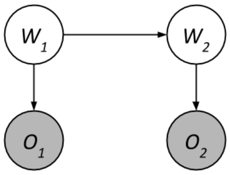
CS 188
Fall 2021

Regular Discussion 9 Solutions

1 HMMs

Consider the following Hidden Markov Model.



W_1	$P(W_1)$
0	0.3
1	0.7

W_t	W_{t+1}	$P(W_{t+1}\|W_t)$
0	0	0.4
0	1	0.6
1	0	0.8
1	1	0.2

W_t	O_t	$P(O_t\|W_t)$
0	a	0.9
0	b	0.1
1	a	0.5
1	b	0.5

Suppose that we observe $O_1 = a$ and $O_2 = b$.
Using the forward algorithm, compute the probability distribution $P(W_2|O_1 = a, O_2 = b)$ one step at a time.

(a) Compute $P(W_1, O_1 = a)$.

$P(W_1, O_1 = a) = P(W_1)P(O_1 = a|W_1)$
$P(W_1 = 0, O_1 = a) = (0.3)(0.9) = 0.27$
$P(W_1 = 1, O_1 = a) = (0.7)(0.5) = 0.35$

(b) Using the previous calculation, compute $P(W_2, O_1 = a)$.

$P(W_2, O_1 = a) = \sum_{w_1} P(w_1, O_1 = a)P(W_2|w_1)$
$P(W_2 = 0, O_1 = a) = (0.27)(0.4) + (0.35)(0.8) = 0.388$
$P(W_2 = 1, O_1 = a) = (0.27)(0.6) + (0.35)(0.2) = 0.232$

(c) Using the previous calculation, compute $P(W_2, O_1 = a, O_2 = b)$.

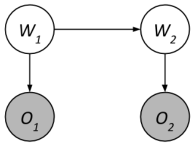
$P(W_2, O_1 = a, O_2 = b) = P(W_2, O_1 = a)P(O_2 = b|W_2)$
$P(W_2 = 0, O_1 = a, O_2 = b) = (0.388)(0.1) = 0.0388$
$P(W_2 = 1, O_1 = a, O_2 = b) = (0.232)(0.5) = 0.116$

(d) Finally, compute $P(W_2|O_1 = a, O_2 = b)$.

Renormalizing the distribution above, we have
$P(W_2 = 0|O_1 = a, O_2 = b) = 0.0388/(0.0388 + 0.116) \approx 0.25$
$P(W_2 = 1|O_1 = a, O_2 = b) = 0.116/(0.0388 + 0.116) \approx 0.75$

2 Particle Filtering

Let's use Particle Filtering to estimate the distribution of $P(W_2|O_1 = a, O_2 = b)$. Here's the HMM again.



W_1	$P(W_1)$
0	0.3
1	0.7

W_t	W_{t+1}	$P(W_{t+1}|W_t)$
0	0	0.4
0	1	0.6
1	0	0.8
1	1	0.2

W_t	O_t	$P(O_t|W_t)$
0	a	0.9
0	b	0.1
1	a	0.5
1	b	0.5

We start with two particles representing our distribution for W_1.
$P_1 : W_1 = 0$
$P_2 : W_1 = 1$
Use the following random numbers to run particle filtering:

$$[0.22, 0.05, 0.33, 0.20, 0.84, 0.54, 0.79, 0.66, 0.14, 0.96]$$

(a) Observe: Compute the weight of the two particles after evidence $O_1 = a$.
$w(P_1) = P(O_t = a|W_t = 0) = 0.9$
$w(P_2) = P(O_t = a|W_t = 1) = 0.5$

(b) Resample: Using the random numbers, resample P_1 and P_2 based on the weights.
We now sample from the weighted distribution we found above. Using the first two random samples, we find:
$P_1 = sample(weights, 0.22) = 0$
$P_2 = sample(weights, 0.05) = 0$

(c) Predict: Sample P_1 and P_2 from applying the time update.
$P_1 = sample(P(W_{t+1}|W_t = 0), 0.33) = 0$
$P_2 = sample(P(W_{t+1}|W_t = 0), 0.20) = 0$

(d) Update: Compute the weight of the two particles after evidence $O_2 = b$.
$w(P_1) = P(O_t = b|W_t = 0) = 0.1$
$w(P_2) = P(O_t = b|W_t = 0) = 0.1$

(e) Resample: Using the random numbers, resample P_1 and P_2 based on the weights.
Because both of our particles have $X = 0$, resampling will still leave us with two particles with $X = 0$.
$P_1 = 0$
$P_2 = 0$

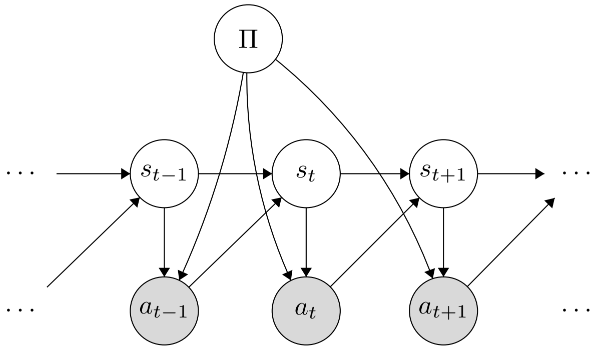
(f) What is our estimated distribution for $P(W_2|O_1 = a, O_2 = b)$?
$P(W_2 = 0|O_1 = a, O_2 = b) = 2/2 = 1$
$P(W_2 = 1|O_1 = a, O_2 = b) = 0/2 = 0$

3 Particle Filtering Apprenticeship

We are observing an agent's actions in an MDP and are trying to determine which out of a set $\{\pi_1, \ldots, \pi_n\}$ the agent is following. Let the random variable Π take values in that set and represent the policy that the agent is acting under. We consider only *stochastic* policies, so that A_t is a random variable with a distribution conditioned on S_t and Π. As in a typical MDP, S_t is a random variable with a distribution conditioned on S_{t-1} and A_{t-1}. The full Bayes net is shown below.

The agent acting in the environment knows what state it is currently in (as is typical in the MDP setting). Unfortunately, however, we, the observer, cannot see the states S_t. Thus we are forced to use an adapted particle filtering algorithm to solve this problem. Concretely, we will develop an efficient algorithm to estimate $P(\Pi \mid a_{1:t})$.

(a) The Bayes net for part (a) is



(i) Select all of the following that are guaranteed to be true in this model for $t > 3$:

□ $S_t \perp\!\!\!\perp S_{t-2} \mid S_{t-1}$

■ $S_t \perp\!\!\!\perp S_{t-2} \mid S_{t-1}, A_{1:t-1}$

□ $S_t \perp\!\!\!\perp S_{t-2} \mid \Pi$

□ $S_t \perp\!\!\!\perp S_{t-2} \mid \Pi, A_{1:t-1}$

■ $S_t \perp\!\!\!\perp S_{t-2} \mid \Pi, S_{t-1}$

■ $S_t \perp\!\!\!\perp S_{t-2} \mid \Pi, S_{t-1}, A_{1:t-1}$

□ None of the above

We will compute our estimate for $P(\Pi \mid a_{1:t})$ by coming up with a recursive algorithm for computing $P(\Pi, S_t \mid a_{1:t})$. (We can then sum out S_t to get the desired distribution; in this problem we ignore that step.)

(ii) Write a recursive expression for $P(\Pi, S_t \mid a_{1:t})$ in terms of the CPTs in the Bayes net above.

$$P(\Pi, S_t \mid a_{1:t}) \propto \sum_{s_{t-1}} P(\Pi, s_{t-1} \mid a_{1:t-1}) P(a_t \mid S_t, \Pi) P(S_t \mid s_{t-1}, a_{t-1})$$

We now try to adapt particle filtering to approximate this value. Each particle will contain a single state s_t and a potential policy π_i.

(iii) The following is pseudocode for the body of the loop in our adapted particle filtering algorithm. Fill in the boxes with the correct values so that the algorithm will approximate $P(\Pi, S_t \mid a_{1:t})$.

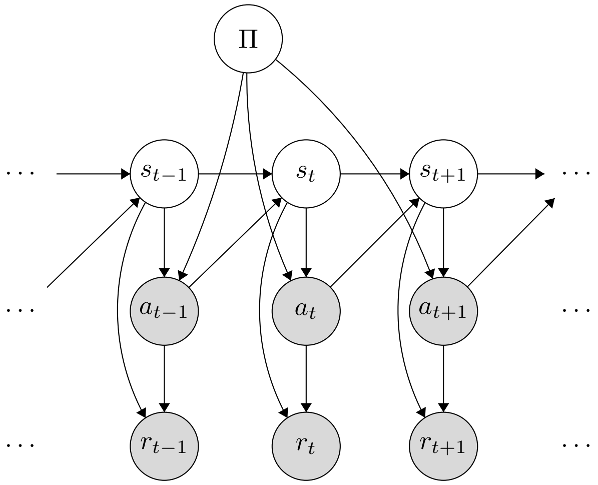
1. Elapse time: for each particle (s_t, π_i), sample a successor s_{t+1} from $P(S_{t+1} \mid s_t, a_t)$.

   The policy π' in the new particle is π_i .

2. Incorporate evidence: To each new particle (s_{t+1}, π'), assign weight $P(a_{t+1} \mid s_{t+1}, \pi')$.

3. Resample particles from the weighted particle distribution.

(b) We now observe the acting agent's actions *and* rewards at each time step (but we still don't know the states). Unlike the MDPs in lecture, here we use a stochastic reward function, so that R_t is a random variable with a distribution conditioned on S_t and A_t. The new Bayes net is given by



Notice that the observed rewards do in fact give useful information since d-separation does not give that $R_t \perp\!\!\!\perp \Pi \mid A_{1:t}$.

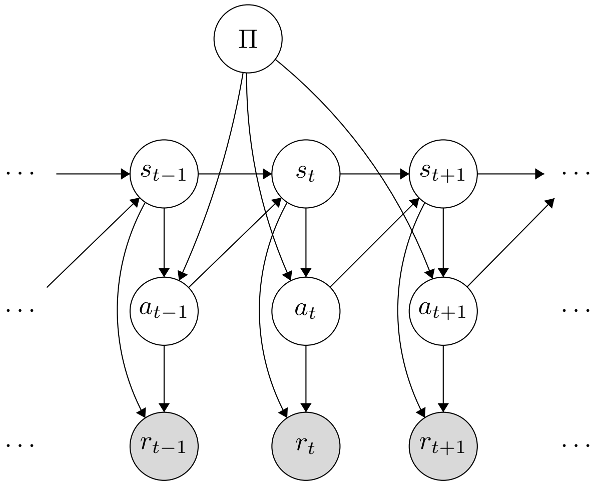
(i) Give an active path connecting R_t and Π when $A_{1:t}$ are observed. Your answer should be an ordered list of nodes in the graph, for example "$S_t, S_{t+1}, A_t, \Pi, A_{t-1}, R_{t-1}$".

R_t, S_t, A_t, Π. This list reversed is also correct, and many other similar (though more complicated) paths are also correct.

(ii) Write a recursive expression for $P(\Pi, S_t \mid a_{1:t}, r_{1:t})$ in terms of the CPTs in the Bayes net above.

$$P(\Pi, S_t \mid a_{1:t}, r_{1:t}) \propto \sum_{s_{t-1}} P(\Pi, s_{t-1} \mid a_{1:t-1}, r_{1:t-1}) P(a_t \mid S_t, \Pi) P(S_t \mid s_{t-1}, a_{t-1}) P(r_t \mid a_t, S_t)$$

(c) We now observe *only* the sequence of rewards and no longer observe the sequence of actions. The new Bayes net is shown on the right.



(i) Write a recursive expression for $P(\Pi, S_t, A_t \mid r_{1:t})$ in terms of the CPTs in the Bayes net above.

$$P(\Pi, S_t, A_t \mid r_{1:t}) \propto \sum_{s_{t-1}} \sum_{a_{t-1}} P(\Pi, s_{t-1}, a_{t-1} \mid r_{1:t-1}) P(A_t \mid S_t, \Pi) P(S_t \mid s_{t-1}, a_{t-1}) P(r_t \mid S_t, A_t)$$

We now try to adapt particle filtering to approximate this value. Each particle will contain a single state s_t, a single action a_t, and a potential policy π_i.

(ii) The following is pseudocode for the body of the loop in our adapted particle filtering algorithm. Fill in the boxes with the correct values so that the algorithm will approximate $P(\Pi, S_t, A_t \mid r_{1:t})$.

1. Elapse time: for each particle (s_t, a_t, π_i), sample a successor state s_{t+1} from $P(S_{t+1} \mid s_t, a_t)$.

Then, sample a successor action a_{t+1} from $P(A_{t+1} \mid s_{t+1}, \pi_i)$.

The policy π' in the new particle is π_i.

2. Incorporate evidence: To each new particle (s_{t+1}, a_{t+1}, π'), assign weight $P(r_{t+1} \mid s_{t+1}, a_{t+1})$.
3. Resample particles from the weighted particle distribution.