# Discussion 12B

The following note is useful for this discussion: Note 18.

## 1. Using a Nonlinear NMOS Transistor for Amplification

Consider the following schematic where $V_{DD} = 1.5\,V$, $R_L = 400\,\Omega$ and the NMOS transistor has threshold voltage $V_{th} = 0.2\,V$. We are interested in analyzing the response of this circuit to input voltages of the form $V_{in}(t) = V_{in,DC} + v_{in,AC}(t)$, where $V_{in,DC}$ is some constant voltage and $v_{in,AC}(t) = 0.001\cos(\omega t)V$ is a sinusoidal signal whose magnitude is much smaller than $V_{in,DC}$.
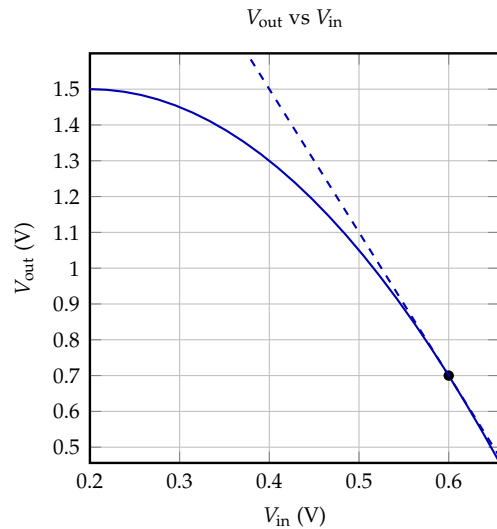
The I-V relationship of an NMOS can be modeled as non-linear functions over different regions of operation. For simplicity, let's just focus on the case when $0 \leq V_{GS} - V_{th} < V_{DS}$. In this regime of interest, the relevant I-V relationship is given by

$$I_{DS}(V_{GS}) = \frac{K}{2}(V_{GS} - V_{th})^2 \tag{1}$$

where $K$ is a constant that depends on the NMOS transistor size and properties.
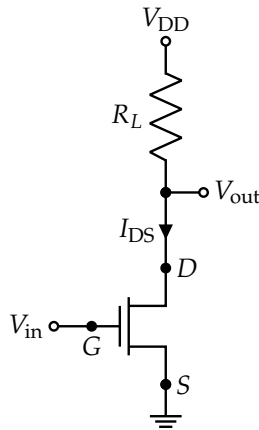


(a) NMOS Transistor circuit

**(b)** $V_{out}$ vs $V_{in}$ in the regime of interest. Tangent is drawn at the operating point $V_{in,DC} = 0.6\,V$, $V_{out,DC} = 0.7\,V$

**Figure 1:** NMOS figures.

From Ohm's law and KCL, we know that

$$V_{out}(t) = V_{DD} - R_L I_{DS}(t). \tag{2}$$

Note from Figure 1a that $V_{in} = V_{GS}$ and $V_{out} = V_{DS}$. In Figure 1b, we can see the curve of $V_{out}$ vs $V_{in}$ in the transistor operating regime of interest.

(a) **Using eq. (1) and eq. (2), express $V_{out}(t)$ as a function of $V_{in}(t)$ symbolically.** (You can use $V_{DD}, R_L, V_{in}, K, V_{th}$ in your answer.)

(b) We can decompose the input into constant (i.e., DC) and time-varying (i.e., AC) components to obtain $V_{\text{in}}(t) = V_{\text{in,DC}} + v_{\text{in,AC}}(t)$. **Linearize $V_{\text{out}}(t)$ as a function of $V_{\text{in}}(t)$ about $V_{\text{in}} = V_{\text{in,DC}}$. What is the slope of the linearized function?**

(c) Next, we can also decompose the output $V_{\text{out}}$ into DC and AC components to obtain $V_{\text{out}} = V_{\text{out,DC}} + v_{\text{out,AC}}(t)$. **What is $V_{\text{out,DC}}$ from the linearized representation in part 1.b? Simplify the linear approximation to be in terms of $v_{\text{out,AC}}(t)$ and $v_{\text{in,AC}}(t)$, for very small $v_{\text{in,AC}}(t)$.**

(d) **For very small $v_{\text{in,AC}}(t)$, what circuit element can we use to represent the I-V relation between $v_{\text{out,AC}}(t)$ and $I_{DS} = g_m v_{\text{in,AC}}(t)$? Draw this circuit element.**

## 2. Feedback Control and Linearization

Consider the problem of balancing a pole on a cart as follows:
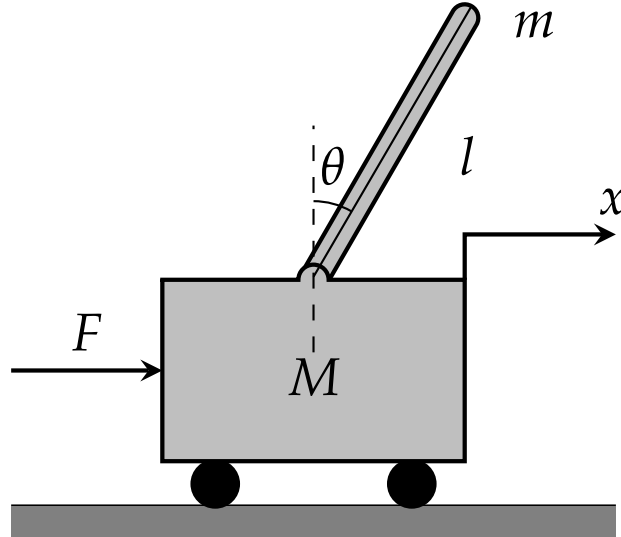


The mass of the cart itself is $M$, the length of the rod is $l$, and the mass of the rod is $m$. The angle $\theta$ is measured with respect to the vertical as shown above, and $x$ is the horizontal translation of the cart (i.e., along the $x$-axis). The force $F$ is the control input to the system. Assume that all of the mass on the rod is concentrated at the very end of the rod. Further assume that there is no friction. The following differential equations are derived from the physics describing the cart-pole system:

$$
\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = \frac{1}{M + m\sin^2(\theta)}\left(F + m\sin(\theta)\left(l\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - g\cos(\theta)\right)\right) \tag{3}
$$

$$
\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} = \frac{1}{l(M + m\sin^2(\theta))}\left(-F\cos(\theta) - ml\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\cos(\theta)\sin(\theta) + (M + m)g\sin(\theta)\right) \tag{4}
$$

*Deriving these equations is out of scope.* **The task is as follows: we would like the pole to remain upright and the cart to be at the origin (i.e., $x = 0$). The cart and pole must also be stationary.** In this problem, we will use Jacobian linearization and state feedback to derive a controller that can achieve this goal for us.

(a) **Show that an appropriate state space would be $\vec{x}(t) = \begin{bmatrix} x \\ \frac{\mathrm{d}x}{\mathrm{d}t} \\ \theta \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} \end{bmatrix}$. Find an appropriate control**

**input $u(t)$, and then find $\vec{f}(\vec{x}, u)$ such that $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = \vec{f}(\vec{x}, u)$. Assume $m = M = l = 1$.**

(b) **What point $\vec{x}_\star$ and $u_\star$ do we want to linearize around?**
*(HINT: Think about what we want the cart-pole system to do. What state do we want the system to converge to?)*

(c) Write $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$. **Find the Jacobian of $\vec{f}(\vec{x}, u)$. That is, find $J_{\vec{x}}\vec{f}$ and $J_u\vec{f}$.** You may leave your answer in terms of the components of $\vec{x}$, $\frac{\partial f_2}{\partial x_3}$, and $\frac{\partial f_4}{\partial x_3}$.

(d) **Linearize the dynamics about the $\vec{x}_\star$ that you found earlier. Explicitly write this linearized system.** You may use the fact that $\frac{\partial f_2}{\partial x_3}(\vec{x}_\star, u_\star) = -g$ and $\frac{\partial f_4}{\partial x_3}(\vec{x}_\star, u_\star) = 2g$. **Is the linearized system stable? How can we accomplish the task mentioned at the beginning of the problem?**

The accompanying Colab notebook shows a demo of how we may want to accomplish the goal of stabilizing the cart-pole system.

**Contributors:**

- Anish Muthali.
- Vladimir Stojanovic.
- Moses Won.
- Ayan Biswas.