Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 4-cycles

We use $G(n,p)$ to denote the distribution of graphs obtained by taking n vertices and for each pair of vertices i,j placing edge $\{i,j\}$ independently with probability p.

(a) Compute the expected number of edges in $G(n,p)$?

(b) Compute the expected number of 4-cycles in $G(n,p)$?

(c) Give a polynomial time randomized algorithm that takes in n as input and in poly(n)-time outputs a graph G such that G has no 4-cycles and the expected number of edges in G is $\Omega(n^{4/3})$.

2 Universal Hashing

Let $[m]$ denote the set $\{0,1,...,m-1\}$. Recall that a family of functions $\mathcal{H}$ is *universal* if for any $x \neq y, \Pr_{h\sim\mathcal{H}}[h(x)=h(y)] \leq 1/m$. That is, the chance that $h(x)=h(y)$ if we sample h uniformly at random from $\mathcal{H}$ is at most $1/m$.

For each of the following families of hash functions, determine whether or not it is universal. If it is universal, determine how many random bits are needed to choose a function from the family.

(a) $H = \{h_{a_1,a_2} : a_1, a_2 \in [m]\}$, where m is a fixed prime and

$$h_{a_1,a_2}(x_1,x_2) = a_1x_1 + a_2x_2 \mod m$$

Notice that each of these functions has signature $h_{a_1,a_2} : [m]^2 \to [m]$, that is, it maps a pair of integers in $[m]$ to a single integer in $[m]$.

(b) H is as before, except that now $m = 2^k$ for $k > 1$ is some fixed power of 2.

(c) H is the set of all functions $f : [m] \to [m-1]$.

3 Polynomial Identity Testing

Suppose we are given a polynomial $p(x_1, \dots, x_k)$ over the reals such that p is not in any normal form. For example, we might be given the polynomial:

$$p(x_1, x_2, x_3) = (3x_1 + 2x_2)(2x_3 - 2x_1)(x_1 + x_2 + 3x_3) + x_2$$

We want to test if p is equal to 0, meaning that $p(a_1, a_2, \dots, a_k) = 0$ on all real inputs $a_1, a_2, \dots a_k$. This suggests a simple algorithm to check if p equals 0: test p on some $a_1, \dots, a_k$ and say p equals 0 if and only if $p(a_1, \dots, a_k) = 0$. However, this algorithm will fail if $p(a_1, \dots, a_k) = 0$ and p does not equal 0. We will show through the following questions that if the degree of p is d, then for any finite set of reals S, by randomly choosing $a_1, \dots, a_k \in S$, we get $Pr[p(a_1, \dots, a_k) = 0] \leq \frac{d}{|S|}$. So by choosing a large enough subset, S, we are very likely to correctly determine whether p is equal to 0 or not. Note that the degree of, say, $p(x, y) = x^2y^2 + x^3$ is 4, since the monomial x^2y^2 has degree $4 = 2 + 2$. You may assume here that all basic operations on the reals take constant time.

(a) First let's see why a probabilistic algorithm might be necessary. Give a naive deterministic algorithm to test if a given polynomial p is equal to 0. What is the worst-case runtime of your algorithm?

(b) Show that if p is univariate and degree d, then $Pr[p(a_1) = 0] \leq \frac{d}{|S|}$

(c) (Challenge question) Show by induction on k that for all nonzero degree-d polynomials p on k variables:

$$Pr[p(a_1, \dots, a_k) = 0] \leq \frac{d}{|S|}$$

Hint: Write $p(x_1, \dots, x_n) = \Sigma_{i=0}^{d} p_i(x_1, \dots, x_{n-1}) x_n^i$

(d) Use the bound you found in the last part to show a way to check if two polynomials p and q are identical (i.e., yield the same outputs on all inputs).

4 Monte Carlo Games

Let's suppose we have a Monte Carlo algorithm (a randomized algorithm which has a deterministic bound on its runtime, but which only outputs the correct answer some of the time). Call this algorithm A; then $A(x, r)$ is the output of A on input x and random bits r. In this question, we will think of A as a distribution over many deterministic algorithms. Convince yourself that this makes sense: after all, if we fix a setting to the random bits r, we get $A_r(x)$, which is a deterministic algorithm (which may be wrong on some inputs). Let's fix a set of algorithms S (say, polynomial-time algorithms). Note that A has whatever property defines S if and only if it is a distribution over only algorithms in S (for example, we say a Monte Carlo algorithm is polynomial time if and only if it runs in polynomial time for all settings to the randomness, which is equivalent to all the deterministic algorithms in its distribution running in polynomial time).

We will define a function $c(a, x)$ which indicates whether the deterministic algorithm $a \in S$ is correct on input x; $c(a, x) = 1$ if a is correct on input x, and 0 if it is incorrect.

Let's use this function to define a zero-sum game; the row player will choose a and the column player will choose x; then a payoff of $c(a, x)$ will go to the row player.

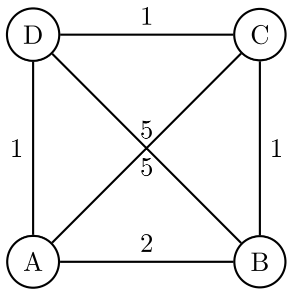
(a) Describe the action and goal of the row and column players. Interpret these in the setting of 'correctness of the randomly chosen algorithm' that we constructed the game from. *Hint: Since $c(a, x)$ is an indicator, $\mathbb{E}[c(a, x)] = \Pr[c(a, x) = 1]$.*

(b) Using zero-sum game duality in conjunction with your interpretation above, what can we say about a problem if we know there exists a polynomial-time randomized algorithm which is correct with probability 2/3 on all inputs? What can we say if we know that there is a distribution of inputs on which no deterministic algorithm is correct with probability 2/3?

Hint: use the fact that a randomized algorithm induces a distribution over deterministic algorithms.

5 Traveling Salesman Problem

In the lecture, we learned an approximation algorithm for the Traveling Salesman Problem based on computing an MST and a depth first traversal. Suppose we run this approximation algorithm on the following graph:



The algorithm will return different tours based on the choices it makes during its depth first traversal.

1. Which DFS traversal leads to the best possible output tour?

2. Which DFS traversal leads to the worst possible output tour?

3. What is the approximation ratio given by the algorithm in the worst case for the above instance? Why is it worse than 2? (*Hint*: Consider the triangle inequality on the graph).