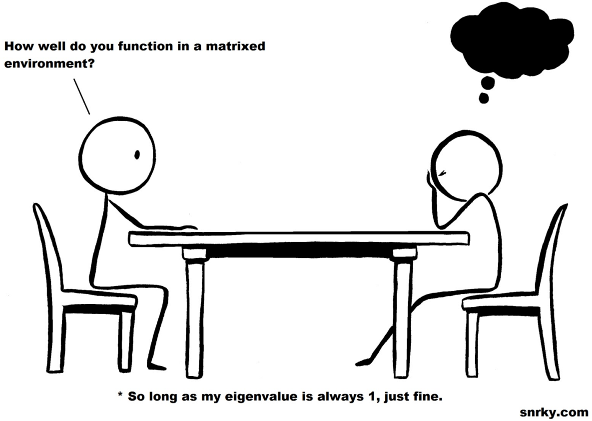
While waiting for Berkeley time, here's a nice video with some new jargon for you:

- https://www.youtube.com/watch?v=Ac7G7xOG2Ag



EE16A

more Page Rank, Eigenvalues and Eigenspaces

Admin

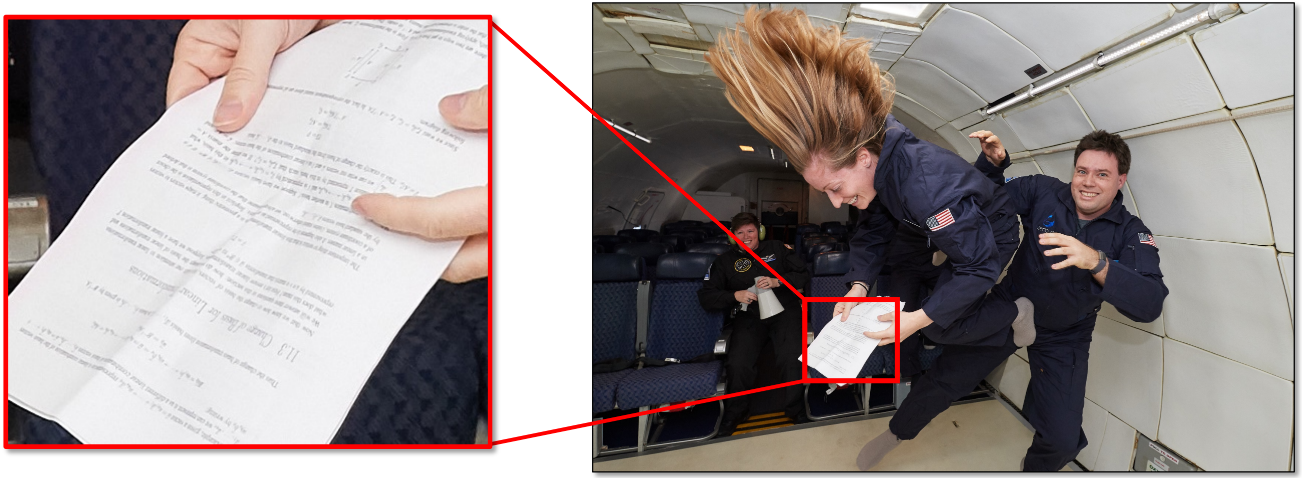
- Q&A is open for students to upvote questions and comment/answer other people's questions

Admin

- Q&A is open for students to upvote questions and comment/answer other people’s questions
- Tues is last lecture of Module 1 (Linear Algebra)
  - Midterm #1 (Monday, Mar. 1, 7-9 PM PT) - start studying now!

Admin

- Q&A is open for students to upvote questions and comment/answer other people's questions
- Tues is last lecture of Module 1 (Linear Algebra)
  - Midterm #1 (Monday, Mar. 1, 7-9 PM PT) - start studying now!
    - Review: Lectures, Discussions, Labs, read the Notes!!



Summary: Eigenvalues and Eigenvectors (and Eigenspaces)

$\vec{x}$

vector space associated w/ $\lambda, \vec{x}$

↳ scaling factor

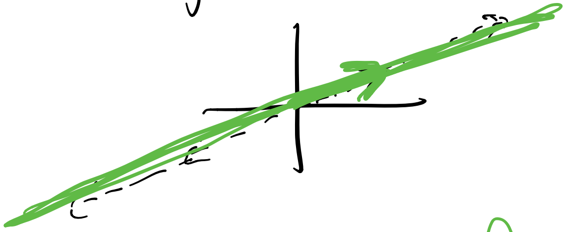
↳ vectors that don't change direction when tranformed by a matrix $A \rightarrow A\vec{x}$

$$A\vec{x} = \lambda\vec{x}$$

matrix

scalar

vector



$$(A - \lambda I)\vec{x} = \vec{0}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\boxed{\det(A) = |A| = ad - bc}$$

nullspace → maps things to 0

eigenspace → preserves vectors (w/ scaling)

Matrix transformations

What does the matrix do?

Scaling by 2x

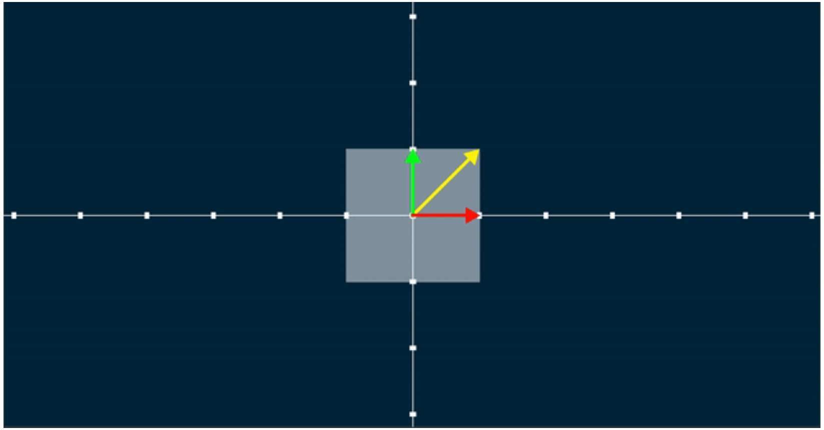
What is the A matrix?

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

What are its eigenvectors?

$\mathbb{R}^2$

What are its eigenvalues?



Matrix transformations

What does the matrix do?

scaling in y only

What is the A matrix?

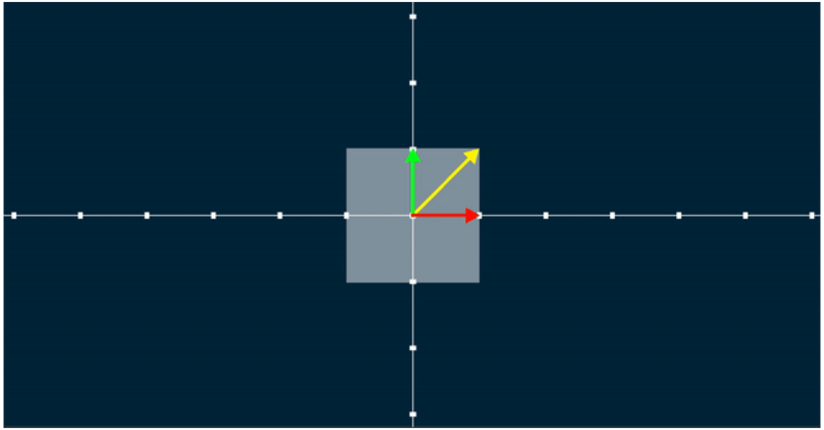
$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

What are its eigenvectors?

red green

What are its eigenvalues?

1 2



Matrix transformations

What does the matrix do?

shear

What is the A matrix?

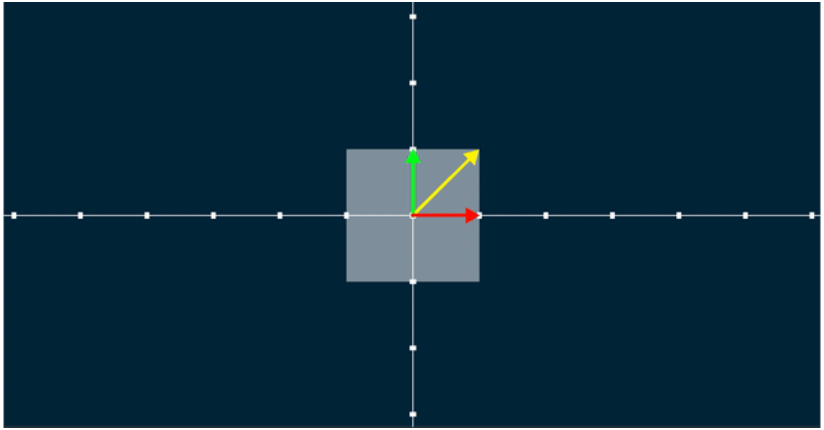
$\begin{bmatrix} ? \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$

What are its eigenvectors?

red

What are its eigenvalues?

1



Matrix transformations

What does the matrix do?

rotating by 90°

What is the A matrix?

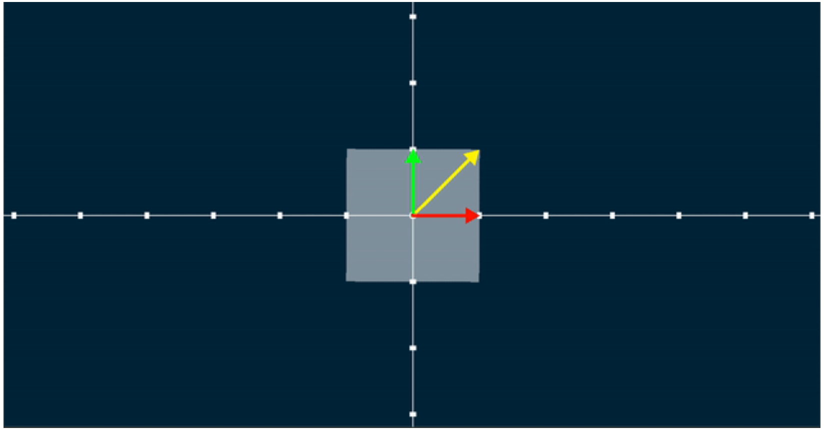
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

What are its eigenvectors?

none

What are its eigenvalues?

none



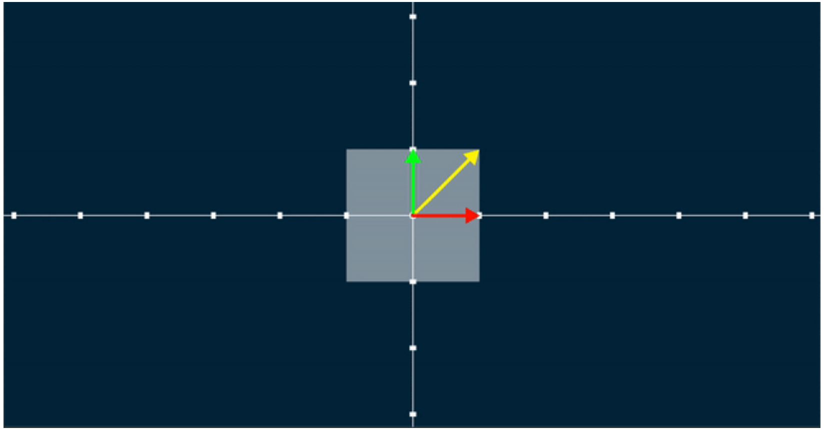
Matrix transformations

What does the matrix do?

What is the A matrix?

What are its eigenvectors?

What are its eigenvalues?

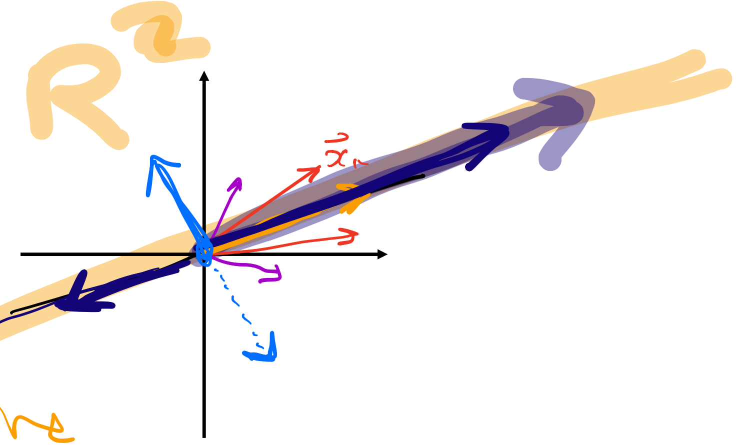


Matrix transformations

For a matrix that flips (reflects) vectors along a line:

What is the A matrix?

$A = [\ ?\ ]$



What are its eigenvectors?

vector perpendicular

any vector along flip line

What are its eigenvalues?

-1 $\quad$ 1

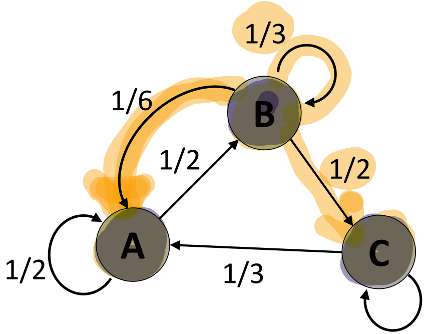
Eigenspace? R1

Last time: PageRank eigenvectors and eigenvalues

THE \$25,000,000,000* EIGENVECTOR
THE LINEAR ALGEBRA BEHIND GOOGLE

KURT BRYAN† AND TANYA LEISE‡

Abstract. Google's success derives in large part from its PageRank algorithm, which ranks the importance of webpages according to an eigenvector of a weighted link matrix. Analysis of the PageRank formula provides a



$$\vec{x} = \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix}$$

$\vec{x}(0)$

$\vec{x}(1)$

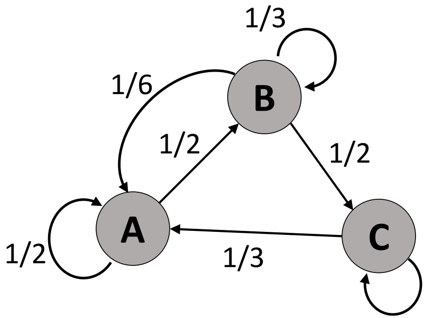
Last time: PageRank eigenvectors and eigenvalues

$25B is good eigenVALUE!

THE $25,000,000,000* EIGENVECTOR
THE LINEAR ALGEBRA BEHIND GOOGLE

KURT BRYAN† AND TANYA LEISE‡

Abstract. Google's success derives in large part from its PageRank algorithm, which ranks the importance of webpages according to an eigenvector of a weighted link matrix. Analysis of the PageRank formula provides a



Last time: PageRank eigenvectors and eigenvalues

$25B is good eigenVALUE!

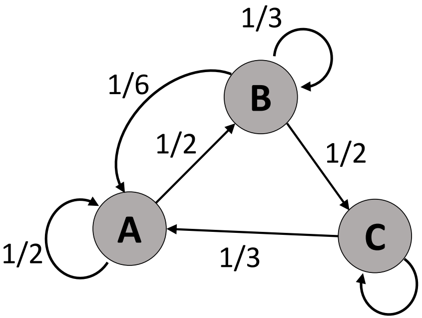
THE \$25,000,000,000* EIGENVECTOR
THE LINEAR ALGEBRA BEHIND GOOGLE

KURT BRYAN† AND TANYA LEISE‡

Abstract. Google's success derives in large part from its PageRank algorithm, which ranks the importance of webpages according to an eigenvector of a weighted link matrix. Analysis of the PageRank formula provides a

$$\vec{x} = \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix} \begin{matrix} \leftarrow 99\% \\ \leftarrow 1\% \\ \leftarrow 0\% \end{matrix}$$

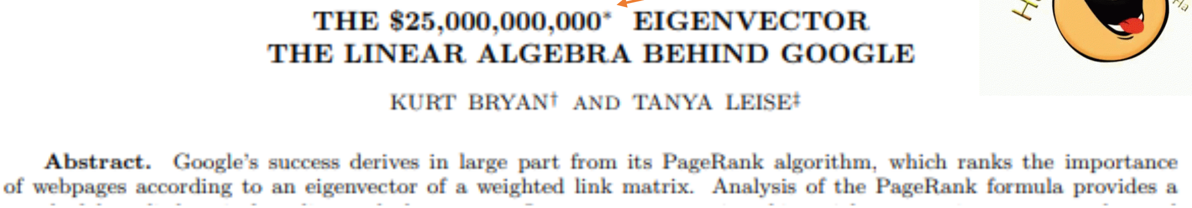
$$A = \underbrace{\begin{bmatrix} \quad ? \quad \end{bmatrix}}_{\text{eig vec/val?}}$$



What do the eigenvectors and eigenvalues for PageRank mean?

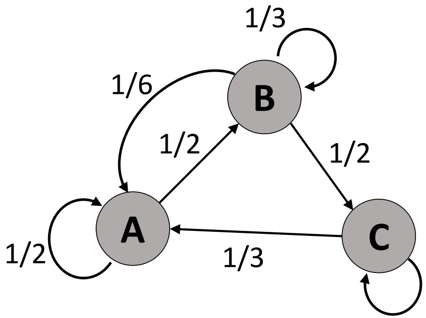
Last time: PageRank eigenvectors and eigenvalues

$25B is good eigenVALUE!


THE $25,000,000,000* EIGENVECTOR
THE LINEAR ALGEBRA BEHIND GOOGLE

KURT BRYAN† AND TANYA LEISE‡

Abstract. Google's success derives in large part from its PageRank algorithm, which ranks the importance of webpages according to an eigenvector of a weighted link matrix. Analysis of the PageRank formula provides a



What do the eigenvectors and eigenvalues for PageRank mean?

Describes behavior of system after many timesteps, in order to find “popularity” of each site.

Eigenvectors make good basis sets

Today we will show that the eigenvectors of a matrix form a basis set, and why it's a useful basis set!

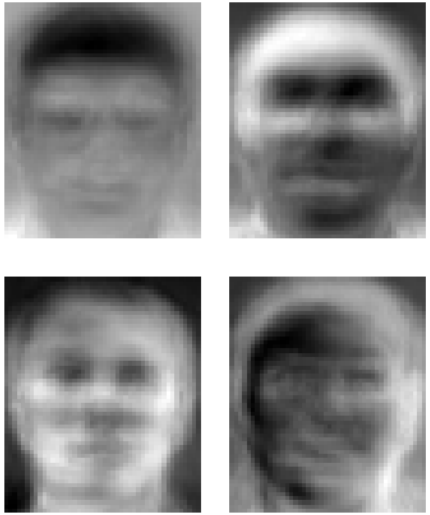
Eigenvectors make good basis sets

Today we will show that the eigenvectors of a matrix form a basis set, and why it's a useful basis set!

Human face recognition uses *eigenfaces*

- Make a vector space of face images
- Find a basis set (eigenvectors) of all images
- This <u>*smaller set*</u> of eigenfaces can be used to represent all faces by linear combinations

Some eigenfaces

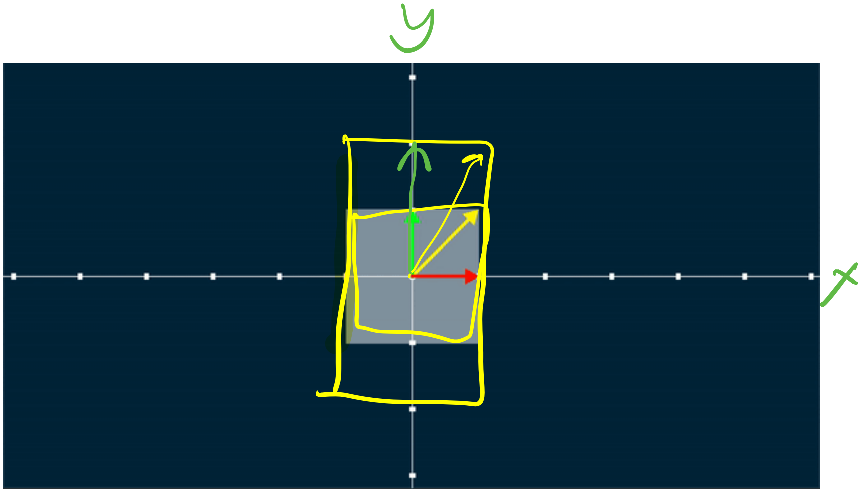


Sirovich and Kirby (1987)

Back to example matrix transformation: scaling y

Transformation is scaling of y component of vector by 2x:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$



eigenvalues

eigenvectors

$$\lambda_1 = 1, \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 2, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Back to example matrix transformation: scaling y

Transformation is scaling of y component of vector by 2x:

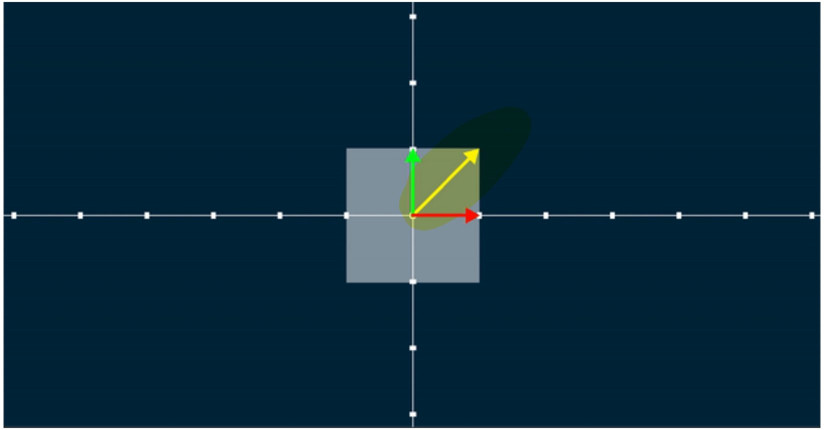
$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

eigenvalues

$$\lambda_1 = 1, \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 2, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

eigenvectors



Don't always have intuition/graph, so need to form a procedure to calculate the Eigenvalues/vectors!

Transformation is scaling of y component of vector by 2x:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A\begin{bmatrix}1\\0\end{bmatrix} = 1 \cdot \begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \quad \checkmark$$

$$\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix} \quad \checkmark$$

$$= 2 \cdot \begin{bmatrix}0\\1\end{bmatrix}$$

$$\rightarrow \boxed{A\vec{v} = \lambda\vec{v}}$$

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix}1-\lambda & 0\\0 & 2-\lambda\end{bmatrix}\right) = 0$$

$$\rightarrow \boxed{(1-\lambda)(2-\lambda) - \cancel{0} = 0}$$

$$\lambda = 1 \qquad \lambda = 2$$

$$\underline{(A - \lambda I)} = \begin{bmatrix}1-\lambda & 0\\0 & 2-\lambda\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

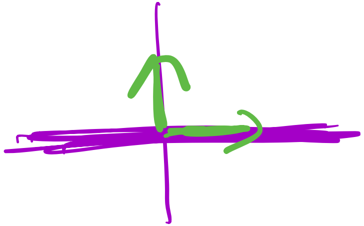
$v_1 = \begin{bmatrix}1\\0\end{bmatrix}$ $\quad v_2 = \begin{bmatrix}0\\1\end{bmatrix}$

$$\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}v\end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{cc|c}0&1&0\\0&0&0\end{array}\right]$$

$\left.v_2 = 0\right\}\ \vec{v}_1 = \begin{bmatrix}\alpha\\0\end{bmatrix}$

eigenspace: $\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$



$$\vec{v}_3 = \begin{bmatrix}1\\1\end{bmatrix} = \underbrace{1\cdot\vec{v}_1 + 1\cdot\vec{v}_2}$$

$$\underline{A\vec{v}_3} = A(\vec{v}_1 + \vec{v}_2)$$

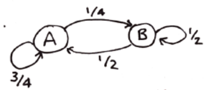
$$= \underbrace{A\vec{v}_1} + A\vec{v}_2$$

$$= 1\cdot\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} \checkmark$$

$A^7\vec{v}_3 = ?$

Let's return to our A matrix representing a Linear Dynamical System (e.g. Page Rank, pumps)

Ex.



we want to look at behaviour over time!

E.g. If there's a steady state, then

$\vec{x}_{ss} = A \cdot \vec{x}_{ss}$ eigenvector with eigenvalue 1

since it matches form $\boxed{A \cdot \vec{x} = \lambda \vec{x}}$ ← eigvalue; eigenvector

state transition matrix $Q = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$ state vector $\vec{x} = \begin{bmatrix} x_A \\ x_B \end{bmatrix}$ ← # people in A, # people in B

Is it conservative? yes

Want to find eigvals/vecs? How many? at most 2 since 2 "directions"

From **first principles:** "from scratch"

Want $\lambda, \vec{x}$ such that $Q \cdot \vec{x} = \lambda \vec{x}$

↑ too many unknowns? hmm...

$Q \cdot \vec{x} = \lambda I \vec{x}$ (match dimensions)

$Q \cdot \vec{x} - \lambda I \vec{x} = \vec{0}$

$(Q - \lambda I)\vec{x} = \vec{0}$ So task is to find Null$(Q - \lambda I)$

$$Q - \lambda I = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (3/4 - \lambda) & 1/2 \\ 1/4 & (1/2 - \lambda) \end{bmatrix}$$

} How do i find nullspace if i don't know λ?

We will use determinant to

① find λ ← may be more than 1

② then find $\vec{x}$

recall: if $\det(A) = 0$ → not invertible & nullspace exists

If there is a non-trivial nullspace, then $\det(Q - \lambda I) = 0$

recall: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$ → $(3/4 - \lambda)(\frac{1}{2} - \lambda) - \frac{1}{4}(\frac{1}{2}) = 0$

expand: $\frac{3}{8} - \frac{3}{4}\lambda - \frac{1}{2}\lambda + \lambda^2 - \frac{1}{8} = 0$

factor: $\frac{1}{4} - \frac{5}{4}\lambda + \lambda^2 = 0$

"characteristic Polynomial" → $\boxed{(\lambda - 1/4)(\lambda - 1) = 0}$ ← 2 sol'ns

So there is a non-trivial nullspace when $\lambda_1 = \frac{1}{4}$ and $\lambda_2 = 1$ ← 2 eigenvalues

Now i can find eigenvectors by plugging each eigenvalue into $Q - \lambda I$ and finding it's nullspace:

$\boxed{\lambda_1 = 1/4}$: $Q - \lambda I = \begin{bmatrix} (3/4 - \lambda) & 1/2 \\ 1/4 & (1/2 - \lambda) \end{bmatrix} = \begin{bmatrix} 3/4 - 1/4 & 1/2 \\ 1/4 & 1/2 - 1/4 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 1/4 \end{bmatrix}$

to find Null$\left(\begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 1/4 \end{bmatrix}\right)$ → $\left[\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 0 \end{array}\right]$ → $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ basic, free

So $x_A + x_B = 0 \to x_A = -x_B$, x_B free

1st Eigenvector $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ spans Nullspace w $\lambda = 1/4$

could write as Null() = span$\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ ← This is called the Eigenspace of Q for λ_1

$\boxed{\lambda_2 = 1}$: $Q - \lambda I = \begin{bmatrix} (3/4 - 1) & 1/2 \\ 1/4 & (1/2 - 1) \end{bmatrix} = \begin{bmatrix} -1/4 & 1/2 \\ 1/4 & -1/2 \end{bmatrix}$

Find Null$(Q - \lambda I)$: $\left[\begin{array}{cc|c} -1/4 & 1/2 & 0 \\ 1/4 & -1/2 & 0 \end{array}\right]$ → $\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ basic, free

so $x_A = 2x_B$, x_B free

2nd Eigenvector $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ Spans Null$(Q - \lambda I)$ when $\lambda = 1$

We have: $\boxed{\begin{array}{l} \lambda_1 = \frac{1}{4}, \ \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ \lambda_2 = 1, \ \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{array}}$ Eigvals/vecs!

Now what should I do w them?

Let's look at some other vector $\vec{x}_0$ (not an eigvec):

Eigenvecs form a basis for $\mathbb{R}^2$, are lin. indep., so I can write $\vec{x}$ as lin comb.

$\vec{x}_0 = \alpha_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2$

What happens after 1 timestep? $\vec{x}(1) = A\vec{x} = A(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2)$ scalars

$= \alpha_1 A\vec{v}_1 + \alpha_2 A\vec{v}_2$

$= \alpha_1 \lambda_1 \vec{v}_1 + \alpha_2 \lambda_2 \vec{v}_2$

after 2 timesteps? $\vec{x}(2) = A\vec{x}(1) = A(\alpha_1 \lambda_1 \vec{v}_1 + \alpha_2 \lambda_2 \vec{v}_2)$

$= \alpha_1 \lambda_1 A\vec{v}_1 + \alpha_2 \lambda_2 A\vec{v}_2$

$= \alpha_1 \lambda_1^2 \vec{v}_1 + \alpha_2 \lambda_2^2 \vec{v}_2$

after t timesteps? $\vec{x}(t) = \alpha_1 \lambda_1^t \vec{v}_1 + \alpha_2 \lambda_2^t \vec{v}_2$

Very easy compute!

What happens as $t \to \infty$?

If $\lvert\lambda\rvert < 1$	$\lambda^\infty \to 0$	that term dies
$\lambda = 1$	$\lambda^\infty \to 1$	steady state
$\lvert\lambda\rvert > 1$	$\lambda^\infty \to \infty$	explode

For our example: $\lambda_1 = \frac{1}{4}$ $\lambda_2 = 1$

So $\vec{x}(t) = \cancel{\alpha_1 \left(\tfrac{1}{4}\right)^t \vec{v}_1} + \alpha_2 (1)^t \vec{v}_2$ $\to 0$ as $t \to \infty$

$$\boxed{\lim_{t \to \infty} \vec{x}(t) = \alpha_2 \vec{v}_2}$$

System converges towards $\vec{v}_2$ so $\vec{x}(\infty) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

website A is 2x more "popular"

We used eiganalysis to find behavior after long time (easily)

Will the system always converge / settle? No! Ex: $A \underset{1}{\overset{1}{\rightleftarrows}} B$ just flip flops

Is steady state @ $t = -\infty$ same as $t = \infty$? No, backwards time is a different system.

Can I have more than 1 steady state? Yes

Recall: Nullspace: turns things to $\vec{0}$ * Nullspace $(Q - \lambda I) \neq$ null (Q)
Eigenspace: keeps things preserved (but scaled)

What about higher dimensions $A^{n \times n}$? ← still must be square.
↳ In general, $A^{N \times N}$ has N roots (eigvals)

What if we get $\lambda = 0$? Then lin. dep., A not invertible.

Are eigenvectors unique? → No, since $\alpha [\vec{v}]$ is also eigvec.
→ Eigvec purpose is to give direction
→ But <u>eigenspace</u> $\text{span}\{[\vec{v}]\}$ is <u>unique</u>

Proof: Properties of eigvals/vecs:

Theorem Let A be an $n \times n$ matrix, and $\lambda_1, \lambda_2 \ldots \lambda_n$ are its (distinct) eigen~~vectors~~values

$\lambda_i \neq \lambda_j$ for all i, j ← none are equal

Let $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ be the corresponding eigenvectors.

$$A \cdot \vec{v}_1 = \lambda_1 \vec{v}_1$$

Then: $\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n$ form a basis for $\mathbb{R}^n$

How to prove? Full proof in notes, but we will do $\mathbb{R}^2$ here

For $\mathbb{R}^2$ case, $n = 2$ so

Theorem $A^{2\times2}$, λ_1, λ_2 are eigvals and $\vec{v}_1, \vec{v}_2$ are eigvecs, $\lambda_1 \neq \lambda_2$

Show: $\vec{v}_1, \vec{v}_2$ form a basis for $\mathbb{R}^2$ Try proof on your own!

Proof:

Known:

$A \cdot \vec{v}_1 = \lambda_1 \vec{v}_1$

$A \cdot \vec{v}_2 = \lambda_2 \vec{v}_2$

To show: $\vec{v}_1, \vec{v}_2$ form a basis ← def'n

do both → ① $\vec{v}_1, \vec{v}_2$ are lin. indep.

(2nd will be easy after 1st) → ② $\vec{v}_1, \vec{v}_2$ span all of $\mathbb{R}^2$ (any vector in $\mathbb{R}^2$ can be written as lin. comb)

① How to show lin. indep.?

Let's assume they are dep. then look for contradiction

Proof by contradiction:

If possible, let $\vec{v}_1, \vec{v}_2$ be lin. dependent:

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 = \vec{0} \quad (\text{not all } \alpha_i = 0)$$

What else do we know?

→ $\vec{v}_1, \vec{v}_2$ are not $\vec{0}$ because they're eig vecs

$$\vec{v}_1 = -\frac{\alpha_2}{\alpha_1} \cdot \vec{v}_2 \; *$$

want to use A now to relate to knowns

multiply eq'n by A

$$A\vec{v}_1 = A\left(-\frac{\alpha_2}{\alpha_1} \cdot \vec{v}_2\right)$$

$$A \cdot \vec{v}_1 = -\frac{\alpha_2}{\alpha_1} A\vec{v}_2$$

$$\boxed{} \; A \cdot \vec{v}_1 = \left(-\frac{\alpha_2}{\alpha_1}\right) \lambda_2 \vec{v}_2$$

But $A \cdot \vec{v}_1 = \lambda_1 \vec{v}_1$ also ← plug in *

$$= \lambda_1 \left(-\frac{\alpha_2}{\alpha_1} \cdot \vec{v}_2\right)$$

$$A \cdot \vec{v}_1 = -\lambda_1 \frac{\alpha_2}{\alpha_1} \cdot \vec{v}_2$$

compare to $\boxed{}$ $A \cdot \vec{v}_1 = -\lambda_2 \frac{\alpha_2}{\alpha_1} \cdot \vec{v}_2$

$\lambda_2 = \lambda_1$ contradiction! yay! ☺

Therefore, $\vec{v}_1$ and $\vec{v}_2$ must be lin. independent! (proves ①)

Now ②: can I have a set of more than 2 vectors in 2D space that are non-zero lin. dep.? No! so $\vec{v}_1, \vec{v}_2$ must span $\mathbb{R}^2$ & form a basis for $\mathbb{R}^2$

QED