EECS 16B Designing Information Devices and Systems II
Summer 2020

Note 5

1 Introduction

In the previous note, we examined a system of differential equations and developed an approach to solve it by changing coordinates in order to create a series of n first-order differential equations. This technique was called **diagonalization** which let us view the matrix A in a different basis made up of eigenvectors in which A had a diagonal representation Λ. When solving these first order differential equations and converting our system back into standard basis coordinates, we saw that each state $x_i(t)$ was a linear combination of exponentials $e^{\lambda_1 t}, \dots, e^{\lambda_n t}$. As a result, we saw the connection between a system of differential equations and the eigenvalues of the matrix A.

In this note, we will look at the uniqueness of our solution to a system of differential equations. Then we will analyze the eigenvalues of the matrix A and the physical effects they have on a system called damping. Lastly, we consider the case in which the system is **not** diagonalizable and develop an approach to tackle this case and another physical phenomena that occurs from this.

2 Uniqueness

2.1 Recalling our previous solution

Given a system of differential equations

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \tag{1}$$

we defined a new coordinate system $\vec{z} = V^{-1}\vec{x}$ we arrived at a series of first order differential equations

$$\begin{aligned}
\frac{d}{dt}z_1(t) = \lambda_1 z_1(t) &\implies z_1(t) = z_1(0)e^{\lambda_1 t} \\
\vdots \qquad\qquad & \qquad\qquad \vdots \\
\frac{d}{dt}z_n(t) = \lambda_n z_n(t) &\implies z_n(t) = z_n(0)e^{\lambda_n t}
\end{aligned}$$

So when we convert our solution back into standard basis coordinates through $\vec{x} = V\vec{z}$, we see that each $x_i(t)$ is a linear combination of $e^{\lambda_1 t}, \dots, e^{\lambda_n t}$.

$$\vec{x}(t) = V\vec{z}(t) = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} z_1(t) \\ \vdots \\ z_n(t) \end{bmatrix} \tag{2}$$

$$= z_1(t)\vec{v}_1 + \ldots + z_n(t)\vec{v}_n \tag{3}$$

$$= z_1(0)e^{\lambda_1 t}\vec{v}_1 + \ldots + z_n(0)e^{\lambda_n t}\vec{v}_n \tag{4}$$

It should come with no surprise that we now question whether this solution is in fact unique. If it were unique, we could solidify a guess to our differential equation $x_i(t) = \alpha_1 e^{\lambda_1 t} + \ldots + \alpha_n e^{\lambda_n t}$. [1] Therefore, the next step of our journey is to show that the solution to this differential equation is indeed unique.

2.2 Proof of Uniqueness

This isn't going to be a formal proof of uniqueness, but after reading this section, we can definitely formalize it if desired. We will be taking a similar approach to how we solved this system of differential equations to prove its uniqueness. Currently we have already proven the uniqueness of scalar differential equations of the form $\frac{d}{dt}x(t) = \lambda x(t) + u(t)$. Therefore, we would like to somehow use this fact when trying to prove uniqueness for vector differential equations.

Based on the uniqueness of scalar differential equations, we know that the vector differential equation

$$\frac{d}{dt}\vec{z}(t) = \Lambda\vec{z}(t);\ \vec{z}(0) = \vec{z}_0 \tag{5}$$

has a unique solution when Λ is a diagonal matrix. Therefore, since we know our diagonal system has a unique solution, it remains to show that changing coordinates from $\vec{x}(t)$ to $\vec{z}(t)$ is indeed unique. Since we know that changing coordinates from one basis to another can be represented as a matrix multiplication by V, our proof boils down to showing the uniqueness of matrix mappings on vectors.

We will not prove this result here but the invertibility of V allows our coordinate representations to be unique henceforth proving the uniqueness of the solution to our differential equation.

2.2.1 Extension to the Nonhomogenous Case

Can we also show that our solution is indeed unique for the nonhomogenous case? Well, we showed that the vector differential equation $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ had a unique solution. The nonhomogenous vector differential equation would be an equation of the form

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \tag{6}$$

We can in fact show that this differential equation will have a unique solution as well since scalar differential equations with an input $u(t)$ had a unique solution. To solidify this, note that after changing coordinates to $\vec{z}(t)$, we get the diagonal system

$$\frac{d}{dt}\vec{z}(t) = \Lambda\vec{z}(t) + V^{-1}\vec{b}u(t) = \Lambda\vec{z}(t) + \vec{w}(t) \tag{7}$$

where $\vec{w}(t)$ is treated as a vector of individual inputs to the scalar differential equations. Again since $V^{-1}\vec{b}$ has a unique representation, we can indeed see that the solutions to our differential equation are unique once we change back to $\vec{x}(t)$ coordinates.
Note that if $u(t)$ is constant, we can guess the solution $x_i(t) = \alpha_1 e^{\lambda_1 t} + \ldots + \alpha_n e^{\lambda_n t} + \beta$ where β is a constant.

[1] Here $x_i(t)$ is referring to the i^{th} entry of the vector $\vec{x}(t)$.

3 Second Order Differential Equations

When setting up and solving a system of differential equations, we must always choose a collection of state-variables. However, it is often possible to represent our system by using only one variable with the tradeoff of higher order derivatives. Despite all of this, we show that the two approaches are interchangeable and we can always create a system of differential equations given a higher order differential equation.

Consider the differential equation

$$\frac{d^2}{dt^2}y(t) + a\frac{d}{dt}y(t) + by(t) = 0 \tag{8}$$

$$y(0) = y_0;\ \frac{d}{dt}y(0) = w_0 \tag{9}$$

This is an example of a second order differential equation. Notice how there are two initial conditions for this problem. An n^{th} order differential equation will require n initial conditions for it to have a unique solution.

3.0.1 Guess and Check

To solve this differential equation, we can either guess and check or convert it into a system of differential equations. We will start by guess the solution $y(t) = ke^{\lambda t}$.

$$y(t) = ke^{\lambda t};\ \frac{d}{dt}y(t) = k\lambda e^{\lambda t};\ \frac{d^2}{dt^2}y(t) = k\lambda^2 e^{\lambda t} \tag{10}$$

$$k\lambda^2 e^{\lambda t} + ka\lambda e^{\lambda t} + kbe^{\lambda t} = 0 \implies k(\lambda^2 + a\lambda + b) = 0 \tag{11}$$

If our initial condition is nonzero, k will be nonzero meaning we have a quadratic equation for λ similar to the characteristic polynomial of our matrix A. Since this quadratic equation has two roots λ_1 and λ_2, our solution $y(t)$ will be a linear combination of the functions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ or of the form

$$y(t) = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} \tag{12}$$

Plugging in the initial conditions $y(0)$ and $\frac{dy}{dt}(0)$, we should be able to solve for the coefficients α_1 and α_2.

3.0.2 Converting to a Vector Differential Equation

Similar to how we converted a system of differential equations into a vector differential equation, we can also turn our second order differential equation into a **first order** vector differential equation. We will do so by defining state variables

$$x_1(t) = y(t),\ x_2(t) = \frac{d}{dt}y(t) \tag{13}$$

Taking the derivative of our states, we see that

$$\frac{d}{dt}x_1(t) = \frac{d}{dt}y(t) = x_2(t) \tag{14}$$

$$\frac{d}{dt}x_2(t) = \frac{d^2}{dt^2}y(t) = -by(t) - a\frac{d}{dt}y(t) = -bx_1(t) - ax_2(t) \tag{15}$$

Therefore, we can write this as a vector differential equation

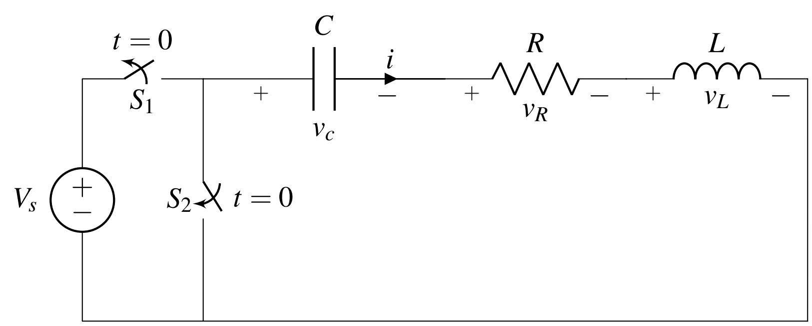
$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{16}$$

Note that the eigenvalues of the A matrix yields the exact characteristic polynomial that we found using guess and check. This is not coincidental and in fact arises since we were looking for eigenvalues of the differentiation operator $\frac{d}{dt}$. There is another underlying connection between higher order and vector differential equations and their eigenvalues, but we explore this in a much later part of the course.

4 RLC Circuits and Higher Order Differential Equations

The LC tank we studied in the previous section was a very ideal case where we assumed there was no resistor in the system. But this is rarely the case, and we will need to understand how adding this third component will modify our differential equations.

To motivate our discussions, consider the following circuit, with component values $V_s = 4\,\text{V}$, $C = 2\,\text{fF}$, $R = 60\,\text{k}\Omega$, and $L = 1\,\mu\text{H}$. Before $t = 0$, switch S_1 is on while S_2 is off. At $t = 0$, both switches flip state (S_1 turns off and S_2 turns on):



First, let's figure out the initial conditions. Since the system had been connected to the battery for a long time, the capacitor would be at steady state meaning $v_c(0) = V_s = 4$ and $i(0) = 0A$. From this, we can also deduce that $\frac{d}{dt}v_c = 0$. Next, let's write our branch equations:

$$i = C\frac{d}{dt}v_c,\ v_L = L\frac{d}{dt}i,\ v_R = i \cdot R \tag{17}$$

$$v_c + v_L + v_R = 0 \tag{18}$$

Using the above equations, and substituting for i from Equation (17) when needed, we can describe our system with the following differential equation:

$$\frac{d^2}{dt^2}v_c(t) + \frac{R}{L}\frac{d}{dt}v_c(t) + \frac{1}{LC}v_c(t) = 0 \tag{19}$$

Here we have chosen the second order differential equation as means of an example. As usual, we can solve this differential equation by computing its eigenvalues

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0 \tag{20}$$

$$\lambda = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \tag{21}$$

Substituting numbers, we get,

$$\lambda_{1,2} = \frac{-6\times 10^4 \pm \sqrt{36\times 10^8 - 20\times 10^8}}{2\times 10^{-6}} \Rightarrow \lambda_1 = -10^{10},\ \lambda_2 = -5\times 10^{10} \tag{22}$$

Next, we can solve for coefficients α_1 and α_2 using any approach [2] to get $v_c(t) = 5e^{-10^{10}t} - e^{-5\times 10^{10}t}$.

From Equation (17), we know that $i = C\frac{d}{dt}v_c = -10\times 10^{-5}e^{-10^{10}t} + 10\times 10^{-5}e^{-5\times 10^{10}t}$. Figure 1 plots our solutions for the current and voltage.

The first thing we observe is that the voltage is expoenetially decaying similar to voltage in a discharging RC circuit. Next, we also observe that the current is a negative value. Why is that? Well, if we go back to the circuit, we can see that when S_1 is opened and S_2 is closed, the current would flow from the positive plate of the capacitor to the negative plate, and this is in the opposite direction of the labelled current. Next, we also see that, the absolute value of the current sharply increases and then begins to decay expoenentially. How can we explain this? Well, as the capacitor discharges, initally, $v_L \approx 0$, hence absolute value of the current will dip to about $\frac{v_c}{R} = 6.66\times 10^{-5}$, and looking at the graph, we see that the absolute current is about 6×10^{-5}. However, at the same time, the increase in current creates a voltage difference across the inductor, which will result in the net voltage across the resistor to decrease, hence the absolute current begins to reduce. To confirm of our intuition, let's also graph the voltage across the inductance according to equation (17). We get $v_L = e^{-10^{10}t} - 5e^{-5\times 10^{10}t}$.

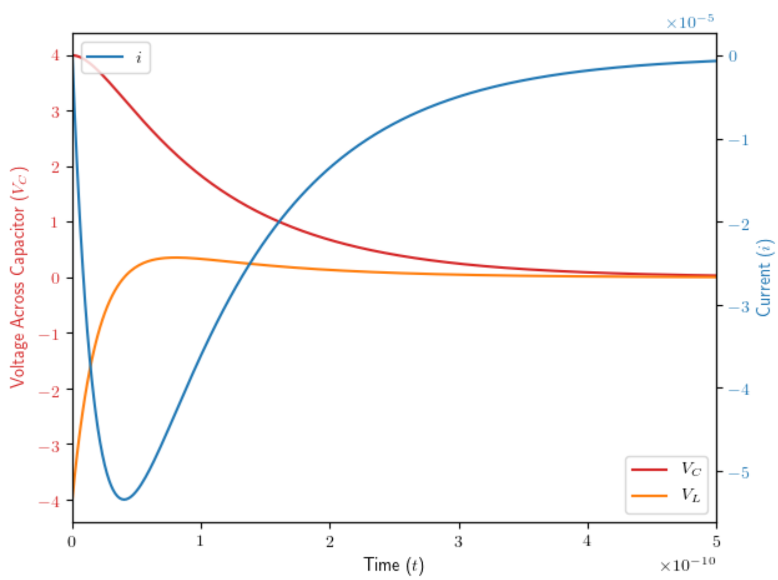
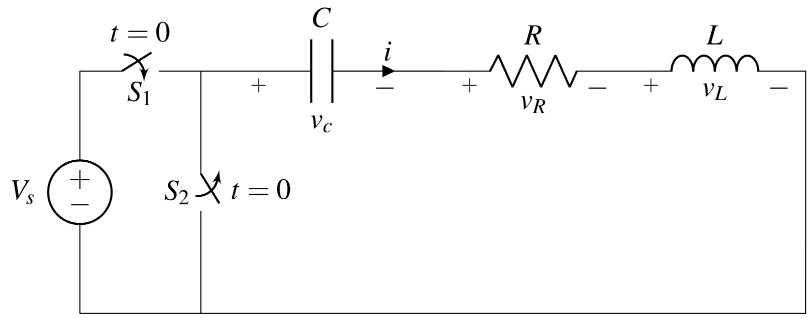


Figure 1: Capacitor, Inductor voltage and current transient response

[2]Guess and Check, Changing Coordinates, we are no longer emphasizing the solving process rather we would like to extrapolate information from the results.

5 Charging an RLC Circuit

Now that we have equipped ourselves with some knowledge on eigenvalues, we will take a look at the nonhomogenous case. Consider the following circuit. Before $t = 0$, switch S_1 is off while S_2 is on. At $t = 0$, both switches flip state (S_1 turns on and S_2 turns off):



Firstly, we must find the initial conditions. Since the capacitor has been discharging for a long time, $v_c(0) = 0$ and $i_L(0) = i(0) = 0$. Next, let's write out the branch equations,

$$i = C\frac{d}{dt}v_c,\ v_L = L\frac{d}{dt}i,\ v_R = i \cdot R \tag{23}$$

$$v_c + v_L + v_R = 0 \tag{24}$$

Using the above equations, and substituting for i from Equation (17) when needed, we can describe our system with the following differential equation:

$$\frac{d^2 v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{v_c}{LC} = V_s \tag{25}$$

A quick technique we can use to homogenize the above equation is a substitution of variables: [3] $\widetilde{v_c} = v_c - V_s$, hence $\frac{d}{dt}\widetilde{v_c} = \frac{d}{dt}v_c$ and $\frac{d^2}{dt^2}\widetilde{v_c} = \frac{d^2}{dt^2}v_c$. Applying this substiution,

$$\frac{d^2 \widetilde{v_c}}{dt^2} + \frac{R}{L}\frac{d\widetilde{v_c}}{dt} + \frac{1}{LC}\widetilde{v_c} = 0 \tag{26}$$

Looking back, Equation (19) close resembles our above equation. Hence, we will find the same eigenvalues and eigenvectors.

$$\widetilde{\lambda_1} = -10^{10},\ \widetilde{\lambda_2} = 5 \times 10^{10} \tag{27}$$

Solving our homogenous differential equation using our method of choice, we see that the solution is

$$\widetilde{v_c}(t) = -5e^{-10^{10}t} + e^{-5\times 10^{10}t} \implies v_c(t) = 4 - 5e^{-10^{10}t} + e^{-5\times 10^{10}t} \tag{28}$$

Then converting back to $\vec{v}_c = \widetilde{v_c} + V_s$, we see that

$$v_c(t) = 4 - 5e^{-10^{10}t} + e^{-5\times 10^{10}t} \tag{29}$$

[3] We could've also guess and checked the solution $v_c(t) = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} + \beta$ where β is a constant. The substitution of variables approach ensures that our eigenvalues $\widetilde{\lambda}$ and λ are indeed the same.

Concept Check: Are the eigenvalues always real? If not, when are they real and when are they complex?
Solution: No, as a sanity check, if $R = 0$, we get the LC tank example from the previous note which had purely imaginary eigenvalues. If we look at the formula for the eigenvalues in terms of R, L, C, we notice that if the discriminant $\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} < 0$, then the eigenvalues will be complex.

We will now take a detour to analyze the eigenvalues of our differential equation and the effect they have on our RLC circuit.

5.1 The Power of Eigenvalues

In this section, we strengthen our understanding of eigenvalues in the context of second order differential equations. Recall that the solution to a second order differential equation is of the form $x(t) = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} + \beta$.

5.1.1 Real Eigenvalues

When eigenvalues are purely real, the solution will be a linear combination of real exponentials. If both eigenvalues are purely real and negative, then $x(t)$ will be a decaying exponential whereas if either eigenvalue λ_i is greater than 0, then $x(t)$ will be an increasing exponential that goes to ∞.

Concept Check: In the context of RLC circuits will the eigenvalues be purely real and positive?
Solution: No, the eigenvalues will always be purely real and negative. Intuitively, the voltage across a capacitor cannot go to infinity. Mathematically, we can observe that the quantities R, L, and C are always positive. In addition, since $\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} < \frac{R}{L}$ we see that the eigenvalues must be negative if they are purely real.

5.1.2 Purely Imaginary Eigenvalues

When the eigenvalues are purely imaginary, like our LC Tank, then our solution will be a linear combination of sines and cosines. This was a result of Euler's Identity $e^{j\theta} = \cos(\theta) + j\sin(\theta)$.

Note: Since our solution is a linear combination of sines and cosines and we have proven its uniqueness, we are allowed to guess a solution $x(t) = \alpha_1 \cos(\omega t) + \alpha_2 \sin(\omega t) + \beta$ where $\lambda = \pm j\omega$.

5.1.3 Complex Eigenvalues

Lastly, when the eigenvalues are complex, they came in complex conjugates. We will call them $\lambda_1 = \sigma + j\omega_d$ and $\lambda_2 = \sigma - j\omega_d$. [a] We can prove that the eigenvectors as well came in complex conjugates meaning the coefficients α_1 and α_2 are complex conjugates. [b]
Therefore, our exponentials $e^{\lambda t} = e^{\sigma t}(\cos(\omega_d t) + j\sin\omega_d t)$ meaning our solution will be a decaying exponential $e^{\sigma t}$ multiplied by a linear combination of sines and cosines.

Note: Again, since we have proved uniqueness, we are allowed to guess a solution $x(t) = e^{\sigma t}(\alpha_1 \cos(\omega_d t) + \alpha_2 \sin(\omega_d t) + \beta$.

[a] The subscript 'd' in ω_d stands for damping. Its meaning will be clear in the next section. Also remember that for an RLC, σ is negative.

[b] We won't prove this here, but it should be relatively straightforward using the properties of Complex Numbers.

6 Damping

Now that we understand how to solve these second order differential equations, let's try to tweak the resistor value and observe how our voltages and currents will behave in the discharging capacitor case described in equation (19). We can think of our resistor as a knob that we can turn up or down to make our eigenvalues real or complex.

6.1 Underdamped Systems

Let $R = 20\,\mathrm{k}\Omega$. Then,

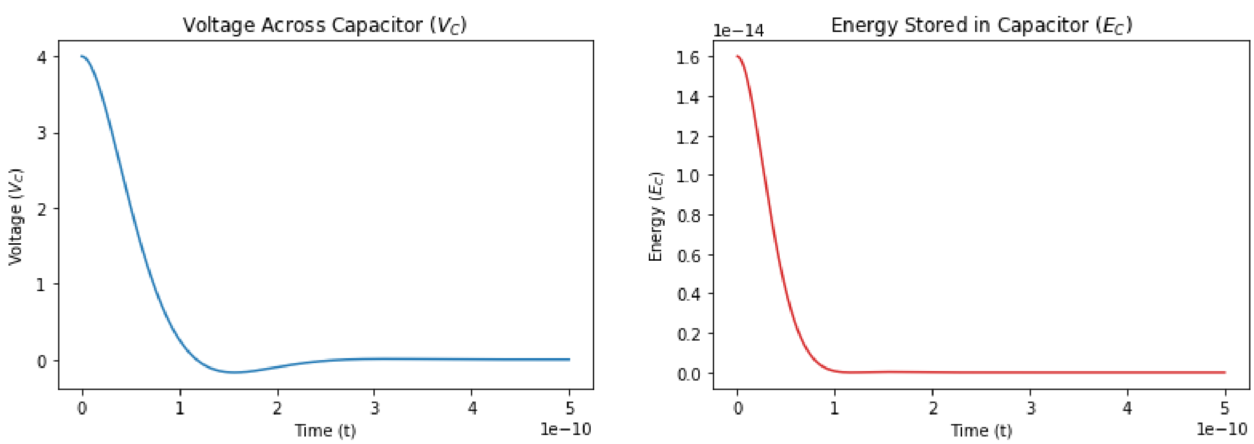
$$\lambda_{1,2} = -\frac{2\times 10^4}{1\times 10^{-6}} \pm \frac{\sqrt{4\times 10^8 - 20\times 10^8}}{2\times 10^{-6}} = 2\times 10^{10}(-1\pm j)$$

meaning the eigenvalues are complex conjugates. Solving our differential equation, we see that

$$v_c(t) = e^{-2\times 10^{10}t}\left(4\cos\left(2\times 10^{10}t\right) + 4\sin\left(2\times 10^{10}t\right)\right) \tag{30}$$

The $\cos(\omega_d t)$ and $\sin(\omega_d t)$ terms create oscillation but as time passes, we see that the amplitude of the oscillation decays exponentially. We can attribute this to the dissapation of energy across the resistor. Fundamentally, compared to the previous cases, we see oscillation here since the resistor value is small enough to accomodate the sloshing of energy we observed in the LC tank. Aptly, this case is called the **underdamped** case. Let's confirm this intuition by plotting the energy across the capacitor in Figure 2b.

$$E_C = \frac{1}{2}Cv_c^2 = 16\times 10^{-15}e^{-4\times 10^{10}t}(1+\sin\left(4\times 10^{10}\right))$$



(a) Voltage across capacitor (v_c)

(b) Energy stored in capacitor (E_C)

Figure 2: RLC Circuit behaviour in Underdamped case.

6.2 Overdamped Systems

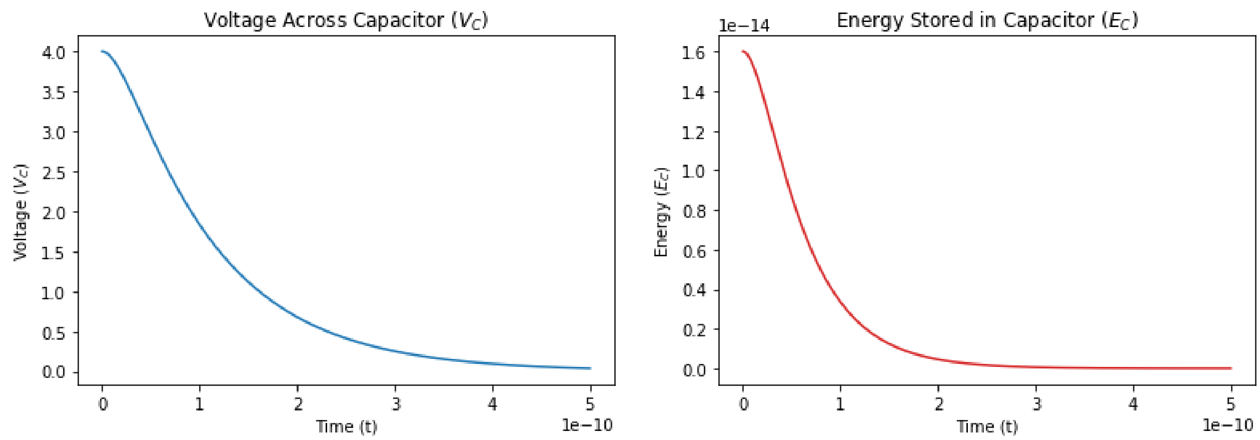
In our previous example from section 4 with $R = 60\,\mathrm{k}\Omega$, we saw that the eigenvalues were purely real and that the solution $v_c(t)$ was a decaying exponential

$$v_c(t) = 5e^{-10^{10}t} - e^{-5\times10^{10}t} \tag{31}$$

Here the resistor value is large enough to limit the current so that the RLC circuit will not face any oscillations. This case is called the **overdamped** case in which the capacitor discharges to 0 in as a decaying exponential. We confirm this effect in Figure 3a

When our circuit was underdamped like in Figure 2a, we saw that the capacitor discharged to a value below 0 and then rose back up to 0. Such an effect is called **overshoot** and the effect of "overdamping" is what prevents this overshoot.

Since we have limited the amount of current in our RLC circuit, notice how it takes longer in the overdamped case for the capacitor to discharge. This is a tradeoff that engineers must make to prevent the circuit from ringing.



(a) Voltage across capacitor (v_c) **(b)** Energy stored in capacitor (E_C)

Figure 3: RLC Circuit behaviour in Overdamped case.

6.3 Damping Ratio (Optional)

We have continually referred to the term **damping** without giving it a formal definition. It is time to introduce the meaning behind damping. For any second order differential equation of the form

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x = 0 \tag{32}$$

ζ is defined to be the **damping ratio** and ω_n[4] is the **natural frequency** of the system.

Note that these constants will apply for physical systems outside the context of circuits as well. We will see more of these examples in a later part of the course. However, for now in the context of RLC, the natural frequency of the system is the frequency that the circuit oscillates at when undamped or $\zeta = 0$. Recall that this is the specific case of the LC tank and the natural frequency will be $\omega_n = \frac{1}{\sqrt{LC}}$.

[4]The subscript 'n' in ω_n stands for "natural".

This means that the damping ratio of an RLC circuit is $\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$. Connecting this back to our eigenvalues, notice that when $0 < \zeta < 1$, the response is underdamped whereas if $\zeta > 1$, then the system is overdamped. Now what happens when $\zeta = 1$? We will finally answer the question of what happens when there is a single real eigenvalue and the matrix representing the system is **not diagonalizable.**

6.4 Critical Damping (Optional)

6.4.1 Repeated Eigenvalues

Our entire process of solving second order differential equations relied on the truth that a $n \times n$ matrix A has n linearly independent eigenvectors. However, if we were to have repeated eigenvalues in our system, then we cannot guarantee that A is diagonalizable. One example of a second order differential equation that is nondiagonalizable when put into matrix form is

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0 \tag{33}$$

By picking state-variables $x_1(t) = y(t)$ and $x_2(t) = \frac{dy}{dt}$, we could set up the following system of differential equations

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{34}$$

There is a single eigenvalue $\lambda = -3$ and the eigenspace of $A + 3I$ is

$$\text{Nul}(A + 3I) = \text{Nul} \begin{bmatrix} 3 & 1 \\ -9 & -3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{35}$$

which is one-dimensional. This was where we left off last time since our matrix was non-diagonalizable.

Concept Check: Why did we use diagonalization when solving a system of differential equations?
Solution: Our motivation behind diagonalization was to find a basis in which the matrix A was diagonal so that we could decompose our system into n first order differential equations.

However, does our system need to be **diagonal** for us to create n first order differential equations? What if it was possible to pick a basis in which A had an **upper-triangular** representation?

$$\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} \lambda & \star \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \tag{36}$$

where $\star$ is any nonzero value. At a first glance, we are not able to uncouple the equations to create two first order equations. However, if were to solve these differential equations from the bottom up exactly like how we performed back-substitution when row-reducing, what would happen?

This would mean we first solve $\frac{d}{dt}z_2(t) = \lambda z_2(t)$ which has unique solution $z_2(t) = z_2(0)e^{\lambda t}$. Now that we have a solution $z_2(t)$, we can plug it back into our first differential equation!

$$\frac{d}{dt}z_1(t) = \lambda z_1(t) + \star \cdot z_2(0)e^{\lambda t} \tag{37}$$

Since $z_1(t)$ is a first order differential equation with an input $u(t) = ke^{\lambda t}$, referring back to Note 3, the solution is $z_1(t) = z_0e^{\lambda t} + kte^{\lambda t}$.

But what is our basis?

At last we have developed a strategy on how to tackle the case in which A cannot be diagonalized. However, we have yet to define the basis $\{\vec{v}_1, \vec{v}_2\}$ that makes A have an upper-triangular representation. So we will now define a basis to make A have an upper triangular representation. [a]

(1) We will start by picking $\vec{v}_1 = \vec{v}$ where $\vec{v}$ is our eigenvector of A.

(2) To form a basis for $\mathbb{R}^2$, our second vector $\vec{v}_2$ can be any vector linearly independent to $\vec{v}_1$. It follows that since the null-space of $A - \lambda I$ was one-dimensional, we can pick any vector $\vec{v}_2$ not in this null-space.

(3) Since $\vec{v}_1$ and $\vec{v}_2$ form a basis, we can represent any vector in $\mathbb{R}^2$ using coordinates. For ease of calculation, we pick $\vec{v}_2$ such that $(A - \lambda I)\vec{v}_2 = \vec{v}_1$. This would mean that $A\vec{v}_2 = \vec{v}_1 + \lambda \vec{v}_2$.

(4) In matrix form we can write this as $AV = \Lambda V$ where $\Lambda = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ proving the existence of a basis V in which A is upper-triangular.

[a] We do this for a 2×2 matrix, but this can be extended to an arbitrary $n \times n$ matrix through induction. We will look at the $n \times n$ case in a later note.

To summarize, we have shown the existence of a basis V for which A has an upper triangular representation and we have also derived a form for our solution

$$x(t) = \alpha_1 e^{\lambda t} + \alpha_2 t e^{\lambda t} \tag{38}$$

To prove its uniqueness, we can use the same arguments from before for vector differential equations.

Going back to our RLC example, we experience critically damping when there is a single eigenvalue $\lambda = -\frac{R}{2L} = -2\sqrt{5} \times 10^{10}$. The value of R that made this possible was $R = 2\sqrt{\frac{L}{C}} = 2\sqrt{\frac{1 \times 10^6}{2 \times 10^{-15}}} = 2\sqrt{5} \times 10^4$. Using our verified solution from Equation 38, we can solve our differential equation by plugging in initial conditions to arrive at the solution

$$v_c(t) = 4e^{-2\sqrt{5} \times 10^{10} t} + 8\sqrt{5} \times 10^{10} t e^{-2\sqrt{5} \times 10^{10} t} \tag{39}$$

In our discussion of damping, recall that an overdamped system took longer to reach steady state while an underdamped system reached steady state quicker with the cost of ringing. In a critically damped system, we reach steady state as quickly as possible without experiencing ringing making it a very desirable outcome. In Figure 4 we show a comparison between the effects of overdamping and critical damping.

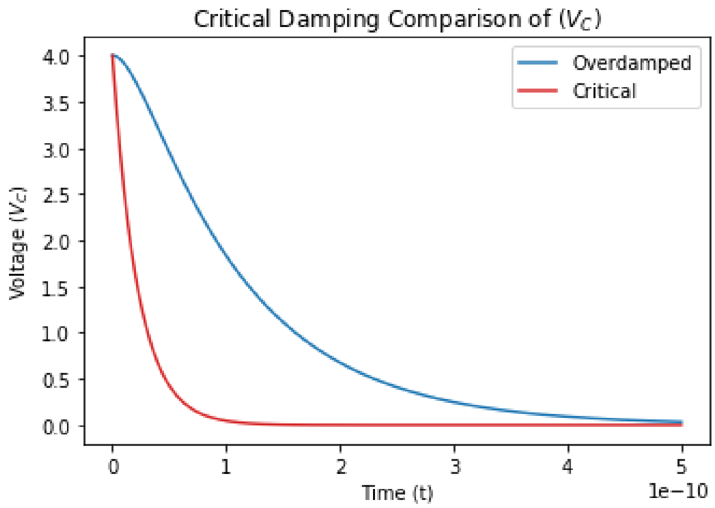


Figure 4: Voltage across capacitor (v_c)

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