
EECS 16A Designing Information Devices and Systems I

Spring 2021

Homework 11

This homework is due April 16, 2020, at 23:59.

Self-grades are due April 19, 2020, at 23:59.

Midterm 2 re-do is due April 16, 2020, at 23:59. This homework is intentionally short to give you time to re-do and review the midterm.

Submission Format

Your homework submission should consist of **one** file.

- `hw11.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned)

Submit each file to its respective assignment on Gradescope.

1. Reading Assignment

For this homework, please read Notes 21 and 22 to learn about inner products, norms, trilateration, and correlation. You are always encouraged to read beyond this as well.

- (a) What does it mean for two vectors \vec{x} and \vec{y} to be orthogonal, in terms of their inner product?

Solution: The inner product of orthogonal vectors is zero, $\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \|\vec{y}\| \cos 90^\circ = \|\vec{x}\| \|\vec{y}\| \cdot 0 = 0$

2. Inner Product Properties

Learning Goal: The objective of this problem is to exercise useful identities for inner products.

Our definition of the inner product in \mathbb{R}^n is:

$$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \vec{x}^T \vec{y}, \quad \text{for any } \vec{x}, \vec{y} \in \mathbb{R}^n$$

Prove the following identities in \mathbb{R}^n :

- (a) $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

Solution: This is seen by direct expansion:

Let $x_i, y_i \in \mathbb{R}$, then

$$\begin{aligned} \left\langle \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \right\rangle &= x_1 \cdot y_1 + x_2 \cdot y_2 + \dots + x_n \cdot y_n \\ &= y_1 \cdot x_1 + y_2 \cdot x_2 + \dots + y_n \cdot x_n \\ &= \left\langle \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right\rangle \end{aligned}$$

So the inner product is commutative.

(b) $\langle \vec{x}, \vec{x} \rangle = \|\vec{x}\|^2$

Solution:

$$\begin{aligned} \left\langle \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right\rangle &= x_1 \cdot x_1 + x_2 \cdot x_2 + \cdots + x_n \cdot x_n \\ &= x_1^2 + x_2^2 + \cdots + x_n^2 \\ &= (\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2})^2 \\ &= (\|\vec{x}\|)^2 \end{aligned}$$

The inner product of a vector with itself is its norm squared.

(c) $\langle -\vec{x}, \vec{y} \rangle = -\langle \vec{x}, \vec{y} \rangle$.

Solution:

$$\begin{aligned} \langle -\vec{x}, \vec{y} \rangle &= \left\langle \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \right\rangle \\ &= -x_1 \cdot y_1 - x_2 \cdot y_2 - \cdots - x_n \cdot y_n \\ &= -(x_1 \cdot y_1 + x_2 \cdot y_2 + \cdots + x_n \cdot y_n) \\ &= -\langle \vec{x}, \vec{y} \rangle \end{aligned}$$

Flipping the sign of one of the vectors in the inner product flips the sign of the inner product, but does not change the magnitude.

(d) $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

Solution:

$$\begin{aligned} \langle \vec{x}, \vec{y} + \vec{z} \rangle &= \vec{x}^T (\vec{y} + \vec{z}) \\ &= \vec{x}^T \vec{y} + \vec{x}^T \vec{z} \\ &= \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle \end{aligned}$$

The inner product is distributive.

$$(e) \langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle = \langle \vec{x}, \vec{x} \rangle + 2\langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{y} \rangle$$

Solution:

$$\begin{aligned} \langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle &= \langle \vec{x}, \vec{x} + \vec{y} \rangle + \langle \vec{y}, \vec{x} + \vec{y} \rangle \\ &= \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle \\ &= \langle \vec{x}, \vec{x} \rangle + 2\langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{y} \rangle \end{aligned}$$

3. Cauchy-Schwarz Inequality

Learning Goal: The objective of this problem is to understand and prove the Cauchy-Schwarz inequality for real-valued vectors.

The Cauchy-Schwarz inequality states that for two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$:

$$|\langle \vec{v}, \vec{w} \rangle| = |\vec{v}^T \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\|$$

In this problem we will prove the Cauchy-Schwarz inequality for vectors in \mathbb{R}^2 .

Take two vectors: $\vec{v} = r \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ and $\vec{w} = t \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix}$, where $r > 0, t > 0, \theta$, and ϕ are scalars. Make sure you understand why any vector in \mathbb{R}^2 can be expressed this way and why it is acceptable to restrict $r, t > 0$.

- (a) In terms of some or all of the variables r, t, θ , and ϕ , what are $\|\vec{v}\|$ and $\|\vec{w}\|$?

Solution: We use the trig identity $\cos^2 x + \sin^2 x = 1$ to show:

$$\begin{aligned} \|\vec{v}\| &= \sqrt{v_1^2 + v_2^2} \\ &= \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta} \\ &= r \end{aligned}$$

Similarly, $\|\vec{w}\| = t$.

- (b) In terms of some or all of the variables r, t, θ , and ϕ , what is $\langle \vec{v}, \vec{w} \rangle$? *Hint: The trig identity $\cos(a)\cos(b) + \sin(a)\sin(b) = \cos(a-b)$ may be useful.*

Solution: We use the trig identity $\cos(x)\cos(y) + \sin(x)\sin(y) = \cos(x-y)$ to show:

$$\begin{aligned} \langle \vec{v}, \vec{w} \rangle &= (r \cos \theta)(t \cos \phi) + (r \sin \theta)(t \sin \phi) \\ &= r \cdot t (\cos \theta \cos \phi + \sin \theta \sin \phi) \\ &= r \cdot t \cos(\theta - \phi) \end{aligned}$$

- (c) Show that the Cauchy-Schwarz inequality holds for any two vectors in \mathbb{R}^2 . *Hint: consider your results from part (b). Also recall $-1 \leq \cos x \leq 1$ and use both inequalities.*

Solution: We use the fact that $\cos x \leq 1$ to show:

$$\begin{aligned}\langle \vec{v}, \vec{w} \rangle &= r \cdot t \cos(\theta - \phi) \\ &= \|\vec{v}\| \|\vec{w}\| \cos(\theta - \phi) \\ &\leq \|\vec{v}\| \|\vec{w}\|\end{aligned}$$

We use the fact that $\cos x \geq -1$ to show:

$$\begin{aligned}\langle \vec{v}, \vec{w} \rangle &= r \cdot t \cos(\theta - \phi) \\ &= \|\vec{v}\| \|\vec{w}\| \cos(\theta - \phi) \\ &\geq -\|\vec{v}\| \|\vec{w}\|\end{aligned}$$

Therefore:

$$-\|\vec{v}\| \|\vec{w}\| \leq \langle \vec{v}, \vec{w} \rangle \leq \|\vec{v}\| \|\vec{w}\|,$$

which gives us that

$$|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \|\vec{w}\|.$$

- (d) Note that the inequality states that the inner product of two vectors must be less than *or equal to* the product of their magnitudes. What conditions must the vectors satisfy for the equality to hold? In other words, when is $\langle \vec{v}, \vec{w} \rangle = \|\vec{v}\| \cdot \|\vec{w}\|$?

Solution:

$$\begin{aligned}\langle \vec{v}, \vec{w} \rangle &= \|\vec{v}\| \|\vec{w}\| \\ \|\vec{v}\| \|\vec{w}\| \cos(\theta - \phi) &= \|\vec{v}\| \|\vec{w}\| \\ \cos(\theta - \phi) &= 1 \\ \theta - \phi &= 0\end{aligned}$$

We see that the equality holds when the angle between the two vectors is zero. Note that when the angle is zero, the vectors would be linearly dependent.

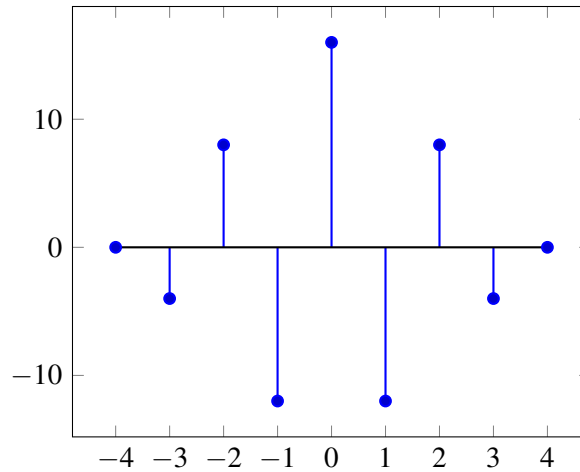
4. Mechanical Linear Correlation

Learning Goal: The objective of this problem is to understand how to compute the linear correlation between signals.

We recall that the linear correlation of signal \vec{y} with signal \vec{x} is given as:

$$\text{corr}_{\vec{x}}(\vec{y})[k] = \sum_{n=-\infty}^{\infty} \vec{x}[n] \vec{y}[n-k]$$

$\vec{s}_1[n]$	0	0	0	2	-2	2	-2	0	0	0								
$\vec{s}_1[n-3]$	0	0	0	0	0	0	2	-2	2	-2								
$\langle \vec{s}_1[n], \vec{s}_1[n-3] \rangle$	0	+	0	+	0	+	0	+	0	+	-4	+	0	+	0	+	0	= -4



- (a) Using the procedure demonstrated above, compute $\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$, the linear cross-correlation of \vec{s}_2 with \vec{s}_1 . Like the example, use tables like the one given below for $k = -3$ and plot the resulting correlation.

$\vec{s}_1[n]$	0	0	0	2	-2	2	-2	0	0	0
$\vec{s}_2[n+3]$	1	2	3	4	0	0	0	0	0	0
$\langle \vec{s}_1[n], \vec{s}_2[n+3] \rangle$										

Solution:

$\vec{s}_1[n]$	0	0	0	2	-2	2	-2	0	0	0										
$\vec{s}_2[n+3]$	1	2	3	4	0	0	0	0	0	0										
$\langle \vec{s}_1[n], \vec{s}_2[n+3] \rangle$	0	+	0	+	0	+	8	+	0	+	0	+	0	+	0	+	0	+	0	= 8

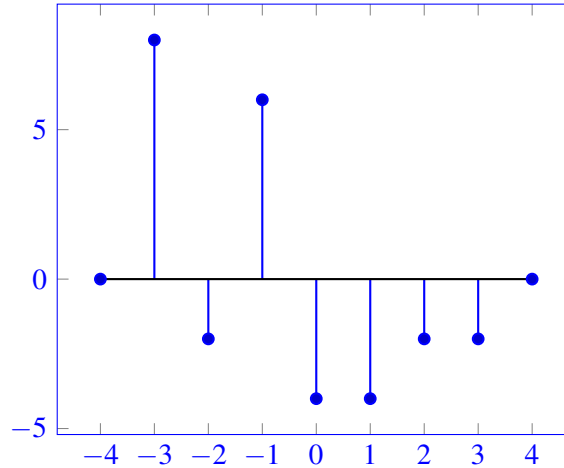
$\vec{s}_1[n]$	0	0	0	2	-2	2	-2	0	0	0										
$\vec{s}_2[n+2]$	0	1	2	3	4	0	0	0	0	0										
$\langle \vec{s}_1[n], \vec{s}_2[n+2] \rangle$	0	+	0	+	0	+	6	+	-8	+	0	+	0	+	0	+	0	+	0	= -2

$\vec{s}_1[n]$	0	0	0	2	-2	2	-2	0	0	0										
$\vec{s}_2[n+1]$	0	0	1	2	3	4	0	0	0	0										
$\langle \vec{s}_1[n], \vec{s}_2[n+1] \rangle$	0	+	0	+	0	+	4	+	-6	+	8	+	0	+	0	+	0	+	0	= 6

$\vec{s}_1[n]$	0	0	0	2	-2	2	-2	0	0	0										
$\vec{s}_2[n+0]$	0	0	0	1	2	3	4	0	0	0										
$\langle \vec{s}_1[n], \vec{s}_2[n+0] \rangle$	0	+	0	+	0	+	2	+	-4	+	6	+	-8	+	0	+	0	+	0	= -4

$\vec{s}_1[n]$	0	0	0	2	-2	2	-2	0	0	0										
$\vec{s}_2[n-1]$	0	0	0	0	1	2	3	4	0	0										
$\langle \vec{s}_1[n], \vec{s}_2[n-1] \rangle$	0	+	0	+	0	+	0	+	-2	+	4	+	-6	+	0	+	0	+	0	= -4

$\vec{s}_1[n]$	0	0	0	2	-2	2	-2	0	0	0								
$\vec{s}_2[n-2]$	0	0	0	0	0	1	2	3	4	0								
$\langle \vec{s}_1[n], \vec{s}_2[n-2] \rangle$	0	+	0	+	0	+	0	+	2	+	-4	+	0	+	0	+	0	= -2
$\vec{s}_1[n]$	0	0	0	2	-2	2	-2	0	0	0								
$\vec{s}_2[n-2]$	0	0	0	0	0	0	1	2	3	4								
$\langle \vec{s}_1[n], \vec{s}_2[n-2] \rangle$	0	+	0	+	0	+	0	+	0	+	-2	+	0	+	0	+	0	= -2



- (b) Will the linear cross-correlation of \vec{s}_2 with \vec{s}_1 ($\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$) be the same as the cross-correlation of \vec{s}_1 with \vec{s}_2 ($\text{corr}_{\vec{s}_2}(\vec{s}_1)[k]$)? You can use the iPython notebook **prob11.ipynb** to figure this out. How are they related to each other?

Solution: See sol12.ipynb. They do not have the same result, but they are related: one is the reverse of the other. If you were able to observe this, give yourself full points.

You were not explicitly required to show why, but a sketch of why this is the case follows. Let us compare $\text{corr}_{\vec{s}_2}(\vec{s}_1)[k]$ and $\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$.

By definition:

$$\begin{aligned}\text{corr}_{\vec{s}_2}(\vec{s}_1)[k] &= \sum_{n=-\infty}^{\infty} \vec{s}_2[n] \vec{s}_1[n-k] \\ \text{corr}_{\vec{s}_1}(\vec{s}_2)[k] &= \sum_{n=-\infty}^{\infty} \vec{s}_1[n] \vec{s}_2[n-k]\end{aligned}$$

Using a substitution of index, $m = n - k$ we have:

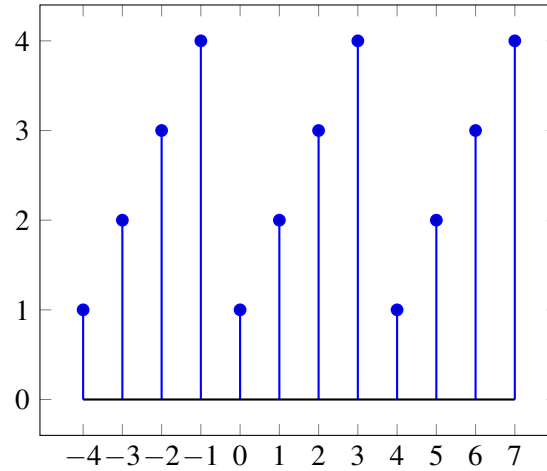
$$\begin{aligned}\text{corr}_{\vec{s}_1}(\vec{s}_2)[k] &= \sum_{m=-\infty}^{\infty} \vec{s}_1[m+k] \vec{s}_2[m] \\ &= \sum_{m=-\infty}^{\infty} \vec{s}_2[m] \vec{s}_1[m-(-k)] \\ &= \text{corr}_{\vec{s}_2}(\vec{s}_1)[-k]\end{aligned}$$

So we can conclude that $\text{corr}_{\vec{s}_1}(\vec{s}_2)[k] = \text{corr}_{\vec{s}_2}(\vec{s}_1)[-k]$.

Now, we will review the procedure to perform linear cross-correlation between one signal that is periodic with a period of 4 and another that is finite length and extended with zeros as in the previous parts. As an

example, we will compute the linear correlation $\text{corr}_{\vec{p}_2}(\vec{s}_1)[k]$ between the periodic signal \vec{p}_2 (with period 4), formed by repeating \vec{s}_2 , and the finite length signal \vec{s}_1 extended with zeros. The result will be a periodic signal with period 4.

The periodic signal, \vec{p}_2 , formed by repeating \vec{s}_2 is plotted below for indices -4 to 7. It is defined and non-zero for all indices from $-\infty$ to $+\infty$.



We compute one period of the result of the cross-correlation by starting at a shift of $k = -1$ and ending at a shift of $k = 2$.

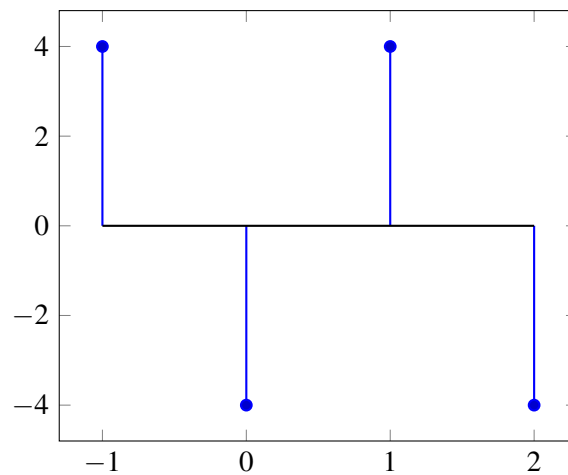
$\vec{p}_2[n]$	2	3	4	1	2	3	4	1	2	3										
$\vec{s}_1[n+1]$	0	0	2	-2	2	-2	0	0	0	0										
$\langle \vec{p}_2[n], \vec{s}_1[n+1] \rangle$	0	+	0	+	8	+	-2	+	4	+	-6	+	0	+	0	+	0	+	0	=4

$\vec{p}_2[n]$	2	3	4	1	2	3	4	1	2	3										
$\vec{s}_1[n+0]$	0	0	0	2	-2	2	-2	0	0	0										
$\langle \vec{p}_2[n], \vec{s}_1[n+0] \rangle$	0	+	0	+	0	+	2	+	-4	+	6	+	-8	+	0	+	0	+	0	=-4

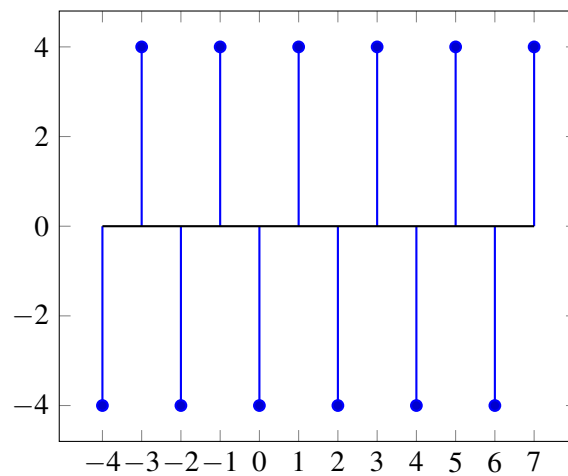
$\vec{p}_2[n]$	2	3	4	1	2	3	4	1	2	3										
$\vec{s}_1[n-1]$	0	0	0	0	2	-2	2	-2	0	0										
$\langle \vec{p}_2[n], \vec{s}_1[n-1] \rangle$	0	+	0	+	0	+	0	+	4	+	-6	+	8	+	-2	+	0	+	0	=4

$\vec{p}_2[n]$	2	3	4	1	2	3	4	1	2	3										
$\vec{s}_1[n-2]$	0	0	0	0	0	2	-2	2	-2	0										
$\langle \vec{p}_2[n], \vec{s}_1[n-2] \rangle$	0	+	0	+	0	+	0	+	0	+	6	+	-8	+	2	+	-4	+	0	=-4

The computed single period of the resulting linear cross correlation is plotted below.



The resulting linear cross correlation for shifts from $k = -4$ to $k = 7$ is plotted below.



- (c) Repeat the procedure described above to compute the correlation $\text{corr}_{\vec{p}_1}(\vec{s}_1)[k]$ between a periodic signal \vec{p}_1 (with period 4), formed by repeating \vec{s}_1 , and the finite-length signal \vec{s}_1 extended with zeros. Like the example, evaluate tables like the one below for $k = -3$ for different shifts and plot a single period of the result.

$\vec{p}_1[n]$	-2	2	-2	2	-2	2	-2	2	-2	2
$\vec{s}_1[n+3]$	2	-2	2	-2	0	0	0	0	0	0
$\langle \vec{p}_1[n], \vec{s}_1[n+3] \rangle$										

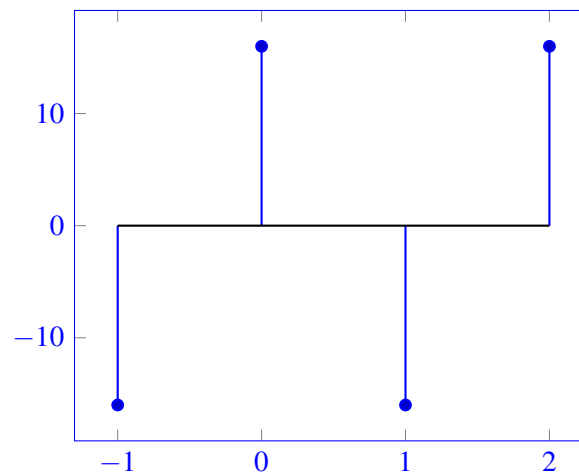
Solution: We have computed below shifts from $k = -3$ to $k = 3$. However, so long as you have enough values for a single period, give yourself full credit.

$\vec{p}_1[n]$	-2	2	-2	2	-2	2	-2	2	-2	2										
$\vec{s}_1[n+3]$	2	-2	2	-2	0	0	0	0	0	0										
$\langle \vec{p}_1[n], \vec{s}_1[n+3] \rangle$	-4	+	-4	+	-4	+	-4	+	0	+	0	+	0	+	0	+	0	+	0	= -16

$\vec{p}_1[n]$	-2	2	-2	2	-2	2	-2	2	-2	2										
$\vec{s}_1[n+2]$	0	2	-2	2	-2	0	0	0	0	0										
$\langle \vec{p}_1[n], \vec{s}_1[n+2] \rangle$	0	+	4	+	4	+	4	+	4	+	0	+	0	+	0	+	0	+	0	= 16

$\vec{p}_1[n]$	-2	2	-2	2	-2	2	-2	2	-2	2										
$\vec{s}_1[n+1]$	0	0	2	-2	2	-2	0	0	0	0										
$\langle \vec{p}_1[n], \vec{s}_1[n+1] \rangle$	0	+	0	+	-4	+	-4	+	-4	+	0	+	0	+	0	+	0	= -16		
$\vec{p}_1[n]$	-2	2	-2	2	-2	2	-2	2	-2	2										
$\vec{s}_1[n+0]$	0	0	0	2	-2	2	-2	0	0	0										
$\langle \vec{p}_1[n], \vec{s}_1[n+0] \rangle$	0	+	0	+	0	+	4	+	4	+	4	+	4	+	0	+	0	+	0	= 16
$\vec{p}_1[n]$	-2	2	-2	2	-2	2	-2	2	-2	2										
$\vec{s}_1[n-1]$	0	0	0	0	2	-2	2	-2	0	0										
$\langle \vec{p}_1[n], \vec{s}_1[n-1] \rangle$	0	+	0	+	0	+	0	+	-4	+	-4	+	-4	+	-4	+	0	+	0	= -16
$\vec{p}_1[n]$	-2	2	-2	2	-2	2	-2	2	-2	2										
$\vec{s}_1[n-2]$	0	0	0	0	0	2	-2	2	-2	0										
$\langle \vec{p}_1[n], \vec{s}_1[n-2] \rangle$	0	+	0	+	0	+	0	+	0	+	4	+	4	+	4	+	4	+	0	= 16
$\vec{p}_1[n]$	-2	2	-2	2	-2	2	-2	2	-2	2										
$\vec{s}_1[n-3]$	0	0	0	0	0	0	2	-2	2	-2										
$\langle \vec{p}_1[n], \vec{s}_1[n-3] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	-4	+	-4	+	-4	+	-4	= -16

Like the example, the period was plotted from $k = -1$ to $k = 2$. Give yourself full credit if you plotted four consecutive values sufficient for a single period, i.e. your plot starts from a shift of $k = k_0$ and ends at $k = k_0 + 3$



5. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.