EECS 16B Designing Information Devices and Systems II
Summer 2020 Note 13

1 Overview

The response of a linear system can be broken down into its natural and force responses. The natural response is the response given zero input while the forced response is the result due to the input $u(t)$.

$$x(t) = x_{natural}(t) + x_{forced}(t) \tag{1}$$

Ideally we would like the natural response to decay to zero. This would mean that $x(t)$ at steady state would be the forced response due to the input.

In this note, we will look into the **stability** of a system. Often times, a model for a system does not fully capture its behavior. Factors such as nonlinearities, noise, or even disturbances can cause our measurements to have large error terms.

Therefore, we include a new term $\vec{e}$ representing the error in our state-space model

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) + \vec{e}(t) \tag{2}$$

These errors are often unpredictable and unobservable meaning a stable system must be robust to all of these types of error. Therefore, to perform our analyses, we will start by assuming our error is bounded by some constant or $\|\vec{e}\| < \varepsilon$.

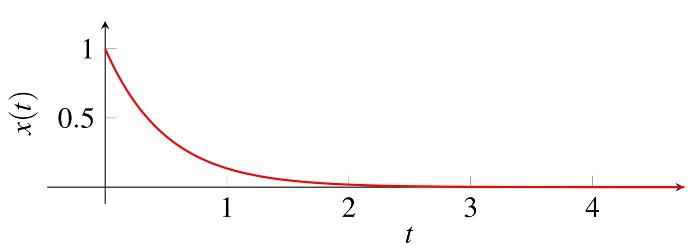
2 Stability

Before analyzing continuous and discrete-time systems, we need to define what it means for a system to be stable. It turns out that there are multiple definitions of stability each depending on the context. A system is **Asymptotically Stable** if the natural response of system converges to 0.

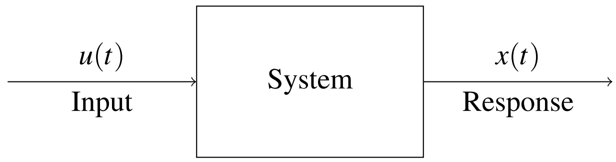
From an intuitive approach, this means that the system naturally decays to 0 given zero input. One example of an asymptotically stable system is the one represented by the differential equation

$$\frac{d}{dt}x(t) = -2x(t) \tag{3}$$

The visual below shows that the response $x(t)$ decays to 0 without an input.



Alternatively, we can view stability through our input $u(t)$. A system is **Bounded Input Bounded Output Stable** or **BIBO Stable** if for every bounded input, the output is also bounded. In this perspective, we think of our system as a box that shapes our input and gives an output response $x(t)$



Note that we can easily generalize all of these scalar stability ideas to matrix vector systems.

Although we won't do it in this note, we can in fact show that asymptotic stability is a stronger claim than BIBO stability. The two definitions are almost identical and if a system is controllable, observable, and asymptotically stable, it is also BIBO stable.

This means that every asymptotically stable system will also be BIBO stable. For the purposes of this class, whenever we refer to the term *stability* we will often be referring to BIBO stability.

3 Continuous-Time Stability

We start by analyzing the stability of a continuous-time system since we are familiar with how to solve differential equations. We will begin with the scalar case and then build it up to the multivariate vector case.

3.1 Scalar Stability

Consider the following continuous-time system with initial condition $x(0) = x_0$

$$\frac{d}{dt}x(t) = \lambda x(t) + w(t) \tag{4}$$

The term $w(t)$ is a combined term which is the sum of the input $u(t)$ and error $e(t)$.

We know that this differential equation has the following solution for $t \geq 0$

$$x(t) = x_0 e^{\lambda t} + \int_0^t w(\tau) e^{\lambda(t-\tau)} d\tau \tag{5}$$

Now we will break this down into cases by looking at the value of λ assuming $\left|w(t)\right| < \varepsilon$.

3.1.1 Complex λ

When λ is a complex number, we can break it down into its real and imaginary parts as $\lambda = \sigma + j\omega$. Upon analyzing our solution, we see that

$$\left|e^{\lambda t}\right| = \left|e^{(\sigma + j\omega)t}\right| = \left|e^{\sigma t} e^{j\omega t}\right| = \left|e^{\sigma t}\right|\left|e^{j\omega t}\right| = e^{\sigma t} \tag{6}$$

The last equality comes from the fact that a complex exponential $e^{j\omega t}$ has magnitude 1. Therefore, this analysis shows that the imaginary part of λ will never affect the boundedness of $x(t)$ hence it won't affect the stability of the system.

Recall from the circuits module that we showed for an underdamped system that the imaginary part of the eigenvalues added oscilations to the system. This understanding is identical and follows from what we have shown above.

3.1.2 Positive λ

Now that we have shown that the imaginary part of λ cannot affect the stability of our system, let's analyze what happens when $\mathfrak{Re}\,\lambda > 0$

We immediately see that the natural response $x_{natural}(t) = x_0 e^{\lambda t}$ will go to infinity as $t \to \infty$. Therefore, we claim that this system is unstable since the repsonse is unbounded regardless of what the input is.

3.1.3 $\lambda = 0$

We have shown that if $\mathfrak{Re}\,\lambda > 0$, the system is unstable since the natural response will go to infinity regardless of what the input is.

Now let's analyze what happens when $\mathfrak{Re}\,\lambda = 0$. For simplicity, we assume $\lambda = 0$ since the imaginary part will not affect stability.

If the system were stable, then the response $x(t)$ would be bounded for any input that is bounded. Therefore, let us analyze the ouptut when $w(t) = 1$

$$x(t) = x_0 e^{\lambda t} + \int_0^t e^{\lambda(t-\tau)} d\tau = x_0 + \int_0^t d\tau = x_0 + t \tag{7}$$

As $t \to \infty$, the response will be unbounded since $x(t) \to \infty$. Since the system is **not** bounded for every bounded input, we again see that this system is unstable.

3.1.4 Negative λ

It seems like all hope is lost, but we have one more case when $\mathfrak{Re}\,\lambda < 0$. It turns out that $x(t)$ will always be bounded as long as $w(t)$ is bounded. To show this, we again look at the solution $x(t)$. This proof is quite involved and does require ideas such as the Triangle Inequality for Sums and Integrals

$$\left|x(t)\right| = \left|x_0 e^{\lambda t} + \int_0^t w(\tau) e^{\lambda(t-\tau)} d\tau\right| \leq |x_0|\, e^{\lambda t} + \left|\int_0^t w(\tau) e^{\lambda(t-\tau)} d\tau\right| \tag{8}$$

$$\leq \left|x_0 e^{\lambda t}\right| + \int_0^t \left|w(\tau) e^{\lambda(t-\tau)} d\tau\right| \leq \left|x_0 e^{\lambda t}\right| + \left|e^{\lambda t} \int_0^t \varepsilon e^{-\lambda\tau} d\tau\right| \tag{9}$$

$$\leq \left|x_0 e^{\lambda t}\right| + \left|\frac{e^{\lambda t}\varepsilon}{\lambda}\right| \left|e^{-\lambda t} - 1\right| = \left|x_0 e^{\lambda t}\right| + \left|\frac{\varepsilon}{\lambda}\right| \left|1 - e^{\lambda t}\right| \tag{10}$$

It follows that $x(t)$ must be bounded since $x(t)$ is less than the sum of two terms, both of which are bounded.

3.2 System Stability

From the scalar case, we have discovered that the real part of the eigenvalue of the differential equation plays a crucial role in determining stability. We saw that when $\mathfrak{Re}\,\lambda < 0$, the scalar system is stable since every bounded input yieled a bounded output.

To extend this to the vector case, we will use go back to our familiar method of changing coordinates or diagonalization

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) + \vec{e}(t) \implies \frac{d}{dt}\vec{z}(t) = \Lambda\vec{z}(t) + V^{-1}B\vec{u}(t) + V^{-1}\vec{e}(t) \tag{11}$$

In our eigenbasis coordinates, we can uncouple all of our equations and see that

$$z_1(t) = \lambda_1(t)z_1(t) + w_1(t) \tag{12}$$

$$\vdots \tag{13}$$

$$z_n(t) = \lambda_n(t)z_n(t) + w_n(t) \tag{14}$$

The term $w_i(t)$ is again a combination of the input and error terms. If all our eigenvalues $\lambda_1, \ldots, \lambda_n$ have real part less than 0, we can say that our system is stable.

3.2.1 Non-Diagonalizable Case

Now what is the matrix A is non-diagonalizable? Just like how we solved differential equations for the critically-damped case, it turns out that we can change coordinates to a basis in which the matrix A has an upper-triangular representation. Let us call this basis U and it turns out that the diagonal entries of our upper-triangular reprsentation, R, has the eigenvalues of A on its diagonal.

$$A = URU^{-1} = U \begin{bmatrix} \lambda_1 & r_{12} & \ldots & r_{1n} \\ 0 & \lambda_2 & \ldots & r_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix} U^{-1} \tag{15}$$

We will prove its existence in a later note since its existence is not the focus of this note.

With this in mind, we can again change coordinates to the basis represented by the columns of U

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) + \vec{e}(t) \implies \frac{d}{dt}\vec{z}(t) = R\vec{z}(t) + U^{-1}B\vec{u}(t) + U^{-1}\vec{e}(t) \tag{16}$$

In our new coordinate system, we can again uncouple all of our equations and see that

$$z_1(t) = \lambda_1(t)z_1(t) + r_{12}z_2(t) + \ldots + r_{1n}z_n(t) + w_1(t) \tag{17}$$

$$\vdots \tag{18}$$

$$z_n(t) = \lambda_n(t)z_n(t) + w_n(t) \tag{19}$$

We immediately see that if $\mathfrak{Re}\,(\lambda_n) < 0$, then the n^{th} differential equation is stable. We can then move up to the $n-1^{st}$ differential equation represented as

$$\frac{d}{dt}z_{n-1}(t) = \lambda_{n-1}z_{n-1}(t) + r_{n-1,n}\,z_n(t) + w_{n-1}(t) \tag{20}$$

Since $z_n(t)$ is stable and hence a bounded function, we can define a new function $\widetilde{w}_{n-1}(t) = r_{n-1,n}\,z_n(t) + w_{n-1}(t)$ as the input to our $n-1^{st}$ differential equation. Through the same logic, $z_{n-1}(t)$ will be bounded if $\mathfrak{Re}\,(\lambda_{n-1}) < 0$.

Therefore, it again follows that if all of the eigenvalues of A have real part less than 0, the system will be stable. Note that **all** of the eigenvalues must have real part less than 0. If even one of the eigenvalues has a real part greater than or equal to zero, the system will be unstable.

4 Discrete-Time Stability

Now that we have shown when a continuous-time system is stable for both the scalar and vector case, we will move onto analyzing stability for discrete-time systems. Note that if we can define a stability condition for the scalar case, extending it to the vector case will be identical to the work we did in the previous section. Therefore, we shall only consider scalar stability for discrete-time systems and easily extend it to the vector case.

4.1 Scalar Case

Consider the following scalar discrete-time system

$$x[n+1] = \lambda x[n] + w[n] \tag{21}$$

where $w[n]$ is a combined term which is the sum of the input $u[n]$ and the error $e[n]$.

Now let's try writing $x[n]$ as a linear combination of the previous states similar to what we did when proving Controllability

$$\begin{aligned}
x[n] &= \lambda x[n-1] + w[n-1] = \lambda\left(\lambda x[n-2] + w[n-2]\right) + w[n-1] && (22)\\
&= \lambda^2 x[n-2] + \lambda w[n-2] + w[n-1] && (23)\\
&\vdots && (24)\\
&= \lambda^n x[0] + (\lambda^{n-1}w[0] + \ldots + \lambda w[n-2] + w[n-1]) && (25)
\end{aligned}$$

If $\left|w[n]\right| < \varepsilon$ for all n then we could bound the summation for $x[n]$

$$\begin{aligned}
\left|x[n]\right| &\leq \left|\lambda^n x[0] + (\lambda^{n-1}w[0] + \ldots + \lambda w[n-2] + w[n-1])\right| && (26)\\
&\leq \left|\lambda^n x[0]\right| + \left|\lambda^{n-1}\varepsilon + \ldots + \lambda\varepsilon + \varepsilon\right| && (27)\\
&= \left|\lambda^n x[0]\right| + |\varepsilon|\left|\lambda^{n-1} + \ldots + \lambda + 1\right| && (28)
\end{aligned}$$

Note how the right hand side is contains a geometric series with common ratio $r = \lambda$. As $n \to \infty$, the sum converges only when $|\lambda| < 1$.

This implies that a scalar discrete-time system is unstable when $|\lambda| \geq 1$ since the summation diverges and is unbounded. We will now continue our analysis assuming $|\lambda| < 1$. If the summation converges to some finite value B, then it must be that

$$\left|x[n]\right| \leq \left|\lambda^n x[0]\right| + |\varepsilon B| \tag{29}$$

Formally the value of B is $\frac{1}{1-\lambda}$ and this implies that $x[n]$ is bounded when $|\lambda| < 1$. Therefore, we conclude by saying a discrete-time system is stable when $|\lambda| < 1$. Note how our analysis does not depend on the fact that λ is real. In fact, for a complex $\lambda = \sigma + j\omega$, the system is stable when $\left|\sigma^2 + \omega^2\right| < 1$. [1]

4.2 Vector Case

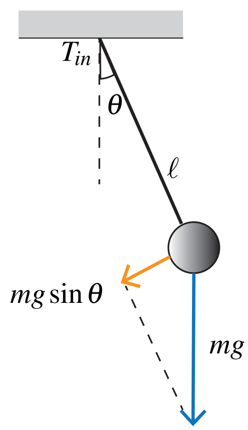
There isn't too much to say about the vector case apart from the fact that all of the eigenvalues must be stable.

$$|\lambda_1| < 1, \ldots, |\lambda_n| < 1 \implies \text{Discrete-Time System is stable} \tag{30}$$

5 Examples

5.1 Downward Pendulum

Let's come back to our favorite linearized pendulum example.



Recall that the linearized system can be represented by the following differential equation

$$\frac{d}{dt}\vec{x}_\ell(t) = \begin{bmatrix} 0 & 1 \\ \frac{-g}{\ell} & \frac{-k}{m} \end{bmatrix} \vec{x}_\ell(t) + \begin{bmatrix} 0 \\ \frac{1}{\ell} \end{bmatrix} u_\ell(t) \tag{31}$$

The eigenvalues of this system can be computed through the characteristic polynomial

$$\lambda^2 + \frac{k}{m}\lambda + \frac{g}{\ell} \implies \lambda = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\left(\frac{k}{m}\right)^2 - \frac{4g}{\ell}} \tag{32}$$

[1] Recall that the magnitude of a complex number $z = a + bj$ is $|z| = \sqrt{a^2 + b^2}$. Since the square-root function is monotonic, or always increasing, showing $|z| < 1$ is equivalent to showing $|z|^2 < 1$.

Since all of the constants k, m, g, ℓ are positive, the real part of the eigenvalues will both be negative implying that the system is stable. Intuitively if we slightly perturb the downward pendulum with a small torque, it returns back to its original equilibrium.

5.2 Inverted Pendulum

Now let's recall the linearized inverted pendulum system represented by the following differential equation

$$\frac{d}{dt}\vec{x}_\ell(t) = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & \frac{-k}{m} \end{bmatrix} \vec{x}_\ell(t) + \begin{bmatrix} 0 \\ \frac{1}{\ell} \end{bmatrix} u_\ell(t) \tag{33}$$

The eigenvalues of this system can be computed through the characteristic polynomial

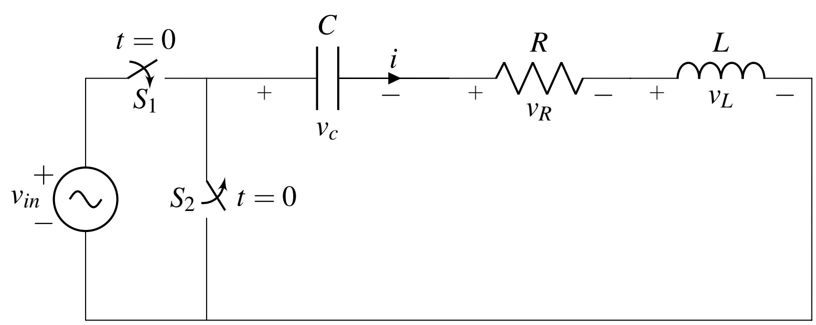
$$\lambda^2 + \frac{k}{m}\lambda - \frac{g}{\ell} \implies \lambda = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\left(\frac{k}{m}\right)^2 + \frac{4g}{\ell}} \tag{34}$$

A quick computation shows that the eigenvalues now both have positive real parts. This means the system is unstable and the response will "blow up." From a physical perspective, if we slightly perturb the inverted pendulum, it will rapidly fall from its upward balanced state.

Since this was a linearized system, we see that $\theta = \pi$ is an unstable equilibrium point and the linear approximation to the system quickly becomes invalid since the angle θ doesn't actually "blow up" and go to infinity.

5.3 RLC Circuit

Let's recall the process of charging an RLC circuit.



We can pick the v_c and i_L to be state-variables and the system can be reprsented as

$$\frac{d}{dt}\begin{bmatrix} v_c(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v_{in}(t) \tag{35}$$

The eigenvalues of this system can be computed through the characteristic polynomial [2]

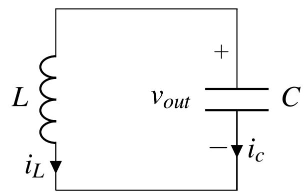
$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} \implies \lambda = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \tag{36}$$

[2] Note the similarities to the pendulum example. This type of second order system is characterized as a harmonic oscillator and is very important when approximating higher-order systems.

We again see that the real part of the eigenvalues must be negative since R, L, and C are all greater than 0. Therefore an RLC circuit will always be stable.

5.4 LC Tank

Now let's consider the LC Tank case in which $R = 0$



The differential equation representing this system was

$$\frac{d}{dt}\begin{bmatrix} v_{out} \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} v_{out} \\ i_L \end{bmatrix} \tag{37}$$

The eigenvalues of this system can be computed as

$$\lambda^2 + \frac{1}{LC} \implies \lambda = \pm j\sqrt{\frac{1}{LC}} \tag{38}$$

Since the real part of the eigenvalues are equal to 0 we characterize this system as unstable. However, from an intuitive presepective, the energy in the capacitor and inductor will continually slosh back and forth meaning the response is always bounded. This is an example of a system that has an always bounded response but is not asymptotically stable since the natural response does not decay to zero.

To show that this system is indeed unstable, try applying a voltage input of $v_{in}(t) = \cos(\omega t)$ where $\omega = \sqrt{\frac{1}{LC}}$. While tedious, a computation can show that the output does indeed blow up and go to infinity.

