EECS 16A   Designing Information Devices and Systems I
## Spring 2022   Discussion 8B

1. Capacitance Equivalence

For the structures shown below, assume that the plates have a depth L into the page and a width W and are always a distance d apart. The dielectric between the plates has absolute permitivity ε. For the following calculations, assume the capacitance is purely parallel plate, i.e. ignore fringing field effects.
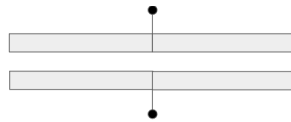
(a) What is the capacitance of the structure shown below?
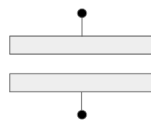
Answer:
The capacitance of two parallel plate conductors is given by $C = \varepsilon \frac{A}{d}$. The cross-sectional area A is WL, so the capacitance is $C = \varepsilon \frac{WL}{d}$.

(b) Suppose that we take two such structures and put them next to each other as shown below. What is the capacitance of this new structure?

Answer:
Here, we have just doubled the width of the capacitor plates. The new capacitance is $C = \varepsilon \frac{2WL}{d}$. Notice that this is just double the capacitance from the first part.

(c) Now suppose that rather than connecting them together as shown above, we connect them with an ideal wire as shown below. What is the capacitance of this structure?
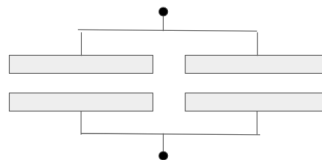
Answer:
Even though we use a wire to connect the plates (vs in part (b) where they are abutted), they are still connected to the same nodes and so can be treated as one capacitor. We have to look at the area with overlapping parallel plates. Hence the total plate area is still $2WL$, and we get $C = \varepsilon \frac{2WL}{d}$.

(d) Suppose that we now take two capacitors and connect them as shown below. What is the capacitance of the structure?
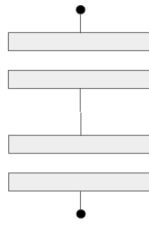
Answer: Physically, parallel plate capacitance comes from charge building up and creating an electric field between 2 parallel plates. The wire in between the two middle plates cannot contribute to the electric field since it has no area and cannot build up charge. Thus, we only look at the distance created between the two pairs of plates.
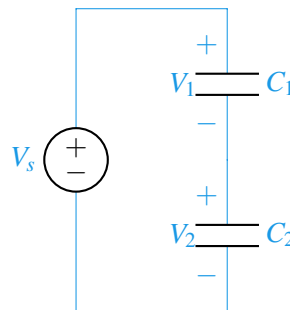
$$d_{total} = d_1 + d_2 = 2d$$

$$C_{eq} = \varepsilon \frac{WL}{d_1 + d_2} = (\frac{d_1}{\varepsilon WL} + \frac{d_2}{\varepsilon WL})^{-1} = (C_1^{-1} + C_2^{-1})^{-1} = \frac{C}{2}$$

Thus we get that putting 2 of the same capacitors in series halves the effective capacitance. Note this is the same result we develop from the "parallel" rule:

$$C_{eq} = C||C = \frac{C \cdot C}{C + C} = \frac{C}{2}$$

Alternate solution: If we apply a voltage source to the two capacitors in this configuration, we can apply KVL to solve:
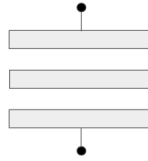

where $C_1 = C_2 = C = \varepsilon \frac{A}{d}$. If the capacitors were initially discharged, then we know that the charge across C_1 and C_2 have to be equal. Then we have:

$$V_s = V_1 + V_2$$
$$\frac{Q_{tot}}{C_{eq}} = \frac{Q_1}{C_1} + \frac{Q_2}{C_2}$$
$$\frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2}$$
$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

This gives an equivalence capacitor to the two in series. This last equation is the same result as the "parallel" rule.

(e) What is the capacitance of the structure shown below?

Answer:

Like in the previous part, we only care about the total area and distance where we have parallel plates and therefore electric fields. The total distance is again $d_1 + d_2 = 2d$, so the total capacitance is:

$$C_{eq} = \varepsilon \frac{WL}{2d} = \frac{C}{2}$$

which is the same result as in part (d).

2. **Current Sources And Capacitors**

Given the circuit below, find an expression for $v_{out}(t)$ in terms of I_s, C, V_0, and t, where V_0 is the initial voltage across the capacitor at $t = 0$.
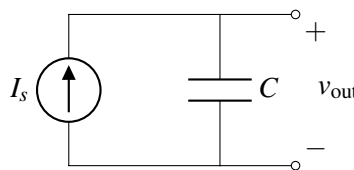
Then plot the function $v_{out}(t)$ over time on the graph below for the following conditions detailed below. Use the values $I_s = 1\text{mA}$ and $C = 2\,\mu\text{F}$.

(a) Capacitor is initially uncharged, with $V_0 = 0$ at $t = 0$.

(b) Capacitor has been charged with $V_0 = +1.5\text{V}$ at $t = 0$.

(c) **Practice:** Swap this capacitor for one with half the capacitance $C = 1\,\mu\text{F}$, which is initially uncharged, with $V_0 = 0$ at $t = 0$.

HINT: Recall the calculus identity $\int_a^b f'(x)dx = f(b) - f(a)$, where $f'(x) = \frac{df}{dx}$.
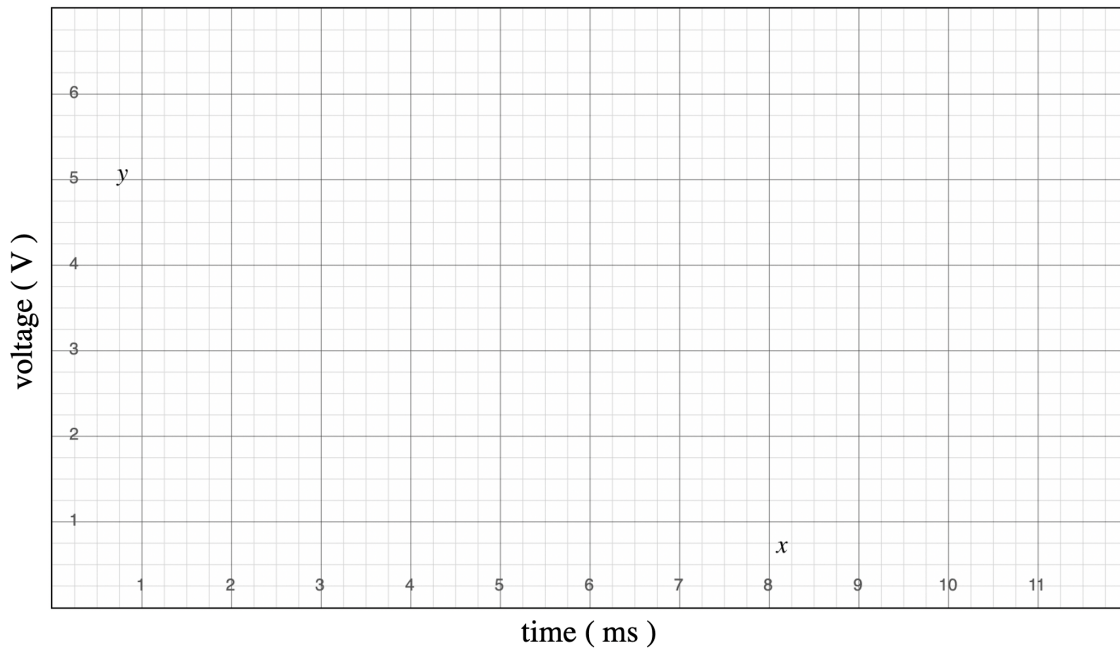
Answer:
The key here is to exploit the capacitor equation by taking its time-derivative

$$Q = C\,V_{out} \quad \longrightarrow \quad \frac{dQ}{dt} \equiv I_s = C\frac{dV_{out}}{dt}.$$

From here we can rearrange and show that $\frac{dV_{out}}{dt} = \frac{I_s}{C}$.
Thus the voltage has a constant slope!
Our solution is

$$V_{out}(t) = V_0 + \left(\frac{I_s}{C}\right) t \qquad \square$$

To be more mathematically formal, we are solving a differential equation that happens to return a linear function for $v_{out}(t)$:

$$\frac{dV_{out}}{dt} = \frac{I_s}{C} \quad \longrightarrow \quad \int_0^t \frac{dV_{out}}{dt}dt \equiv V_{out}(t) - V_{out}(0) = \int_0^t \frac{I_s}{C}dt \equiv \frac{I_s}{C}\int_0^t 1dt \equiv \frac{I_s}{C}t$$

Thus we arrive at the same statement as seen earlier $V_{out}(t) = V_{out}(0) + \left(\frac{I_s}{C}\right)t$.

For all parts, we have $I_s = 1\text{mA}$. For part (a), we have $V_{out}(0) = 0\text{V}$ and $C = 2\mu\text{F}$. Plugging this into our equation, we get:

$$V_{out}(t) = 0\text{V} + \frac{1\text{mA}}{2\mu\text{F}} = \left(\frac{1}{2}\frac{\text{V}}{\text{ms}}\right)t$$

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For part (b), this only changes by an intercept / initial condition $V_{out}(0) = 1.5\text{V}$:

$$V_{out}(t) = 1.5\text{V} + \frac{1\text{mA}}{2\mu\text{F}} = 1.5\text{V} + \left(\frac{1}{2}\frac{\text{V}}{\text{ms}}\right)t$$

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For part (c), we have $V_{out}(0) = 0V$ and $C = 1\mu F$.

$$V_{out}(t) = 0V + \frac{1mA}{1\mu F} = \left(1\frac{V}{ms}\right)t$$

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With half the capacitance and the same current, we get that the slope is twice as large. Some physical intuition: capacitors are like buckets, and charge is like the total water. Voltage is a measure of the height of the water in the bucket. Capacitance is the area of the opening of the bucket, i.e. the area of top hole or the lid. If the opening, like the capacitance, is halved, and a hose filling the bucket, like a current source, has constant flow, then the height of the water will grow twice as fast.

When plotting, make sure to recall (a) and (c) start at the origin, while (b) has initially charged plates by $V_0 = 1.5V$. Results are shown below