Discussion 2B

The following notes are useful for this discussion: Note 3 (sections 1 and 2)

1. **Changing Coordinates and Systems of Differential Equations, I**

   Recall from lecture that matrix differential equations follow the form

   $$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \tag{1}$$

   where $\vec{x}(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. Note that $\frac{d}{dt}\vec{x}(t)$ is equivalent to $\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \vdots \\ \frac{dx_n(t)}{dt} \end{bmatrix}$ where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$.

   In other words, taking the derivative of the vector is the same as taking the derivative elementwise of its components. When A is diagonal, we can treat the matrix differential equation as a system of n separate, scalar differential equations. In this discussion, we will use **change of variables** to tackle a matrix differential equation where A is not diagonal. This will help us model the behavior of more complex circuits where A will usually be non-diagonal.

   First, we can practice by solving a matrix differential equation with diagonal A. Suppose we have the following differential equation (valid for $t \geq 0$)

   $$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix} \vec{x}(t) \tag{2}$$

   with initial condition $\vec{x}(0) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

   (a) **Write the matrix differential equation as a system of individual, scalar differential equations and solve for $\vec{x}(t)$ for $t \geq 0$.**
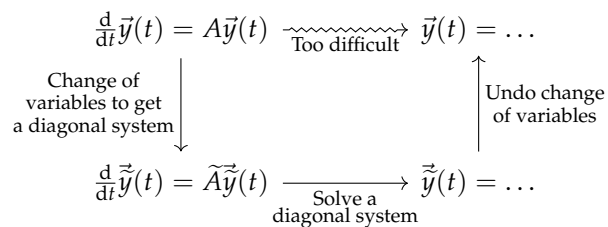
Now, suppose we are actually interested in a different set of variables with the following differential equations:

$$\frac{dy_1(t)}{dt} = -5y_1(t) + 2y_2(t) \tag{3}$$

$$\frac{dy_2(t)}{dt} = 6y_1(t) - 6y_2(t) \tag{4}$$

(b) **Write out the above system of differential equations in matrix form**. *HINT: Define the matrix differential equation in terms of $\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$. What is your "A" matrix here?* Can we solve this system in a similar way as we did above?

As you may have noticed, it is not possible to solve the differential equation using methods we have already covered in this class. We can try to use change of variables to turn this problem into one with a diagonal system since we know how to solve these types of equations. Consider the strategy outlined in fig. 1. We want to change variables to $\vec{\tilde{y}}(t)$, such that we end up with a differential equation where $\tilde{A}$ will be diagonal. This is especially important when there is no clear path to a solution with just the system involving $\vec{y}(t)$ (as is the case here).

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t) \xrightsquigarrow[\text{Too difficult}]{} \vec{y}(t) = \ldots$$

Change of variables to get a diagonal system ↓
Undo change of variables ↑

$$\frac{d}{dt}\vec{\tilde{y}}(t) = \tilde{A}\vec{\tilde{y}}(t) \xrightarrow[\substack{\text{Solve a} \\ \text{diagonal system}}]{} \vec{\tilde{y}}(t) = \ldots$$

Figure 1: A Strategy to Solve for $\vec{y}(t)$

(c) We can define $\vec{\tilde{y}}(t) = \begin{bmatrix} \tilde{y}_1(t) \\ \tilde{y}_2(t) \end{bmatrix}$ to achieve the goal described above. Consider the following relationship between $\vec{y}(t)$ and $\vec{\tilde{y}}(t)$:

$$y_1(t) = -\tilde{y}_1(t) + 2\tilde{y}_2(t) \tag{5}$$
$$y_2(t) = 2\tilde{y}_1(t) + 3\tilde{y}_2(t) \tag{6}$$

(For now, take this change of variables as a given. We will explain how to come up with transformations like this a bit later.)

Write out this transformation in matrix form ($\vec{y}(t) = V\vec{\tilde{y}}(t)$ for some V). This will give us a representation for $\vec{y}(t)$ in terms of $\vec{\tilde{y}}(t)$. **Then, find a way to represent $\vec{\tilde{y}}(t)$ in terms of $\vec{y}(t)$.** What conditions need to hold for this representation to work? Recall that, the inverse of a 2×2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{7}$$

(d) Suppose that the following initial conditions are given: $\vec{y}(0) = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$. Now that we changed variables, we also have to appropriately change our initial condition for the new variables we defined. **How do these initial conditions for $\vec{y}(t)$ translate into the initial conditions for $\vec{\tilde{y}}(t)$?** *HINT: Use the representation for $\vec{\tilde{y}}(t)$ in terms of $\vec{y}(t)$ from part **1.c**.*

(e) Now, we are ready to write a new differential equation for $\vec{\tilde{y}}(t)$. **Incorporate $\vec{\tilde{y}}(t)$, your new variable, into the matrix differential equation from part 1.b to come up with a differential equation for $\vec{\tilde{y}}(t)$.** *HINT: How can we substitute the $\vec{y}(t)$ terms with terms involving $\vec{\tilde{y}}(t)$? Also, recall that, since the derivative operator is linear, we can write $\frac{d}{dt}M\vec{x}(t) = M\frac{d}{dt}\vec{x}(t)$ where M is a matrix of constants.* Can we solve this system of differential equations?

(f) **Solve the differential equation for $\vec{\tilde{y}}(t)$. Then, "undo" the change of variables from the previous parts to find a solution for $\vec{y}(t)$.** *HINT: How can we "recover" $\vec{y}(t)$ from $\vec{\tilde{y}}(t)$?* **Then, fill in the strategy diagram in fig. 2 with the following:**

   (i) Mathematically, how did we define our change of variables?

   (ii) In terms of $\vec{\tilde{y}}(t)$, A, and V, what matrix differential equation did we solve?

   (iii) Mathematically, how did we "undo" our change of variables?

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t) \xrightarrow[\text{Too difficult}]{} \vec{y}(t) = \ldots$$

(i) $\downarrow$ (iii) $\uparrow$

(ii) $\xrightarrow[\substack{\text{Solve a}\\\text{diagonal system}}]{} \vec{\tilde{y}}(t) = \ldots$
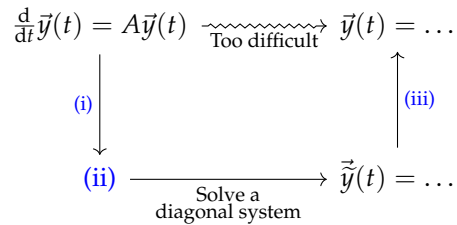
Figure 2: Mathematical Description of Our Strategy to Solve for $\vec{y}(t)$

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