Homework 3

This homework is due on Friday, February 11, 2022 at 11:59PM. Self-grades and HW Resubmissions are due the following Friday, February 18, 2022 at 11:59PM.

1. **Reading Lecture Notes**

   Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 3 Note 4

   (a) When writing differential equations, what should you choose as your state variables?

   Solution: Any variable that has its derivative taken should be chosen as a state variable.

   (b) When is a matrix diagonalizable?

   Solution: For a square matrix of size $N \times N$, when it has N linearly independent eigenvectors.

   (c) How do we solve differential equations when a derivative of a state variable depends on other state variables besides itself? (This is called coupling between states.)

   Solution: Diagonalize the matrix of coefficients in the differential equation and express a new state vector in terms of the eigenbasis. We can expect each of the new state variables to have derivatives that will be functions of only themselves and the corresponding eigenvalue. Having such a system of scalar differential equations in each new state variable allows us to use our approach for solving scalar first order differential equations, ater which we can convert this solution back to our original state vector.

   (d) When inductors feature as an element in circuit, what must be taken as a state variable?

   Solution: The current through an inductor has its derivative taken. This means that we should treat the current through it, I_L, as one of our state variables.

2. **Solving the Differential Equation with Input**

Recall that in Discussion 2A we tried to solve the differential equation with input:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + bu_c(t) \tag{1}$$

$$x(0) = x_0. \tag{2}$$

for some continuous input $u_c(t)$.

The general strategy we employ is:

- First we replace our continuous input $u_c(t)$ with an input $u(t)$ which is piecewise constant on the intervals $[i\Delta, (i+1)\Delta)$, that is,

$$u(t) = u(i\Delta) = u[i] \qquad t \in [i\Delta, (i+1)\Delta) \qquad i \in \{0, 1, 2, \dots\} := \mathbb{N}. \tag{3}$$

Using this assumption, in discussion we:
  - solved the differential equation on each interval $[i\Delta, (i+1)\Delta)$ and got a solution expressing $x(t)$ in terms of $x_d[i] := x(i\Delta)$ and $u[i]$, for $t \in [i\Delta, (i+1)\Delta)$;
  - arrived at a formula for $x_d[i+1]$ in terms of $x_d[i]$ and $u[i]$;
  - used this to get a formula for $x_d[i]$ in terms of x_0 and the inputs $u[0], u[1], \dots, u[i-1]$;
  - approximated $x(t) \approx x_d[\lfloor \frac{t}{\Delta} \rfloor]$ to recover an approximate value for $x(t)$, that is,

$$x(t) \approx \left(e^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor} x_0 + b\frac{e^{\lambda\Delta} - 1}{\lambda} \sum_{k=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} \left(e^{\lambda\Delta}\right)^{(\lfloor \frac{t}{\Delta} \rfloor - 1) - k} u[k]. \tag{4}$$

- In this homework, we will take the limit $\Delta \to 0$. This transfers back from u to u_c – we saw in discussion that piecewise constant functions on very small intervals, i.e., our u, approximate general continuous functions u_c arbitrarily well. Using Riemann sums and calculus, we will turn the sum into an integral and show that, if u approximates u_c as $\Delta \to 0$, then

$$x(t) = e^{\lambda t}x_0 + b\int_0^t e^{\lambda(t-\tau)}u_c(\tau)\,\mathrm{d}\tau. \tag{5}$$

(a) We first need to relate $u[i]$ to u_c. Suppose that the $u[i]$ is a sample of $u_c(t)$, namely,

$$u[i] = u_c(i\Delta). \tag{6}$$

To clarify where this fits in with the earlier notation:
- $u(t)$ is a piecewise constant function;
- $u[i]$ is the discrete input that constructs $u(t)$ based on eq. (3);
- and $u_c(t)$ is the underlying input $u[i]$ is sampled from based on eq. (6).

This is one good way to get a piecewise constant approximator of a continuous function.

Substitute an appropriate value of u_c for $u[k]$ in eq. (4) from the discussion.

NOTE: Don't take any limits in this part of the problem; just do the substitution.

Solution: Using the substitution $u_c(j\Delta)$ for $u[j]$, we get

$$x(t) \approx \left(e^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor} x_0 + b\frac{e^{\lambda\Delta} - 1}{\lambda} \sum_{k=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} (e^{\lambda\Delta})^{(\lfloor \frac{t}{\Delta} \rfloor - 1) - k} u[k] \tag{7}$$

$$= \left(e^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor} x_0 + b\frac{e^{\lambda\Delta} - 1}{\lambda} \sum_{k=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} (e^{\lambda\Delta})^{(\lfloor \frac{t}{\Delta} \rfloor - 1) - k} u_c(k\Delta). \tag{8}$$

(b) To simplify our (discrete-time) eq. (4) so we can take $\Delta \to 0$, we would like to make some approximations which are valid for small Δ.

By using the following two estimates for small Δ:[1]

  i. $\left\lfloor \frac{t}{\Delta} \right\rfloor \approx \frac{t}{\Delta}$;

  ii. $\frac{e^{\lambda\Delta}-1}{\lambda} \approx \Delta$;[2]

show that

$$x(t) \approx e^{\lambda t}x_0 + be^{-\lambda\Delta} \sum_{k=0}^{\frac{t}{\Delta}-1} e^{\lambda(t-k\Delta)}u_c(k\Delta)\Delta. \tag{9}$$

Solution: The first estimate justifies getting rid of the "floor" terms. We have a lot of those terms, so it's good to use it here.

Plugging in $\left\lfloor \frac{t}{\Delta} \right\rfloor \approx \frac{t}{\Delta}$ gives

$$x(t) \approx \left(e^{\lambda\Delta}\right)^{\left\lfloor \frac{t}{\Delta} \right\rfloor} x_0 + b\frac{e^{\lambda\Delta}-1}{\lambda} \sum_{k=0}^{\left\lfloor \frac{t}{\Delta} \right\rfloor - 1} (e^{\lambda\Delta})^{(\left\lfloor \frac{t}{\Delta} \right\rfloor - 1)-k}u_c(k\Delta) \tag{10}$$

$$\approx \left(e^{\lambda\Delta}\right)^{\frac{t}{\Delta}} x_0 + b\frac{e^{\lambda\Delta}-1}{\lambda} \sum_{k=0}^{\frac{t}{\Delta}-1} (e^{\lambda\Delta})^{(\frac{t}{\Delta}-1)-k}u_c(k\Delta) \tag{11}$$

$$\approx e^{\lambda t}x_0 + b\frac{e^{\lambda\Delta}-1}{\lambda} \sum_{k=0}^{\frac{t}{\Delta}-1} e^{\lambda t - \lambda\Delta - \lambda\Delta k}u_c(k\Delta) \tag{12}$$

$$\approx e^{\lambda t}x_0 + b\frac{e^{\lambda\Delta}-1}{\lambda}e^{-\lambda\Delta} \sum_{k=0}^{\frac{t}{\Delta}-1} e^{\lambda(t-k\Delta)}u_c(k\Delta). \tag{13}$$

Then plugging in $\frac{e^{\lambda\Delta}-1}{\lambda} \approx \Delta$ gives

$$x(t) \approx e^{\lambda t}x_0 + b\frac{e^{\lambda\Delta}-1}{\lambda}e^{-\lambda\Delta} \sum_{k=0}^{\frac{t}{\Delta}-1} e^{\lambda(t-k\Delta)}u_c(k\Delta) \tag{14}$$

$$\approx e^{\lambda t}x_0 + be^{-\lambda\Delta} \sum_{k=0}^{\frac{t}{\Delta}-1} e^{\lambda(t-k\Delta)}u_c(k\Delta)\Delta. \tag{15}$$

Note that the dependence of $x(t)$ on both x_0 and the input u_c is the same; it's been preserved, and perhaps made more clear, through our approximations.

NOTE: This may seem like a long solution, but the main idea is to just use the estimates one by one, and simplify as much as possible.

(c) **Take the limit of $x(t)$ as $\Delta \to 0$, and show that $x(t)$ is given by eq. (5).**

Recall that the definite integral is defined from Riemann sums as

$$\int_0^t f(\tau)\,d\tau = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(\tau_k^*)\Delta_k \tag{16}$$

where $0 = \tau_0 < \tau_1 < \cdots < \tau_n = t$, $\tau_k^* \in [\tau_k, \tau_{k+1}]$, and $\Delta_k = \tau_{k+1} - \tau_k$. The Δ_k is the length of the base of the rectangles and the $f(\tau_k^*)$ are the heights. As n goes to infinity, the rectangles get skinnier and skinnier, but there are more and more of them.

(*HINT: Start with eq. (9) and take limits on both sides. What is n? What is τ_k and τ_k^*? What is Δ_k? What is f?*)

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[1]Both these approximations become equalities in the limit $\Delta \to 0$.
[2]We can see this approximation using Taylor's theorem from calculus.

(*HINT: We chose the form of eq.* (9) *carefully; it turns out that* Δ_k *is one particular term involving* Δ *that goes to* 0 *as* $\Delta \to 0$, *and also that it is independent of k.*)

Solution: We are evaluating

$$\lim_{\Delta \to 0} x(t) = \lim_{\Delta \to 0} \left[e^{\lambda t} x_0 + b e^{-\lambda \Delta} \sum_{k=0}^{\frac{t}{\Delta}-1} e^{\lambda(t-k\Delta)} u_c(k\Delta) \Delta \right] \tag{17}$$

$$= e^{\lambda t} x_0 + b \lim_{\Delta \to 0} \left(e^{-\lambda \Delta} \sum_{k=0}^{\frac{t}{\Delta}-1} e^{\lambda(t-k\Delta)} u_c(k\Delta) \Delta \right) \tag{18}$$

$$= e^{\lambda t} x_0 + b \left(\lim_{\Delta \to 0} e^{-\lambda \Delta} \right) \left(\lim_{\Delta \to 0} \sum_{k=0}^{\frac{t}{\Delta}-1} e^{\lambda(t-k\Delta)} u_c(k\Delta) \Delta \right) \tag{19}$$

$$= e^{\lambda t} x_0 + b \lim_{\Delta \to 0} \sum_{k=0}^{\frac{t}{\Delta}-1} e^{\lambda(t-k\Delta)} u_c(k\Delta) \Delta. \tag{20}$$

Here, we want to evaluate the sum on the right side; by pattern matching with the Riemann integration template and the fact that Δ_k should shrink to 0 in the limit, we have

$$n = \frac{t}{\Delta} \qquad \Delta_k = \Delta. \tag{21}$$

This implies that

$$\tau_k = k\Delta. \tag{22}$$

To recover τ_k^* and f from what we already have, one notes that $\tau_k^* \in [k\Delta, (k+1)\Delta]$ and that we must have

$$f(\tau_k^*) = e^{\lambda(t-k\Delta)} u_c(k\Delta). \tag{23}$$

From here we see that

$$\tau_k^* = k\Delta \qquad f(\tau) = e^{\lambda(t-\tau)} u_c(\tau). \tag{24}$$

We have

$$\lim_{\Delta \to 0} x(t) = e^{\lambda t} x_0 + b \lim_{\Delta \to 0} \sum_{k=0}^{\frac{t}{\Delta}-1} e^{\lambda(t-k\Delta)} u_c(k\Delta) \Delta \tag{25}$$

$$= e^{\lambda t} x_0 + b \lim_{n \to \infty} \sum_{k=0}^{n-1} e^{\lambda(t-\tau_k^*)} u_c(\tau_k^*) \Delta_k \tag{26}$$

$$= e^{\lambda t} x_0 + b \lim_{n \to \infty} \sum_{k=0}^{n-1} f(\tau_k^*) \Delta_k \tag{27}$$

$$= e^{\lambda t} x_0 + b \int_0^t f(\tau) \, d\tau \tag{28}$$

$$= e^{\lambda t} x_0 + b \int_0^t e^{\lambda(t-\tau)} u_c(\tau) \, d\tau \tag{29}$$

which is our final answer. We can't simplify further because we don't know the form of $u_c(\tau)$. Note that the dependence of $x(t)$ on both x_0 and the input u_c is the same. This is a special case of a crucial point: *sums of small quantities behave roughly the same as integrals.* This is one of the main ways to fluently transfer between discrete and continuous time.

This problem brings together many different concepts and uses a lot of notation. As such, it may be difficult to fully comprehend everything the first time. Being able to grind through complex mathematical problems like this is part of the vaunted "mathematical maturity" that this class helps you foster. As the semester continues, you will find that these kinds of problems will seem progressively easier, both to understand quickly and to solve. But it won't happen without practice.

3. Tracking Terry

Terry is a mischievous child, and his mother is interested in tracking him.

(a) Terry texts his current location as a vector $\vec{x}_v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, but there is a problem! These coordinates are *not* in the standard basis, but rather in the basis $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$. That is to say that the first number 2 above is how many multiples of $\vec{v}_1$ to use and the second number 3 is how many multiples of $\vec{v}_2$ to use in computing his actual location. Here, both $\vec{v}_1$ and $\vec{v}_2$ are vectors in the standard basis. **Let Terry's location in the standard basis be $\vec{x}$. Write $\vec{x}$ in terms of $\vec{v}_1$ and $\vec{v}_2$.**

Solution: By definition, the first coordinate in the V basis is the coefficient of $\vec{v}_1$ and second coordinate in the V basis is the coefficient for $\vec{v}_2$. Hence

$$\vec{x} = V\vec{x}_v = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\vec{v}_1 + 3\vec{v}_2. \tag{30}$$

(b) Terry's friend tells you that Terry's location in the standard basis is $\vec{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Using this along with the previous info that Terry's location in the V basis is $\vec{x}_v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, **is it possible to determine the basis vectors $\vec{v}_1, \vec{v}_2$ Terry is using. If it is impossible to do so, explain why.**

(HINT: How many unknowns do you have? How many equations?)

Solution: Solving for the basis vectors Terry is using (or in other words the axes in his coordinate space) is the same as solving for V in the change of basis equation:

$$V\vec{x}_v = \vec{x} \tag{31}$$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \tag{32}$$

There are four unknowns and only two equations, so this task is impossible.

(c) Terry's basis vectors $\vec{v}_1, \vec{v}_2$ get leaked to his mom on accident, so she knows they are

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}. \tag{33}$$

To hide his location, Terry wants to switch to a new coordinate system with the basis vectors

$$\vec{p}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{p}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{34}$$

In order to do this, he needs a way to convert coordinates from the V basis to the P basis. Thus, **find the matrix T such that if $\vec{x}_v$ is a location expressed in V coordinates and $\vec{x}_p$ is the same location expressed in P coordinates, then $\vec{x}_p = T\vec{x}_v$.**

Solution: The problem can be formulated as a change of basis problem. Since both $\vec{x}_v$ and $\vec{x}_p$ correspond to the same point, converting them to the standard basis gives us

$$V\vec{x}_v = P\vec{x}_p \tag{35}$$

Since we want to find T such that $\vec{x}_p = T\vec{x}_v$, we have:
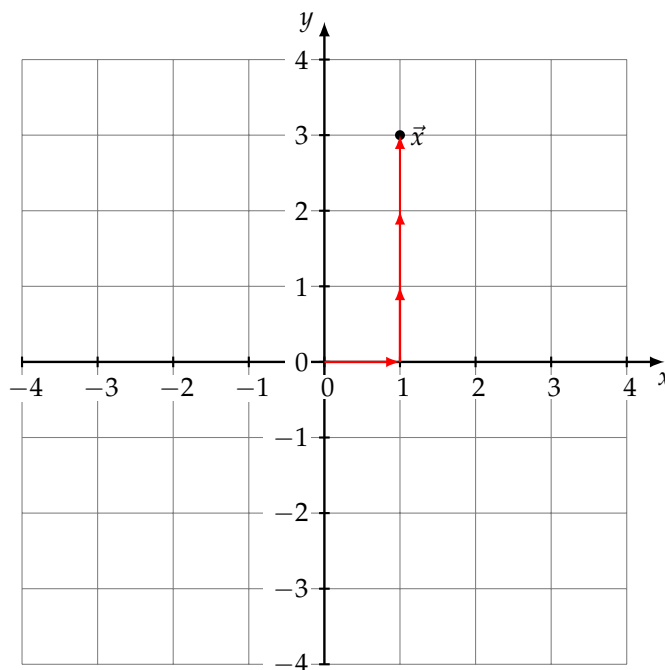
$$\vec{x}_p = P^{-1}V\vec{x}_v \tag{36}$$

$$T = P^{-1}V \tag{37}$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \tag{38}$$

$$= \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \tag{39}$$

$$= \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \tag{40}$$

(d) Terry now wants to make a map and route to where he currently is, $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. **For both the P and V bases from part 3.c, illustrate the sum of scaled basis vectors that are necessary to go from the origin to $\vec{x}$.** An example is shown below when using the standard basis. This illustrates that the same location can be represented by many different coordinate systems/bases.

Solution:

Since we know $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$, and $\vec{x} = V\vec{x}_v$, we can derive:

$$\vec{x}_v = V^{-1}\vec{x} \tag{41}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{42}$$

$$= \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{43}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{44}$$
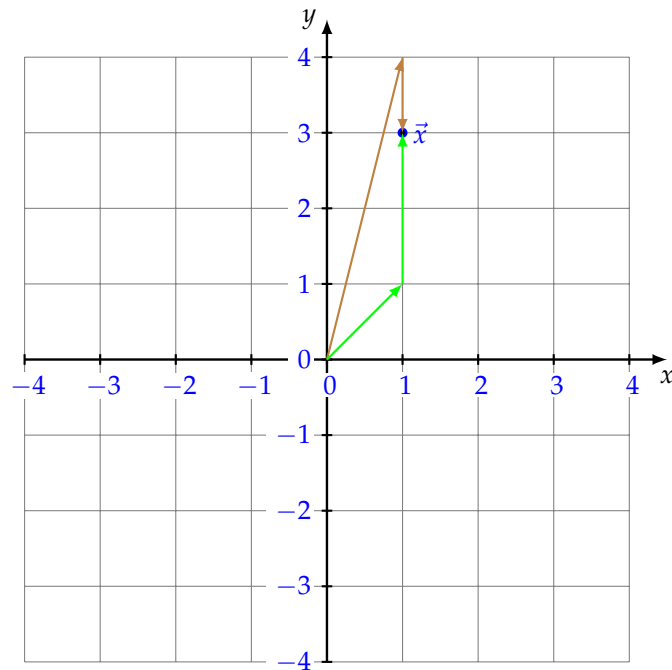
Similarly, in order to compute $\vec{x}_p$, we have:

$$\vec{x}_p = P^{-1}\vec{x} \tag{45}$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{46}$$

$$= \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{47}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{48}$$

Therefore, we can illustrate the sum of scaled basis vectors according to $\vec{x}_v$ (green path), and $\vec{x}_p$ (brown path).

4. Eigenvectors and Diagonalization

(a) Let A be an $n \times n$ matrix with n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Define V to be a matrix with $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ as its columns, $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$.

Show that $AV = V\Lambda$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, a diagonal matrix with the eigenvalues of A as its diagonal entries.

Solution:

$$AV = A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \tag{49}$$
$$= \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots, & A\vec{v}_n \end{bmatrix} \tag{50}$$
$$= \begin{bmatrix} \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \end{bmatrix} \tag{51}$$
$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \tag{52}$$
$$= V\Lambda \tag{53}$$

(b) **Argue that V is invertible, and therefore**

$$A = V\Lambda V^{-1}. \tag{54}$$

(Hint: Why is V invertible? It is fine to cite the appropriate result from 16A.)

Solution: Columns of V are eigenvectors of A which are known to be linearly independent. Since V has linearly independent columns, it has full column rank, and therefore, is invertible.

$$AV = V\Lambda \tag{55}$$
$$AVV^{-1} = V\Lambda V^{-1} \tag{56}$$
$$A = V\Lambda V^{-1} \tag{57}$$

(c) **Write Λ in terms of the matrices A, V, and V^{-1}.**

Solution: We take $A = V\Lambda V^{-1}$ and apply invertible operations to both sides of the equality:

$$A = V\Lambda V^{-1} \tag{58}$$
$$V^{-1}A = V^{-1}V\Lambda V^{-1} \tag{59}$$
$$V^{-1}AV = V^{-1}V\Lambda V^{-1}V \tag{60}$$
$$V^{-1}AV = I\Lambda I \tag{61}$$
$$V^{-1}AV = \Lambda. \tag{62}$$

(d) A matrix A is deemed diagonalizable if there exists a square matrix U so that A can be written in the form $A = UDU^{-1}$ for the choice of an appropriate diagonal matrix D.

Show that the columns of U must be eigenvectors of the matrix A, and that the entries of D must be eigenvalues of A.

(HINT: What does it mean to be an eigenvector? What is $U^{-1}U$? How does matrix multiplication work column-wise?)

Solution: We start with a calculation which is essentially the reverse of the calculation in part (b):

$$A = UDU^{-1} \tag{63}$$
$$AU = UDU^{-1}U \tag{64}$$
$$AU = UD. \tag{65}$$

Now let's expand the definitions of U as a square matrix and D as a diagonal matrix:

$$AU = A \begin{bmatrix} \vec{u}_1 & \dots & \vec{u}_n \end{bmatrix} \tag{66}$$
$$= \begin{bmatrix} A\vec{u}_1 & \dots & A\vec{u}_n \end{bmatrix} \tag{67}$$
$$UD = \begin{bmatrix} \vec{u}_1 & \dots & \vec{u}_n \end{bmatrix} \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \tag{68}$$
$$= \begin{bmatrix} d_1\vec{u}_1 & \dots & d_n\vec{u}_n \end{bmatrix}. \tag{69}$$

Comparing columns, we see that $A\vec{u}_i = d_i\vec{u}_i$. This is exactly the eigenvector-eigenvalue equation! In particular, this says that $\vec{u}_i$ is an eigenvector of A, with eigenvalue d_i.

The previous part shows that the *only* way to diagonalize A is using its eigenvalues/eigenvectors. Now we will explore a payoff for diagonalizing A – an operation that diagonalization makes *much* simpler.

(e) For a matrix A and a positive integer k, we define the exponent to be

$$A^k = \underbrace{A \cdot A \cdots \cdot A \cdot A}_{k \text{ times}} \tag{70}$$

Let's assume that matrix A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ (i.e. the n eigenvectors are all linearly independent).

Show that A^k has eigenvalues $\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k$ and eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. Conclude that A^k is diagonalizable.

Solution: Consider the ith eigenvector of A, $\vec{v}_i$ and the corresponding eigenvalue λ_i.

$$A^k\vec{v}_i = A^{k-1} \cdot A\vec{v}_i \tag{71}$$
$$= A^{k-1}\lambda_i\vec{v}_i \tag{72}$$
$$= \lambda_i A^{k-2} \cdot A\vec{v}_i \tag{73}$$
$$= \lambda_i^2 A^{k-3} \cdot A\vec{v}_i \tag{74}$$
$$\vdots \tag{75}$$
$$= \lambda_i^k\vec{v}_i \tag{76}$$

Thus by definition, v_i is an eigenvector of A^k with corresponding eigenvalue λ_i^k.

Alternate solution: Since A is diagonalizable, we can express A as

$$A = V\Lambda V^{-1} \tag{77}$$

Substituting A as shown in Equation 77 in 70, we get

$$A^k = \underbrace{A \cdot A \cdots \cdot A \cdot A}_{k \text{ times}} \tag{78}$$

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$$= \underbrace{V\Lambda V^{-1} \cdot V\Lambda V^{-1} \cdots \cdots V\Lambda V^{-1} \cdot V\Lambda V^{-1}}_{k \text{ times}} \tag{79}$$

$$= \underbrace{V\Lambda \left(V^{-1} \cdot V\right) \Lambda V^{-1} \cdots \cdots V\Lambda \left(V^{-1} \cdot V\right) \Lambda V^{-1}}_{k \text{ times}} \tag{80}$$

$$= V \underbrace{\Lambda \cdot \Lambda \cdots \cdots \Lambda \cdot \Lambda}_{k \text{ times}} V^{-1} \tag{81}$$

$$= V\Lambda^k V^{-1} \tag{82}$$

Since Λ is a diagonal matrix,

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix} \tag{83}$$

Thus, A^k is clearly diagonalizable, where the eigenvectors of A^k are just the eigenvectors of A, and the eigenvalues of A^k are $\lambda_1^k, \ldots, \lambda_n^k$.

5. **Vector Differential Equations**

   Note: it's recommended to finish Question 4 (Eigenvectors and Diagonalization) before this problem.

   Consider a system of ordinary differential equations that can be written in the form

$$\frac{d}{dt}\vec{x}(t) := \begin{bmatrix} \frac{d}{dt}x_1(t) \\ \frac{d}{dt}x_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\vec{x}(t) \tag{84}$$

   where $x_1, x_2 : \mathbb{R} \to \mathbb{R}$ are scalar functions of time t, and $A \in \mathbb{R}^{2\times 2}$ is a 2×2 matrix with constant coefficients. We call eq. (84) a vector differential equation.

   (a) Suppose we have a system of ordinary differential equations

$$\frac{dx_1}{dt} = 7x_1 - 8x_2, \tag{85}$$

$$\frac{dx_2}{dt} = 4x_1 - 5x_2, \tag{86}$$

   Here, we denote $x_1(t), x_2(t)$ as x_1, x_2 for notational simplicity.

   Find an appropriate matrix A to write this system in the form of eq. (84). Compute the eigenvalues of A. Denote the smaller and larger eigenvalues by λ_1 and λ_2 respectively, so that $\lambda_1 \leq \lambda_2$.

   Solution:

$$\frac{d}{dt}\vec{x} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 & -8 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{87}$$

   The characteristic polynomial of A is

$$\det\left(\begin{bmatrix} 7-\lambda & -8 \\ 4 & -5-\lambda \end{bmatrix}\right) = (7-\lambda)(-5-\lambda) + 32 \tag{88}$$

$$= \lambda^2 - 7\lambda + 5\lambda - 35 + 32 \tag{89}$$

$$= \lambda^2 - 2\lambda - 3 \tag{90}$$

$$= (\lambda + 1)(\lambda - 3). \tag{91}$$

   Thus the eigenvalues of A are $\lambda_1 = -1, \lambda_2 = 3$.

   (b) **Compute the eigenvectors of A. Is A diagonalizable? Why?**

   Solution: We will use the standard null space approach.

$$(A - \lambda_1 I)\vec{v}_1 = (A + I)\vec{v}_1 = \begin{bmatrix} 8 & -8 \\ 4 & -4 \end{bmatrix}\vec{v}_1 = 0 \tag{92}$$

$$(A - \lambda_2 I)\vec{v}_2 = (A - 3I)\vec{v}_2 = \begin{bmatrix} 4 & -8 \\ 4 & -8 \end{bmatrix}\vec{v}_2 = 0 \tag{93}$$

$$\tag{94}$$

   From the first equation, we see that any vector $\vec{v}_1 = (v_{11}, v_{12})^\top$ satisfies $v_{11} = v_{12}$. From the second equation, we see that any vector $\vec{v}_2 = (v_{21}, v_{22})^\top$ satisfies $v_{21} = 2v_{22}$. Thus we have eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{95}$$

   Since these are linearly independent, the columns of $V = [\vec{v}_1, \vec{v}_2]$ are linearly independent. Thus V^{-1} exists and is well-defined, so A can be written $A = V\Lambda V^{-1}$, where Λ is the diagonal matrix of eigenvalues. Thus A is diagonalizable. This can also be seen from the fact that the eigenvalues were calculated to be distinct.

(c) We now transform our system (eq. (85) and eq. (86)) in x coordinates, to new coordinates z to simplify our system of differential equations. **What basis V should we use so that in the new coordinates $\vec{z} = V^{-1}\vec{x}$, the Λ matrix in the equation $\frac{d\vec{z}(t)}{dt} = \Lambda\vec{z}(t)$ is diagonal? Write out this new system in the $\vec{z}$ coordinates.**

Solution: Let our basis V be the eigenvector matrix $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$. Then, since A can be diagonalized, we have $A = V\Lambda V^{-1}$. Then

$$\frac{d}{dt}\vec{x} = A\vec{x} = V\Lambda V^{-1}\vec{x} = V\Lambda\vec{z} \tag{96}$$

$$V^{-1}\frac{d}{dt}\vec{x} = V^{-1}V\Lambda\vec{z} \tag{97}$$

$$\frac{d}{dt}V^{-1}\vec{x} = \frac{d}{dt}\vec{z} = \Lambda\vec{z} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\vec{z} \tag{98}$$

In the last line, we have $V^{-1}\frac{d}{dt}\vec{x} = \frac{d}{dt}V^{-1}\vec{x}$ because for

$$V^{-1} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

we have

$$V^{-1}\frac{d}{dt}\vec{x} = \begin{bmatrix} v_{11}\frac{d}{dt}x_1 + v_{12}\frac{d}{dt}x_2 \\ v_{21}\frac{d}{dt}x_1 + v_{22}\frac{d}{dt}x_2 \end{bmatrix} \tag{99}$$

$$\frac{d}{dt}V^{-1}\vec{x} = \frac{d}{dt}\begin{bmatrix} v_{11}x_1 + v_{12}x_2 \\ v_{21}x_2 + v_{22}x_2 \end{bmatrix} \tag{100}$$

$$= \begin{bmatrix} v_{11}\frac{d}{dt}x_1 + v_{12}\frac{d}{dt}x_2 \\ v_{21}\frac{d}{dt}x_1 + v_{22}\frac{d}{dt}x_2 \end{bmatrix}. \tag{101}$$

(d) **Solve the new system in the $\vec{z}$ coordinates, using the initial conditions that $x_1(0) = 1, x_2(0) = -1$.**

Solution: Now z_1 and z_2 are their own separated differential equations so $\frac{d}{dt}z_1 = -z_1$ and $\frac{d}{dt}z_2 = 3z_2$. We know the general form solution of these differential equations:

$$z_1(t) = k_1e^{-t} \tag{102}$$

$$z_2(t) = k_2e^{3t} \tag{103}$$

We now find the initial conditions in the $\vec{z}$ coordinates with

$$\vec{z}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \tag{104}$$

This means that $k_1 = z_1(0) = -3$ and $k_2 = z_2(0) = 2$. Thus, our final solutions for $z(t)$ are

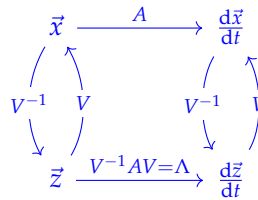$$z_1(t) = -3e^{-t} \tag{105}$$

$$z_2(t) = 2e^{3t} \tag{106}$$

(e) Now **convert your solution from the $\vec{z}$ coordinates back to the original $\vec{x}$ coordinates.** In other words, give us the functions $x_1(t)$ and $x_2(t)$.

Solution: We can just use the coordinate transformation we defined in the beginning, that $\vec{x} = B\vec{z}$. Then,

$$\vec{x}(t) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -3e^{-t} \\ 2e^{3t} \end{bmatrix} = \begin{bmatrix} -3e^{-t} + 4e^{3t} \\ -3e^{-t} + 2e^{3t} \end{bmatrix} \tag{107}$$

At this point we can summarize the picture with the following diagram:

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$$\vec{x} \xrightarrow{\quad A \quad} \frac{d\vec{x}}{dt}$$

$$V^{-1} \quad V \qquad\qquad V^{-1} \quad V$$

$$\vec{z} \xrightarrow{\quad V^{-1}AV = \Lambda \quad} \frac{d\vec{z}}{dt}$$

(f) It turns out that we can actually turn all higher-order differential equations with constant coefficients into vector differential equations of the style of eq. (84).

Consider a second-order ordinary differential equation

$$\frac{d^2 y(t)}{dt^2} + a\frac{dy(t)}{dt} + by(t) = 0, \tag{108}$$

where $a, b \in \mathbb{R}$.

Write this differential equation in the form of (eq. (84)), by choosing appropriate variables $x_1(t)$ and $x_2(t)$.

(HINT: Your original unknown function $y(t)$ has to be one of those variables. The heart of the question is to figure out what additional variable can you use so that you can express eq. (113) without having to take a second derivative, and instead just taking the first derivative of something. This is another manifestation of the larger thought pattern of "lifting.")

Solution:

If we set $x_1(t) = y(t)$, $x_2(t) = \frac{dy(t)}{dt}$, then we have

$$\frac{dx_1(t)}{dt} = \frac{dy(t)}{dt} = x_2(t) \tag{109}$$

$$\frac{dx_2(t)}{dt} = \frac{d^2 y(t)}{dt^2} = -a\frac{dy(t)}{dt} - by(t) = -ax_2(t) - bx_1(t) \tag{110}$$

We can write this in the form of eq. (84) as follows

$$\frac{d}{dt}\vec{x} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{111}$$

(g) It turns out that all two-dimensional vector linear differential equations with distinct eigenvalues have a solution in the general form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{\lambda_1 t} + c_1 e^{\lambda_2 t} \\ c_2 e^{\lambda_1 t} + c_3 e^{\lambda_2 t} \end{bmatrix} \tag{112}$$

where c_0, c_1, c_2, c_3 are constants, and λ_1, λ_2 are the eigenvalues of A (this can be proven by just repeating the same steps in the previous parts and using the fact that distinct eigenvalues implies linearly independent eigenvectors). Thus, an alternate way of solving this type of differential equation in the future is to now use your knowledge that the solution is of this form and just solve for the constants c_i.

Now let $a = -1$ and $b = -2$ in eq. (108), i.e.

$$\frac{d^2 y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = 0, \tag{113}$$

Verify that eq. (113) has a solution in the general form eq. (112). Solve eq. (113) with the initial conditions $y(0) = 1$, $\frac{dy}{dt}(0) = 1$, using this method.

(HINT: You get two equations using the initial conditions above. How many unknowns are here?) (SECOND HINT: Given your specific choice of x_1 and x_2 in part (f), how many unknowns are there really?)

Solution: We have

$$\begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \tag{114}$$

First, we calculate the eigenvalues of this matrix. The characteristic polynomial is

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix}\right) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1). \tag{115}$$

Thus the eigenvalues are $\lambda_1 = -1, \lambda_2 = 2$. Since they are distinct, we can proceed with this method.

We know the solution for $x_1(t), x_2(t)$ is of the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{-t} + c_1 e^{2t} \\ c_2 e^{-t} + c_3 e^{2t} \end{bmatrix}. \tag{116}$$

At $t = 0$, we have $y(0) = 1$ and $\frac{dy}{dt}(0) = 1$. Using our differential equation (eq. (113)), we can get $\frac{d^2y}{dt^2}(0) = \frac{dy}{dt}(0) + 2y(0) = 3$. Plugging these in,

$$x_1(0) = y(0) = 1 = c_0 + c_1 \tag{117}$$

$$x_2(0) = \frac{dy}{dt}(0) = 1 = c_2 + c_3 \tag{118}$$

$$\frac{dx_1}{dt}(0) = \frac{dy}{dt}(0) = 1 = -c_0 + 2c_1 \tag{119}$$

$$\frac{dx_2}{dt}(0) = \frac{d^2y}{dt^2}(0) = 3 = -c_2 + 2c_3 \tag{120}$$

This gives $c_0 = \frac{1}{3}, c_1 = \frac{2}{3}, c_2 = -\frac{1}{3}, c_3 = \frac{4}{3}$. Alternatively, you could've seen that $c_2 = -c_0$ and $c_3 = 2c_1$ since $x_2(t)$ is the derivative of $x_1(t)$ which makes it solvable with just the first 2 equations. Thus we have

$$x_1(t) = y(t) = \frac{1}{3}e^{-t} + \frac{2}{3}e^{2t} \tag{121}$$

$$x_2(t) = \frac{dy(t)}{dt} = -\frac{1}{3}e^{-t} + \frac{4}{3}e^{2t} \tag{122}$$

6. Op-Amp Integrators: A continuation from the previous HW

Op-amp integrators are popular analog circuits where we take help of active and passive electronic components to implement a mathematical function (integration in this case). You might already know from your basic circuit knowledge that capacitors integrate the current flowing through it. If somehow, the current injected in a capacitor can be made proportional with the input voltage, the capacitor would be able to integrate the input voltage. That is achieved by means of an operational amplifier (op-amp). Look at the Op-Amp integrator circuit shown in Figure 2. Considering an ideal op-amp (i.e. gain A is infinite), due to negative feedback the voltage difference between the positive and negative input terminals of the op-amp is $0V$ (i.e. $V_- = V_+ = 0V$). Which means the current through the resistor R is proportional to the input voltage V_{in}. This current gets integrated on the capacitor C to produce the output voltage V_{out}. Hence, the output voltage is obtained by integrating the input voltage with some constant factor.

Now, in this question we will continue on from our analysis in Homework 2 and look at the eigenvalues of the integrator circuit in Figure 2 in both non-ideal and ideal situations. Note that in our analysis we will be dealing with a non-ideal op-amp where the op-amp gain A is finite, which means $V_- \neq V_+$
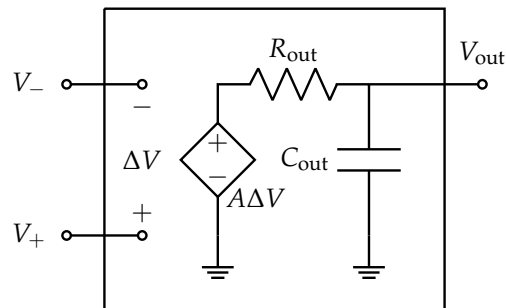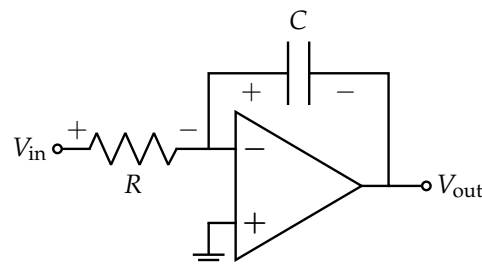

Figure 1: Op-amp model: $\Delta V = V_+ - V_-$

Figure 2: Integrator circuit
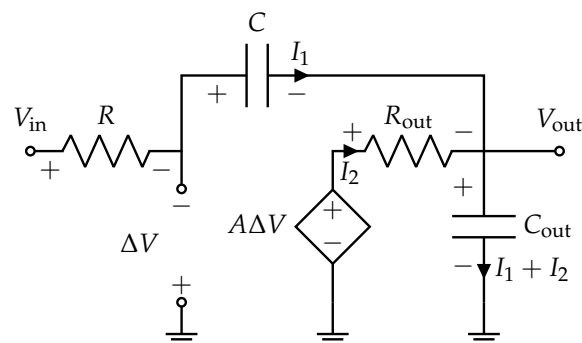

Figure 3: Integrator circuit with op-amp model.

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(a) Recall from Homework 2 we had the following analysis to the integrator circuit shown in Figure 3.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} V_{\text{out}} \\ V_C \end{bmatrix} = \begin{bmatrix} -\left(\frac{A+1}{R_{\text{out}}C_{\text{out}}} + \frac{1}{RC_{\text{out}}}\right) & -\left(\frac{1}{RC_{\text{out}}} + \frac{A}{R_{\text{out}}C_{\text{out}}}\right) \\ -\frac{1}{RC} & -\frac{1}{RC} \end{bmatrix}\begin{bmatrix} V_{\text{out}} \\ V_C \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_{\text{out}}} \\ \frac{1}{RC} \end{bmatrix} V_{\text{in}} \tag{123}$$

Let us assume V_{in} is constant for this problem. Note that eq. (123) is a vector differential equation which can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = M\vec{x} + B \tag{124}$$

where $\vec{x} = \begin{bmatrix} V_{\text{out}} \\ V_C \end{bmatrix}$. **Find the eigenvalues of the matrix M.**

For simplicity, assume $C_{\text{out}} = C = 0.01\,\text{F}$ and $R = 1\,\Omega$ and looking at the datasheet for the TI LMC6482 (the op-amps used in lab), we have $A = 10^6$ and $R_{\text{out}} = 100\,\Omega$.

Feel free to assume $A + 1 \approx 10^6$ whenever it's reasonable to do so, but do not make any other approximations. (Of course, such an approximation is not valid if you have a $A + 1 - A$ term showing up somewhere.) Feel free to use a scientific calculator or Jupyter to find the eigenvalues. You don't have to grind this out by hand.

Solution: To find the eigenvalues, we can find the characteristic polynomial of matrix M:

$$\det(M - \lambda I) = \det\left(\begin{bmatrix} -\left(\frac{A+1}{R_{\text{out}}C_{\text{out}}} + \frac{1}{RC_{\text{out}}}\right) & -\left(\frac{1}{RC_{\text{out}}} + \frac{A}{R_{\text{out}}C_{\text{out}}}\right) \\ -\frac{1}{RC} & -\frac{1}{RC} \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \tag{125}$$

$$= \det\left(\begin{bmatrix} -\left(\frac{A+1}{R_{\text{out}}C_{\text{out}}} + \frac{1}{RC_{\text{out}}}\right) - \lambda & -\left(\frac{1}{RC_{\text{out}}} + \frac{A}{R_{\text{out}}C_{\text{out}}}\right) \\ -\frac{1}{RC} & -\frac{1}{RC} - \lambda \end{bmatrix}\right) \tag{126}$$

$$= \left(-\left(\frac{A+1}{R_{\text{out}}C_{\text{out}}} + \frac{1}{RC_{\text{out}}}\right) - \lambda\right)\left(-\frac{1}{RC} - \lambda\right) - \left(-\left(\frac{1}{RC_{\text{out}}} + \frac{A}{R_{\text{out}}C_{\text{out}}}\right)\right)\left(-\frac{1}{RC}\right) \tag{127}$$

$$= \lambda^2 + \left(\frac{A+1}{R_{\text{out}}C_{\text{out}}} + \frac{1}{RC_{\text{out}}} + \frac{1}{RC}\right)\lambda + \frac{1}{R_{\text{out}}C_{\text{out}}RC}. \tag{128}$$

Solving for the eigenvalues,

$$0 = \lambda^2 + \left(\frac{A+1}{R_{\text{out}}C_{\text{out}}} + \frac{1}{RC_{\text{out}}} + \frac{1}{RC}\right)\lambda + \frac{1}{R_{\text{out}}C_{\text{out}}RC} \tag{129}$$

$$= \lambda^2 + (10^6 + 2 \times 10^2)\lambda + 10^2 \tag{130}$$

Hence, the eigenvalues are

$$\lambda_\pm = \frac{-(10^6 + 2 \times 10^2) \pm \sqrt{(10^6 + 2 \times 10^2)^2 - 400}}{2} \tag{131}$$

$$\implies \lambda_+ = -1 \times 10^{-4} \tag{132}$$

$$\lambda_- = -1 \times 10^6 \tag{133}$$

You should see that one eigenvalue corresponds to a slowly dying exponential and is close to 0. The other corresponds to a much faster dying exponential. The very slowly dying exponential is what corresponds to the desired integrator-like behavior. This is what lets it "remember." (If you don't understand why, think back to the HW problem you saw in a previous HW where you proved the uniqueness of the integral-based solution to a scalar differential equation with an input waveform.)

(b) Again, assume we have an ideal op-amp, i.e., $A \to \infty$. **Find the eigenvalues of M under this limit.** Feel free to make any reasonable approximations.

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Here, you should see that the eigenvalue that used to be a slowly dying exponential stops dying out at all — corresponding to the ideal integrator's behavior of remembering forever. On the other hand, the eigenvalue that was very strongly negative goes straight to $-\infty$, showing the lack of interference from the other eigenvector in the ideal integrator's behavior.

Solution: The given assumption is that A is very large. Thus we can approximate

$$A + 1 \approx A \tag{134}$$

and also notice that $\frac{A}{R_{\text{out}}C_{\text{out}}}$ is much larger in size than all other coefficients of λ in (128), so that

$$\frac{A+1}{R_{\text{out}}C_{\text{out}}} + \frac{1}{RC_{\text{out}}} + \frac{1}{RC} \approx \frac{A}{R_{\text{out}}C_{\text{out}}}. \tag{135}$$

With this approximation, we can rewrite our characteristic equation as

$$\lambda^2 + \frac{A}{R_{\text{out}}C_{\text{out}}}\lambda + \frac{1}{R_{\text{out}}C_{\text{out}}RC} = 0 \tag{136}$$

Solving eq. (136) we can find the expressions for the eigenvalues as

$$\lambda_{\pm} = -\frac{A}{2R_{\text{out}}C_{\text{out}}} \pm \sqrt{\frac{A^2}{4R_{\text{out}}^2 C_{\text{out}}^2} - \frac{1}{R_{\text{out}}C_{\text{out}}RC}} \tag{137}$$

$$= -\frac{A}{2R_{\text{out}}C_{\text{out}}} \pm \frac{A}{2R_{\text{out}}C_{\text{out}}}\sqrt{1 - \frac{4R_{\text{out}}C_{\text{out}}}{A^2 RC}}. \tag{138}$$

Now, considering the fact that $A \to \infty$ we can safely assume $(1 - \frac{4R_{\text{out}}C_{\text{out}}}{A^2 RC}) \approx 1$. Hence, under this limit the eigenvalues can be found as

$$\lambda_+ = 0 \tag{139}$$

$$\lambda_- = -\frac{A}{R_{out}C_{out}} \tag{140}$$

Note that as A goes to $+\infty$, the eigenvalue λ_+ goes to $-\infty$. .

7. (PRACTICE) Solar cell

In EE16A's imaging labs, you used an electronic component that responded to light, i.e. a photoelectric device. To fully understand these devices requires more advanced courses like EE130 and EE134. For now, let's look at a simplified model of a common photoelectric device, the solar cell, and see how we can model its physical behavior with our knowledge of differential equations.
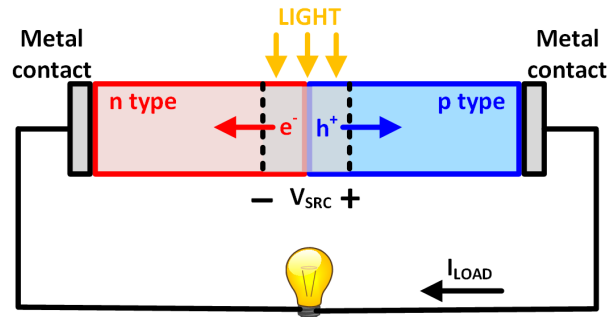

Figure 4: Simplified diagram of a solar cell.

As shown in Fig. 4, a solar cell consists of two types of semiconductor material pressed together: an n-type and a p-type. An n-type semiconductor has some free negative charges (electrons); a p-type has some free positive charges ("holes"). When we press p- and n-types together, an imbalance of charge forms across the device at equilibrium which generates an electric field and thus a voltage. When light shines on the material, photons release extra electrons and holes, and these free charges travel to the metal contacts. What results is a sheet of charge that flows through the device, i.e. a current. The brighter the light, the more free electrons, and higher the current. We can model a solar cell as a voltage source supplying light-dependent current: it acts as an electrical power source for our load.
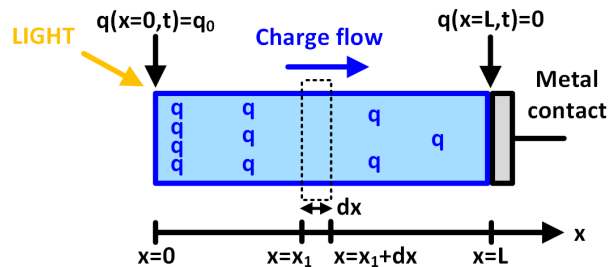

Figure 5: Our model for determining steady-state charge distribution in half of a solar cell.

We want to model how charge is distributed in one of the solar cell's semiconductors at steady-state equilibrium. For this problem, let's look at the right half and create a simplified model as shown in Figure 5. Let $q(x, t)$ define the charge at position x and at time t. At $x = 0$, where light strikes the semiconductor, we have a constant fixed charge q_0. At the metal contact at $x = L$, we have no charge. The physics creating these edge cases are beyond the scope of this course, so assume these to be true and model them as boundary conditions. Note that charge flows from left to right over time, but the amount of charge at a position x at any given time is fixed at steady-state (think of a river). We want to solve for $q(x, t)$ for $0 < x < L$.

Notice a difference in charge concentration: there is more charge on the left than on the right. Free carriers move to try and balance out this concentration difference due to the electric field created by such. This process is called "diffusion", and a diffusion current results in the material. We can use this concentration difference, or *gradient*, to solve for $q(x, t)$. Let the gradient of charge at position x and

time t be $g(x,t)$:

$$g(x,t) = \frac{\mathrm{d}}{\mathrm{d}x}q(x,t). \tag{141}$$

Note that the gradient is positive if the concentration increases from left to right (opposite in our case). Now consider the small dotted black box in Fig. 5. It spans from $x = x_1$ to an infinitesimally small width $\mathrm{d}x$ away, $x = x_1 + \mathrm{d}x$. In any span of time, the charge that enters the left side of the box must equal to the charge that leaves the right side of the box plus the *change* in charge inside the box in that time (just like your bank account). Mathematically, we can write this as:

$$\text{IN} - \text{OUT} = \Delta(\text{INSIDE}). \tag{142}$$

At time t and spanning an infinitesimally small time period $\mathrm{d}t$, we can write the left-hand side as:

$$\text{IN} - \text{OUT} = [-K \cdot g(x_1,t) \cdot \mathrm{d}t] - [-K \cdot g(x_1 + \mathrm{d}x, t) \cdot \mathrm{d}t] \approx K \cdot \left[\frac{\mathrm{d}}{\mathrm{d}x}g(x,t) \Big|_{x=x_1} \right] \cdot \mathrm{d}x \cdot \mathrm{d}t. \tag{143}$$

This approximation is exact at the limit $\mathrm{d}x \to 0$.

Similarly, for the right-hand side, the change in charge inside the box over a time period $\mathrm{d}t$ is:

$$\Delta(\text{INSIDE}) = \left[\frac{\mathrm{d}}{\mathrm{d}t}q(x,t) \Big|_{x=x_1} \right] \cdot \mathrm{d}t \cdot \mathrm{d}x \tag{144}$$

Thus, combining 143 and 144 into 142:

$$K \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x}g(x,t) \Big|_{x=x_1} \right) = \frac{\mathrm{d}}{\mathrm{d}t}q(x,t) \Big|_{x=x_1}. \tag{145}$$

Since this holds for all positions x and all times t, it follows that

$$K\frac{\mathrm{d}}{\mathrm{d}x}g(x,t) = \frac{\mathrm{d}}{\mathrm{d}t}q(x,t). \tag{146}$$

Since we are interested in the steady-state equilibrium solution, our charge distribution is not dependent on time. In other words, $q(x,t) = q(x)$ and $\frac{\mathrm{d}}{\mathrm{d}t}q(x,t) = 0$. Likewise, $g(x,t) = g(x)$. We can therefore re-write equations 141 and 146 as:

$$\frac{\mathrm{d}}{\mathrm{d}x}q(x) = g(x), \tag{147}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}g(x) = 0. \tag{148}$$

This is a system of differential equations we know how to solve. So let's solve for both $q(x)$ and $g(x)$ from Equations (147) and (148).

(a) Assume that we know both $q(0)$ and $g(0)$. **Solve for $q(x)$ and $g(x)$ in terms of these initial conditions.**

(Hint: solve first for $g(x)$. What's its derivative? What does that mean?)

Solution: First, notice that $\frac{\mathrm{d}}{\mathrm{d}x}g(x) = 0$ implies that $g(x) = c_1$ for some constant c_1. However, since we know $g(0)$, this implies that

$$g(x) = g(0). \tag{149}$$

Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}q(x) = g(0) \tag{150}$$

which implies that $q(x) = g(0) \cdot x + c_2$ for some constant c_2. Since we know $q(0)$, this implies

$$q(0) = g(0) \cdot 0 + c_2 = c_2. \tag{151}$$

Hence, we conclude that $c_2 = q(0)$, and that

$$q(x) = g(0) \cdot x + q(0). \tag{152}$$

(b) The challenge is that the physical story does not tell us anything immediately about $g(0)$. Instead, we just know about the free carrier density at both endpoints. **Solve for $q(x)$ and $g(x)$ with boundary conditions $q(0) = q_0$ and $q(L) = 0$ instead.** *(Note: Solutions should be in terms of q_0, L, and x.)*

(Hint: If you knew $g(0)$, what would $q(L)$ be in terms of $q(0)$ and $g(0)$? But you know $q(L)$ so what does that imply?)

Solution: We know from the previous part and the given initial condition $q(0) = q_0$ that

$$q(x) = g(0) \cdot x + q(0) \tag{153}$$
$$= g(0) \cdot x + q_0. \tag{154}$$

Now we plug in the other condition we know, $q(L) = 0$, and get

$$0 = q(L) \tag{155}$$
$$= g(0) \cdot L + q_0 \tag{156}$$
$$\implies -\frac{q_0}{L} = g(0). \tag{157}$$

Thus we have found the value of $g(0)$, and thus we know the value of $q(x)$. In the previous part we found that g is a constant, so we also know $g(x)$.

$$q(x) = -\frac{q_0}{L} \cdot x + q_0 \tag{158}$$
$$g(x) = -\frac{q_0}{L}. \tag{159}$$

Hence $q(x)$ is just a constant slope line that starts at q_0 and then reaches zero at L.

(c) The gradient $g(L)$ is related to the current flowing through the semiconductor. **What is $g(L)$?**

Solution: This can be computed explicitly from our solution $g(x)$.

$$g(L) = -\frac{q_0}{L}. \tag{160}$$

(d) **Use the provided Jupyter notebook to plot your solution so that you can visualize the charge distribution. If you wanted to increase the current coming out of the solar cell, should you make L bigger or smaller?**

Solution: The graph looks like this:
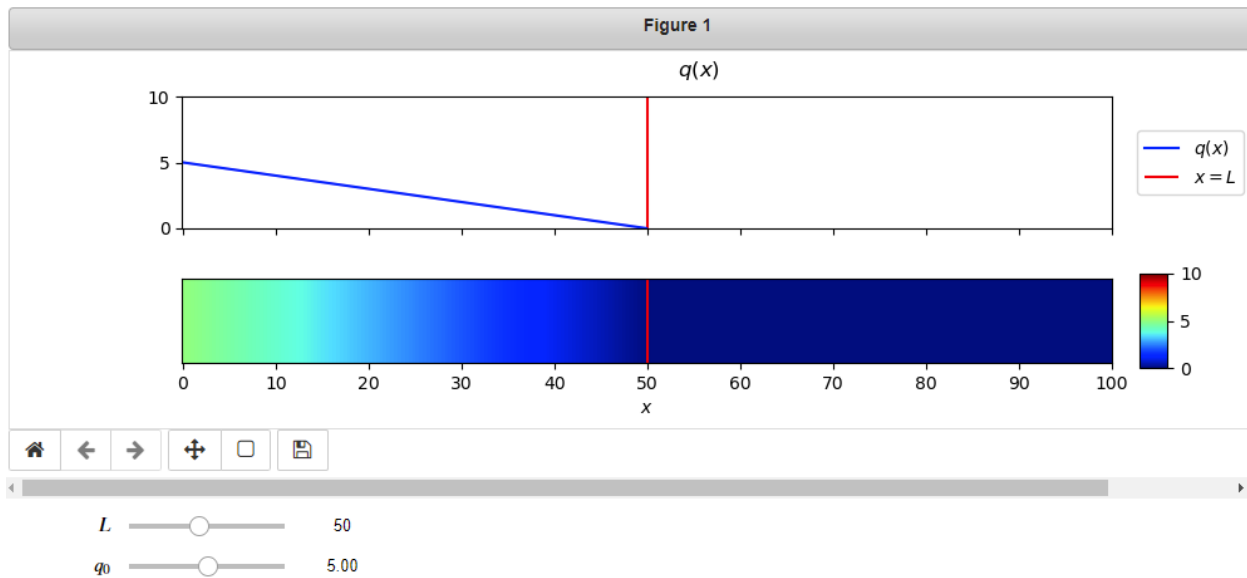
If we want to increase the current, we make L smaller. This is so that $\frac{q_0}{L}$, which is the magnitude of the current, larger.

(e) **Write out the differential equation in matrix/vector form:**

$$\frac{d}{dx}\begin{bmatrix} q(x) \\ g(x) \end{bmatrix} = \underbrace{\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}}_{A} \cdot \begin{bmatrix} q(x) \\ g(x) \end{bmatrix} \tag{161}$$

In other words, solve for matrix A.

Solution: Note that $\frac{d}{dx}q(x) = g(x)$ and $\frac{d}{dx}g(x) = 0$. Hence, A can be written as

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{162}$$

(f) **Find the eigenvalues and eigenvectors of A.**

Solution: Note that

$$\det(A - \lambda \cdot I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}\right) = (-\lambda)^2 - 1 \cdot 0 = \lambda^2. \tag{163}$$

Solving $\det(A - \lambda \cdot I) = \lambda^2 = 0$, we see that both of the eigenvalues of A are 0. The eigenvectors corresponding to eigenvalue 0 are then $k \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Notice that this problem only has a one-dimensional eigenspace despite being a 2x2 matrix.

(g) The model presented in this problem is extremely simplified. To be more realistic, you could also model the random recombination of free charge carriers within the semiconductor outside the depletion region. This recombination is proportional to the local density of free charge carriers and thus modifies (146) to instead be $\frac{d}{dt}q(x,t) = K\frac{d}{dx}g(x,t) - K_2 q(x,t)$. At steady-state equilibrium, we still have $\frac{d}{dt}q(x,t) = 0$, and this changes (148) to be $\frac{d}{dx}g(x) = \frac{K_2}{K}q(x)$ where $K_2 > 0$ is another physical constant that depends on the material. **For this part, assume $K_2 = 1$ and $K = 16$. What is your solution for $q(x)$ in this case?**

(HINT: You don't need to calculate for eigenvectors here, just the eigenvalues. The solution will have two terms — one corresponding to each of the distinct eigenvalues. Just get the eigenvalues and then fit to the boundary conditions. Also note that $g(x,t) = \frac{d}{dx}q(x,t)$ from equation 141 by definition.)

Solution: Note that in this case the matrix A for this problem is

$$A = \begin{bmatrix} 0 & 1 \\ \frac{K_2}{K} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{16} & 0 \end{bmatrix} \tag{164}$$

Writing out $\det(A - \lambda I) = 0$ gives us

$$\lambda^2 - \frac{1}{16} = 0 \tag{165}$$

which implies

$$\lambda_1 = \sqrt{\frac{1}{16}} = \frac{1}{4} \qquad \lambda_2 = -\sqrt{\frac{1}{16}} = -\frac{1}{4}. \tag{166}$$

Here, it appears that one of the eigenvalues λ_1 is negative and stable (corresponding to an exponential that will decay) while the other is positive and unstable (corresponding to a growing exponential). Now, we write out the general solution for $q(x)$:

$$q(x) = a_1 \cdot e^{\frac{1}{4}x} + a_2 \cdot e^{-\frac{1}{4}x}. \tag{167}$$

By our boundary conditions $q(0) = q_0$ and $q(L) = 0$, we solve for a_1 and a_2 via the following equations:

$$a_1 + a_2 = q_0, \tag{168}$$

$$a_1 \cdot e^{\frac{1}{4}L} + a_2 \cdot e^{-\frac{1}{4}L} = 0. \tag{169}$$

which gives us the solutions

$$a_1 = \frac{q_0}{1 - e^{\frac{1}{2}L}} \qquad \text{and} \qquad a_2 = \frac{q_0 \cdot e^{\frac{1}{2}L}}{e^{\frac{1}{2}L} - 1}. \tag{170}$$

Consequently, we have

$$q(x) = \frac{q_0}{1 - e^{\frac{1}{2}L}} \cdot e^{\frac{1}{4}x} + \frac{q_0 \cdot e^{\frac{1}{2}L}}{e^{\frac{1}{2}L} - 1} \cdot e^{-\frac{1}{4}x}. \tag{171}$$

Finally, since $g(x) = \frac{\mathrm{d}}{\mathrm{d}x}q(x)$, we get the expression

$$g(x) = \frac{1}{4}\left(\frac{q_0}{1 - e^{\frac{1}{2}L}} \cdot e^{\frac{1}{4}x} - \frac{q_0 \cdot e^{\frac{1}{2}L}}{e^{\frac{1}{2}L} - 1} \cdot e^{-\frac{1}{4}x} \right). \tag{172}$$

When the charge recombination rate K_2 is very small, then the above $q(x)$ ends up looking very much like the straight line solution that we got earlier and the current flowing out of the solar cell is large. If the recombination rate is significant, the charge-carrier density dies more quickly with distance x and the current flowing out of the solar cell is reduced.

(h) **Use the provided Jupyter notebook to explore what happens. What does the solution tend to as $\frac{K_2}{K} \to 0$?**

Solution: The solution tends to the straight line from the earlier parts with no recombination.

This is interesting because qualitatively, the two solutions look very different as formulae. One is a polynomial and the other is the weighted sum of two exponentials.

If you want to learn more about these kinds of things, take EE130 (Integrated Circuit Devices) and EE134 (Fundamentals of Photovoltaic Devices). Device physics is a beautiful subject in which physical intuition plus quantum mechanics combine with differential equations and careful experimentation (for which, EE143: Microfabrication Technology is a great introduction — you will literally fabricate

your own devices in the clean room) to give rise to the fundamental devices that are the foundation of our contemporary information era.

Whether we're interested in providing cheap renewable energy to everyone on this planet, in building cybernetic implants to help people with disabilities, or in creating faster and more energy efficient computing platforms for machine intelligences — advancements in device physics lead the way. For almost $\frac{3}{4}$ of a century now, our ability to solve problems has improved exponentially with most of that improvement coming from improved devices, another large fraction coming from improved theory/algorithms, and the rest dominated by our ability to scale to large-scale systems like cloud computing and the like. Along the way, computer architectures, analog circuits, and programming languages/platforms need to constantly adapt to leverage these underlying advances and enable them to keep working together.

8. **(OPTIONAL) Make Your Own Problem.**

   Write your own problem about content covered in the course thus far, and provide a thorough solution to it.

   NOTE: This can be a totally new problem, a modification on an existing problem, or a Jupyter part for a problem that previously didn't have one. Please cite all sources for anything (including course material) that you used as inspiration.

   NOTE: High-quality problems may be used as inspiration for the problems we choose to put on future homeworks or exams.

9. **Homework Process and Study Group**

   Citing sources and collaborators are an important part of life, including being a student!

   We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

   (a) **What sources (if any) did you use as you worked through the homework?**

   (b) **If you worked with someone on this homework, who did you work with?**
       List names and student ID's. (In case of homework party, you can also just describe the group.)

   (c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

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