EECS 16B Designing Information Devices and Systems II
Summer 2020

Note 3

1 Introduction

In the last note, we further developed our understanding of RC Circuits and differential equations by treating our voltage source $v_s(t)$ as an "input." When the input was piecewise constant, we took the approach of breaking down the input into windows at which we treated $v_s(t)$ as a constant and were able to analyze the response.

In this note, we will develop a method to solve a differential equation of the form

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t) \tag{1}$$

where $u(t)$ is any function of time. Doing so will allow us to solve responses for an RC Circuit with any arbitrary input and by the end of the note, we will have gained some insight on our response when the input $u(t)$ is exponential or sinusoidal.

2 Solving our Differential Equation

As stated in the introduction, our goal is to try and find a solution for the differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t)$$
$$x(0) = x_0$$

At the moment, this seems like a very daunting task since $u(t)$ can be any arbitrary function. We would like to rephrase this problem into one that we know how to solve. To do this, we will introduce a trick called the Integrating Factor.

The Product Rule from Calculus tells us how to take derivatives of a product of two functions.

$$\frac{d}{dt}[x(t)\cdot y(t)] = y(t)\frac{d}{dt}x(t) + x(t)\frac{d}{dt}y(t)$$

It might not seem obvious at first, but we can in fact use this product rule relation to rephrase our differential equation as it relates $x(t)$ to $\frac{d}{dt}x(t)$. Let's start by observing what happens when $y(t) = e^{-\lambda t}$.

$$\frac{d}{dt}[x(t)e^{-\lambda t}] = e^{-\lambda t}\frac{d}{dt}x(t) - x(t)\lambda e^{-\lambda t} = e^{-\lambda t}(\frac{d}{dt}x(t) - \lambda x(t)) = e^{-\lambda t}u(t)$$

This clever trick has allowed us to rephrase our problem by rewritting the product of $u(t)$ and an exponential function $e^{-\lambda t}$ as the derivative of a function that we can integrate. We will rename our functions to be in terms of τ to avoid confusion when taking the integral from 0 to t. [1]

$$\int_0^t \frac{d}{d\tau}[x(\tau)e^{-\lambda\tau}]d\tau = \int_0^t e^{-\lambda\tau}u(\tau)d\tau \tag{2}$$

$$x(t)e^{-\lambda t} - x(0)e^{-\lambda\cdot 0} = \int_0^t e^{-\lambda\tau}u(\tau)d\tau \tag{3}$$

Simplifying and rearranging terms, we can write the solution to our differential equation as

$$x(t) = x_0 e^{\lambda t} + \int_0^t u(\tau)e^{\lambda(t-\tau)}d\tau \tag{4}$$

2.1 Checking our solution

Our approach using an Integrating Factor has given us a candidate solution for $x(t)$. However, we need to check to see if our solution makes any sense and then understand if it is indeed correct.

2.1.1 Plug in a known function

In order to check our solution to the differential equation, the first thing to do is plug in an input whose solution we already know and trust. Let us plug in a constant input that is 1 for time $t \geq 0$. Recall that the differential equation will be $\frac{d}{dt}v(t) = -\frac{1}{RC}v(t) + \frac{1}{RC}$ and using our solution for $v(t)$ we get:

$$v(t) = v_0 e^{\lambda t} + \int_0^t \frac{1}{RC}e^{\lambda(t-\tau)}d\tau$$

where for our capacitor circuit $\lambda = \frac{-1}{RC}$ and the initial condition $v_0 = 0$.

$$\begin{aligned}
v(t) &= v_0 e^{-\frac{1}{RC}t} + \int_0^t \left(\frac{1}{RC}\right) e^{-\frac{1}{RC}(t-\tau)}d\tau \\
&= e^{-\frac{1}{RC}(t-\tau)}\Big|_0^t \\
&= e^{-\frac{1}{RC}(t-t)} - e^{-\frac{1}{RC}t} \\
&= 1 - e^{-\frac{1}{RC}t}
\end{aligned}$$

This is exactly the equation for a charging capacitor: $v(t) = V_{DD}\left(1 - e^{\frac{-1}{RC}t}\right)$ where $V_{DD} = 1$. Which is exactly what we expect with this constant input! So this makes sense. The solution also clearly makes sense for a zero input.

[1]This approach assumes that $u(t)$ is a continuous and integrable function. There are many pathological examples of $u(t)$ where this approach would break down. However, all of the functions we will be dealing with in this class will be integrable and the main purpose behind this proof is to see what happens when $u(t)$ is an exponential function.

2.1.2 Plug into the original differential equation

We can further verify this by plugging the solution $x(t) = x_0 e^{\lambda(t)} + \int_0^t u(\tau)e^{\lambda(t-\tau)}d\tau$ into the original differential equation:

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t);\ x(0) = x_0$$

Starting with the initial condition, $x(0) = x_0 e^{\lambda(0)} + \int_0^0 u(\tau)e^{\lambda(t-\tau)}d\tau = x_0 e^0 + 0 = x_0$, as expected.

Then taking the derivative of our solution,

$$\frac{d}{dt}x(t) = \frac{d}{dt}\left[x_0 e^{\lambda t} + e^{\lambda t}\int_0^t u(\tau)e^{-\tau)}d\tau\right]$$

We can then use the fundamental theorem of calculus to compute the derivative.[2]:

$$\begin{aligned}\frac{d}{dt}x(t) &= \lambda x_0 e^{\lambda t} + \left[1e^{\lambda(t-t)}u(t) + \int_0^t u(\tau)\lambda e^{\lambda(t-\tau)}d\tau\right] \\ &= \lambda x_0 e^{\lambda t} + \left[u(t) + \lambda\int_0^t u(\tau)e^{\lambda(t-\tau)}d\tau\right] \\ &= \lambda\left[x_0 e^{\lambda t} + \int_0^t u(\tau)e^{\lambda(t-\tau)}d\tau\right] + u(t).\end{aligned}$$

This shows that our proposed solution does indeed satisfy the original differential equation!

2.1.3 Uniqueness

Now that we have showed a solution to the differential equation, it is important to consider uniqueness. You will do this in your homework! The key trick is to consider the difference $z(t) = x(t) - y(t)$ of two candidate solutions $x(t)$ and $y(t)$. If you take the derivative $\frac{d}{dt}z(t)$, you will see that this must solve the differential equation $\frac{d}{dt}z(t) = \lambda z(t)$ with no input, together with the initial condition $z(0) = x(0) - y(0) = 0$. Since this differential equation has a unique solution $0e^{\lambda t} = 0$ for all $t \geq 0$, it must be the case that $z(t) = 0$ and hence $x(t) = y(t)$. So solutions must be unique. Because we have found one, we have found the only one!

[2]Recall that the fundamental theorem can be used to apply the derivative to the integral in a chain rule like fashion. We first take the derivative of the upper limit of the integral times the upper limit plugged into the inside of the integral. To this, we add the integral of the derivative of the inside of the integral. The latter term can be viewed as corresponding to bringing the derivative inside a summation. The first term corresponds to understanding that the number of terms essentially depends on t, and so the "last term" in the sum has to do with the derivative with respect to the upper limit of the integral. If you don't remember this, look up the Fundamental Theorem of Calculus in Leibniz form.

3 Exponential Inputs

Alternate Form of Inputs

Before we begin applying any inputs, we introduce an alternate form to our differential equation that has an input $u(t)$.

$$\frac{d}{dt}x(t) = \lambda(x(t) - u(t)) \tag{5}$$

This form will come especially in handy when looking at circuits. Compare this with the differential equation

$$\frac{d}{dt}v_c(t) = -\frac{1}{RC}v_c(t) + \frac{v_s(t)}{RC} \tag{6}$$

Notice the advantages of this form with $\lambda = -\frac{1}{RC}$ and $u(t) = v_s(t)$. Looking back at our solution from equation (4), we can deduce that the solution to equation (5) is

$$x(t) = x_0 e^{\lambda t} - \lambda \int_0^t u(\tau) e^{\lambda(t-\tau)} d\tau \tag{7}$$

Now that we have a verified unique solution to our differential equation, let us examine some interesting inputs. We start with the input $u(t) = e^{st}$ which from equation (13) yields the solution

$$x(t) = x_0 e^{\lambda t} - \frac{\lambda}{s-\lambda}(e^{st} - e^{\lambda t}) \tag{8}$$

However, notice that this assumes that $s \neq \lambda$. We can verify that if $s = \lambda$, the solution will be

$$x(t) = x_0 e^{\lambda t} - \lambda t \cdot e^{\lambda t} \tag{9}$$

The result of equation (9) turns out to be important later, but for now, it is just an interesting example.

Now going back to our original exponential input $u(t) = e^{st}$ with $s \neq \lambda$, we can notice that if $\lambda < 0$ as seen in RC circuits, then as $t \to \infty$, the $e^{\lambda t}$ terms will approach 0 and are henceforth **transient.** This means that the steady state of $x(t)$ based on equation (4) is

$$x_{ss}(t) = \frac{-\lambda}{s-\lambda}e^{st} \tag{10}$$

Again the result of this equation (10) will be of importance later, but notice the "eigenfunction" property of our input $u(t)$. We input $u(t) = e^{st}$, the response of our system is again a multiple of e^{st}. The eigenvalue associated with this eigenfunction is $\frac{-\lambda}{s-\lambda}$. Specifically for an RC circuit with $\lambda = -\frac{1}{RC}$ we can see that this eigenvalue μ is

$$\mu = \frac{-\lambda}{s-\lambda} = \frac{\frac{1}{RC}}{s+\frac{1}{RC}} = \frac{1}{1+sRC} \tag{11}$$

4 Sinusoidal Inputs

Another type of input we can look at are sinusoidal inputs. In the context of RC circuits, this would represent an AC waveform also known as alternating current. Using the same alternate form of our differential equation from equation (5), if our input is sinusoidal or $u(t) = A\sin(\omega t)$, our solution will be of the form

$$x(t) = x_0 e^{\lambda t} - \lambda \int_0^t A\sin(\omega\tau) e^{\lambda(t-\tau)} d\tau \tag{12}$$

This is a rather involved integral and will require a clever trick by using Integration by Parts twice. [3] Upon solving the integral, we will see that the solution is

$$x(t) = x_0 e^{\lambda t} - \frac{A}{\lambda^2 + \omega^2} \left(\lambda^2 \sin(\omega t) + \omega\lambda \cos(\omega t) - e^{\lambda t} \omega\lambda \right) \tag{13}$$

Plugging in for $\lambda = -\frac{1}{RC}$, we can simplify our expression to be

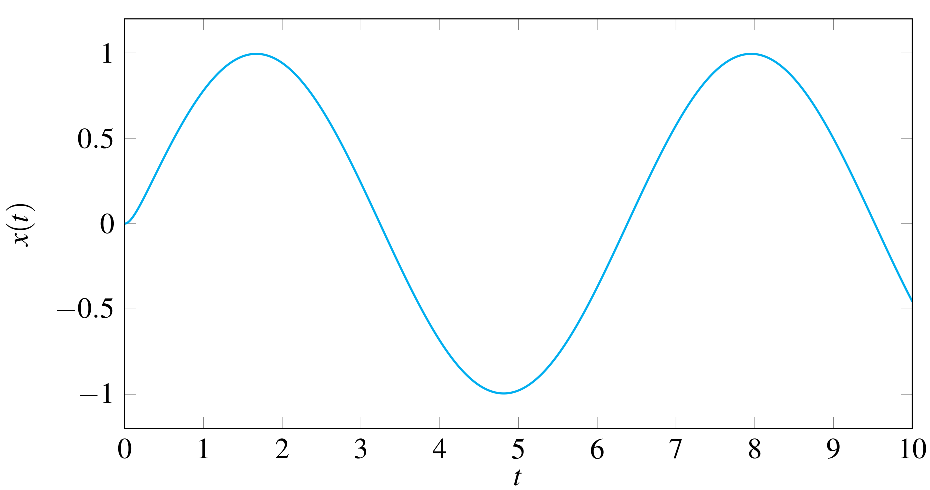
$$x(t) = x_0 e^{\lambda t} + A \frac{\sin(\omega t) - \omega RC\cos(\omega t) - e^{-\frac{1}{RC}t}\omega RC}{1 + (\omega RC)^2} \tag{14}$$

As $t \to \infty$, the exponential terms will go to zero again and the steady state value of $x(t)$ will be

$$x_{ss}(t) = A \frac{\sin(\omega t) - \omega RC\cos(\omega t)}{1 + (\omega RC)^2} \tag{15}$$

Notice how the response to a sinusoidal input is also sinusoidal at steady state. We show a plot of $x(t)$ for $A = 1, \omega = 1, RC = 0.1$ to illustrate this sinusoidal behavior.

Responses for assorted values of $|a|$



We conclude this note by claiming that sinusoidal inputs are in fact a special case of exponential inputs! Once we build an understanding of complex numbers, we will come back and re-explore the idea that $u(t) = e^{st}$ is an eigenfunction for these differential equations.

[3] The emphasis here is to not on solving the integral, rather understanding its complexity and realizing how involved the calculation is.

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