

EECS 16A Designing Information Devices and Systems I

Summer 2020 Homework 0B

This homework is due Sunday June 28, 2020, at 23:59 PT.

Self-grades are due Wednesday July 1, 2020, at 23:59 PT.

Please note that future homeworks will have more problems than HW1 and we ask you to plan accordingly.

Submission Format

Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

Please attach a PDF of your Jupyter notebook for all the problems that involve coding. Make sure the results of your plots (if any) are visible. Please assign the PDF of the notebook to the correct problems on Gradescope — we will be unable to grade the problems without this assignment or submission.

Homework Learning Goals: *The objective of this homework is to introduce systems of linear equations. This homework additionally serves as an introduction to working with the Python environment through IPython/Jupyter notebooks.*

1. Counting Solutions

Learning Goal: *(This problem is meant to illustrate the different types of systems of equations. Some have a unique solution and others have no solutions or infinitely many solutions. We will learn in this class how to systematically figure out which of the three above cases holds.)*

For each of the following systems of linear equations, determine if there is a unique solution, no solution, or an infinite number of solutions. If there is a unique solution, find it. If there is an infinite number of solutions state this and give one solution. If there is no solution, explain why. **Use augmented matrices and show your work.**

(a)

$$\begin{array}{rcrcrcrcl} 2x & + & 3y & = & 5 \\ x & + & y & = & 2 \end{array}$$

Solution:

$$\begin{aligned} \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 1 & 1 & 2 \end{array} \right] &\rightarrow \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{5}{2} \\ 1 & 1 & 2 \end{array} \right] \text{ using } R_1 \leftarrow \frac{1}{2}R_1 \\ &\rightarrow \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{array} \right] \text{ using } R_2 \leftarrow R_2 - R_1 \\ &\rightarrow \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 1 \end{array} \right] \text{ using } R_2 \leftarrow -2R_2 \\ &\rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right] \text{ using } R_1 \leftarrow R_1 - \frac{3}{2}R_2 \end{aligned}$$

Unique solution, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b)

$$\begin{array}{rrrr} x & + & y & + & z & = & 3 \\ 2x & + & 2y & + & 2z & = & 5 \end{array}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 5 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{array} \right] \text{ using } R_2 \leftarrow R_2 - 2R_1$$

No solution. The fact that there are fewer equations than there are unknowns immediately means that it is not possible to have a unique solution; however, this does not guarantee that there is a solution to begin with. From Gaussian Elimination, we can see that these equations are inconsistent and we have a row of zero equating to a nonzero value. In other words, no values of x , y , and z can satisfy both equations simultaneously.

(c)

$$\begin{array}{rrrr} & - & y & + & 2z & = & 1 \\ 2x & & & + & z & = & 2 \end{array}$$

Solution:

Because there are two equations and three unknowns, we immediately see that there can be no unique solution. The question then becomes if there are an infinite number of solutions, or no solution at all.

$$\begin{aligned} \left[\begin{array}{ccc|c} 0 & -1 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{array} \right] &\rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 2 \\ 0 & -1 & 2 & 1 \end{array} \right] \text{ swapping } R_1 \text{ and } R_2 \\ &\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 1 \\ 0 & -1 & 2 & 1 \end{array} \right] \text{ using } R_1 \leftarrow \frac{1}{2}R_1 \\ &\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & -2 & -1 \end{array} \right] \text{ using } R_2 \leftarrow -R_2 \end{aligned}$$

We have arrived at reduced row echelon form. In this way we can explicitly see that z is a free variable (x and y depend on z and there are no constraints on the value of z). Thus there are an infinite number of solutions. The set of infinite solutions has the form (for some $z \in \mathbb{R}$):

$$\begin{aligned} x &= 1 - \frac{1}{2}z \\ y &= 2z - 1 \end{aligned}$$

To get full credit it is enough to state "Infinite solutions" and give one possible solution that fits the form above.

(d)

$$\begin{array}{rrrr} x & + & 2y & = & 3 \\ 2x & - & y & = & 1 \\ 3x & + & y & = & 4 \end{array}$$

Solution:

$$\begin{aligned}
\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{array} \right] &\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 3 & 1 & 4 \end{array} \right] \text{ using } R_2 \leftarrow R_2 - 2R_1 \\
&\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -5 & -5 \end{array} \right] \text{ using } R_3 \leftarrow R_3 - 3R_1 \\
&\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -5 \end{array} \right] \text{ using } R_2 \leftarrow -\frac{1}{5}R_2 \\
&\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right] \text{ using } R_3 \leftarrow R_3 + 5R_2 \\
&\rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right] \text{ using } R_1 \leftarrow R_1 - R_2
\end{aligned}$$

Unique solution, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ The system of linear equations at the end of the Gaussian Elimination above simply reads out

$$x = 1$$

$$y = 1$$

$$0 = 0$$

(e)

$$\begin{aligned}
x + 2y &= 3 \\
2x - y &= 1 \\
x - 3y &= -5
\end{aligned}$$

Solution:

$$\begin{aligned}
\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 1 & -3 & -5 \end{array} \right] &\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 1 & -3 & -5 \end{array} \right] \text{ using } R_2 \leftarrow R_2 - 2R_1 \\
&\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -5 & -8 \end{array} \right] \text{ using } R_3 \leftarrow R_3 - R_1 \\
&\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -8 \end{array} \right] \text{ using } R_2 \leftarrow -\frac{1}{5}R_2 \\
&\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{array} \right] \text{ using } R_3 \leftarrow R_3 + 5R_2
\end{aligned}$$

No solution. We can think of this to mean that there are no values of x and y which satisfy the conditions in all three equations simultaneously, because in order to satisfy all three equations, the last row $0 = -3$ would need to be true. Even though we have more equations than unknowns, that does not guarantee that a unique solution, or any solutions, exist.

2. Sikder's Optimal Boba

Solution: #SystemsOfEquations #GaussianElimination

Learning Goal: Recognize a problem that can be cast as a system of linear equations.

Sikder's Optimal Boba has a unique way of serving its customers. To ensure the best customer experience, each customer gets a combination drink personalized to their tastes. Professor Sikder knows that a lot of customers don't know what they want, so when customers walk up to the counter, they are asked to taste four standard combination drinks that each contain a different mixture of the available pure teas.

Each combination drink (Classic, Roasted, Mountain, and Okinawa) is made of a mixture of pure teas (Black, Oolong, Green, and Earl Grey), with the total amount of pure tea in each combination drink always the same, and equal to one cup. The table below shows the quantity of each pure tea (Black, Oolong, Green, and Earl Grey) contained in each of the four standard combination drinks (Classic, Roasted, Mountain, and Okinawa).

Tea [cups]	Classic	Roasted	Mountain	Okinawa
Black	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$
Oolong	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$
Green	0	$\frac{1}{3}$	$\frac{1}{3}$	0
Earl Grey	$\frac{1}{3}$	0	0	0

Initially, the customer's ratings for each of the pure teas are unknown. Professor Sikder's goal is to determine how much the customer likes each of the pure teas, so that an optimal combination drink can then be made. By letting the customer taste and score each of the four standard combination drinks, Professor Sikder can use linear algebra to determine the customer's initially unknown ratings for each of the pure teas. After a customer gives a score (all of the scores are real numbers) for each of the four standard combination drinks, Professor Sikder then calculates how much the customer likes each pure tea and mixes up a special combination drink that will maximize the customer's score.

The score that a customer gives for a combination drink is a linear combination of the ratings of the constituent pure teas, based on their proportion. For example, if a customer's rating for black tea is 6 and oolong tea is 3, then the total score for the Okinawa boba drink would be $6 \cdot \frac{2}{3} + 3 \cdot \frac{1}{3} = 5$ because Okinawa has $\frac{2}{3}$ black tea and $\frac{1}{3}$ Oolong tea.

Professor Kuo was thirsty after giving the first lecture, so Professor Kuo decided to take a drink break at Sikder's Optimal Boba. Professor Kuo walked in and gave the following ratings:

Combination Drink	Score
Classic	7
Roasted	7
Mountain	$7\frac{2}{3}$
Okinawa	$6\frac{1}{3}$

- (a) What were Professor Kuo's ratings for each tea? **Work this problem out by hand in terms of the steps. You may use a calculator to do algebra.**

Solution:

Using Professor Kuo's ratings, Professor Sikder mentally records the following system of equations. Let x_b be the customer's rating of black tea, x_o be the customer's rating of oolong tea, x_g be the customer's rating of green tea, and x_e be the customer's rating of earl grey tea.

$$\begin{array}{ll} \text{Classic:} & 7 = \frac{1}{3}x_b + \frac{1}{3}x_o + \frac{1}{3}x_e \\ \text{Roasted:} & 7 = \frac{1}{3}x_b + \frac{1}{3}x_o + \frac{1}{3}x_g \\ \text{Mountain:} & 7\frac{2}{5} = \frac{2}{5}x_o + \frac{3}{5}x_g \\ \text{Okinawa:} & 6\frac{1}{3} = \frac{2}{3}x_b + \frac{1}{3}x_o \end{array}$$

Multiply each equation by the denominator of the fraction (in order to make them easier to read):

$$\begin{aligned} 21 &= x_b + x_o + x_e \\ 21 &= x_b + x_o + x_g \\ 37 &= 2x_o + 3x_g \\ 19 &= 2x_b + x_o \end{aligned}$$

The above equations can be written as an augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 21 \\ 1 & 1 & 1 & 0 & 21 \\ 0 & 2 & 3 & 0 & 37 \\ 2 & 1 & 0 & 0 & 19 \end{array} \right].$$

Row reduce the matrix into reduced row echelon form as follows. (It's fine if you solved the system of equations by hand a different way. Here, however, we will demonstrate how to do it using Gaussian elimination.)

Noting that there is a 1 in the upper left hand corner, subtract Row 1 from Row 2 and $2 \times$ Row 1 from Row 4.

$$\begin{array}{l} \text{Row 2: subtract Row 1} \\ \text{Row 4: subtract } 2 \times \text{Row 1} \end{array} \implies \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 21 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 3 & 0 & 37 \\ 0 & -1 & 0 & -2 & -23 \end{array} \right]$$

Since Row 2 has a 0 in the diagonal element, multiply Row 4 by -1 and then switch Rows 2 and 4.

$$\begin{array}{l} \text{Multiply Row 4 by } -1 \\ \text{Switch Row 2 and Row 4} \end{array} \implies \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 21 \\ 0 & 1 & 0 & 2 & 23 \\ 0 & 2 & 3 & 0 & 37 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right]$$

Subtract Row 2 from Row 1 and $2 \times$ Row 2 from Row 3.

$$\begin{array}{l} \text{Row 1: subtract Row 2} \\ \text{Row 3: subtract } 2 \times \text{Row 2} \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 2 & 23 \\ 0 & 0 & 3 & -4 & -9 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right]$$

Switch Row 3 and Row 4 and subtract $3 \times$ the new Row 3 from the new Row 4.

$$\begin{array}{l} \text{Switch Row 3 and Row 4} \\ \text{Row 4: subtract } 3 \times \text{Row 3} \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 2 & 23 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -9 \end{array} \right]$$

Finally, multiply Row 4 by -1 and add Row 4 to Row 1 and Row 3 and subtract $2 \times$ Row 4 from Row 2.

$$\begin{array}{l} \text{Multiply Row 4 by } -1 \\ \text{Row 1, Row 3: add Row 4} \\ \text{Row 2: subtract } 2 \times \text{Row 4} \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & 9 \end{array} \right]$$

Professor Kuo's ratings for each tea are

Tea	Score
Black	7
Oolong	5
Green	9
Earl Grey	9

- (b) What mystery tea combination could Professor Sikder put in Professor Kuo's personalized drink to maximize the customer's score? If there is more than one correct answer, state that there are many answers, and give one such combination. What score would Professor Kuo give for the answer you wrote down?

Solution:

There are many answers. *Any* combination of green tea and earl grey is acceptable as they have equal ratings. More precisely, for any $0 \leq a \leq 1$, a combination with a cups of green tea and $1 - a$ cups of earl grey will yield a score of $9a + 9(1 - a) = 9$.

As an example, Professor Sikder could choose $a = \frac{1}{2}$ so that Professor Kuo's drink has $\frac{1}{2}$ cup of green tea and $\frac{1}{2}$ cup of earl grey.

How to see this? Green tea and earl grey are tied for Professor Kuo's favorite tea, so it doesn't make a difference if Professor Sikder substitutes one for the other in any quantity - the score remains 9. It also doesn't make sense to substitute a less preferred tea like black tea for Professor Kuo's favorite tea, or to add a less preferred tea at the expense of the most preferred teas, as this will lead to scores less than 9.

3. Filtering Out The Troll

Solution: #SystemsOfEquations #LinearCombination

Learning Goal: *(The goal of this problem is to represent a practical scenario using a simple model of directional microphones. Students will tackle the problem of sound reconstruction through solving a system of linear equations.)*

You attended a very important public speech and recorded it using a recording device that consists of two directional microphones. However, there was this particular person in the audience who was trolling around, adding noise to the recording. When you went back home to listen to the recording, you realized that the two recordings were dominated by the troll's noise and you could not hear the speech. Fortunately, since your recording device contained two microphones, you realized there is a way to combine the two individual microphone recordings so that the troll's noise is removed. You remembered the locations of the speaker and the troll and created the diagram shown in Figure 1. You (and your two microphones) are located at the origin.

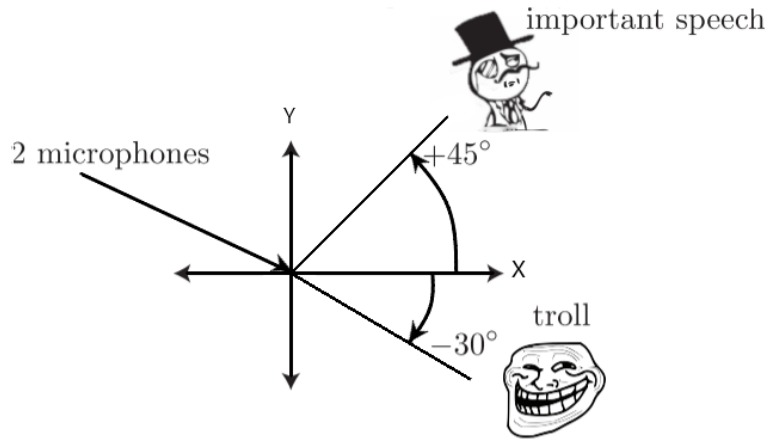


Figure 1: Locations of the speaker and the troll.

Each directional microphone records signals differently based on where they are coming from. For the first microphone, when a signal is coming from an angle θ with respect to the x-axis, it is weighted by the factor $f_1(\theta) = \cos(\theta)$. If there are two signals simultaneously playing (as is the case with the speech and the troll noise), then both are recorded as a linear combination, each weighted by the respective $f_1(\theta)$ for their angles). For the second microphone, if the signal is coming from an angle θ with respect to the x-axis, then the signal is weighted by the factor $f_2(\theta) = \sin(\theta)$. The linear combination also applies to the second microphone. Graphically, the directional characteristics of the two microphones are given in Figure 2.

We can now refer to the diagram in Figure 1 and develop a mathematical model of the microphone recordings. Let the person who gave the important speech and the troll be speakers A and B , respectively. The person who gave the important speech (speaker A) was located at angle $\alpha = +45^\circ$ relative to the x-axis, and the troll (speaker B) was located at angle $\beta = -30^\circ$ relative to the x-axis. Speaker A produced an audio signal represented by the vector $\vec{a} \in \mathbb{R}^n$. That is, the i -th entry of vector \vec{a} was the signal at the i -th time step. Similarly, speaker B produced an audio signal $\vec{b} \in \mathbb{R}^n$, where the i -th entry of vector \vec{b} was the signal at the i -th time step.

Therefore, the first microphone recorded the signal

$$\vec{m}_1 = f_1(\alpha) \cdot \vec{a} + f_1(\beta) \cdot \vec{b},$$

and the second microphone recorded the signal

$$\vec{m}_2 = f_2(\alpha) \cdot \vec{a} + f_2(\beta) \cdot \vec{b}.$$

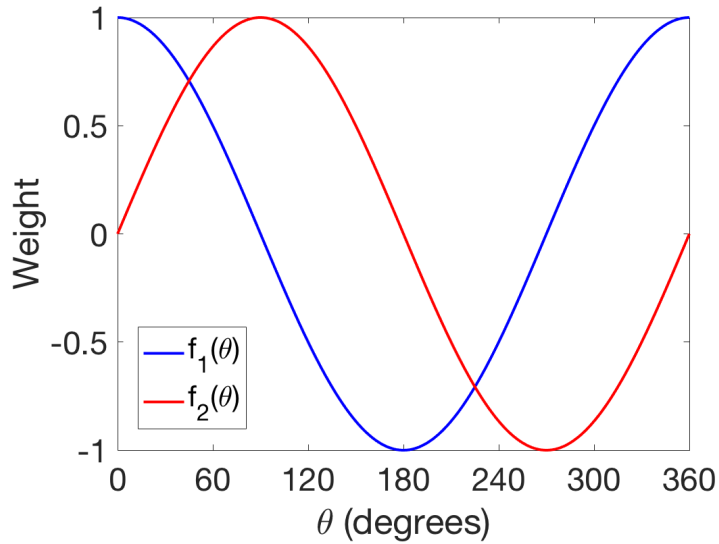


Figure 2: Weights for recorded audio signals for each of the two microphones, as a function of audio source angle θ . Microphone 1 is blue and microphone 2 is red. Note that a weight can be negative as well as positive.

- (a) Using the notation above, express the recordings of the two microphones \vec{m}_1 and \vec{m}_2 (i.e. the signals recorded by the first and the second microphones, respectively) as a linear combination (i.e. a weighted sum) of \vec{a} and \vec{b} .

Solution:

$$\begin{aligned}\vec{m}_1 &= \cos\left(\frac{\pi}{4}\right) \cdot \vec{a} + \cos\left(-\frac{\pi}{6}\right) \cdot \vec{b} \\ &= \frac{1}{\sqrt{2}} \cdot \vec{a} + \frac{\sqrt{3}}{2} \cdot \vec{b} \\ \vec{m}_2 &= \sin\left(\frac{\pi}{4}\right) \cdot \vec{a} + \sin\left(-\frac{\pi}{6}\right) \cdot \vec{b} \\ &= \frac{1}{\sqrt{2}} \cdot \vec{a} - \frac{1}{2} \cdot \vec{b}\end{aligned}$$

- (b) Recover the important speech \vec{a} , as a weighted combination of \vec{m}_1 and \vec{m}_2 . In other words, write $\vec{a} = u \cdot \vec{m}_1 + v \cdot \vec{m}_2$ (where u and v are scalars). What are the values of u and v ?

Solution:

Solving the system of linear equations yields

$$\vec{a} = \frac{\sqrt{2}}{1 + \sqrt{3}} \cdot (\vec{m}_1 + \sqrt{3}\vec{m}_2).$$

Therefore, the values are $u = \frac{\sqrt{2}}{1 + \sqrt{3}}$ and $v = \frac{\sqrt{6}}{1 + \sqrt{3}}$.

It is fine if you solved this either using IPython or by hand using any valid technique. The easiest approach is to subtract either of the two equations from the other and immediately see that

$\vec{b} = \frac{2}{\sqrt{3}+1}(\vec{m}_1 - \vec{m}_2)$. Substituting \vec{b} back into the second equation and multiplying through by $\sqrt{2}$ gives that $\vec{a} = \sqrt{2}(\vec{m}_2 + \frac{1}{\sqrt{3}+1}(\vec{m}_1 - \vec{m}_2))$, which simplifies to the expression given above.

Notice that subtracting one equation from the other is natural given the symmetry of the microphone patterns and the fact that the patterns intersect at the 45 degree line where the important speech is happening, and the fact that $\sin(45^\circ) = \cos(45^\circ)$. So we know that the result of subtracting one microphone recording from the other results in only the trolls contribution. Once we have the troll contribution, we can remove it and obtain the important speakers sole content.

- (c) Partial IPython code can be found in `prob0B.ipynb`, which you can access through the datahub link associated with this assignment on the course website. Complete the code to get the signal of the important speech. Write out what the speaker says. (Optional: Where is the speech taken from?)

Note: You may have noticed that the recordings of the two microphones sound remarkably similar. This means that you could recover the real speech from two “trolled” recordings that sound almost identical! Leave out the fact that the recordings are actually different, and have some fun with your friends who aren’t lucky enough to be taking EECS16A.

Solution:

The solution code can be found in `sol1.ipynb`. The speaker says: “All human beings are born free and equal in dignity and rights.” and the speech was taken from the Universal Declaration of Human Rights.

The idea of using multiple microphones to isolate speech is interesting and is increasingly used in practice. Furthermore, similar techniques are used in wireless communication both by cellular systems like LTE and increasingly by WiFi hotspots. (This is why they often have multiple antennas).

4. Fountain Codes

Learning Goal: *Linear algebra shows up in many important engineering applications. Wireless communication and information theory heavily rely on principles of linear algebra. This problem illustrates some of the techniques used in wireless communication.*

Alice wants to send a message to her friend Bob. Alice sends her message \vec{m} across a wireless channel in the form of a transmission vector \vec{w} . Bob receives a vector of symbols denoted as \vec{r} . Alice knows some of the symbols in the transmission vector that she sends may be corrupted, so she needs a way to protect her message. A schematic of how this can potentially be done is shown in Figure 3. Transmission corruptions occur commonly in real wireless communication systems, for instance, when your laptop connects to a WiFi router, or your cellphone connects to the nearest cell tower.) Ideally, Alice can come up with a transmission vector such that if some of the symbols get corrupted, Bob can still figure out what Alice is trying to say!

One way to accomplish this goal is to use fountain codes, which are part of a broader family of codes called error correcting codes. Fountain codes are based on principles of linear algebra, and were actually developed right here at Berkeley! The company that commercialized them, Digital Fountain, (started by a [Berkeley grad, Mike Luby](#)), was later acquired by Qualcomm. In this problem, we will explore some of the underlying principles that make fountain codes work in a very simplified setting.

The message that Alice wants to send to Bob are the three numbers a , b , and c . The message vector representing these numbers is $\vec{m} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

- (a) Since Alice has three numbers she wishes to transmit, she could transmit six symbols in her transmission vector for redundancy. One of the most naive codes is called the repetition code. If Alice uses the

repetition code, her transmission vector is $\vec{w} = \begin{bmatrix} a \\ b \\ c \\ a \\ b \\ c \end{bmatrix}$.

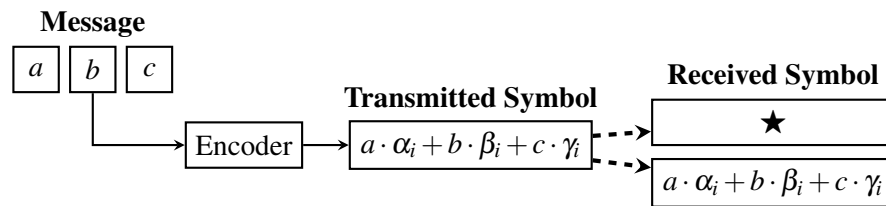


Figure 3: Each symbol in a transmission vector \vec{w} is a linear combination of a , b , and c . Each transmitted symbol is either received exactly as it was sent, or it is corrupted. A corrupted symbol is denoted by \star . The i th row of a symbol generating matrix G determines the values of α_i , β_i , and γ_i .

If only the first symbol is corrupted, Bob would receive the vector $\vec{r} = \begin{bmatrix} \star \\ b \\ c \\ a \\ b \\ c \end{bmatrix}$ (the \star symbol represents

a corrupted symbol). Using this repetition scheme, give an example of a received vector \vec{r} such that a is unrecoverable but b and c are still recoverable.

Solution:

If Bob receives the vector $\vec{r} = \begin{bmatrix} \star \\ b \\ c \\ \star \\ b \\ c \end{bmatrix}$ then there is no way for him to recover a . Any \vec{r} that has \star for

both a symbols and at least one received b symbol and one received c symbol is also a correct answer.

- (b) Alice can generate \vec{w} by multiplying her message \vec{m} by a matrix. Write a matrix-vector multiplication that Alice can use to generate \vec{w} . Specifically, find a symbol generating matrix G_R such that $G_R \vec{m} = \vec{w}$

Solution:

Alice can use the follow symbol generating matrix G_R to generate \vec{w} :

$$G_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- (c) Instead of a repetition code, it is also possible to use other codes (e.g. fountain codes). Alice and Bob can choose any symbol generating matrix, as long as they agree upon it in advance. Each different matrix represents a different "code." Alice's TA recommends using the symbol generating matrix $G_F =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Alice then uses the symbol generating matrix G_F to produce a new transmission vector \vec{w} .

Suppose Bob receives the vector $\vec{r} = \begin{bmatrix} 7 \\ \star \\ \star \\ 3 \\ 4 \\ \star \\ \star \end{bmatrix}$.

Write a system of linear equations that Bob can use to recover the message vector \vec{m} . Solve it to recover the three numbers that Alice sent.

Solution:

In order to recover Alice's message \vec{m} , Bob needs to solve the following equation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \vec{m} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 4 \end{bmatrix}$$

Bob can solve this system by writing the system in augmented matrix form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 4 \end{array} \right]$$

Now using back substitution:

$$\begin{aligned} a &= 7 \\ b &= 3 - a = -4 \\ c &= 4 - a = -3 \end{aligned}$$

- (d) If Alice and Bob agree to use G_F , can Bob recover the message vector \vec{m} from any three uncorrupted symbols he receives? What about any four received uncorrupted symbols? Justify your answer by giving examples and/or counter-examples. No need to provide a rigorous proof.

Solution:

In order for Bob to be able to recover Alice's message \vec{m} , he needs a system of equations with a unique solution.

Not any choice of three rows (i.e. three received symbols) from G_F will provide a unique solution for Alice's message.

For example, an augmented matrix made from rows 1, 2, and 4 of G_F (and their corresponding uncorrupted symbols) and simplified using Gaussian elimination will always produce a row of all 0s. This signifies that there are infinite solutions (i.e. Alice's message cannot be recovered).

Because the third column does not have a leading 1, there will not be a unique solution.

Any four rows (i.e. four uncorrupted received symbols) from the symbol generating matrix G_F are sufficient for recovering Alice's message \vec{m} .

More specifically, an augmented matrix with any four rows from the symbol generating matrix G_F (and their corresponding received symbols) will produce the following after performing Gaussian elimination and back substitution:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 \end{array} \right]$$

(e) Which symbol generating matrix do you prefer? G_R or G_F ? Justify why.

Solution:

G_F is superior to G_R . Using G_F allows Bob to recover Alice's message from *any* four symbols (i.e. the message can be recovered with *any* three corrupted symbols). If Bob uses G_R , then he cannot recover the message if any of the following are true: the first and fourth symbols are corrupted, the second and fifth symbols are corrupted, or the third and sixth symbols are corrupted.

5. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.