = \sum_{k=1}^N y_k \Phi(x - k\Delta)$$

Example:

- Linear Interpolation



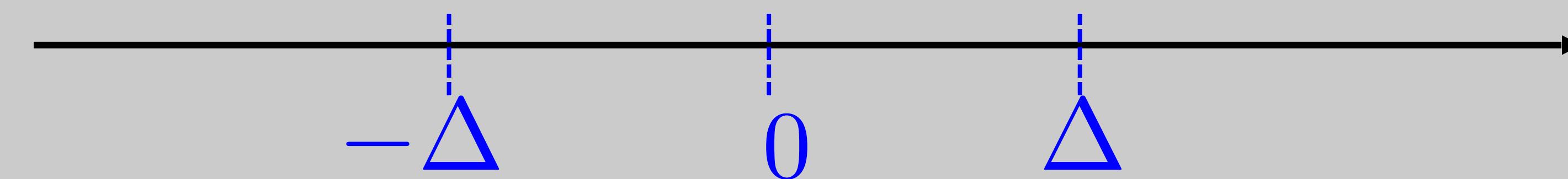
$$y(x) = \sum_{k=1}^N y_k \Phi(x - k\Delta)$$



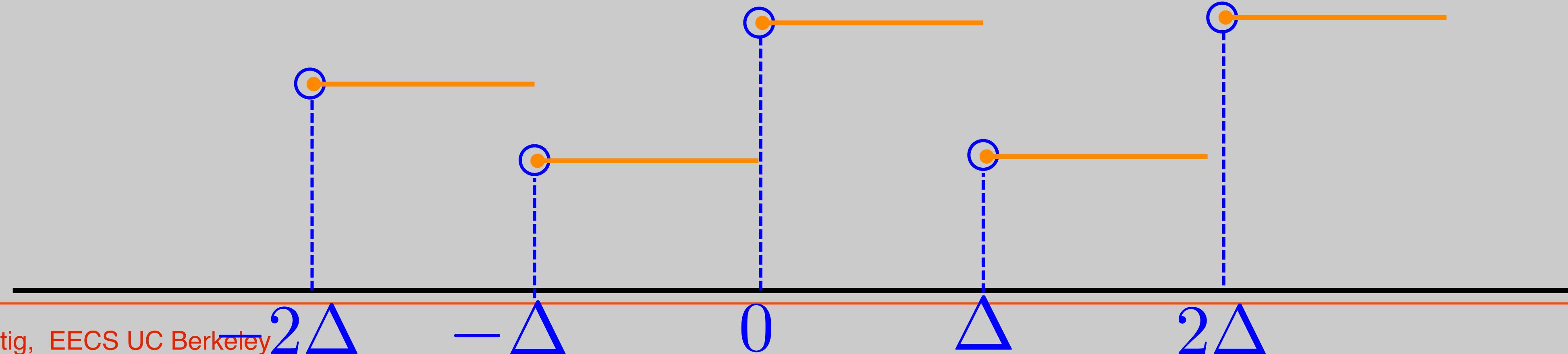
Example:

- Zero-Order Hold

$$1 \cdot \Phi(x)$$

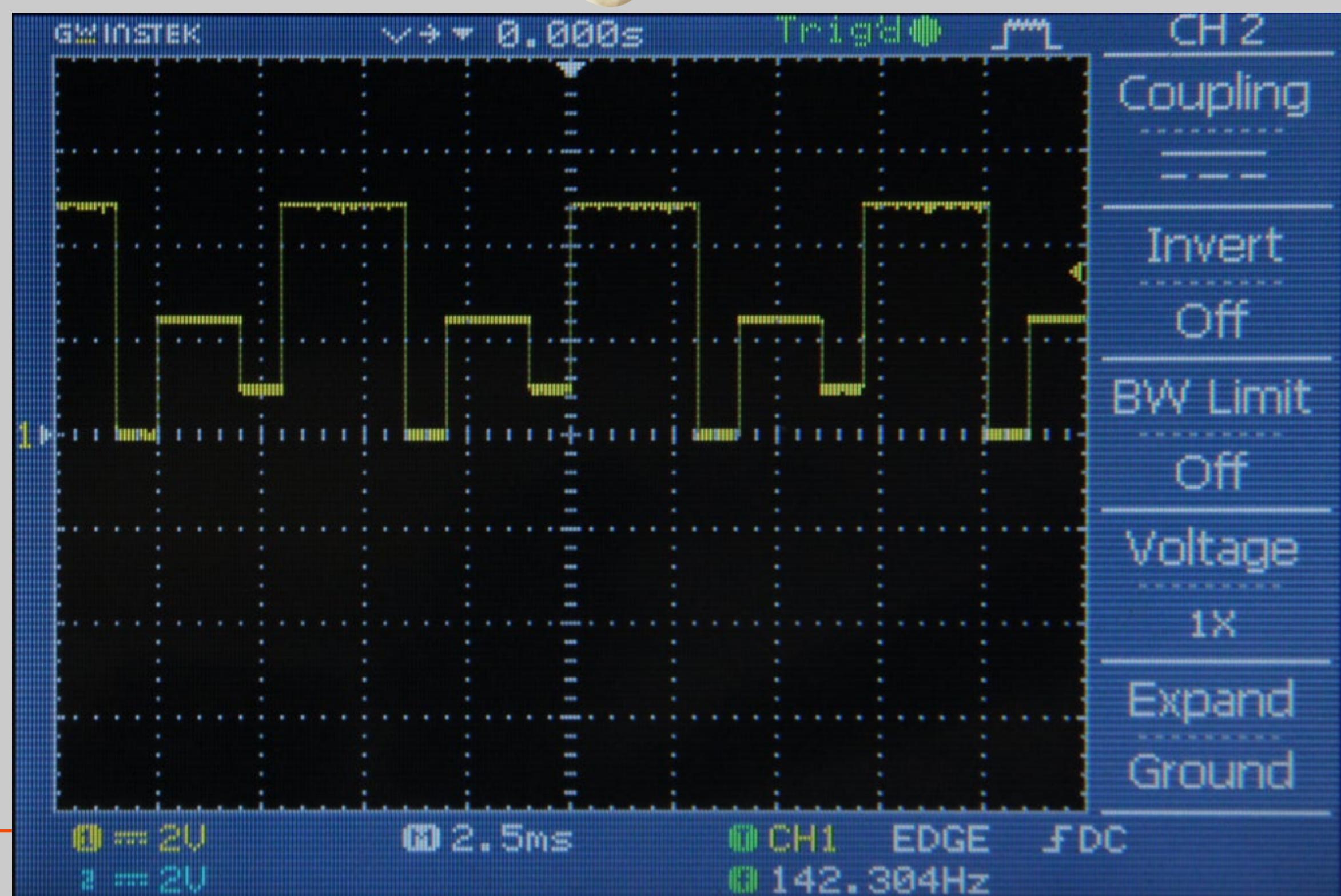
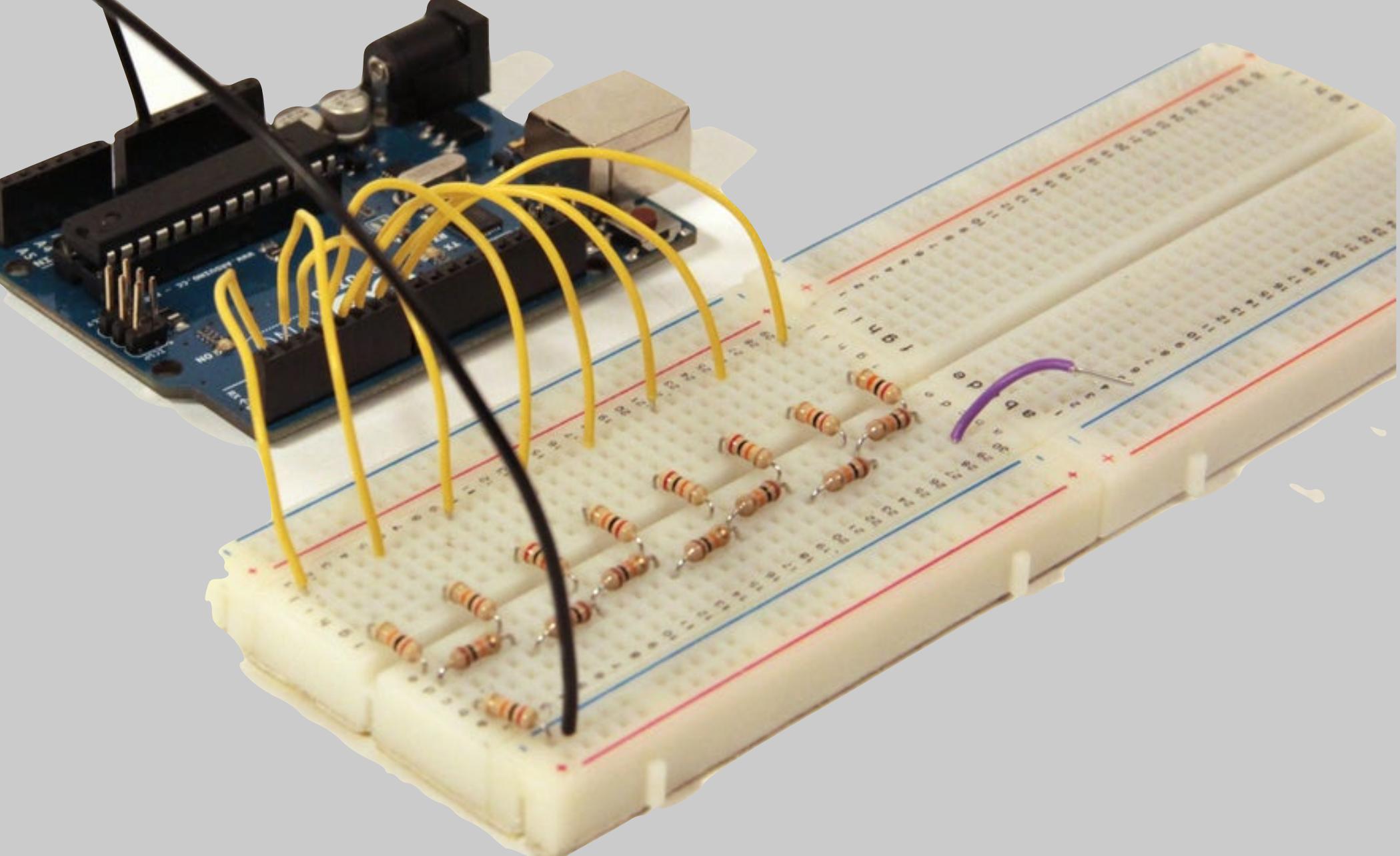


$$y(x) = \sum_{k=1}^N y_k \Phi(x - k\Delta)$$



DAC – interpolation!

```
void setup(){  
 //set digital pins 0-7 as outputs  
 for (int i=0;i<8;i++){  
   pinMode(i,OUTPUT);  
 }  
  
}  
  
void loop(){  
 PORTD = 0;//send (0/255)*5 = 0V out DAC  
 delay(1);//wait 1ms  
 PORTD = 127;//send (127/255)*5 = 2.5V out DAC  
 delay(2);//wait 2ms  
 PORTD = 51;//send (51/255)*5 = 1V out DAC  
 delay(1);//wait 1ms  
 PORTD = 255;//send (255/255)*5 = 5V out DAC  
 delay(3);//wait 3ms  
}
```



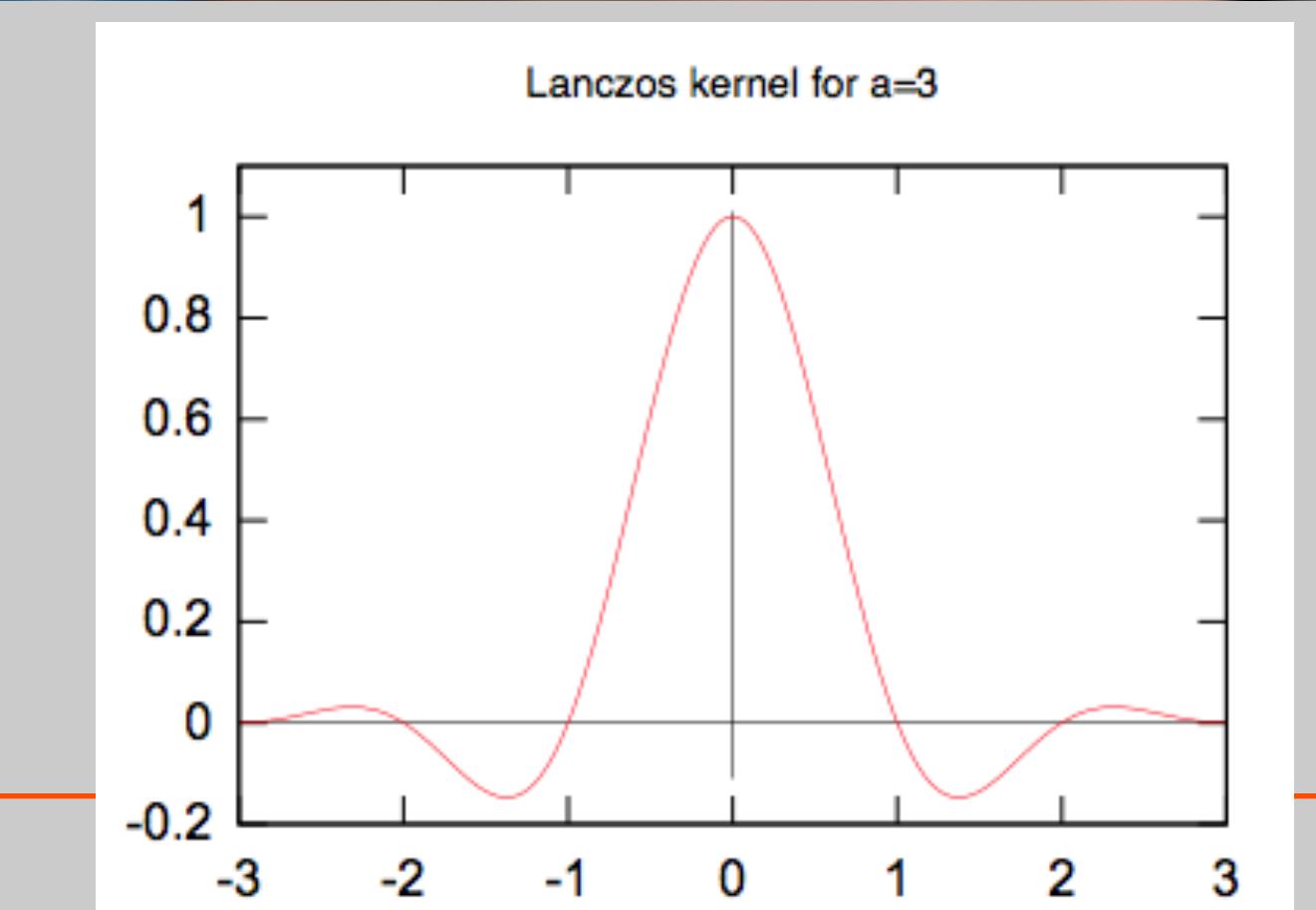
Interpolating Seth



Zero-order-hold



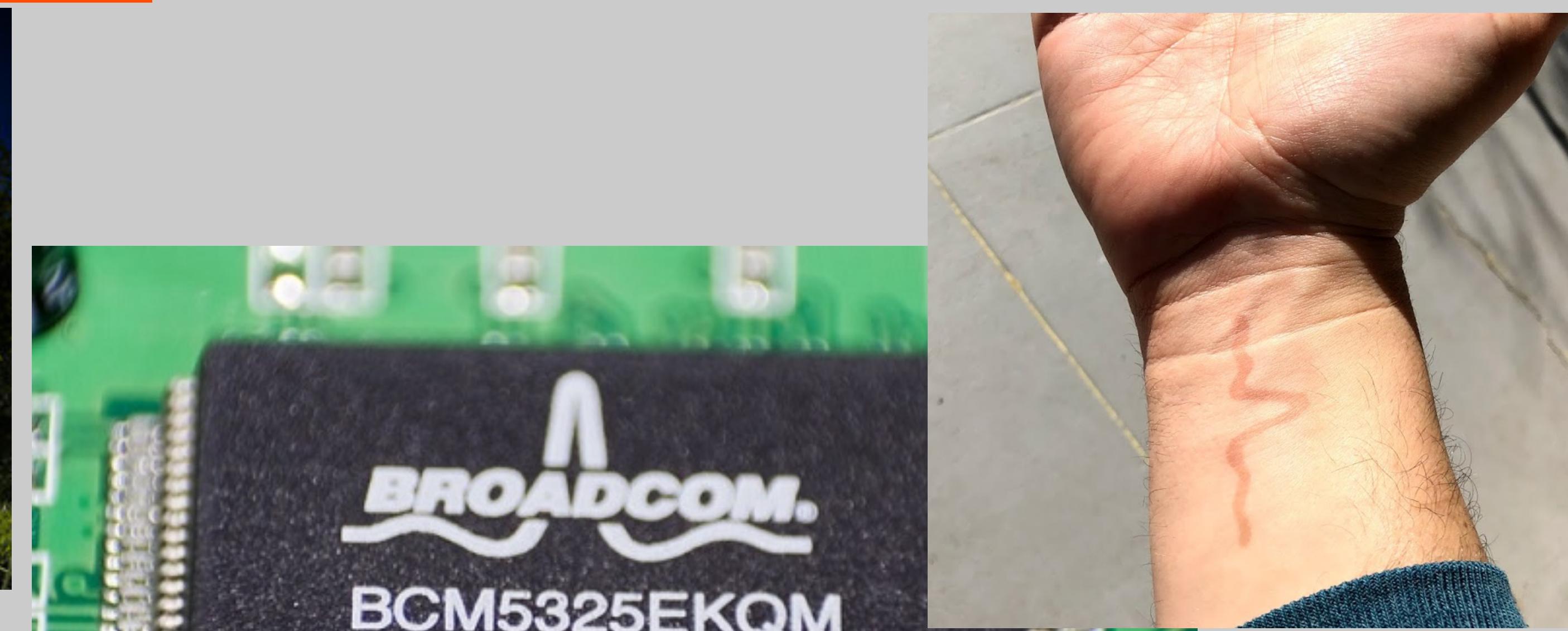
linear



Commercial Break



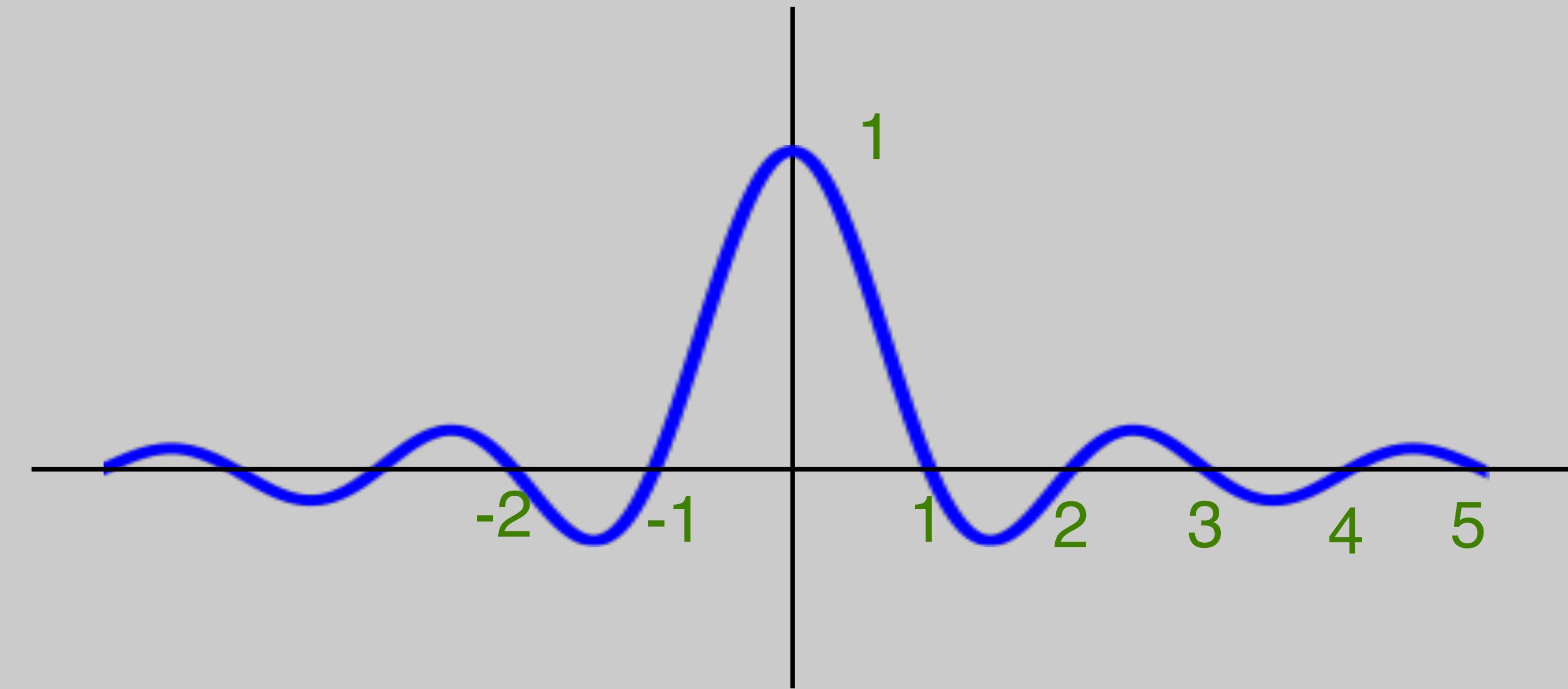
What is common to all these logos?



The sinc function

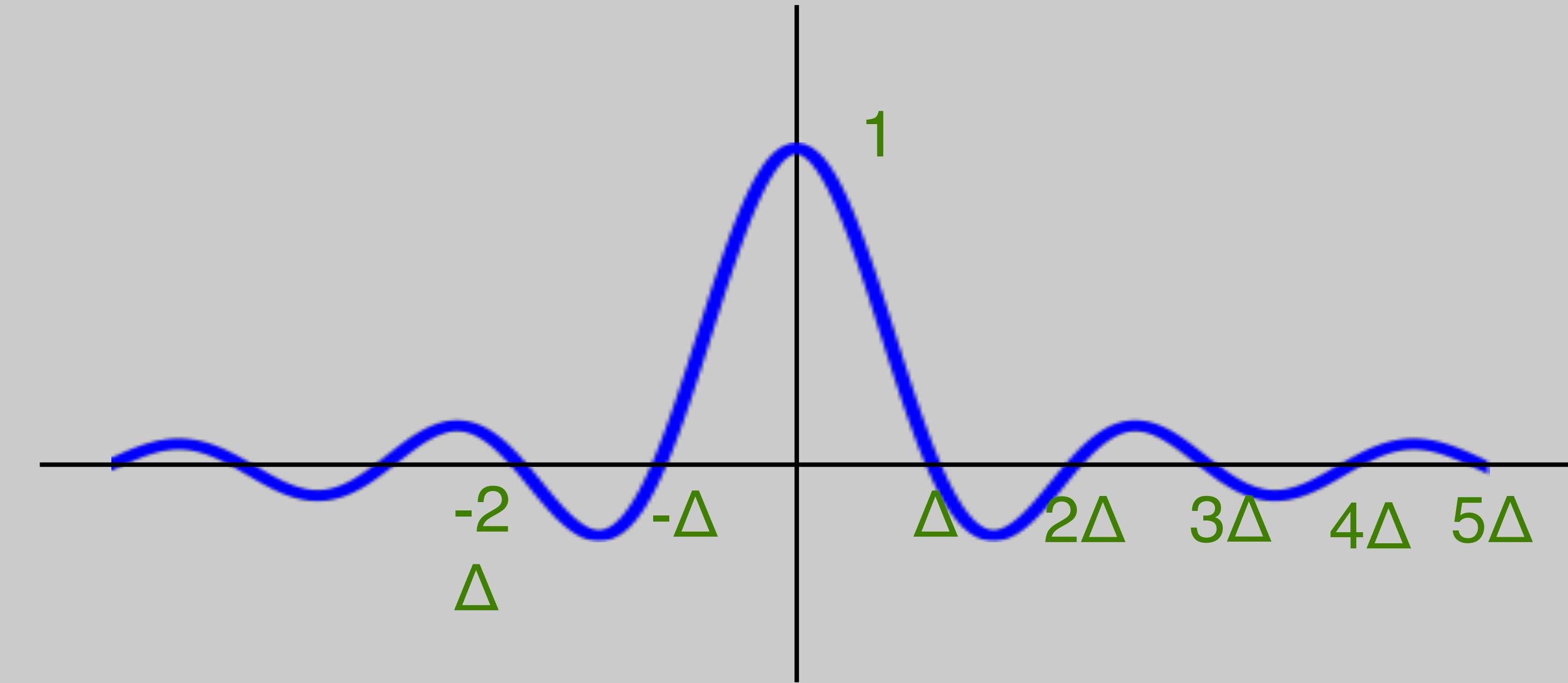
- Sinc:

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \Rightarrow \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$



The sinc function

- Let $\Phi(x) = \text{sinc}\left(\frac{x}{\Delta}\right)$
then $\Phi(k\Delta) = \text{sinc}(k) = ?$



The sinc function

- Let $\Phi(x) = \text{sinc}\left(\frac{x}{\Delta}\right)$

then $\Phi(k\Delta) = \text{sinc}(k) = \begin{cases} 0 & k \neq 0 \\ 1 & k = 0 \end{cases}$

Interpolation with sinc: $y(x) = \sum_{k=-\infty}^{\infty} y_k \Phi(x - k\Delta)$

The sinc function

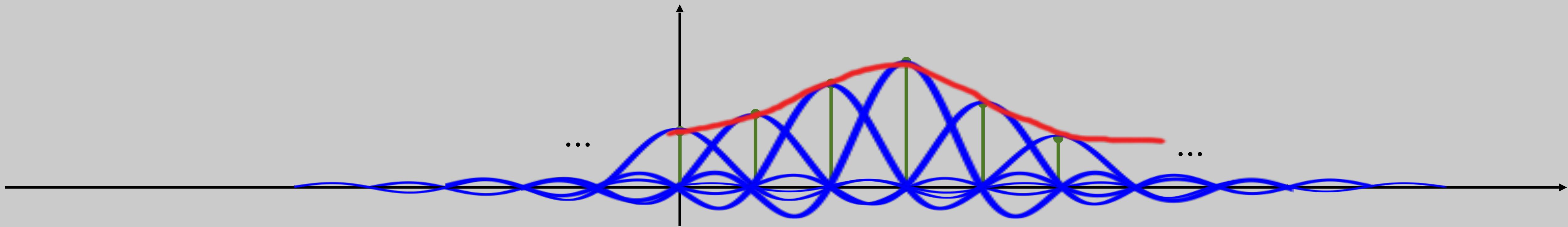
- Let

$$\Phi(x) = \text{sinc}\left(\frac{x}{\Delta}\right)$$

then

$$\Phi(k\Delta) = \text{sinc}(k) = \begin{cases} 0 & k \neq 0 \\ 1 & k = 0 \end{cases}$$

Interpolation with sinc: $y(x) = \sum_{k=-\infty}^{\infty} y_k \Phi(x - k\Delta)$



Bandlimitedness

- The sinc function does not contain frequencies beyond a certain bandwidth

$$\text{sinc}(x) = \frac{1}{\pi} \int_0^{\pi} \cos(\omega x) d\omega$$
$$\left. \frac{\sin(\omega x)}{\pi x} \right|_0^\pi = \frac{\sin \pi x}{\pi x} \quad x \neq 0$$

Sinc is an infinite sum of cosine functions with frequencies in the range $\omega \in [0, \pi]$

More in EE120, EE123!

Sampling and Recovery

- Due to Shannon – Nyquist

CLAUDE SHANNON, THE FATHER OF THE INFORMATION AGE, TURNS 1100100

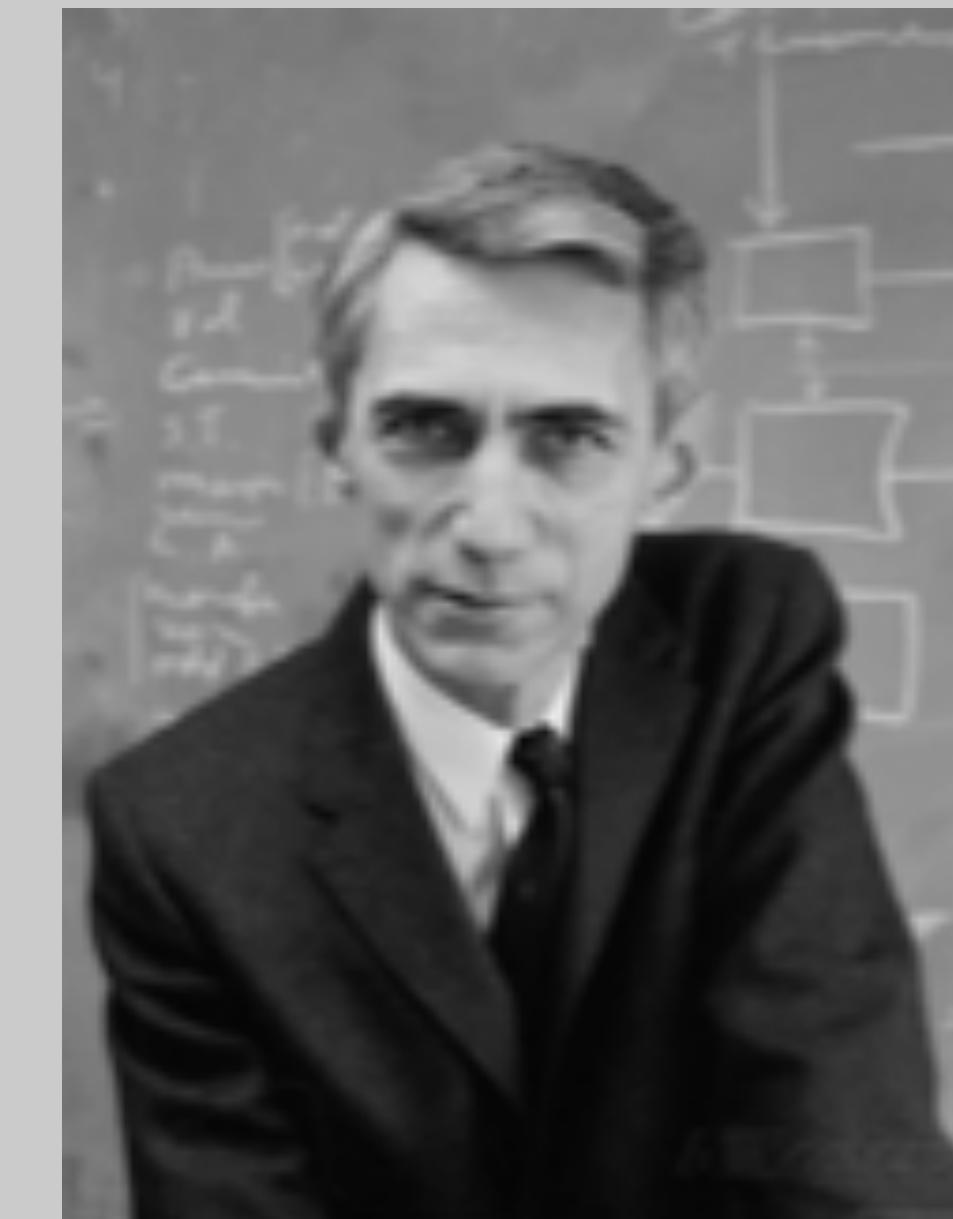
By Siobhan Roberts April 30, 2016

Twelve years ago, Robert McEliece, a mathematician and engineer at Caltech, won the Claude E. Shannon Award, the highest honor in the field of information theory. During his acceptance lecture, at an international symposium in Chicago, he discussed the prize's namesake, who died in 2001. Someday, McEliece imagined, many

A black and white portrait of Claude Shannon, an older man with glasses, wearing a suit and tie, looking slightly to the side.

A vertical column of four small circular icons representing social media platforms: Facebook, Twitter, LinkedIn, and Google+.

rarely. Yet he still tinkered, in the time he might have spent cultivating the big reputation that scientists of his stature tend to seek. In 1973, the Institute of Electrical and Electronics Engineers christened the Shannon Award by bestowing it on the man himself, at the International Symposium on Information Theory in Ashkelon, Israel. Shannon had a bad case of nerves, but he pulled himself together and delivered a fine lecture on feedback, then dropped off the scene again. In 1985, at the International Symposium in



Claude Shannon 1916-2001

Harry Nyquist 1889-1976

<https://www.newyorker.com/tech/elements/clause-shannon-the-father-of-the-information-age-turns-1100100>

Sampling and Recovery

- Can we perfectly recover an analog signal from its samples?

Analog signal:

$$y(x) = f(x)$$

Sample:

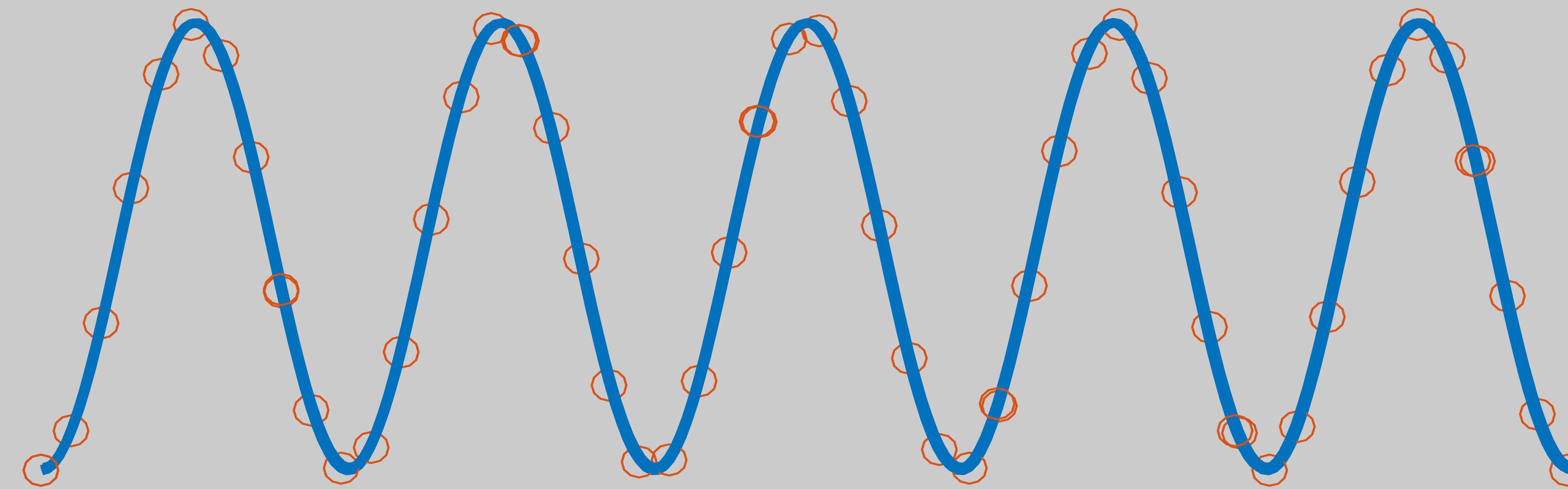
$$y[n] = f(n\Delta)$$

Interpolate:

$$\hat{f}(x) = \sum_{n=-\infty}^{\infty} y[n] \Phi(x - n\Delta) \quad =? f(x)$$

Sampling a sinusoid

- What rate should you be sampling a sinusoid?



Sampling Theorem

- If $f(x)$ is bandlimited by frequency ω_{\max} , then

$$f(x) = \hat{f}(x) = \sum_{n=-\infty}^{\infty} y[n] \Phi(x - n\Delta) \quad \Phi(x) = \text{sinc}\left(\frac{x}{\Delta}\right)$$

As long as,

$$\omega_{\max} < \frac{\pi}{\Delta}$$

$$\frac{\omega_{\max}}{\pi} < \frac{1}{\Delta}$$

$$2\frac{\omega_{\max}}{2\pi} < \frac{1}{\Delta}$$

$$2f_{\max} < f_s$$

$$\omega_s > 2\omega_{\max}$$

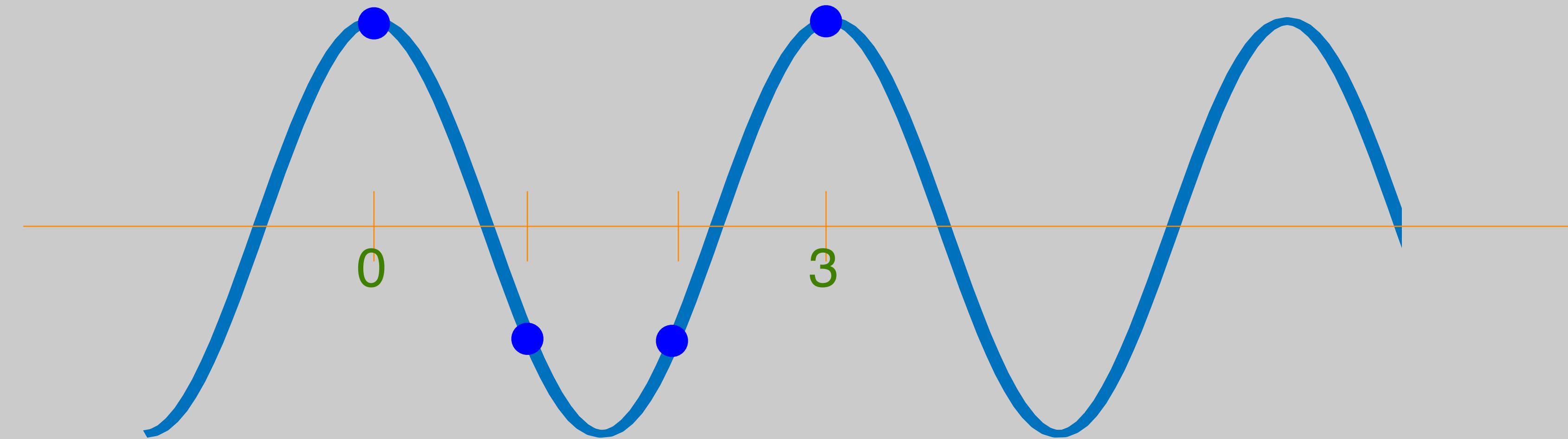
Proof: EE120, EE123

Example 1

$$f(x) = \cos\left(\frac{2\pi}{3}x\right)$$

$$\Delta = 1$$

$$\omega_{\max} < ? \frac{\pi}{\Delta}$$

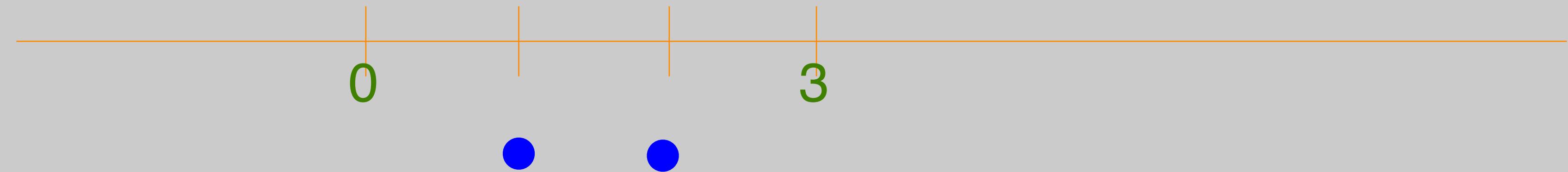


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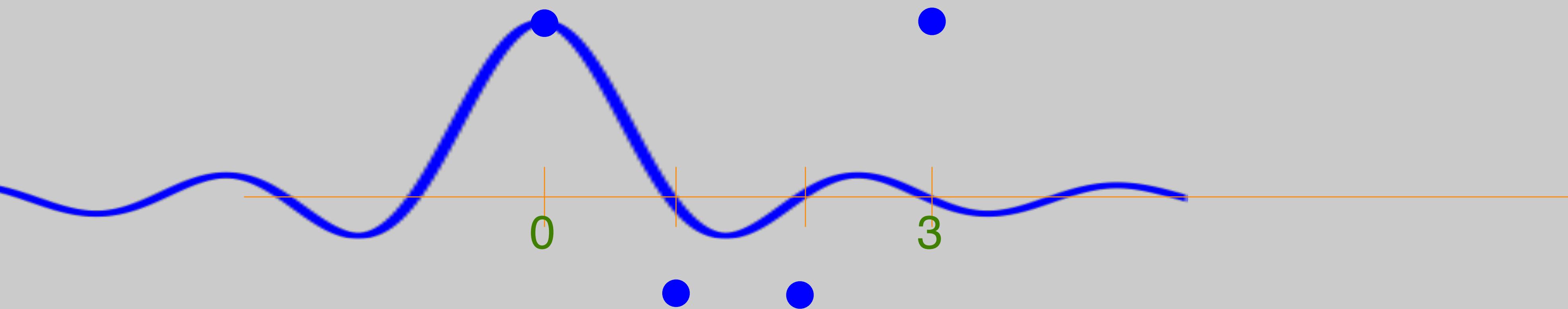


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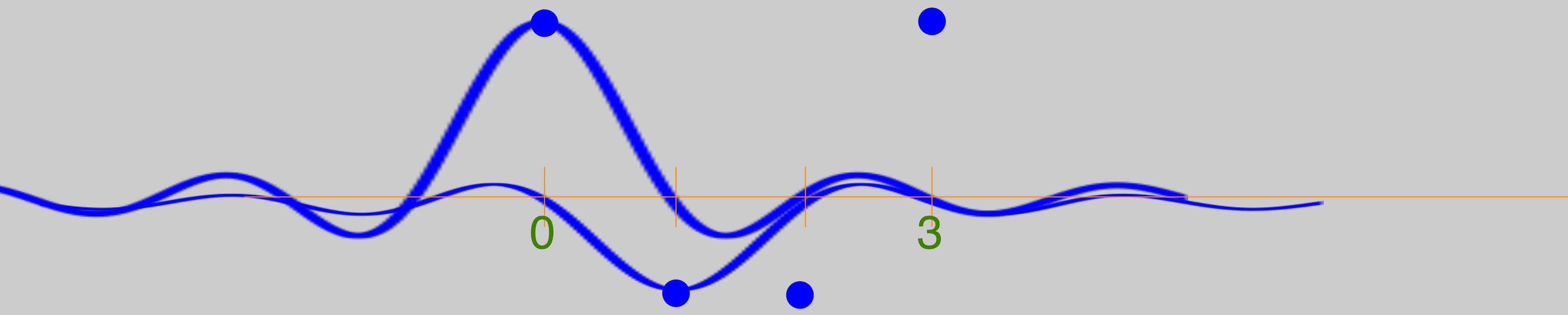


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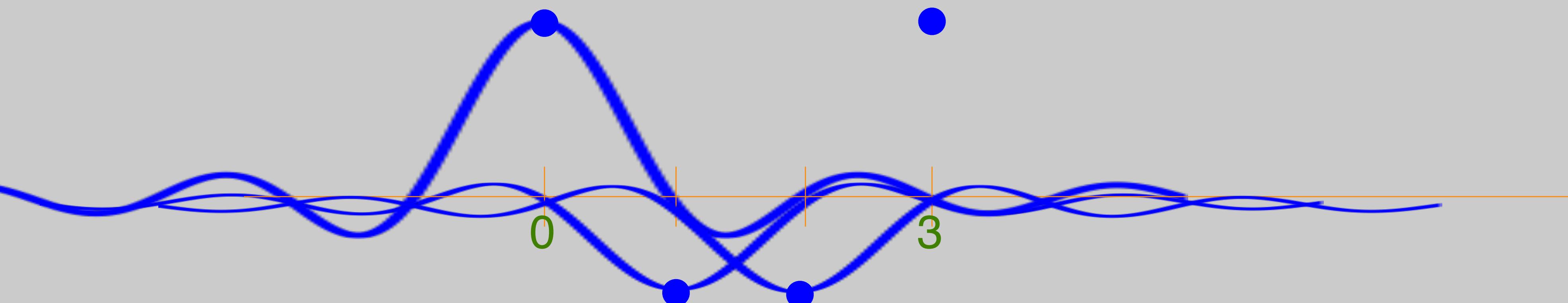


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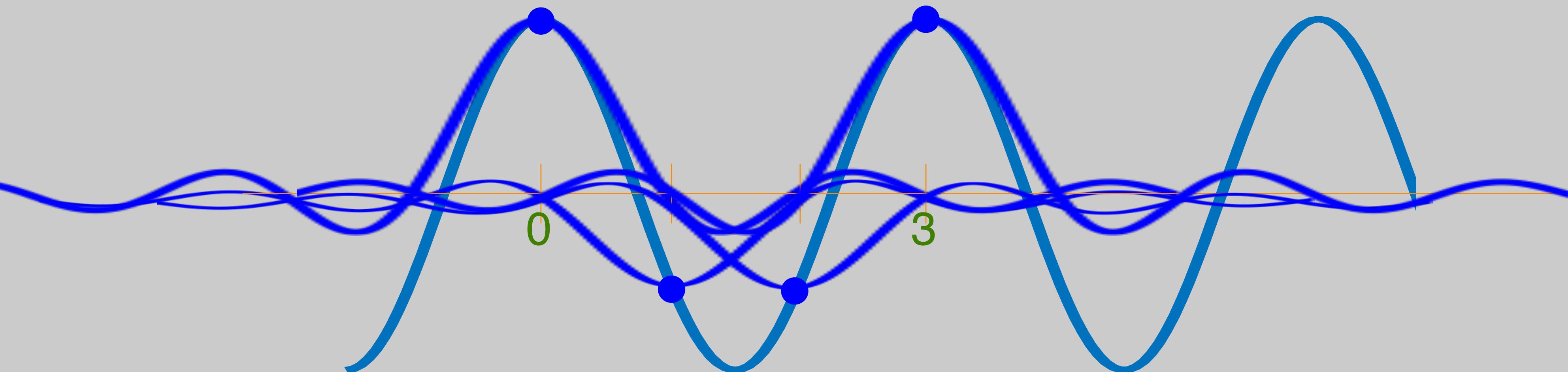


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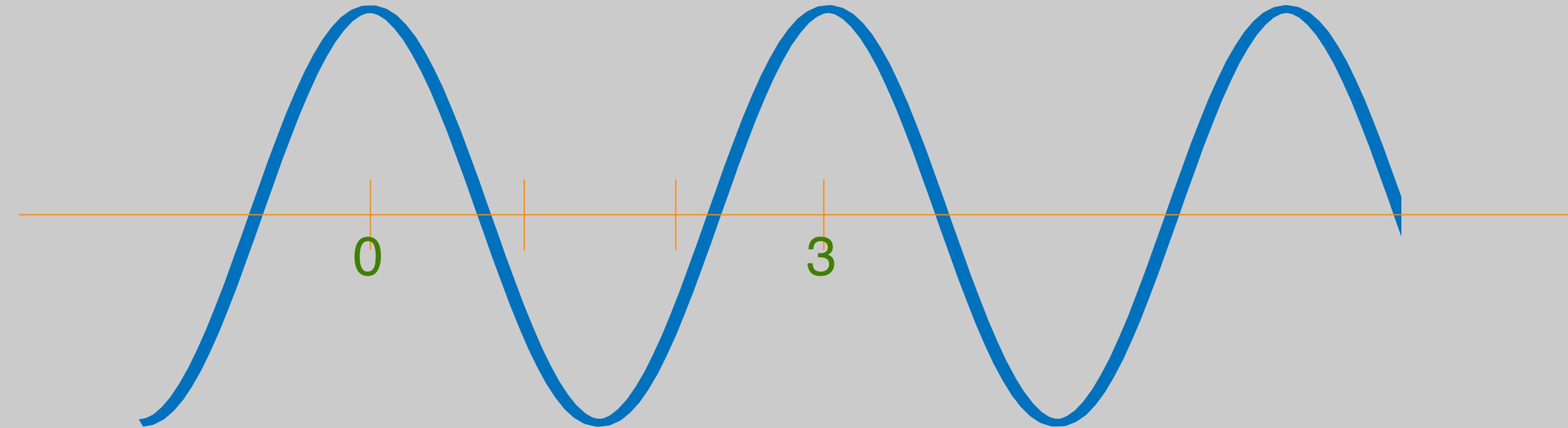


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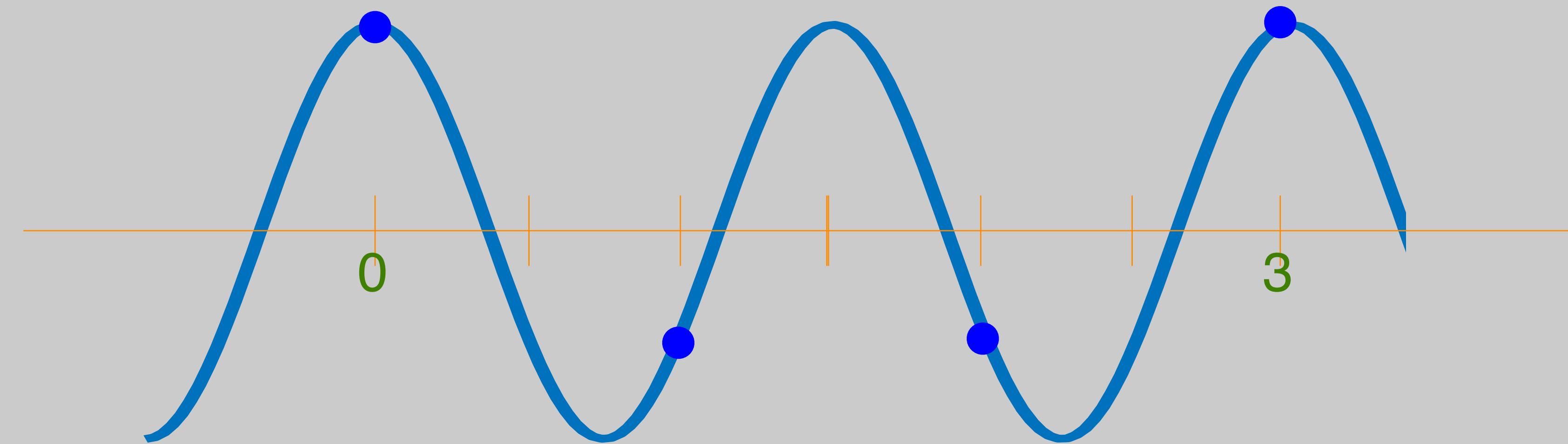


Example 2

$$f(x) = \cos\left(\frac{4\pi}{3}x\right)$$

$$\Delta = 1$$

$$\omega_{\max} < ? \frac{\pi}{\Delta}$$

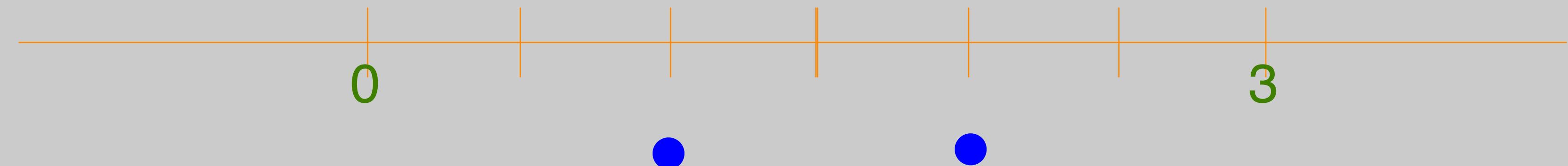


Example 2

$$f(x) = \cos\left(\frac{4\pi}{3}x\right)$$

$$\Delta = 1$$

$$\omega_{\max} < ? \frac{\pi}{\Delta}$$

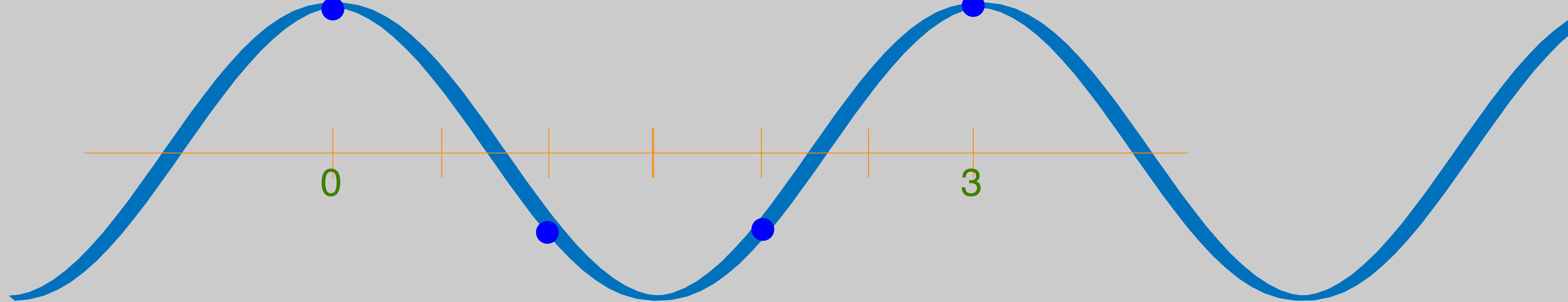


Example 2

$$f(x) = \cos\left(\frac{4\pi}{3}x\right)$$

$$\Delta = 1$$

$$\omega_{\max} < ? \frac{\pi}{\Delta}$$



- Sinc interpolation gives: $\hat{f}(x) = \cos\left(\frac{2\pi}{3}x\right)$

Aliasing of high frequencies
into lower ones!

Aliasing and Phase Reversal

$$f(x) = \cos(\omega x + \phi) \quad \Delta = 1$$

$$y[n] = \cos(\omega n + \phi)$$

- Highest interpolated frequency will not be higher than π

$$y[n] = \cos(\omega n + \phi) = \cos(2\pi n - (\omega n + \phi)) = \cos((2\pi - \omega)n - \phi)$$

$$\cos(2\pi n - \theta) = \cos(\theta)$$

If $\pi < \omega < 2\pi$ and $\Delta=1$, there's an equivalent lower frequency signal with the same samples!

$$\hat{f}(x) = \cos((2\pi - \omega)x - \phi)$$

Example 2

$$f(x) = \cos(\omega x + \phi) \quad \Delta = 1 \quad \omega = \frac{4\pi}{3} \quad \phi = 0$$

$$\hat{f}(x) = \cos((2\pi - \omega)x - \phi)$$

$$= \cos\left(\frac{2\pi}{3}x\right)$$

Example 3

$$f(x) = \sin(1.9\pi x) \quad \Delta = 1$$

$$= \cos(1.9\pi x - \frac{\pi}{2})$$

$$\hat{f}(x) = \cos(0.1\pi x + \frac{\pi}{2})$$

$$= -\sin(0.1\pi x)$$