

# EECS 16B    Designing Information Devices and Systems II

## Summer 2020

# Note 2

## 1 Introduction

In the previous note, we developed an introduction to understanding the speed of digital computers by modeling a ring oscillator using a chain of CMOS inverters. We then broke this model down using circuit analysis to get a circuit with a single resistor and capacitor and created a mathematical model to understand how quickly the output would switch from a logical 1 to a 0, a differential equation arose.

This differential equation was the key to modeling why computers can't run infinitely fast. The logical states of  $V_{DD} = 1$  and 0 are finite values representing the amount of charge stored on the capacitors from the CMOS transistors. In order to move this charge, it must travel through a resistor as current and since this current is finite, we saw that there indeed was a limit on the amount of charge that can be pulled out of the capacitor.

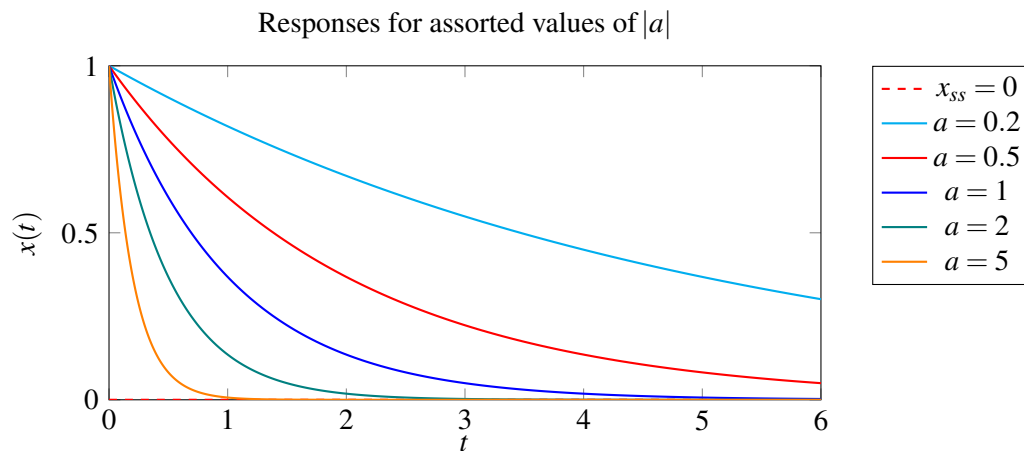
We then marched forward and provided a solution for the differential equation  $x(t) = x_0 e^{at}$  and were able to plot the response as a decaying exponential. Similarly when analyzing the case of switching between 0 and  $V_{DD}$  we created a non-homogenous differential equation  $\frac{d}{dt}x(t) = ax(t) + b$  with solution  $x(t) = x_0 e^{at} + \frac{b}{a}(e^{at} - 1)$ . Its plot was a rising exponential curve that grew slower as time went on and eventually reached a steady state of  $V_{DD}$ .

In this note, we'll further investigate how long the process of switching between 0 and 1 takes and also provide some insight into how we can speed up our circuits. We will then model the switching between 0 and  $V_{DD}$  as a voltage source  $v(t)$  with piecewise constant voltages. Our understanding of rise time and delay will play a crucial role in understanding the piecewise behavior of  $v(t)$ . Lastly, we introduce and analyze another circuit component called the **inductor** to model a circuit with a single resistor and inductor.

## 2 Time Constants

The differential equation  $\frac{d}{dt}x(t) + ax(t) = 0$  with initial condition  $x(0) = x_0$  is an exponential function  $x(t) = x_0 e^{-at}$ . Yet, when analyzing the switching between low and high states in our CMOS Inverter and writing out the differential equation in the form  $\frac{d}{dt}x(t) = \lambda x(t)$ , the constant  $\lambda$  is always negative. This means the term in the exponent  $-at$  will be negative for  $t \geq 0$ . When  $\lambda > 0$ , we can see that as  $t \rightarrow \infty$ , the response would blow up and  $x(t) \rightarrow \infty$ . We will further investigate into this phenomenon later in the controls module of the course, but for the time being, we can understand and realize that for an RC circuit,  $\lambda$  will always be negative meaning as  $t \rightarrow \infty$ , the exponential term will decay and the response  $x(t)$  eventually reaches a steady state.

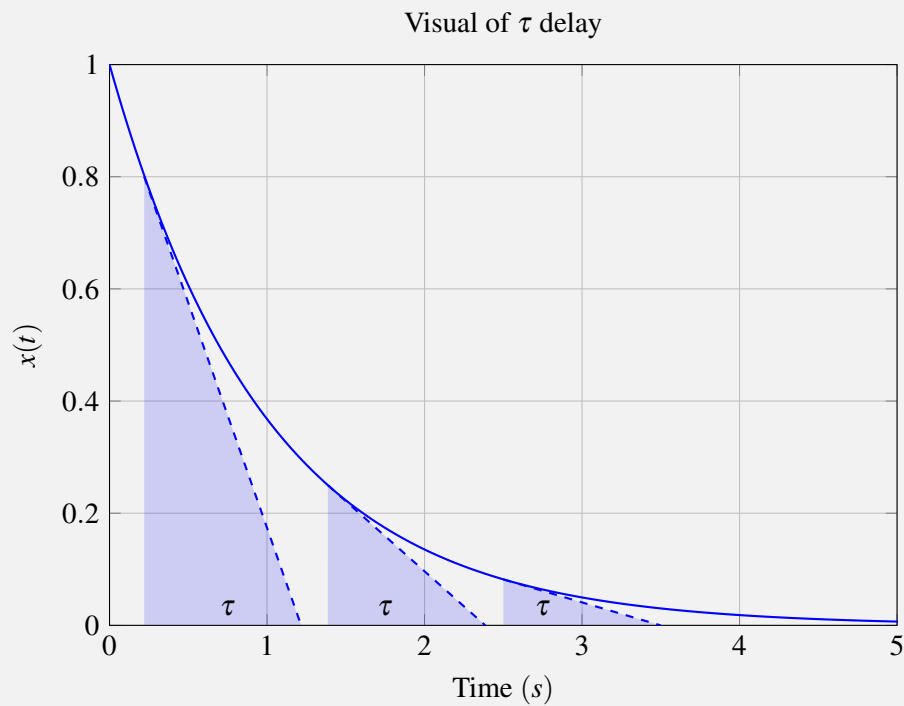
When discharging the capacitor, this steady state was 0 while in the charging case, this steady state was  $V_{DD}$ . Upon graphing and analyzing the function  $x(t) = x_0 e^{-at}$  where  $a > 0$ , we can make the key observation that if  $|a|$  larger, then the response reaches steady state quicker whereas if  $|a|$  is smaller, the response takes longer to reach steady state. We provide some plots below to illustrate this with multiple reference values for  $a$ .



Upon inspection, the  $|a|$  has a huge impact on the speed of the response and the time it takes for it to reach steady state. Therefore we will define the **time constant** denoted by the Greek letter  $\tau$  as a measure of how long our differential equation takes to reach steady state. Mathematically it is defined as the time it takes for be within  $\frac{1}{e}$  of its steady state value. For an exponential that decays to 0, this would be the time at which the response decays to  $\frac{1}{e} = 36.8\%$  of its initial value whereas for a rising exponential, this would be the time at which the response rises to  $1 - \frac{1}{e} = 63.2\%$  of its steady state value.

## The Secret Behind $e$

Now you might question why we specifically use the value of  $\frac{1}{e}$  as opposed to a more tangible value such as 50%. An amazing observation we can make is that if we were to take the instantaneous rate of decay at any point in time,  $\frac{d}{dt}v(t_0)$ , it just happens that if the decay stayed constant after  $t_0$ , it would take exactly  $\tau$  seconds from the point  $t_0$  for the response to decay to 0.



Looking back at our  $RC$  circuit modeled by the differential equation  $\frac{d}{dt}v(t) = -\frac{1}{RC}v(t)$ , with initial condition  $v(0) = V_{DD}$ , we can solve for the time constant by finding the time  $\tau$  at which  $v(\tau) = \frac{V_{DD}}{e}$ .

$$v(\tau) = V_{DD}e^{-\frac{\tau}{RC}} = \frac{V_{DD}}{e}$$

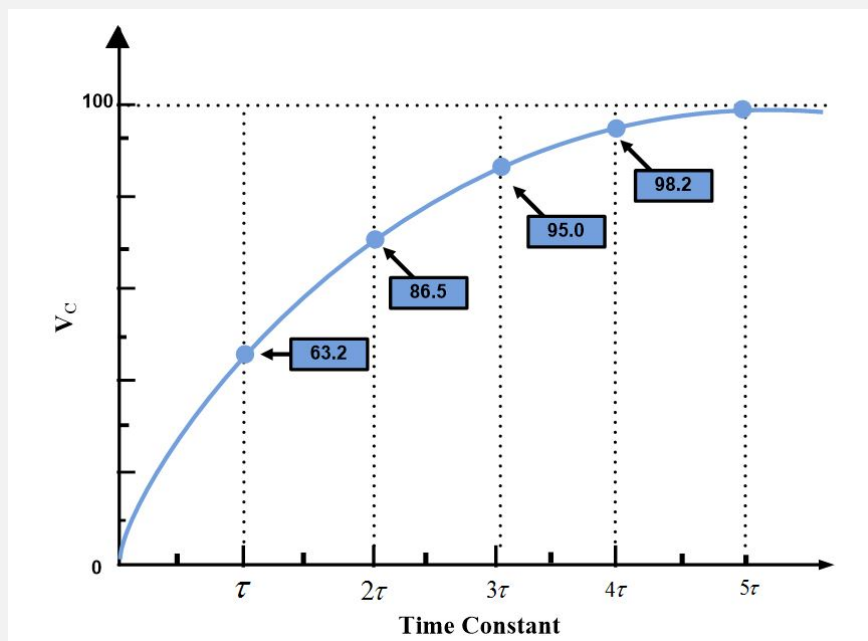
A quick computation tells us that our time constant  $\tau = RC$ . This immediately tells us that the physical values of the resistor and capacitor are what affect the speed at which our response decays to 0 or rises to  $V_{DD}$ . If we wanted to speed up the response by lowering  $\tau$ , we would have to either lower the value of our resistor or capacitor.

An important concept to note from looking at the exponential function is that  $x(t)$  will never truly reach its steady state 0 or  $V_{DD}$ . In the context of  $RC$  Circuits, this would mean that the capacitor will never fully charge or discharge. However, when switching between logical high and low values in a CMOS Inverter, we can make the crucial realization that the capacitor will stop charging / discharging once the voltage  $V_{out}$  becomes greater than the threshold values  $|V_{tp}|$  and  $V_{tn}$  for the transistors to switch on or off.

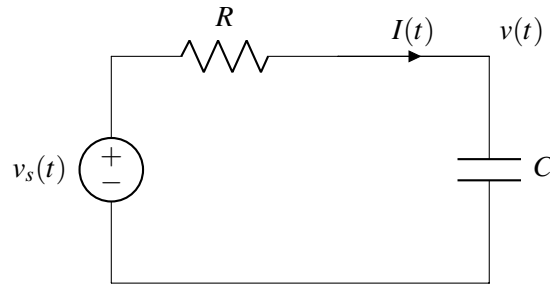
We've shown a diagram below as a reference depicting how much a capacitor charges after a certain number of time constants. From the diagram we see that after  $3\tau$ , the capacitor has charged up to 95% and after  $5\tau$  the response will be within 1% of its steady state value.

## How many $\tau$ will it take?

With our new definition of a time constant  $\tau$  we have not only understood how long it takes for our differential equation to reach steady state, but we have also created a metric by which we can measure how close to steady state our response will be after a specified period of time.



### 3 Piecewise Constant Time Varying Inputs



**Figure 1:** Capacitor charging through a circuit with a resistor. We can imagine that the voltage source is changing with time.

In the previous note we learned to solve for the transient voltage  $v(t)$  on a capacitor charging up through a resistor. Recall that we solved the following differential equation:

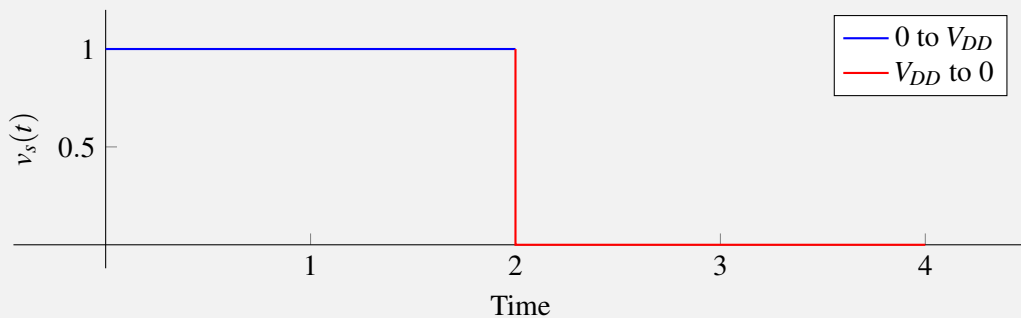
$$\frac{d}{dt}v(t) = -\frac{v(t)}{RC} + \frac{V_{DD}}{RC}$$

to get the solution  $\forall t \geq 0$ :

$$v(t) = V_{DD}(1 - e^{-\frac{t}{RC}}). \quad (1)$$

We had viewed the “input” as coming from transistor switches reconfiguring themselves. When switches changed from one state to another, what held steady across that instantaneous switch was the charge (and hence voltage) on capacitors. The previous configuration’s end state provided the initial condition for the next configuration. Alternatively however, we can also think of the voltage that we used to charge up the capacitor as an input to our circuit and allow that voltage  $v_s(t)$  to change with time. As far as the capacitor was concerned in the previous note, it might as well have faced a piecewise constant input that had  $v_s(t) = 0$  for  $t < 0$  and  $v_s(t) = V_{DD}$  for  $t \geq 0$ .

#### Illustration of Piecewise Constant Inputs

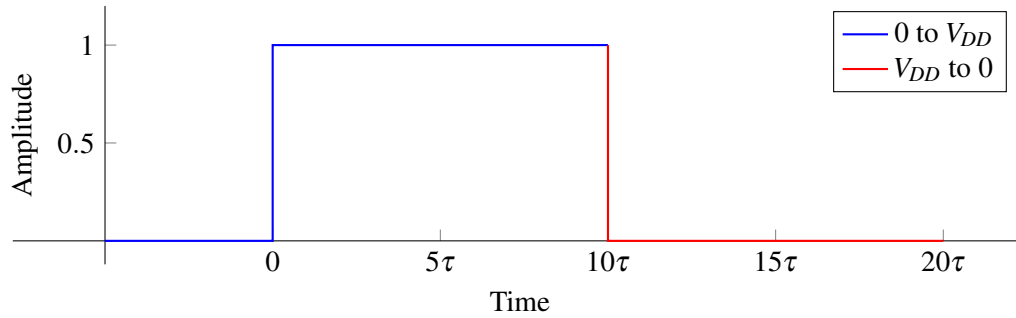


We provide a piecewise constant example of  $v_s(t)$  where  $v_s(t) = 1$  for  $t \in [0, 2)$  and  $v_s(t) = 0$  for  $t \in [2, 4]$ . Using this information, we can write a differential equation for  $v_c(t)$  in each interval and treat  $v_s(t)$  as a constant 1 or 0. Can we take what we know to build to understand more interesting inputs?

### 3.1 Two illustrative cases

Having analyzed these basic cases we want to consider how to deal with inputs that change over time in a more interesting fashion. We have a strategy that we think should work — treat piecewise constant inputs the same way that we dealt with circuits with switches changing configuration. Make the state (charge on the capacitor) be instantaneously constant across the configuration change, and just solve the differential equation with that initial condition.

Let us start by considering the most basic changing input that we can think of: A voltage turning on to some value  $V_{DD}$  and then turning off.



**Figure 2:** On and Off input: On for  $10\tau$ . Here  $\tau = RC$  is the RC time constant for the circuit.

As always when analyzing these more complex problems we try to phrase them in terms of problems that we already know how to solve. We can look at this case as a combination of two piecewise constant cases: A constant zero input steady till some time  $T$  switching instantly to a steady constant 1 input till time  $T + D$  (here  $D$  is some constant representing how long we hold at  $V_{DD}$ ), falling back to zero again for the rest of time beyond  $T + D$ .

We saw in the previous section, that if  $D \gg \tau$  then the circuit has the opportunity to settle to steady state. We treat the circuit in 2 different time intervals. The first with initial condition at 0 and the second with initial condition at  $V_{DD}$ : the value that the circuit settled to in the first interval (from  $T$  to  $T + D$ ).

Before we continue let us establish some notation. Let us use  $V_i(t_{int})$  to denote the voltage on the capacitor during the  $i^{th}$  time interval that we are analyzing. Let  $t$  be absolute time starting at 0 while let  $t_{int}$  be the time from the beginning of the  $i^{th}$  interval till now. This latter time interval to the interval is useful conceptually.

Let us start by analyzing the first interval:

Analyzing the circuit for time  $t \in [0, 10\tau]$  with initial condition  $V(0) = 0$  and constant input  $V_{DD}$  starting at time  $t = 0$ , we get the differential equation:

$$\begin{aligned} \frac{d}{dt}V(t) &= -\frac{V(t)}{RC} + \frac{V_{DD}}{RC} \\ V(0) &= 0 \end{aligned}$$

Recall the solution to this type of differential equation is:

$$V(t) = V_{DD}(1 - e^{-\frac{t}{RC}})$$

However, remember that this solution is not valid for all  $t$  and will only be valid for  $t \in [0, 10\tau]$ .

$$V_1(t_{int}) = V(t) = V_{DD}(1 - e^{-\frac{t}{RC}}) \quad t \in [0, 10\tau]$$

Furthermore, from the previous section we established that after a large number of time constants, the circuit will have the time to settle at a steady state. Plugging in  $t = 10\tau$ , we see that:

$$\begin{aligned} V(10\tau) &= V_1(10\tau) = V_{DD}(1 - e^{-10}) \\ &\approx V_{DD}(1 - 0.00004539992) \\ &\approx V_{DD}. \end{aligned}$$

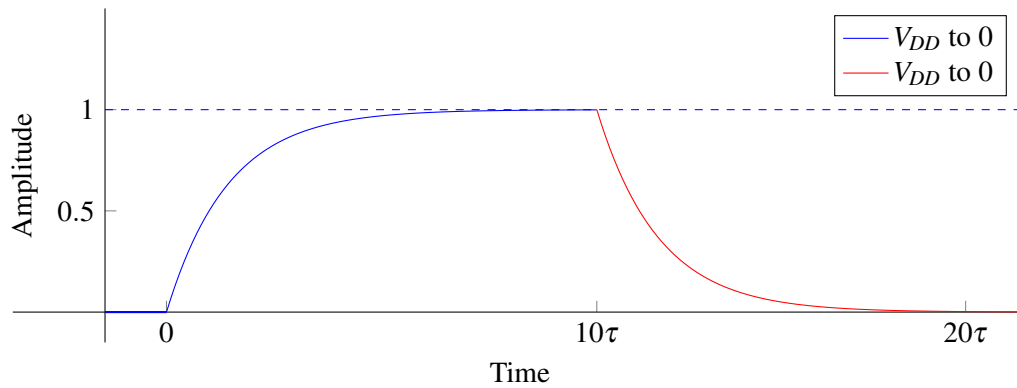
We can now think about what happens for the next chunk of time  $t \in [10\tau, 20\tau]$ . Having solved for  $V(10\tau)$ , we now have a new initial condition for the second interval  $V(10\tau) = V_1(10\tau) = V_2(0) \approx V_{DD}$ .

Using this information, the definition  $t_{int} = t - 10\tau$ , and the steps above we can solve for  $V_2(t_{int})$ :

$$\begin{aligned} \frac{d}{dt}V(t) &= -\frac{V(t)}{RC} + 0 \\ V_2(0) &= V_1(10\tau) \approx V_{DD} \end{aligned}$$

Recall the solution to this type of differential equation is  $V(t) = ke^{-\frac{t}{RC}}$ . Plugging in the initial condition we get  $V_2(t_{int}) = V_{DD}(e^{-\frac{t_{int}}{RC}})$ . And so  $V(t) = V_{DD}(e^{-\frac{t-10\tau}{RC}})$   $t \in [10\tau, 20\tau]$ . Here we can again verify that after  $10\tau$  has elapsed, the voltage  $V(t)$  again seems to reach steady state.

$$\begin{aligned} V(20\tau) &= V_2(10\tau) = V_{DD}(e^{-\frac{20\tau-10\tau}{RC}}) \\ &= V_{DD}(e^{-10}) \\ &\approx V_{DD}(0.00004539992) \approx 0. \end{aligned}$$

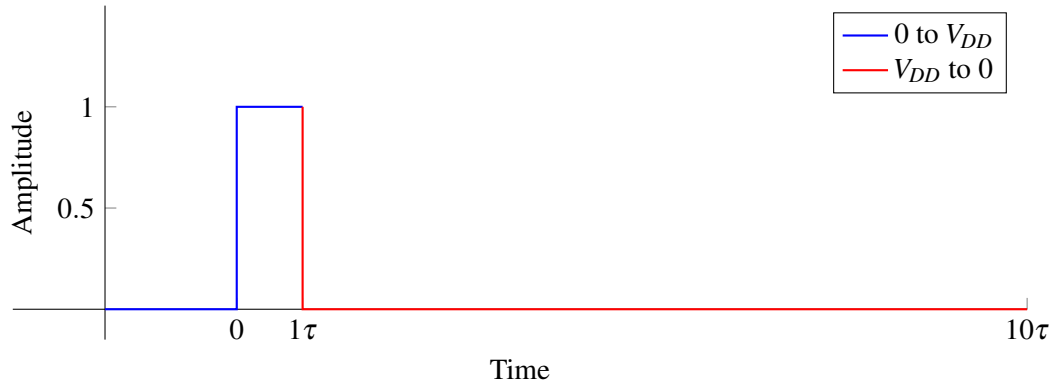


**Figure 3:**  $V(t)$  for On and Off input: On for  $10\tau$  and then off

We can see what is happening in Figure 3. This is one kind of behavior — when the transients are isolated from each other. However there is also the case when the duration  $D < \tau$  or  $D$  is not too much greater than  $\tau$ . In such a case our circuit does not have the opportunity to settle into steady state before the input changes back to 0. In such a case, we would need to calculate the exact voltage at the time our input changes to a 0 so that we could use an accurate initial condition for the second interval. Consider the case illustrated in Figure 4 where the input is only  $V_{DD}$  for a duration of one  $\tau = RC$  time constant.

Since the conditions for time  $t \in [0, 1\tau]$  are the same as the case before we end up with the same equation for  $V_1(t)$ :

$$V_1(t_{int}) = V(t) = V_{DD}(1 - e^{-\frac{t}{RC}}) \quad t \in [0, 1\tau]$$



**Figure 4:** On and Off input: On for  $1\tau$

However now since the input  $V_{DD}$  is only for  $1\tau$  the circuit does not get a chance to reach steady state and will only charge up to 63.2% of  $V_{DD}$  before transitioning to the next stage: when the input shifts from  $V_{DD}$  to 0.

$$\begin{aligned} V(1\tau) &= V_{DD}(1 - e^{-1}) \\ &\approx V_{DD}(1 - 0.368) \neq V_{DD}. \end{aligned}$$

Thus, now we can no longer use  $V_{DD}$  as our initial condition. Instead now we have to explicitly calculate our initial condition using the information we got from solving for  $V_1(t_{int})$  in the first time interval. As defined above let the function for the voltage in the second interval be  $V_2(t)$  such that  $V_2(t_{int}) = V(t)$   $t \in [1\tau, 10\tau]$  where  $t_{int} = t - 1\tau$ . Having solved for  $V_1(1\tau)$  we now have a new initial condition  $V(1\tau) = V_2(0) = V_{DD}(1 - e^{-1})$ .

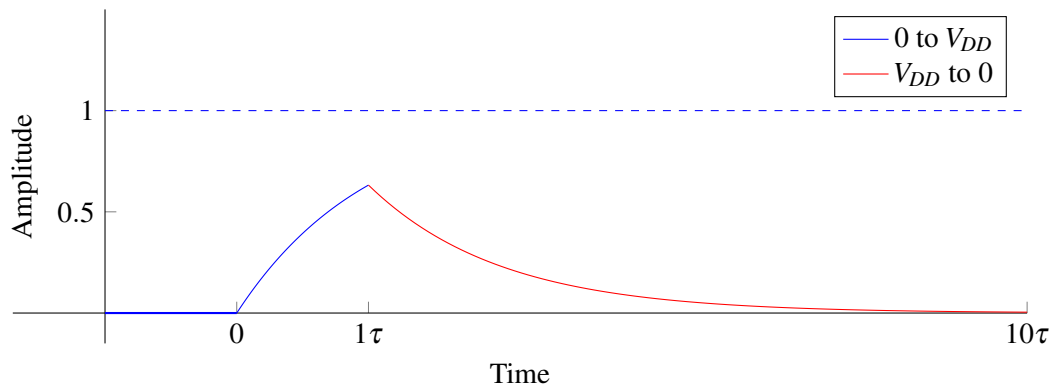
Solving the differential equation for the second interval and plugging in our initial condition we get:

$$V_2(t_{int}) = V_{DD}(1 - e^{-1})(e^{-\frac{t_{int}}{RC}})$$

in terms of time interval to that interval or in terms of absolute time:

$$V(t) = V_{DD}(1 - e^{-1})(e^{-\frac{t-1\tau}{RC}}) \quad t \in [1\tau, 10\tau]$$

This is illustrated in Figure 5.

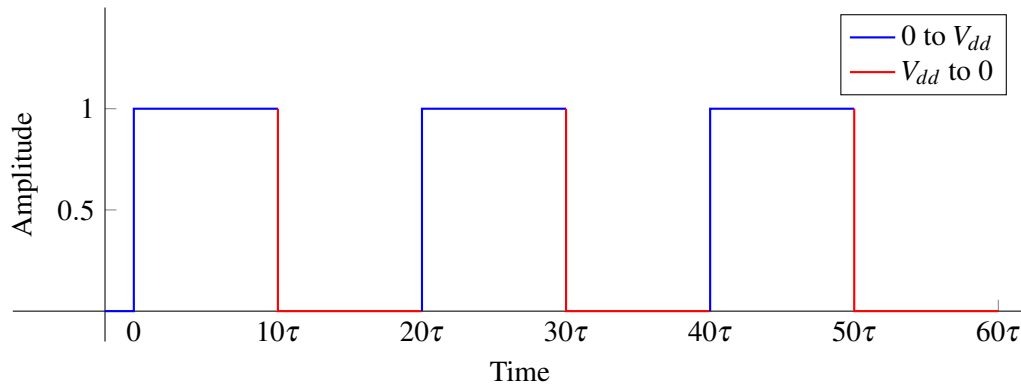


**Figure 5:**  $V(t)$  for On and Off input: On for  $1\tau$

## 3.2 More examples of what can happen

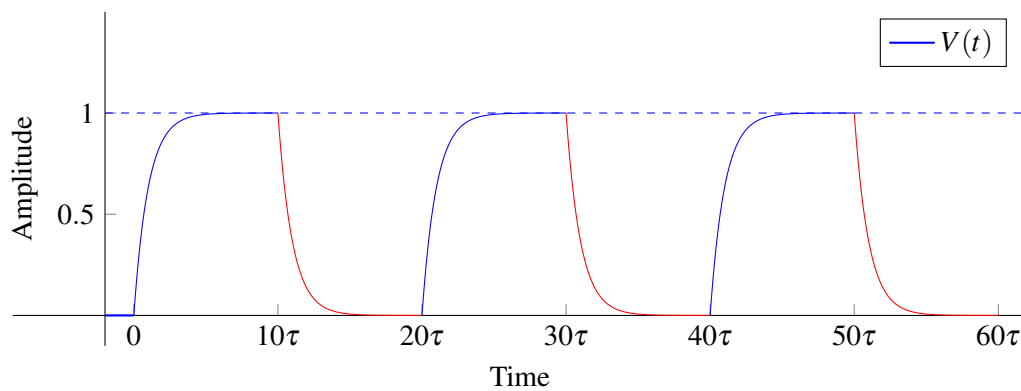
At this point, we can use what we know to see many different examples.

### 3.3 Case 1: Input is at 0 and $V_{dd}$ long enough to reach steady state



**Figure 6:** Case 1 Input: Input where both states reach steady state

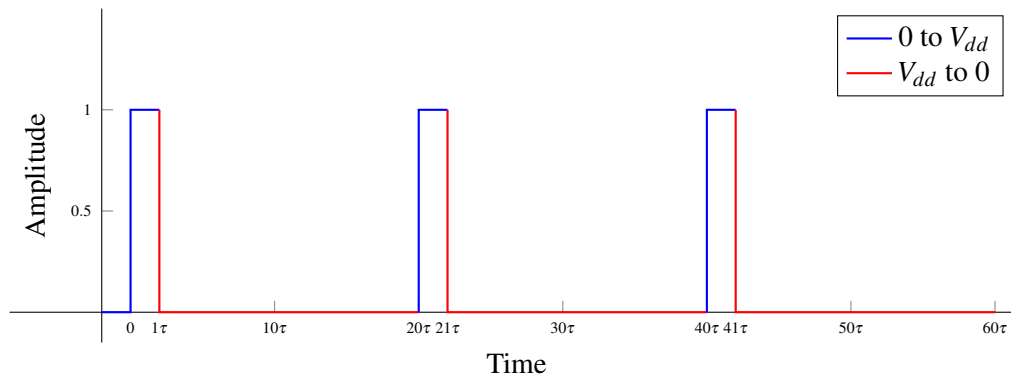
The first case to consider is when our repeated time varying input is at  $V_{dd}$  and 0 long enough to reach steady state. This is illustrated in Figure 6. The output voltage is illustrated in Figure 7.



**Figure 7:** Case 1 Output: Transient voltage for repeated switch when both reach steady state

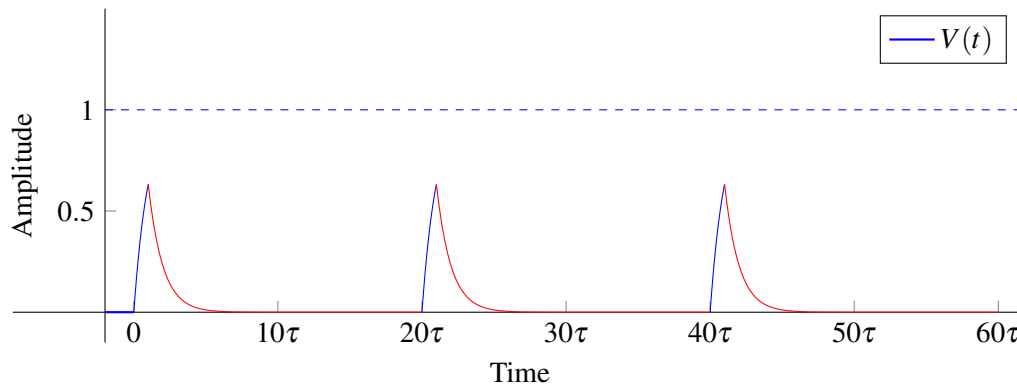


### 3.4 Case 2: Input is at 0 long enough to settle and does not settle at $v_{dd}$



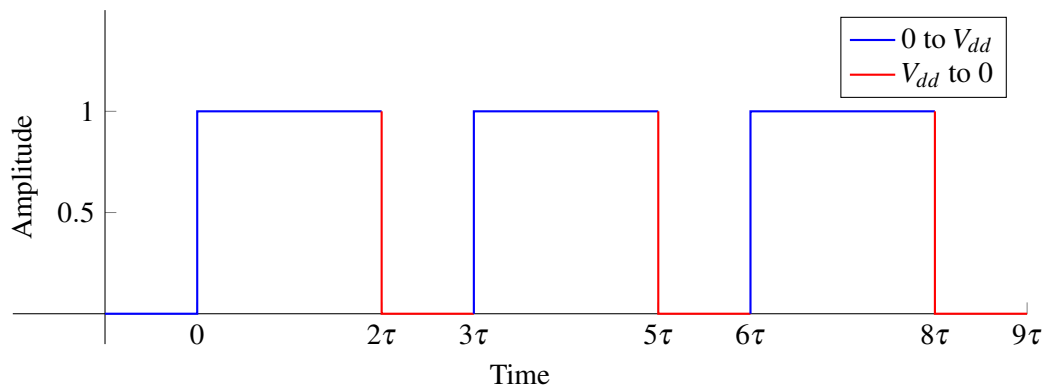
**Figure 8:** Case 2 Input: Input where only one state reaches settles

The second case to consider is when our repeated time varying input is at 0 long enough to reach steady state but not at  $V_{dd}$  long enough to do so (or vice versa). This is illustrated in Figure 8. The output voltage is illustrated in Figure 9.



**Figure 9:** Case 2 Output: Output where only one state of the input settles

### 3.5 Case 3: The input is at neither 0 nor $v_{dd}$ long enough to settle.



**Figure 10:** Case 3 Input: Input where both states of the input do not settle

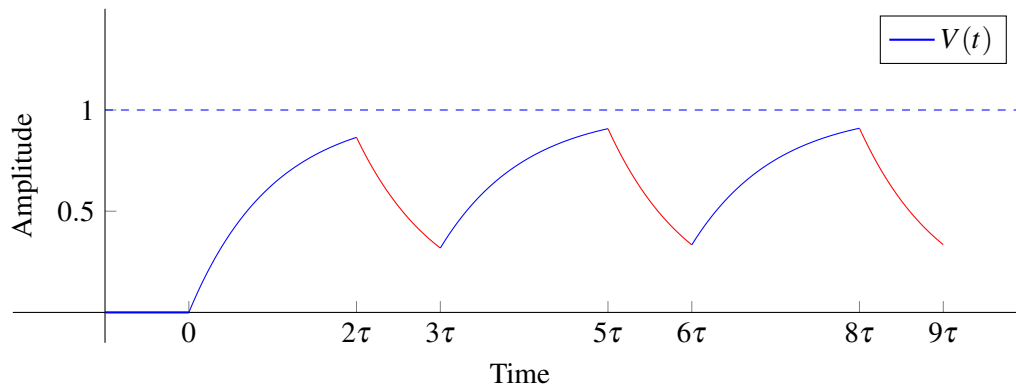
The third case to consider is when our repeated time varying input is not at 0 or  $V_{dd}$  long enough to reach steady state for either extreme. This is illustrated in Figure 10. The output voltage is illustrated in Figure 11.

For this kind of case, we had no choice but to go interval by interval:

- Solve the differential equation to get a function for voltage changing with time.
- Solve for the initial condition using the previous interval's solution.
- Plug in the initial condition to the solution of the differential equation for this interval.

Notice that in this case, the magnitude of the voltage on the capacitor seems to have a slight upward trajectory.

Can you figure out what this sawtooth shape will eventually start looking like? It will stay a sawtooth, and you know that it will have each tooth being  $3\tau$  long. But where will the top and bottom of the teeth be? This is an interesting exercise to think about.

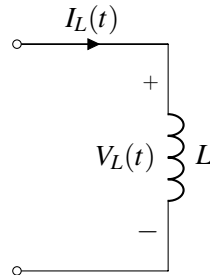


**Figure 11:** Case 3 Output: Transient voltage for repeated switch when both states of the input do not settle: Notice how the peak voltage goes gently up over time.

## 4 Inductors

Let's introduce a new passive component, an inductor. This new component will help us design more interesting circuits and introduce oscillations within our circuits.

### 4.1 Basics



**Figure 12:** Example Inductor Circuit

The voltage across the inductor is related to its current as follows:

$$V_L(t) = L \frac{dI_L(t)}{dt}. \quad (2)$$

where  $L$  is the *inductance* of the inductor. The SI unit of inductance is Henry (H). Looking at equation (2) we can observe that inductors have a dual relationship in terms of voltage and current (I-V) as compared to capacitors (i.e., V being proportional to change in I as opposed to I being proportional to change in V).

**Concept Check:** The current across the inductor cannot change instantaneously. Why?

**Solution:** If our current changes instantaneously, then  $\frac{d}{dt}I_L \rightarrow \infty$ , and from equation (2) the voltage across the inductor  $V_L \rightarrow \infty$ , which is not possible. Hence, our current cannot change instantaneously

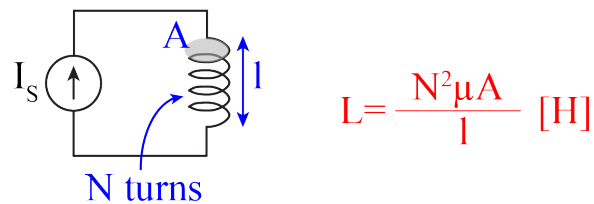
In steady state, when the current flowing through an inductor is constant, there is no voltage drop across the inductor. This makes sense, since an inductor is essentially a spool of wire wrapped around a conductor. Similarly, if the current through the inductor is changing, there will be a voltage drop across the inductor.

The energy stored in the inductor turns out to be  $E_L = \frac{1}{2}LI^2$ , but we won't be using this very much in EECS16B. We are only mentioning it here because it helps us interpret what is happening later.

### 4.2 Physics behind Inductors (not in scope, just for information)

Inductors store energy in a magnetic field. In the same way that a capacitor separates charge ( $Q$ ) and this leads to an electric field ( $\vec{E}$ ), anytime current flows down a conductor, it creates a magnetic field ( $\vec{B}$ ). Likewise, the magnetic field can store energy. Their behavior can be described using **Faraday's Law of Induction**.

The magnitude of magnetic field created by a straight wire is pretty small, so we usually use other geometries if we are trying to create a useful inductance on purpose. A **solenoid** is a good example, where we wind a wire usually around a conductor:



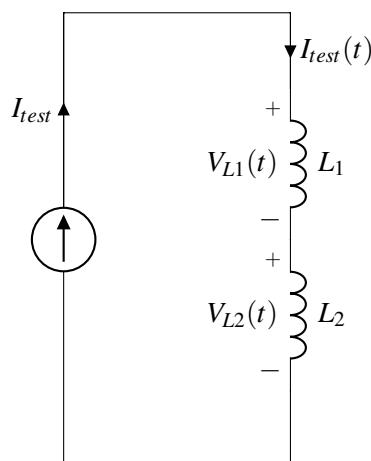
**Figure 13:** The Inductance of a Solenoid: a wire coiled around something

Note that the inductance ( $L$ ) depends on **geometry** and a material property called **magnetic permeability** ( $\mu$ ) of the solenoid core material. In the case of the solenoid in 13, the inductance depends on the number of turns ( $N$ ), the length of the solenoid ( $l$ ) and the area ( $A$ ) of the loops. Inductors are useful in many applications such as wireless communications, chargers, DC-DC converters, key card locks, transformers in the power grid, etc. But in many high speed applications, their presence might be undesirable as they create delays in the time response of the circuit.

## 4.3 Equivalence Relations

Now that we have the basics, let's derive the equivalence relations for series and parallel combinations of inductors. We will find that these are similar to those of resistors. Why? Because the law governing an inductor  $V_L = L \frac{d}{dt} I_L$  involves a proportionality constant  $L$  that multiplies a current-like quantity to give a voltage. In a resistor, the resistance  $R$  multiplies current to give a voltage.

### 4.3.1 Series Equivalence



**Figure 14:** Series Inductor Circuit

Let's apply a  $\frac{dI_{test}}{dt}$  through the two inductors, then

$$V_{L1}(t) + V_{L2}(t) = V_L$$

where,  $V_L$  is the voltage across the two inductors. From VI relationship for inductors, we get

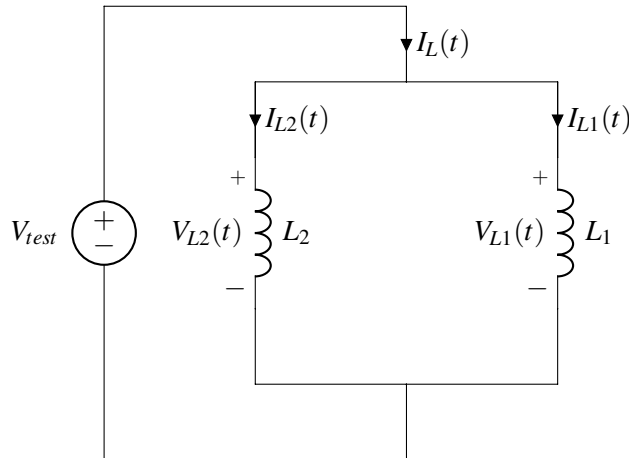
$$L_1 \frac{dI_{test}}{dt} + L_2 \frac{dI_{test}}{dt} = V_L$$

$$(L_1 + L_2) \frac{dI_{test}}{dt} = V_L$$

$$L_{eq} \frac{dI_{test}}{dt} = V_L$$

where,  $L_{eq} = L_1 + L_2$ .

### 4.3.2 Parallel Equivalence



**Figure 15:** Parallel Inductor Circuit

We apply at  $V_{test}$  across the parallel combination. We have

$$V_{L1} = V_{L2} = V_{test}$$

$$L_1 \frac{dI_{L1}}{dt} = L_2 \frac{dI_{L2}}{dt} = L_{eq} \frac{dI_L}{dt}$$

and from KCL, we have

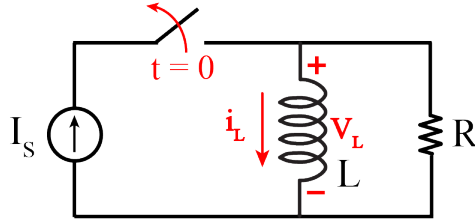
$$I_L(t) = I_{L1}(t) + I_{L2}(t)$$

Differentiating with respect to time, and substituting from above equality,

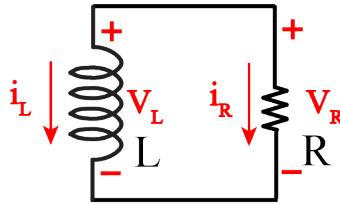
$$\begin{aligned} \frac{dI_L}{dt} &= \frac{dI_{L1}}{dt} + \frac{dI_{L2}}{dt} \\ \frac{dI_L}{dt} &= \frac{L_{eq}}{L_1} \frac{dI_L}{dt} + \frac{L_{eq}}{L_2} \frac{dI_L}{dt} \\ \frac{1}{L_{eq}} &= \frac{1}{L_1} + \frac{1}{L_2}. \end{aligned}$$

## 5 RL Circuits

In the same way that one resistor and one capacitor in a circuit can lead to a differential equation and a solution with an  $e^{-1/RC \cdot t}$  term in it, an RL circuit results in an differential equation and  $e^{-R/L \cdot t}$  terms. We will sketch an example below:



Note  $i_L$  for  $t < 0$  is  $I_S$  because an inductor at steady state resembles a short. For  $t > 0$ , the circuit looks like:



Let's solve this circuit for  $i_L(t)$  for  $t > 0$  by writing a KCL equation:

$$i_L + i_R = 0 \quad (3)$$

We also know that  $v_L = v_R$  since the resistor and inductor are in parallel.

$$v_L = v_R \quad (4)$$

$$i_R \cdot R = -i_L R \quad (5)$$

$$\Rightarrow v_L + i_L R = 0 \quad (6)$$

Finally, substituting in the voltage-current relationship of an inductor, we see that

$$L \frac{di_L}{dt} + i_L R = 0 \quad (7)$$

$$\frac{di_L}{dt} + \frac{R}{L} i_L = 0 \quad (8)$$

This equation should be familiar to you! This is a 1st order DE ( $\dot{x} + ax = 0$  with  $a = R/L$ ). The solution is then:

$$i_L(t) = i_L(0) e^{-\frac{R}{L}t} \quad (9)$$

And since  $i_L(0) = I_S$ , thus:

$$i_L(t) = I_S e^{-\frac{R}{L}t} \quad (10)$$

If we plot and analyze the time it takes to charge up an inductor to steady state, we realize that the time constant  $\tau = \frac{L}{R} = R \cdot \frac{1}{L}$ . This ties back to the dual relationship of voltage and current across an inductor and capacitor since a larger inductor would correspond to a smaller time constant.

**Contributors:**

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