

EECS 16B Designing Information Devices and Systems II

Summer 2020

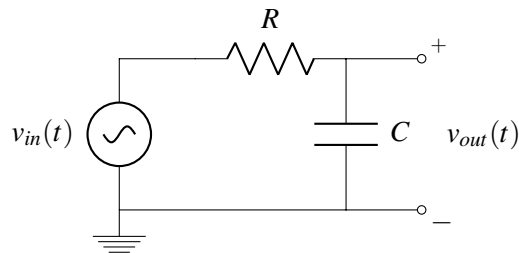
Note 7

Overview

In the previous note, we developed the method of Phasor Analysis to analyze the steady state behavior of sinusoidal voltages and currents. In this note, we will consider some applications of these circuits, and begin to explore techniques for designing these circuits to fit a set of requirements called a **filter**.

1 Circuit Example and Transfer Functions

Recall the circuit from the previous notes:



Let $v_{in}(t) = 5 \cos\left(10^5 t - \frac{\pi}{6}\right)$, $R = 1 \text{ k}\Omega$, $C = 10 \text{ nF}$. We saw that this circuit was very similar to a voltage divider and that we could find \tilde{V}_{in} by looking at the ratio of the impedances. The input voltage phasor is

$$\tilde{V}_{in} = 5e^{-j\frac{\pi}{6}}$$

Applying the equation for a voltage divider, we find the voltage phasor at $v_{out}(t)$ to be

$$\tilde{V}_{out} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \tilde{V}_{in} = \frac{1}{1 + j\omega RC} \tilde{V}_{in}.$$

We will not substitute in our known value for \tilde{V} just yet and instead look at the ratio of the output and input phasor. We call this ratio the **transfer function** of the circuit denoted by $H(\omega)$.

$$H(\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j}.$$

More broadly, transfer functions map an angular frequency ω to a ratio of two phasors - an input and an output - and are used to characterize the behavior of a system across a range of frequencies. Here, our transfer function relates two voltage phasors, but they can also relate two current phasors, or a voltage phasor with a current phasor, in a similar manner.

Observe that as transfer functions are the ratio of two phasors, and will be a complex number at a given frequency. Thus, we can write

$$H(\omega) = M(\omega)e^{j\theta},$$

where $M(\omega)$ is the magnitude of the transfer function, and θ as the phase of the transfer function.

In this example, we can compute $M(\omega)$ by dividing the magnitude of the numerator and denominator of $H(\omega)$.

$$M(\omega) = |H(\omega)| = \frac{|1|}{|1+j|} = \frac{1}{\sqrt{2}},$$

We can then compute the phase θ by subtracting the phase of the numerator and denominator of $H(\omega)$.

$$\theta = \angle H(\omega) = \angle 1 - \angle 1 + j = -\frac{\pi}{4}$$

To solve for v_{out} , we can compute multiply the input phasor \tilde{V}_{in} by $H(\omega)$.

$$\tilde{V}_{out} = H(\omega)\tilde{V}_{in} = \frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}\tilde{V}_{in} = \frac{5}{\sqrt{2}}e^{-j\frac{5\pi}{12}} \quad (1)$$

Then performing the inverse transform,

$$v_{out}(t) = \frac{5}{\sqrt{2}}\cos\left(10^5t - \frac{5\pi}{12}\right)$$

This example is practically identical to the one we did in the note on phasors, but we are now equipped with the notion of a transfer function. Since the transfer function is a ratio of two phasors, using voltage dividers will often simplify our calculations greatly.

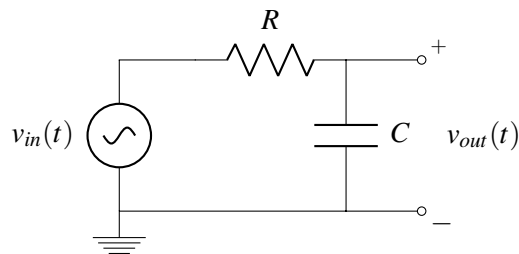
2 First Order Filters

In this section, we will develop the concept of a **filter** using our knowledge of transfer functions. The key idea here is that a filter lets signals at some frequencies through and will block out other frequencies.

2.1 Low-Pass Filters

Let's say we had some high frequency noise greater than 100kHz in our input signal that we would like to attenuate using a filter. If we were to design an ideal filter to get rid of this noise, we would want to pick a cutoff where all frequencies below 100kHz through while blocking out any frequencies above 100kHz. Mathematically this would mean $H(\omega) = 1$ for all $\omega < 100\text{kHz}$ and $H(\omega) = 0$ for $\omega \geq 100\text{kHz}$.

While designing this ideal filter is difficult, we could certainly design a low-pass filter using an RC circuit.



Recall from the previous example that

$$H(\omega) = \frac{1}{1 + j\omega RC}$$

Now how can we show that this filter is in fact a low-pass filter?

As a quick intuition check, if $\omega = 0$, $H(\omega) = 1$ and if $\omega \rightarrow \infty$, then $H(\omega) \rightarrow 0$. Therefore, our filter seems to behave as expected, but what happens in between 0 and ∞ ? Let's take a look at a couple of values around $\omega_c = \frac{1}{RC}$:

| ω | $H(\omega)$ | $M(\omega)$ | θ |
|---------------|----------------------|-------------|-------------|
| $0.1\omega_c$ | $\frac{1}{1 + 0.1j}$ | 0.995 | -6° |
| ω_c | $\frac{1}{1 + j}$ | 0.71 | -45° |
| $10\omega_c$ | $\frac{1}{1 + 10j}$ | 0.1 | -84° |

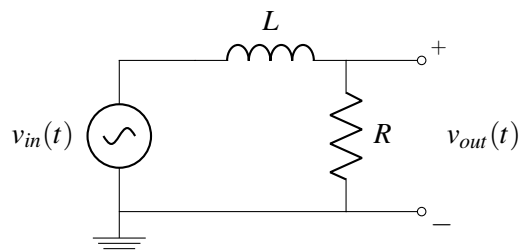
This should show that $\omega_c = \frac{1}{RC}$ is a very important frequency to look at since this is around the frequency where the behavior of the filter starts to qualitatively change.¹ In fact, this is so important that we call this the **cutoff** or **corner** frequency. Below this frequency, the filter seems to let everything through, while much above this frequency, the filter blocks everything.

Mathematically the **cutoff frequency**, ω_c , is defined as the point at which

$$H(\omega_c) = \frac{\max_{\omega} |H(\omega)|}{\sqrt{2}}$$

Where $\max_{\omega} H(\omega)$ is the maximum magnitude of $H(\omega)$ over all frequency. For a **passive circuit**, one without an external power supply, this will usually be 1.

The above circuit was shown to be a low-pass filter, but there are a potpourri of filter topologies and tradeoffs between each design. We won't look into too many of these different examples, but an easy example that we can analyze is an *LR* low-pass filter.



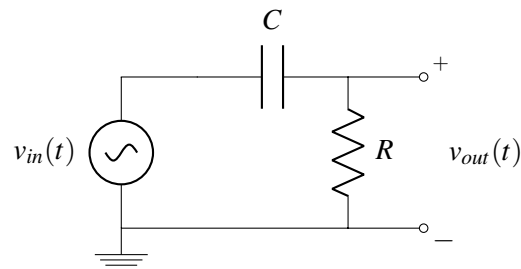
Try to compute its transfer function and find its cutoff frequency.²

¹Note how this is the reciprocal of the time constant. We'll explore its significance in a later section.

² $H(\omega) = \frac{1}{1 + j\omega L/R}$, $\omega_c = \frac{R}{L}$

2.2 High-Pass Filters

Now let's say we were building a sound system but the bass was too strong. We would like to filter out lower frequencies while keeping the remaining higher frequencies the same. To do this, we should start thinking about how we can build a *high-pass* filter. All of our principles that we have developed in the previous section apply here as well, so let's verify that the following *CR* circuit is a high-pass filter.

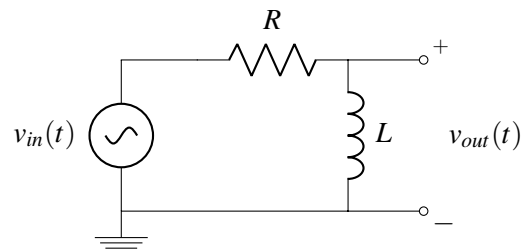


$$H(\omega) = \frac{j\omega RC}{1 + j\omega RC} \implies \omega_c = \frac{1}{RC}$$

| ω | $H(\omega)$ | $M(\omega)$ | θ |
|---------------|-------------------------|-------------|------------|
| $0.1\omega_c$ | $\frac{0.1j}{1 + 0.1j}$ | 0.1 | 84° |
| ω_c | $\frac{j}{1 + j}$ | 0.71 | 45° |
| $10\omega_c$ | $\frac{10j}{1 + 10j}$ | 0.995 | 6° |

Note how the cutoff for a low-pass and high-pass filter are identical.

Another example of a high-pass filter that we can look at is an *RL* filter



Try to compute its transfer function and find its cutoff frequency.³

³ $H(\omega) = \frac{j\omega L/R}{1 + j\omega L/R}, \omega_c = \frac{R}{L}$

3 Bode Plots

Having analyzed our first order filters, we will now plot the frequency response $H(\omega)$. Throughout this section, we will be using various numerical approximations that circuit designers use when building filters. These ideas will prove to be useful when trying to plot the response of a specific filter and will help us better understand its behavior.

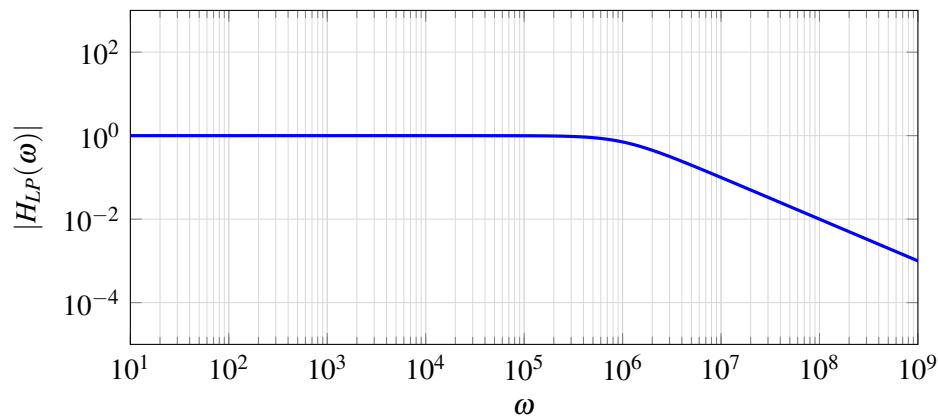
When we make Bode plots, we plot the frequency and magnitude on a logarithmic scale, and the angle in either degrees or radians. We use the logarithmic scale because it allows us to break up complex transfer functions into its constituent components. We start with two simple examples that will build intuition to tackle more complicated transfer functions later.

3.1 Low-pass Filter

Going back to our RC model of the low-pass filter,

$$H_{LP}(\omega) = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega/\omega_c}$$

we plot the magnitude of the frequency response assuming $\omega_c = 10^6$.

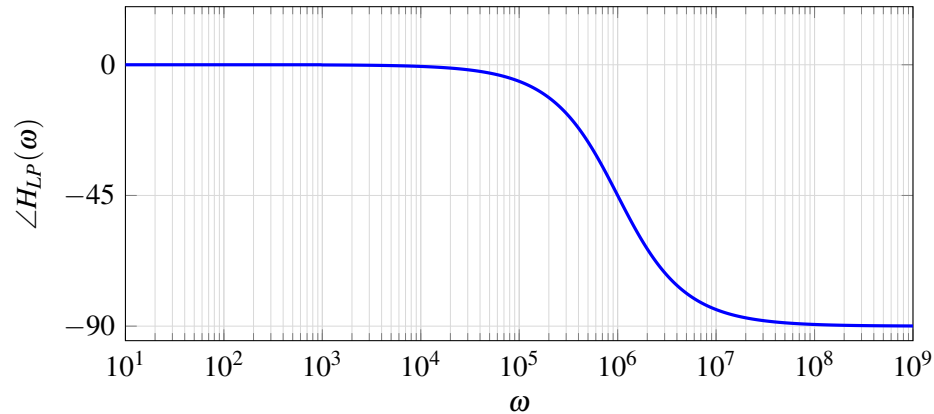


Note how $|H_{LP}(\omega)|$ is very close to 1 for $\omega < \omega_c$ and $|H_{LP}(\omega)|$ starts dropping off with slope 1 after ω_c . To formalize this analysis, we will break it down into cases

- $\omega \ll \omega_c$, then $j\omega/\omega_c \approx 0$. Therefore $H_{LP}(\omega) \approx 1$ which implies $|H_{LP}(\omega)| \approx 1$.
- $\omega = \omega_c$, then $H(\omega) = \frac{1}{1+j}$ meaning $|H_{LP}(\omega)| = \frac{1}{\sqrt{2}}$.
- $\omega \gg \omega_c$, then $\omega/\omega_c \gg 1$. Therefore $H_{LP}(\omega) \approx -j\frac{\omega_c}{\omega}$ which implies $|H_{LP}(\omega)| \approx \frac{\omega_c}{\omega}$. On a log scale, this means that $\log|H_{LP}(\omega)| \approx \log \omega - \log \omega_c$ explaining behavior of dropping off with slope 1.⁴

⁴If you aren't sure why this is the case, recall that the line $y = mx + b$ has slope m . In this case $y = \log|H_{LP}(\omega)|$ and $x = \log|\omega|$.

Now let's plot the phase of the transfer function $H_{LP}(\omega)$.



$\angle H_{LP}(\omega)$ is very close to 0 for $\omega < 0.1\omega_c$ and $\angle H_{LP}(\omega)$ is approximately $-\frac{\pi}{2}$ for $\omega > 10\omega_c$.

Recall the results of our low-pass filter table. These approximations should illustrate the behavior of the magnitude and phase of our low-pass filter.

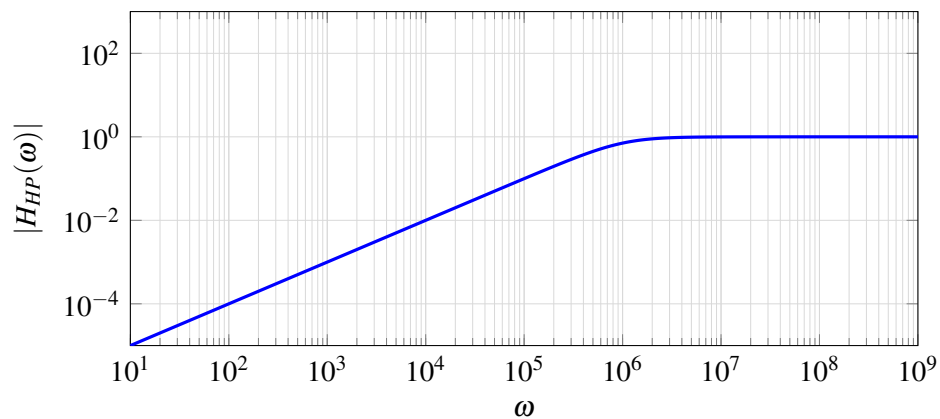
| ω | $H(\omega)$ | $M(\omega)$ | θ |
|---------------|--------------------|-------------|-------------|
| $0.1\omega_c$ | $\frac{1}{1+0.1j}$ | 0.995 | -6° |
| ω_c | $\frac{1}{1+j}$ | 0.71 | -45° |
| $10\omega_c$ | $\frac{1}{1+10j}$ | 0.1 | -84° |

3.2 High-pass Filter

Going back to our RC model of the low-pass filter,

$$H_{HP}(\omega) = \frac{j\omega RC}{1 + j\omega RC} = \frac{j\omega/\omega_c}{1 + j\omega/\omega_c}$$

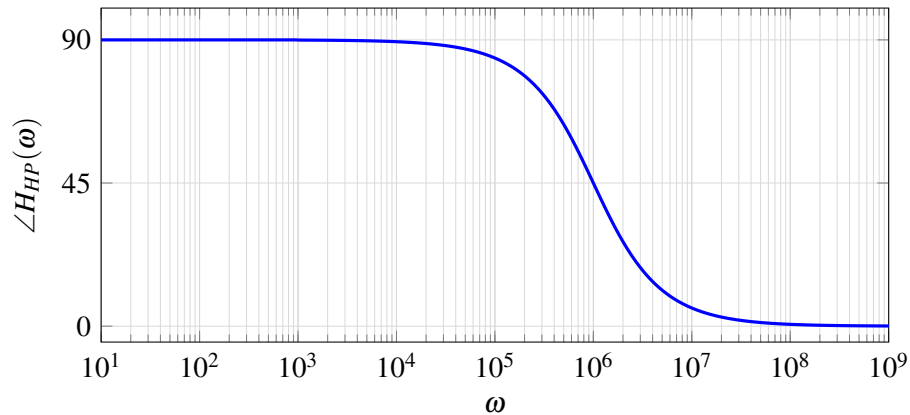
we plot the magnitude of the frequency response assuming $\omega_c = 10^6$.



Note how $|H_{HP}(\omega)|$ rises with slope 1 for $\omega < \omega_c$ and $|H_{HP}(\omega)|$ is approximately 1 after ω_c . To formalize this analysis, we will break it down into cases

- $\omega \ll \omega_c$, then $\omega/\omega_c \ll 1$. Therefore $H_{HP}(\omega) \approx j\frac{\omega}{\omega_c}$ which implies $|H_{HP}(\omega)| \approx \frac{\omega}{\omega_c}$. On a log scale, this means that $\log|H_{HP}(\omega)| \approx \log \omega - \log \omega_c$ explaining behavior of rising with slope 1.
- $\omega = \omega_c$, then $H(\omega) = \frac{j}{1+j}$ meaning $|H_{HP}(\omega)| = \frac{1}{\sqrt{2}}$
- $\omega \gg \omega_c$, then $\omega/\omega_c \gg 1$. Therefore $H_{HP}(\omega) \approx 1$ which implies $|H_{HP}(\omega)| \approx 1$.

Now let's plot the phase of the transfer function $H_{HP}(\omega)$.



$\angle H_{HP}(\omega)$ is very close to $\frac{\pi}{2}$ for $\omega < 0.1\omega_c$ and $\angle H_{HP}(\omega)$ is approximately 0 for $\omega > 10\omega_c$.

Recall the results from our high-pass filter table. These approximations should illustrate the behavior of the magnitude and phase of our high-pass filter.

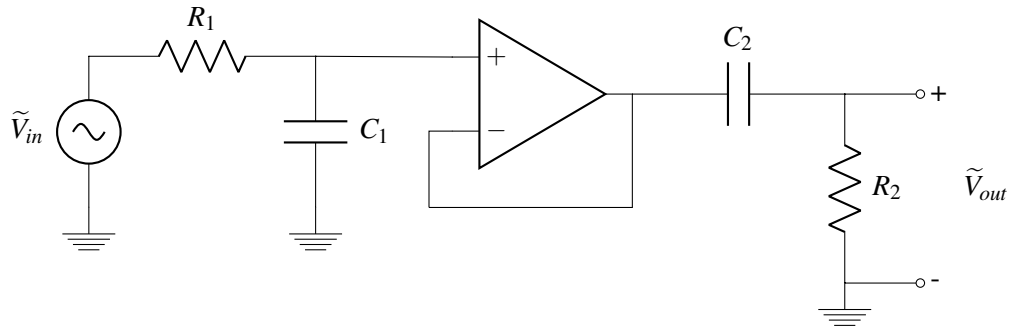
| ω | $H(\omega)$ | $M(\omega)$ | θ |
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| $0.1\omega_c$ | $\frac{0.1j}{1+0.1j}$ | 0.1 | 84° |
| ω_c | $\frac{j}{1+j}$ | 0.71 | 45° |
| $10\omega_c$ | $\frac{10j}{1+10j}$ | 0.995 | 6° |

4 Second Order Filters

Using the intuition that we've gained from analyzing first order filters and their Bode plots, we will move onto more complicated examples.

4.1 Band-Pass Filters

With the knowledge of low-pass filters that block out higher frequencies and high-pass filters that block out lower frequencies, how could we build a filter that lets a range of frequencies through? One idea could be to take the output of the low-pass filter and treat it as an input to the high-pass filter.



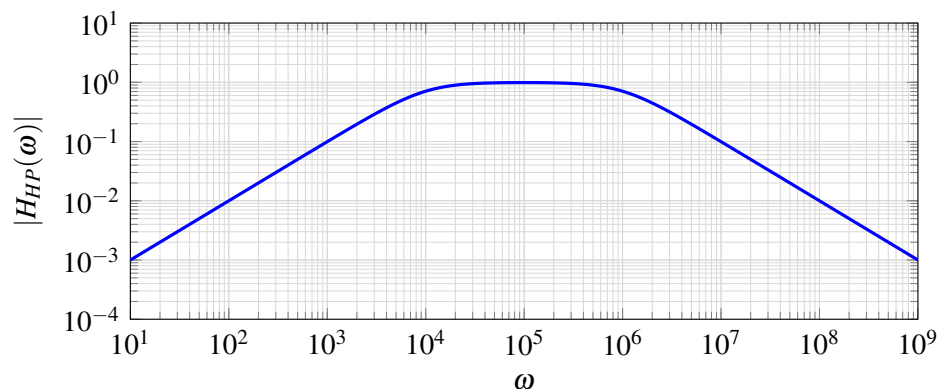
If we were to pick cutoff frequencies such that our desired range is smaller than our low-pass cutoff while being larger than our high-pass cutoff, then we could compute the transfer function of the following circuit and analyze its cutoff frequency.

The transfer function of this circuit is the product of the low-pass and high-pass transfer functions

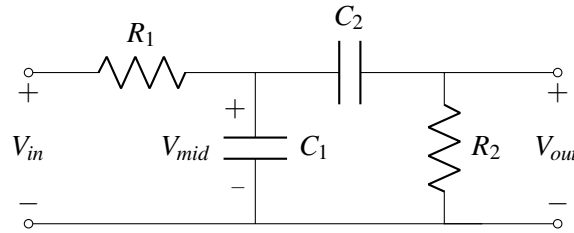
$$H_{BP}(\omega) = H_{LP}(\omega) \cdot H_{HP}(\omega) = \frac{1}{1 + j\omega R_L C_L} \cdot \frac{j\omega R_H C_H}{1 + j\omega R_H C_H}$$

To find the cutoff frequencies of this filter, we can look at the points at which $H(\omega_c) = \frac{1}{\sqrt{2}}$. However, recall that $\omega_{LP} = \frac{1}{R_L C_L}$ and $\omega_{HP} = \frac{1}{R_H C_H}$ and assuming that the low-pass and high-pass frequencies are spaced apart, we can approximate $|H(\omega_L)| \approx \frac{1}{\sqrt{2}} \cdot 1$ and $|H(\omega_H)| \approx 1 \cdot \frac{1}{\sqrt{2}}$.

Therefore, we conclude by saying that the cutoffs for the band-pass filter are identical to the individual cutoffs for the low and high-pass filter. We show a plot of $H(\omega)$ with $\omega_{LP} = 10^{-6}$ and $\omega_{HP} = 10^{-4}$ to give a visual explanation of this idea.



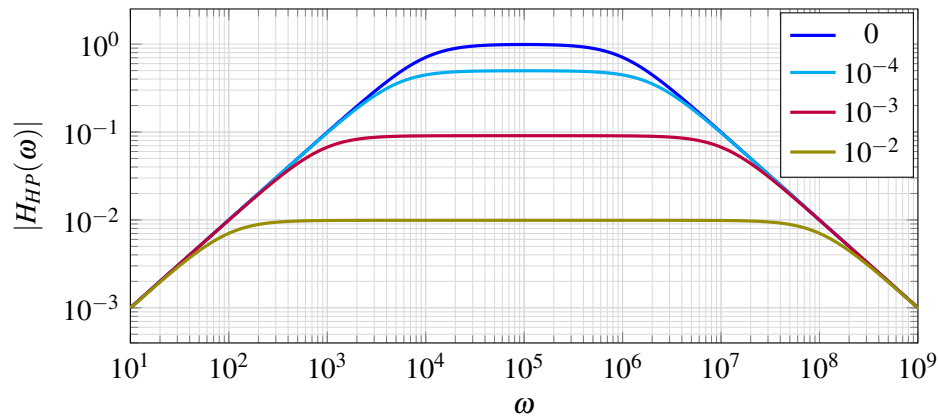
Now the band-pass filter that we built above requires the use of an op-amp. However, what would happen if we instead cascaded the two filters causing a loading effect?



We leave the derivation as an exercise, but computing the transfer function yields

$$H(\omega) = \frac{j\omega R_H C_H}{(1 + j\omega R_L C_L)(1 + j\omega R_H C_H) + j\omega R_L C_H}$$

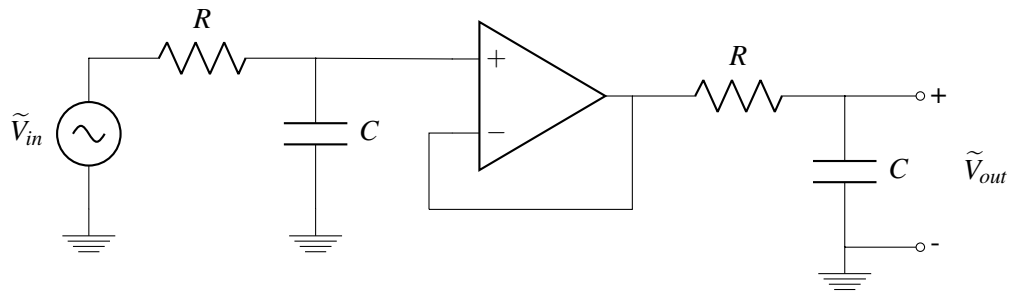
The difference due to loading is a denominator term of $j\omega R_L C_H$. Depending on how large $R_L C_H$ this could have a small or large effect on the circuit. We plot some examples of the band-pass filter with identical low and high cutoff frequencies but different $R_L C_H$ values to show this loading effect.



Note how the maximum value of $H(\omega)$ decreases as $R_L C_H$ increases. In addition, the cutoff frequencies move further and further apart from the original $\omega_{LP} = \frac{1}{R_L C_L}$ and $\omega_{HP} = \frac{1}{R_H C_H}$.

4.2 Low-Pass Filters

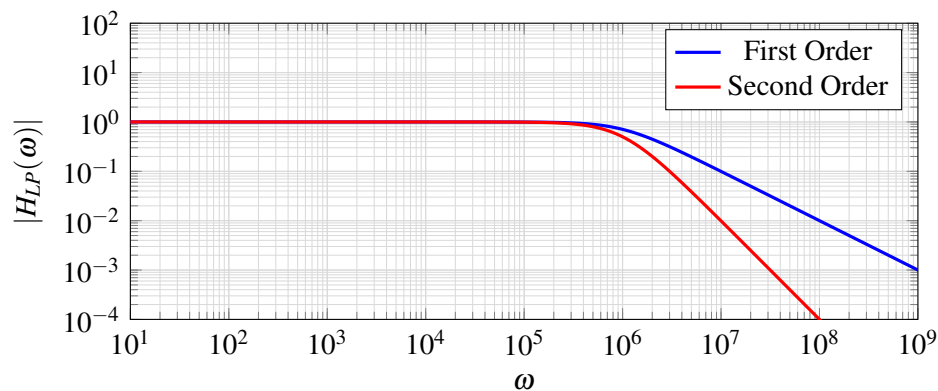
From our analysis of low-pass filters, we saw that the magnitude of the transfer function $H(\omega)$ dropped off by a factor of 10 for each decade of frequency after the cutoff ω_c . While this is a desirable effect, in the ideal case, we would like to build a filter that drops off at a quicker rate after ω_c . Therefore, let's try cascading two low-pass filters of identical cutoff with a buffer in between.



We can compute the transfer function as

$$H_{LP}(\omega) = \frac{1}{(1 + j\omega RC)^2} \quad (2)$$

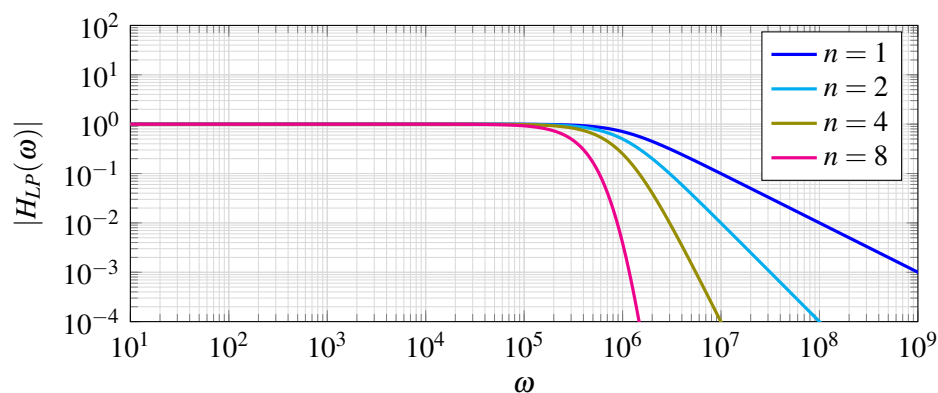
Plotting the magnitude of $H_{LP}(\omega)$, we see that $H(\omega)$ does indeed drop off at a quicker rate with slope 2 after the cutoff ω_c .



In fact, if we were to cascade even more low-pass filters, we approach an ideal low-pass filter in which

$$H(\omega) = \begin{cases} 1 & \omega < \omega_c \\ 0 & \omega \geq \omega_c \end{cases} \quad (3)$$

We show a plot of this effect below. For an n^{th} order filter, we see a dropoff of slope n after the cutoff. We will explore this effect in more detail in the next section.



5 Higher Order Bode Plots

We will now consider a slightly more complex system and, in doing so, see the value in appropriate visualizations for transfer function behavior.

5.1 Rational Transfer Functions

When we write the transfer function of an arbitrary circuit, it always takes the following form. This is called a “rational transfer function.” We also like to factor the numerator and denominator, so that they become easier to work with and plot:

$$H(\omega) = K \cdot \frac{N(\omega)}{D(\omega)} = K \frac{(j\omega)^{N_{z0}} \left(1 + j\frac{\omega}{\omega_{z1}}\right) \left(1 + j\frac{\omega}{\omega_{z2}}\right) \cdots \left(1 + j\frac{\omega}{\omega_{zn}}\right)}{(j\omega)^{N_{p0}} \left(1 + j\frac{\omega}{\omega_{p1}}\right) \left(1 + j\frac{\omega}{\omega_{p2}}\right) \cdots \left(1 + j\frac{\omega}{\omega_{pm}}\right)} \quad (4)$$

Here, we define the constants ω_z as “zeros” and ω_p as “poles.” Note that in standard literature, zeros are defined to be the roots of $N(\omega)$ while poles are the roots of $D(\omega)$.⁵

5.2 “Adding” Bode Plots

For two transfer functions $H_1(\omega)$ and $H_2(\omega)$, if $H(\omega) = H_1(\omega) \cdot H_2(\omega)$,

$$\log|H(\omega)| = \log|H_1(\omega) \cdot H_2(\omega)| = \log|H_1(\omega)| + \log|H_2(\omega)| \quad (5)$$

$$\angle H(\omega) = \angle(H_1(\omega) \cdot H_2(\omega)) = \angle H_1(\omega) + \angle H_2(\omega) \quad (6)$$

As a consequence, when plotting $|H(\omega)|$ on a log-log plot, we can simply plot $|H_1(\omega)|$ and $|H_2(\omega)|$ and add them up. This implies that we will be able to add the slopes of each zero and pole to provide a complete plot. In the next section we provide a further analysis on the meaning of zeros and poles and the idea of adding slopes.

We must be careful, however, to note that in most of our plots, the x -axis does *not* correspond to 0, so we can’t simply “stack” the two plots.

5.3 Decibels

We define the decibel as the following:

$$20\log_{10}(|H(\omega)|) = |H(\omega)| \text{ [dB]}$$

The origin of the decibel comes from looking at the ratio of the output and input power of the system.

$$|H(\omega)| \text{ [dB]} = 10\log\left|\frac{P_{out}}{P_{in}}\right| = 10\log\left|\frac{V_{out}}{V_{in}}\right|^2 = 20\log\left|\frac{V_{out}}{V_{in}}\right|$$

This means that 20dB per decade is equivalent to one order of magnitude. **We won’t be using dB when plotting, but understanding the conversion to dB will help when reading other resources on Bode plots.**

⁵Technically if $s = j\omega$, then the roots of $N(s)$ and $D(s)$ are $-\omega_z$ and $-\omega_p$. However, when plotting Bode plots, we refer to ω_z and ω_p as the zero and pole frequencies.

5.4 Poles and Zeros

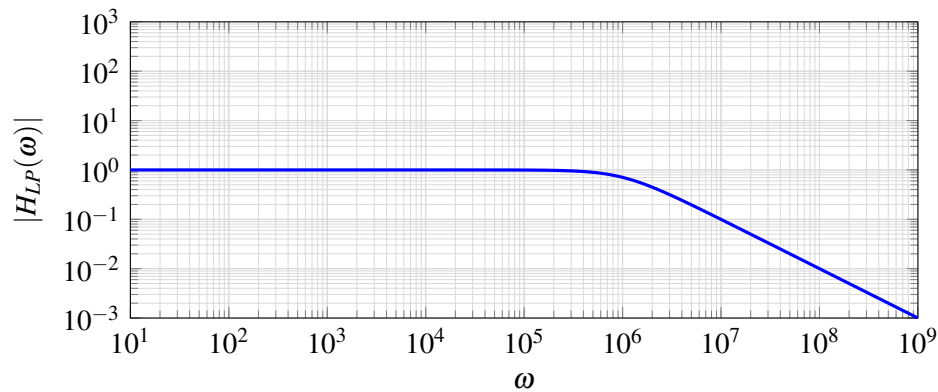
The notion of a **pole** and **zero** frequency is a generalization of the term cutoff frequency.

$$H(\omega) = K \cdot \frac{N(\omega)}{D(\omega)} = K \frac{(j\omega)^{N_{z0}} \left(1 + j\frac{\omega}{\omega_{z1}}\right) \left(1 + j\frac{\omega}{\omega_{z2}}\right) \cdots \left(1 + j\frac{\omega}{\omega_{zn}}\right)}{(j\omega)^{N_{p0}} \left(1 + j\frac{\omega}{\omega_{p1}}\right) \left(1 + j\frac{\omega}{\omega_{p2}}\right) \cdots \left(1 + j\frac{\omega}{\omega_{pm}}\right)} \quad (7)$$

As mentioned in the previous section, frequencies in the numerator ω_{zi} are referred to as zero frequencies whereas the frequencies in the denominator ω_{pi} are the pole frequencies. Let's first look back at a plot of our RC low-pass filter:

$$H(\omega) = \frac{1}{1 + j\omega \cdot 10^6} \quad (8)$$

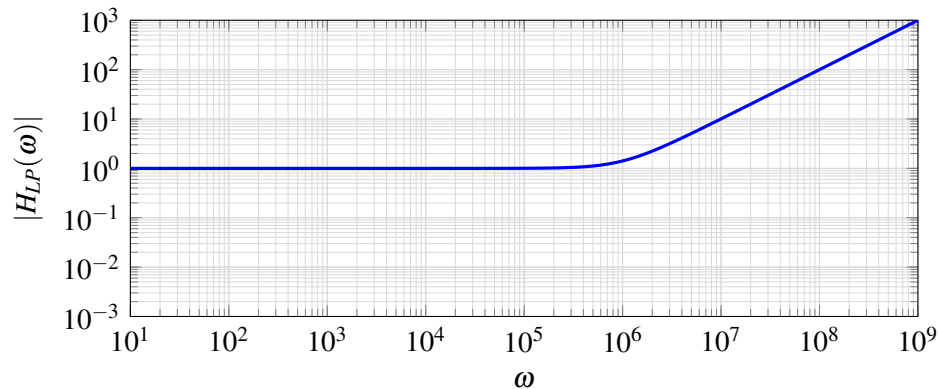
When drawing Bode plots, we claim that the plot drops off with a slope of 1 after a pole ω_p . This transfer function has a single pole at $\omega_p = 10^{-6}$ and as expected, its magnitude Bode plot is of the form



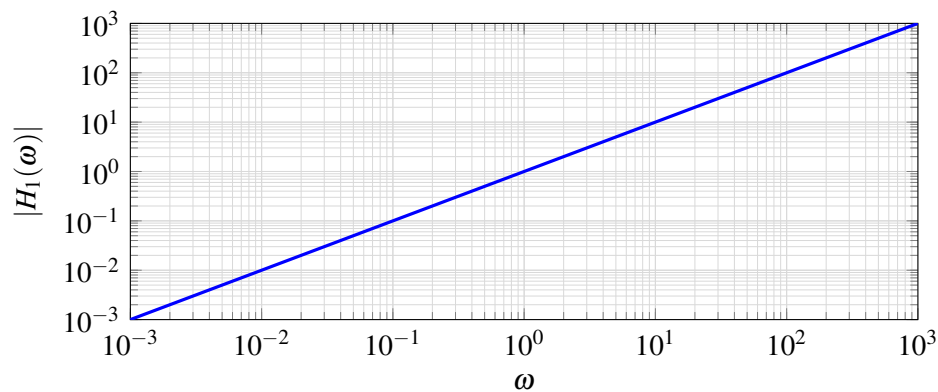
Now let's take a look at a single zero.

$$H(\omega) = 1 + j\omega/\omega_z \quad (9)$$

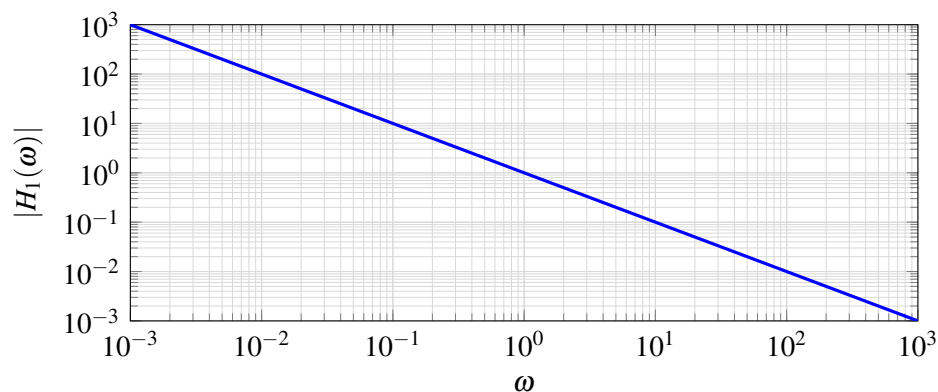
We show that this Magnitude Bode plot rises with a slope of 1 after the zero at ω_z .



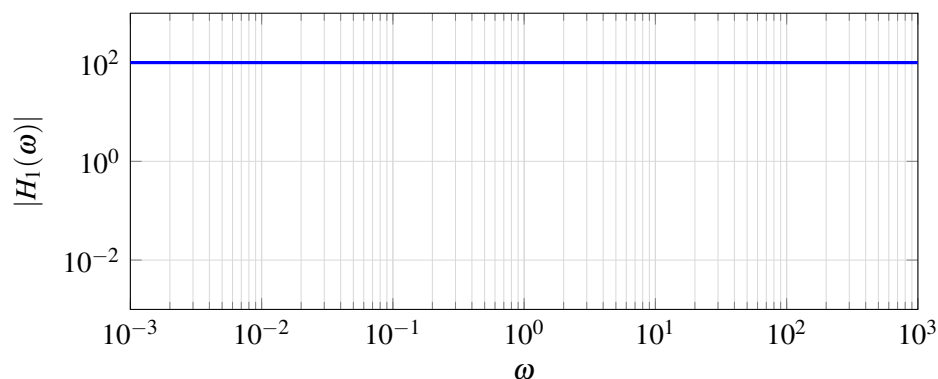
To plot a zero at the origin, recall that $H(\omega) = j\omega$ has magnitude ω and phase 90° . If our transfer function has a zero at the origin, it will start off with a slope of 1.



To plot a pole at the origin, recall that $H(\omega) = \frac{1}{j\omega}$ has magnitude ω and phase -90° . If our transfer function has a zero at the origin, it will start off with a slope of -1 .



Lastly, we show the plot of a constant $K = 100$. As expected, the plot remains constant. This implies that multiplication by K will shift up the entire bode plot up by K .



5.5 Examples

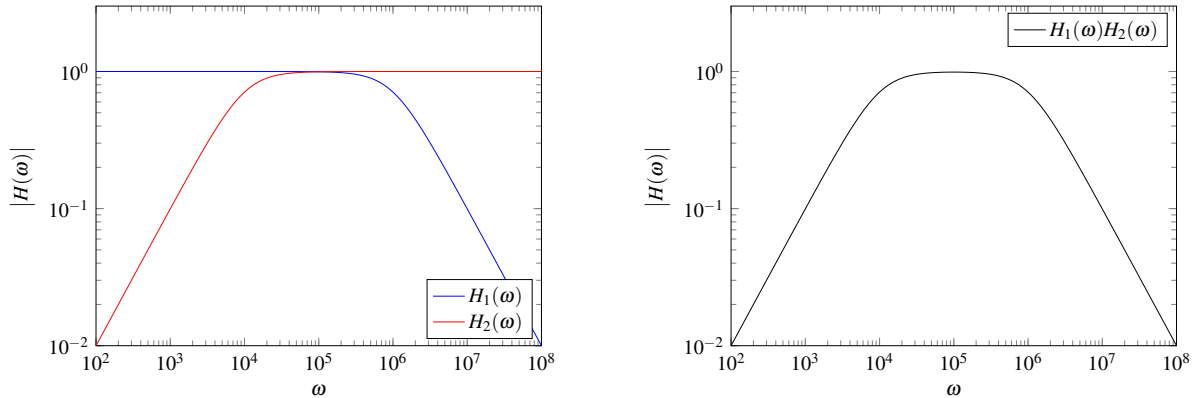
Band-pass Filter

Let's look at some numerical examples. Suppose we have a band-pass filter with the following transfer function

$$H(\omega) = H_{LP}(\omega) \cdot H_{HP}(\omega) = \frac{1}{1 + j\omega/10^6} \cdot \frac{j\omega/10^4}{1 + j\omega/10^4} \quad (10)$$

The cutoff frequency for the high-pass is 10^4 while the cutoff for the low-pass is 10^6 .

Following this procedure of adding plots (with the individual filters on the left and the result on the right), we obtain

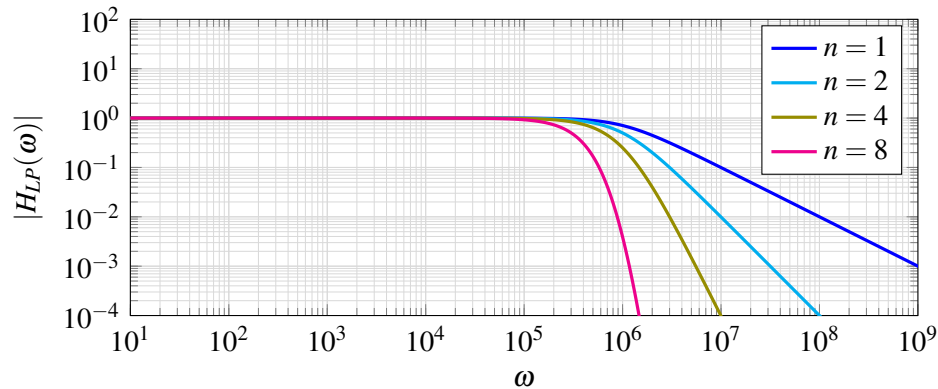


Low-pass Filter

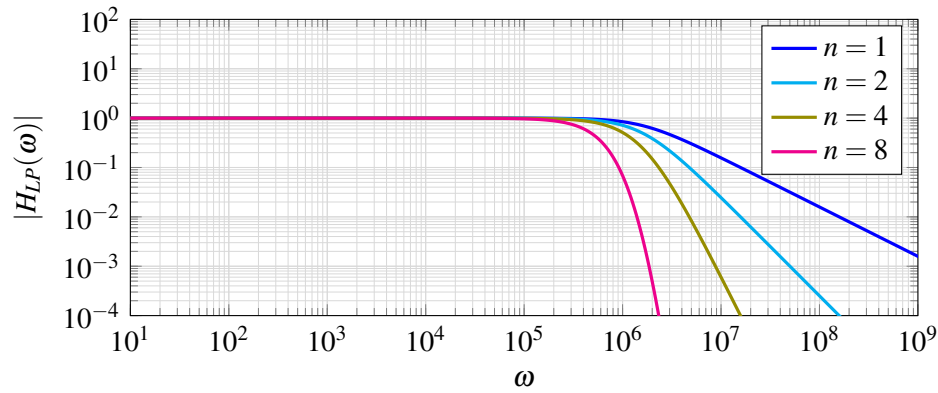
Now let's consider the n^{th} order low-pass filter with transfer function

$$H(\omega) = \frac{1}{(1 + j\omega RC)^n} \quad (11)$$

This transfer function has a pole at ω_{RC} of order n meaning in our Bode-approximation, the slope will start dropping off with a slope of n .



When analyzing the Bode plot however, note that dropoff occurs before $\omega_p = \frac{1}{RC}$. This is because each time we cascade a low-pass filter, the magnitude drops off by a factor of $\frac{1}{\sqrt{2}}$ at ω_p . The Bode approximation is unable to capture this behavior. If we wanted to build something closer to the ideal low-pass filter, we need to shift $\frac{1}{RC}$ to be slightly greater than ω_c . We show a plot below where we set $\omega_p = \frac{1}{RC} = 10^{-6.2}$.



With this slight shift, we see a slight performance improvement, but it is quite expensive with all of the op-amps especially at the $n = 8$ case. There are an entire class of different filter designs each with its own tradeoffs. Some examples that you can look up are the **Butterworth**, **Lattice**, and **Sallen-Key** topologies.

Transfer Function Example

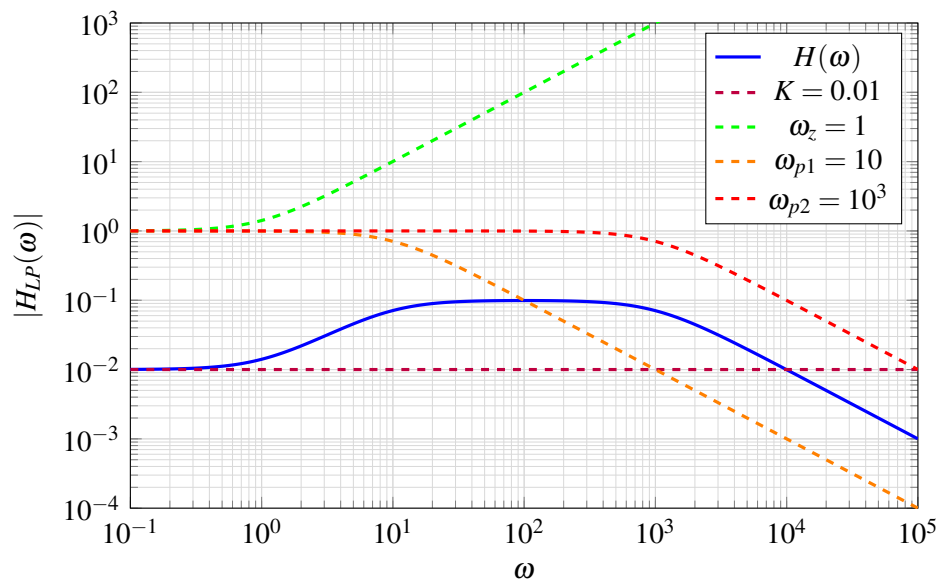
Now let's take a look at the Bode plot of a new transfer function.

$$H(\omega) = 100 \frac{(1 + j\omega)}{(j\omega)^2 + 1010(j\omega) + 10^4} \quad (12)$$

We must first factor it into its rational transfer function form:

$$H(\omega) = 0.01 \frac{(1 + j\omega)}{(1 + j\omega/10)(1 + j\omega/10^3)} \quad (13)$$

With the transfer function in its rational form, we see that $K = 0.01$, $\omega_z = 1$, $\omega_{p1} = 10$, $\omega_{p2} = 10^3$.



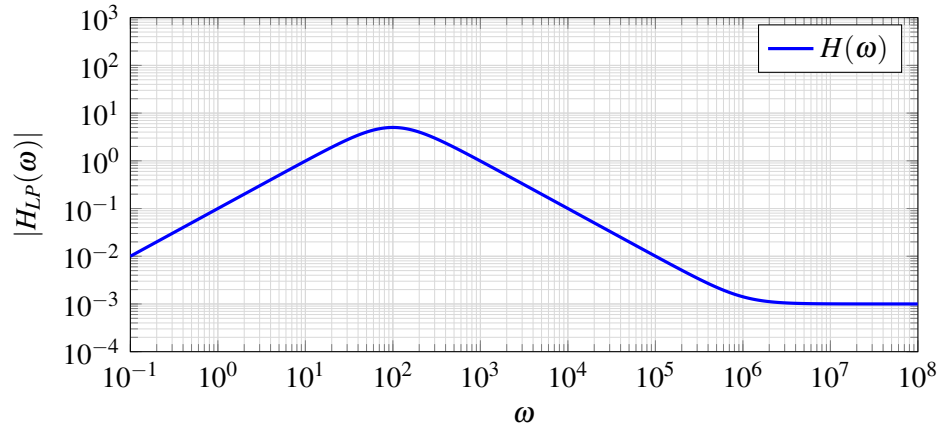
To provide an analysis for this Bode plot, we see that the plot starts off at $K = 0.01$. Then at $\omega_z = 1$, it starts rising with slope 1. When it hits the pole at $\omega_{p1} = 10$, the slope of 1 is cancelled out by the -1 slope that

the pole provides. Then the Bode plot stays constant until $\omega_{p2} = 10^3$ at which it drops off with a slope of 1. We've provided Bode plots of the individual terms to give you a sense of how we "add" Bode plots together.

5.5.1 Zero at the Origin

In our final example, we examine the effects of a zero at the origin. Consider the following transfer function in rational form.

$$H(\omega) = 0.1 \frac{(j\omega)(1 + j\omega/10^6)}{(1 + j\omega/10^2)^2} \quad (14)$$

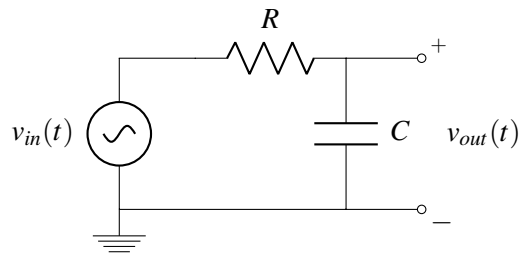


Since there is a zero at the origin, the plot will initially start with a slope of 1. There are no additional zeros or poles before $\omega = 1$, so we can approximate $|H(1)| = K = 0.1$. Then the double pole at $\omega_p = 10^2$ provides a slope of -2 that will cancel out the slope of 1 making the overall slope after ω_p equal to -1 . Lastly, there is a zero at $\omega_z = 10^6$ and we indeed see that the addition of a slope of 1 makes $|H(\omega)|$ remains constant after ω_z .

6 Time Constant

When computing the cutoff frequency for a first order low-pass filter, we noticed that the $\omega_c = \frac{1}{RC} = \frac{1}{\tau}$. In this final section of the note, we draw the connection between time constants and cutoff frequencies.

Recall from the note on differential equations that we defined the time constant of a first-order circuit to be the point at which the response $v_c(t)$ to a constant input was $1 - e^{-1}$ away from its steady state value. With this in mind, let's try plugging in an exponential input $v_{in}(t) = V_0 e^{j\omega t}$ into an RC circuit and see what happens.⁶



The differential equation for this circuit is

$$\frac{d}{dt}v_{out}(t) = \lambda(v_{out}(t) - e^{j\omega t}) \quad (15)$$

for $\lambda = -\frac{1}{\tau}$. In Note 3 we showed that the steady state value of this differential equation is

$$v_{ss}(t) = \frac{-\lambda}{j\omega - \lambda} V_0 e^{j\omega t} \quad (16)$$

Therefore, plugging in for $\lambda = -\frac{1}{\tau}$, it follows that

$$v_{ss}(t) = \frac{1}{1 + j\omega\tau} V_0 e^{j\omega t} \quad (17)$$

Notice that $H(\omega) = \frac{1}{1 + j\omega\tau}$ and the cutoff arises naturally as $\omega_c = \frac{1}{\tau}$. We can also realize that at steady state, $H(\omega)$ is in fact the eigenvalue for the differential equation with eigenfunction $e^{j\omega t}$. This is a crucial connection between differential equations and the frequency response of a linear system that you will see in later half of the course and in courses like EE120.

⁶We should be inputting $v_{in}(t) = V_0 \cos(\omega t)$ but we choose $e^{j\omega t}$ since it provides the same result while simplifying the math.

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