

EECS 16B Designing Information Devices and Systems II Note 10

1 Overview

In the previous note, we discussed different types of state-space systems and ended on the note that linear systems are desirable. Linear models are advantageous because their solutions and control can be studied using Linear Algebra. The methods applicable to nonlinear models are limited; therefore it is common practice to approximate a nonlinear model with a linear one that is valid around a desired operating point.

While many systems in the real world are nonlinear due to disturbances, noise, or internal resistive forces, we will look into a type of analysis called **Linearization** in which we approximate the system around a desired operating point.

2 Taylor Expansion

The key to linearization is understanding the Taylor Expansion of a function f . Recall from Calculus that we can expand a function $f(x)$ around a point x_0 through its Taylor Expansion as follows

$$f(x) = f(x_0) + f'(x_0) \cdot (x - x_0) + \frac{1}{2} f''(x_0) \cdot (x - x_0)^2 + \dots \quad (1)$$

To create a linear approximation f_l to the function f , we keep the first two terms and truncate the rest:

$$f_l(x) = f(x_0) + f'(x_0) \cdot (x - x_0) \quad (2)$$

The function $f_l(x)$ will approximate f well for values of x very close to x_0 . This is because the higher order terms in the Taylor Series of f , $\frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$ are close to zero.

Now that we've seen how Taylor series work in the scalar case, let's try to extend it to the multivariate case. If we have 2 multivariate functions f_1 and f_2 with respect to the variables x and y , we can expand each one around the point x_0, y_0 to see that

$$f_1(x, y) = f_1(x_0, y_0) + \frac{\partial f_1}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f_1}{\partial y}(x_0, y_0) \cdot (y - y_0) + \dots \quad (3)$$

$$f_2(x, y) = f_2(x_0, y_0) + \frac{\partial f_2}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f_2}{\partial y}(x_0, y_0) \cdot (y - y_0) + \dots \quad (4)$$

Expressing our linearized function f in matrix-vector form,

$$\begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} = \begin{bmatrix} f_1(x_0, y_0) \\ f_2(x_0, y_0) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x}(x_0, y_0) & \frac{\partial f_1}{\partial y}(x_0, y_0) \\ \frac{\partial f_2}{\partial x}(x_0, y_0) & \frac{\partial f_2}{\partial y}(x_0, y_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \quad (5)$$

$$= \begin{bmatrix} f_1(x_0, y_0) \\ f_2(x_0, y_0) \end{bmatrix} + J(x_0, y_0) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \quad (6)$$

where J refers to the Jacobian of the functions f_1, f_2 . The Jacobian is a matrix of all of the partial derivatives of f_1 and f_2 with respect to x_1 and x_2 .

Extending the Jacobian to n functions f_1, \dots, f_n , in terms of n variables, x_1, \dots, x_n , we see that

$$J_f(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

and we expand the linearized function f_l using its Taylor Series around the point \vec{x}^*

$$f_l(\vec{x}) = f(\vec{x}^*) + J_f(\vec{x}^*) \cdot (\vec{x} - \vec{x}^*) = f(\vec{x}^*) + J_f(\vec{x}^*) \cdot \vec{x}_l(t) \quad (7)$$

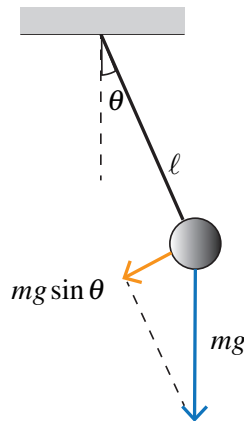
$$\boxed{\vec{x}_l(t) \triangleq \vec{x}(t) - \vec{x}^*} \quad (8)$$

3 Equilibrium Points

Which points should we linearize the system around? Are some points better than others? In this section, we discuss the idea of an equilibrium point. An **equilibrium point** is a point \vec{x}^* where the system remains at rest. This means that $f(\vec{x}^*) = \vec{0}$.

3.1 Pendulum Example

Recall the pendulum system from the previous note:



For state variables $x_1 = \theta$ and $x_2 = \omega$, we wrote out the state-space equations for the pendulum as¹

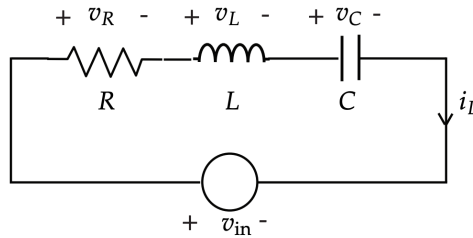
$$\frac{d}{dt}\vec{x} = f(\vec{x}(t)) = \begin{bmatrix} x_2(t) \\ -\frac{g}{l}\sin x_1(t) \end{bmatrix}. \quad (9)$$

There are two equilibrium points for this system: $(x_1, x_2) = (0, 0)$ and $(\pi, 0)$. Note that this is when the pendulum is at rest in a downward position or when the pendulum is turned upside down.

¹ ω is the angular velocity and is equivalent to $\frac{d\theta}{dt}$.

3.2 RLC Circuit

Consider the RLC circuit depicted on the right where u denotes the input voltage.



Then picking state variables $x_1 = v_c$, $x_2 = i_L$ with input $u = v_{in}$, we can write the state-equations

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t), u(t)) = \begin{bmatrix} \frac{1}{C}x_2(t) \\ \frac{1}{L}(-x_1(t) - Rx_2(t) + u(t)) \end{bmatrix} \quad (10)$$

The equilibrium point for this system is when $x_1 = u$ and $x_2 = 0$. For a constant u , this is when the RLC circuit has reached steady state and the capacitor is fully charged.

3.3 Taylor Expansion

What would happen if we expanded a non-linear function f around an equilibrium point x^* ? In the scalar, case we see that

$$f_l(\vec{x}) = f(\vec{x}^*) + J_f(\vec{x}) \cdot (\vec{x} - \vec{x}^*) = J_f(\vec{x}) \cdot x_l \quad (11)$$

From a first glance, we see that our constant term $f(x^*)$ goes away and we have a linear system $\frac{d}{dt}\vec{x} = A\vec{x}(t)$. From a physical perspective, an equilibrium point is one where the system will remain at this point so often times, we would like to see what happens when we make small perturbations to our state.

4 Linearization

Now that we've explored Taylor Expansion of a function and its equilibrium points, we will develop the approach to linearizing a system.

If we have a scalar-system represented by the differential equation,

$$\frac{d}{dt}x = f(x) \quad (12)$$

then we can linearize the system around a point x^* to get the linearized system in terms of $x_\ell = x - x^*$:

$$\frac{d}{dt}x_\ell = f(x^*) + f'(x^*) \cdot x_\ell \quad (13)$$

If the point x^* was an equilibrium point then we would have a linear system with $\alpha = f'(x^*)$

$$\frac{d}{dt}x_\ell = \alpha \cdot x_\ell \quad (14)$$

Extending this idea to the vector case, we see that a linearized system around an equilibrium \vec{x}^* is

$$\frac{d}{dt}\vec{x}_\ell = J_f(\vec{x}^*) \cdot \vec{x}_\ell \quad (15)$$

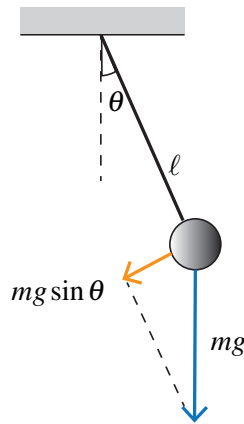
The discrete analog of a linearized system should be nearly identical

$$\vec{x}_\ell[n+1] = f(\vec{x}_\ell[n]) \implies \vec{x}_\ell[n+1] = J_f(\vec{x}^*[n]) \cdot \vec{x}_\ell[n] \quad (16)$$

With all of this in mind, let's take a look at a few examples

4.1 Linearized Pendulum

For the following pendulum system, we found that there were two equilibrium points $(0,0)$ and $(\pi,0)$.



Let us now linearize this system around each equilibrium point. We start by computing the Jacobian matrix

$$J_f(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} \cos x_1 & -\frac{k}{m} \end{bmatrix} \quad (17)$$

Plugging in the equilibrium point $x_1 = x_2 = 0$, we see that our linearized system is

$$\frac{d}{dt}\vec{x}_\ell(t) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{k}{m} \end{bmatrix} \vec{x}_\ell(t) \quad (18)$$

Plugging in the equilibrium point $x_1 = \pi, x_2 = 0$ we see that our linearized system is

$$\frac{d}{dt}\vec{x}_\ell(t) = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{k}{m} \end{bmatrix} \vec{x}_\ell(t) \quad (19)$$

At the moment, this linearized system may not mean much, but if we applied a small torque as an input to the system, we can better understand the meaning of this linearized system. We will do so in the next section.

4.2 Linearization with Inputs

Using our knowledge of Taylor series for multivariate functions, we can quickly extend our knowledge of linearization to systems with inputs. For the following scalar system represented by a differential equation with input u ,

$$\frac{dx}{dt} = f(x, u) \quad (20)$$

we could linearize the system around an equilibrium point (x^*, u^*)

$$\frac{dx_\ell}{dt} = f(x^*, u^*) + \frac{\partial f}{\partial x}(x^*, u^*) \cdot x_\ell + \frac{\partial f}{\partial u}(x^*, u^*) \cdot u_\ell \quad (21)$$

$$= \alpha x_\ell + \beta u_\ell \quad (22)$$

Extending this to the multivariate case with the following differential equation with input \vec{u} ,

$$\frac{d}{dt} \vec{x} = f(\vec{x}, \vec{u}) \quad (23)$$

note that we will need two Jacobians, one with partial derivatives with respect to \vec{x} and one with partial derivatives with respect to \vec{u}

$$\frac{d}{dt} \vec{x}_\ell = A \vec{x}_\ell + B \vec{u}_\ell \quad (24)$$

Writing the Jacobians out explicitly, we see that

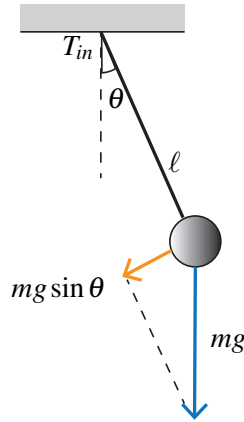
$$A = J_{\vec{x}}(\vec{x}^*, \vec{u}^*) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \bigg|_{\vec{x}^*, \vec{u}^*} \quad (25)$$

$$B = J_{\vec{u}}(\vec{x}^*, \vec{u}^*) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \dots & \frac{\partial f_1}{\partial u_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \dots & \frac{\partial f_n}{\partial u_p} \end{bmatrix} \bigg|_{\vec{x}^*, \vec{u}^*} \quad (26)$$

Note that A and B will have different shapes. The matrix A is an $n \times n$ matrix while the B matrix will be an $n \times p$ matrix if the input \vec{u} is length $p \times 1$ vector.

4.3 Linearized Pendulum with Torque

Now let's revisit the pendulum system that has a small torque T_{in} as an input to the system.



The new differential equation to this system is

$$m\ell \frac{d^2\theta(t)}{dt^2} + k\ell \frac{d\theta(t)}{dt} + mg \sin \theta(t) = \frac{T_{in}(t)}{\ell} \quad (27)$$

We can write out the state equations using the same state variables $x_1 = \theta$, $x_2 = \omega$ and input $u = T_{in}$

$$\frac{d}{dt}\vec{x} = f(\vec{x}(t), u) = \begin{bmatrix} x_2(t) \\ -\frac{k}{m}x_2(t) - \frac{g}{\ell} \sin x_1(t) + \frac{1}{\ell}u(t) \end{bmatrix} \quad (28)$$

Note that if we apply zero torque to the system, there are two equilibrium points at $\theta = 0$ and $\theta = \pi$.

The Jacobian matrices with respect to \vec{x} and u are

$$J_{\vec{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \quad J_u = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} \quad (29)$$

Linearizing around the equilibrium point $x_1 = x_2 = u = 0$, the linearized system is

$$\frac{d}{dt}\vec{x}_\ell(t) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{k}{m} \end{bmatrix} \vec{x}_\ell(t) + \begin{bmatrix} 0 \\ \frac{1}{\ell} \end{bmatrix} u_\ell(t) \quad (30)$$

Applying a small torque to the system, we can show that the system returns back to its equilibrium $\vec{x}^* = \vec{0}$.

Linearizing around the equilibrium point $x_1 = \pi$, $x_2 = 0$, $u = 0$, the linearized system is

$$\frac{d}{dt}\vec{x}_\ell(t) = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{k}{m} \end{bmatrix} \vec{x}_\ell(t) + \begin{bmatrix} 0 \\ \frac{1}{\ell} \end{bmatrix} u_\ell(t) \quad (31)$$

Now if we now apply a small torque to the system, we see that the pendulum rapidly drops to its downward position. In a small neighborhood around $\theta = \pi$, we see that this inverted pendulum is *unstable*. This can also be explained by eigenvalues, but we will revisit the notion of stability later in the course.

5 Conditions for Equilibria

Continuous-Time Systems

Let us take a closer look at the conditions for a linear system represented by the differential equation

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \quad (32)$$

From the get-go we see that $(\vec{x}^*, \vec{u}^*) = (\vec{0}, \vec{0})$ must be an equilibrium point. This is since the system is at rest. Now if we put in a constant input \vec{u}^* then to solve for equilibria, we get the following system of equations

$$A\vec{x} + B\vec{u}^* = \vec{0} \quad (33)$$

To solve for the states \vec{x} in which the system would be in equilibrium, our analysis boils down to whether the square matrix A is invertible.²

1. If A is invertible, then there is a unique equilibrium point $\vec{x}^* = -A^{-1}B\vec{u}^*$.
2. If A is non-invertible, depending on the range of A , we have two scenarios.
 - If $B\vec{u} \in \text{Col}(A)$ then we will have infinitely many equilibrium points.
 - If $B\vec{u} \notin \text{Col}(A)$ then the system has no solution and we will have no equilibrium points.

Discrete-Time Systems

Now let's take a look at the discrete-time system

$$\vec{x}[n] = A\vec{x}[n] + B\vec{u}[n] \quad (34)$$

Again we see that $(\vec{0}, \vec{0})$ is an equilibrium point but notice that the conditions for equilibria are different for discrete-time systems. A system is in equilibrium if it is not changing. In other words, this means that $\vec{x}^*[n+1] = \vec{x}^*[n]$ therefore, for a constant input \vec{u}^* we get the following system of equations

$$\vec{x} = A\vec{x} + B\vec{u}^* \implies (I - A)\vec{x} = B\vec{u}^* \quad (35)$$

The conditions for equilibria now depend on the matrix $I - A$ being invertible instead of the matrix A .

²This should be reviewed from 16A/54, but we restate it here since it isn't quite obvious when A is singular or non-invertible. Normally a singular matrix has infinite solutions but take the system $A\vec{x} = \vec{b}$ with $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This leads to a contradiction that $x_1 = 0 \neq 1$.