

## 1 Diagonalization

Consider an  $n \times n$  matrix  $A$  that has  $n$  linearly independent eigenvalue/eigenvector pairs  $(\lambda_1, \vec{v}_1), \dots, (\lambda_n, \vec{v}_n)$  that can be put into a matrices  $V$  and  $\Lambda$ .

$$V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & & | \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

a) Show that  $AV = V\Lambda$ .

b) Use the fact in part (a) to conclude that  $A = V\Lambda V^{-1}$ .

## 2 Systems of Differential Equations

Consider a system of differential equations (valid for  $t \geq 0$ )

$$\frac{d}{dt}x_1(t) = -4x_1(t) + x_2(t) \quad (1)$$

$$\frac{d}{dt}x_2(t) = 2x_1(t) - 3x_2(t) \quad (2)$$

with initial conditions  $x_1(0) = 3$  and  $x_2(0) = 3$ .

a) Write out the differential equations and initial conditions in matrix/vector form.

b) Find the eigenvalues  $\lambda_1, \lambda_2$  and eigenspaces for the differential equation matrix above.

- c) Use the diagonalization of  $A = V\Lambda V^{-1}$  to express the differential equation in terms of a new variables  $z_1(t)$ ,  $z_2(t)$ . Remember to find the new initial conditions  $z_1(0)$ ,  $z_2(0)$ . (These variables represent eigenbasis-aligned coordinates.)

- d) Solve the differential equation for  $z_i(t)$  in the eigenbasis.

e) Convert your solution back into the original coordinates to find  $x_i(t)$ .

f) We can solve this equation using a slightly shorter approach by observing that the solutions for  $x_i(t)$  will all be of the form

$$x_i(t) = \sum_k c_k e^{\lambda_k t}$$

where  $\lambda_k$  is an eigenvalue of our differential equation relation matrix  $A$ .

Since we have observed that the solutions will include  $e^{\lambda_i t}$  terms, once we have found the eigenvalues for our differential equation matrix, we can guess the forms of the  $x_i(t)$  as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} \\ \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t} \end{bmatrix}$$

where  $\alpha_1, \alpha_2, \beta_1, \beta_2$  are all constants.

Take the derivative to write out

$$\begin{bmatrix} \frac{d}{dt} x_1(t) \\ \frac{d}{dt} x_2(t) \end{bmatrix}.$$

and connect this to the given differential equation.

Solve for  $x_i(t)$  from this form of the derivative.

