

## 1 Controllability

We are given a discrete-time state space system, where  $\vec{x}$  is our state vector,  $A$  is the state space model,  $B$  is the input matrix, and  $\vec{u}$  is the control input.

$$\vec{x}[t + 1] = A\vec{x}[t] + B\vec{u}[t]$$

We want to know if this system is “controllable”; if given set of inputs, we can reach any state in our state-space after a finite number time steps. This has an important physical meaning; if a system is controllable, we can travel anywhere in the system it is living in given enough control inputs.

### Constructing the Controllability Matrix

To figure out if a system is controllable, we start by looking at a simplified problem. Suppose the initial state  $\vec{x}[0] = \vec{0}$  and we would like to reach a certain state  $\vec{x}_0$ . If a system were controllable, we can give a set of control inputs  $\vec{u}[t]$  to reach this state  $\vec{x}_0$ . Let’s start by analyzing what happens after one time step.

$$\vec{x}[1] = A\vec{x}[0] + B\vec{u}[0] = A\vec{0} + B\vec{u}[0] = B\vec{u}[0] \quad (1)$$

This shows us that we can go anywhere spanned by  $B$  in the first time step. Now let us push the system another time step.

$$\vec{x}[2] = A\vec{x}[1] + B\vec{u}[1] = A(A\vec{x}[0] + B\vec{u}[0]) + B\vec{u}[1] \quad (2)$$

$$= AB\vec{u}[0] + B\vec{u}[1] \quad (3)$$

Similarly, at this time step, we can go anywhere spanned by  $\begin{bmatrix} B & AB \end{bmatrix}$ . Every time step adds another degree of freedom to the system.

To generalize this relation, after  $n$  time steps, we get the following:

$$\vec{x}[n] = A\vec{x}[n - 1] + B\vec{u}[n - 1] \quad (4)$$

$$= A^{n-1}B\vec{u}[0] + A^{n-2}B\vec{u}[1] + \dots + AB\vec{u}[n - 2] + B\vec{u}[n - 1] \quad (5)$$

This shows that after  $n$  time steps, we can go anywhere spanned by the columns of the matrix  $C$  defined below. This is called the “controllability” matrix.

$$C = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-2}B & A^{n-1}B \end{bmatrix} \quad (6)$$

If this matrix is of rank  $n$  (the dimension of our state space), then our system is controllable. But what if these aren’t enough steps and the system can be controlled only in  $n + 1$  steps? What is the maximal number of steps we need to take to have a long sequence of control inputs that  $\{B, AB, A^2B, \dots\}$  spans the state space?

### Cayley-Hamilton Theorem

These questions are answered by the Cayley-Hamilton theorem. The Cayley-Hamilton theorem says that higher order powers of  $A$  can be expressed as a linear combination of lower order matrix powers of  $A$ . Specifically if  $A$  is an  $n \times n$ , matrix, the highest order unique power of  $A$  is  $A^{n-1}$ . Thus, if we keep applying control inputs past  $n$  time steps, our control inputs will be a linear combination of the previous control inputs and cannot increase the rank of the controllability matrix.

## Definition of Controllability

This also works for continuous time systems; instead of incrementing the time steps in our system by 1 every time, we increment by  $\Delta t$ ,  $2\Delta t$ , etc. The math and our controllability test work out to be exactly the same! Putting all of this together, we get the following:

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t) \quad \text{or} \quad \vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$$

$$C = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-2}B & A^{n-1}B \end{bmatrix}$$

Given a continuous or discrete time system  $\vec{x}$  of dimension  $n$ , the system is controllable if its controllability matrix  $C$  is of rank  $n$ . If a system is controllable, then given any starting position  $\vec{x}[0]$ , it takes a maximum of  $n$  control inputs over  $n$  time steps for the system to reach any final state  $\vec{x}_0$ .

## 2 Deadbeat Control

Consider the system

$$\vec{x}[t+1] = A\vec{x}[t] + Bu[t] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[t].$$

a) Is this system controllable?

b) For which initial states  $\vec{x}[0]$  is there a control that will bring the state to zero in a single time step?

c) For which initial states  $\vec{x}[0]$  is there a control that will bring the state to zero in two time steps?

d) Now let  $\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  be the initial state. Give a set of control inputs  $u[0]$  and  $u[1]$  to bring to system to  $\vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  in two time steps.

### 3 Discretization

Consider a cart of mass  $M$ , pushed with a force  $u(t)$  with position,  $x(t)$ , and velocity,  $v(t)$ . Hence, we have:

$$\begin{aligned}\frac{d}{dt} x(t) &= v(t) \\ \frac{d}{dt} v(t) &= \frac{u(t)}{M}\end{aligned}$$

We will apply a constant input between any time  $t \in [nT, nT + T)$ . Here  $T$  is our time between samples.

a) Find a discretized system of equations for this system.

b) Is the discretized system controllable?