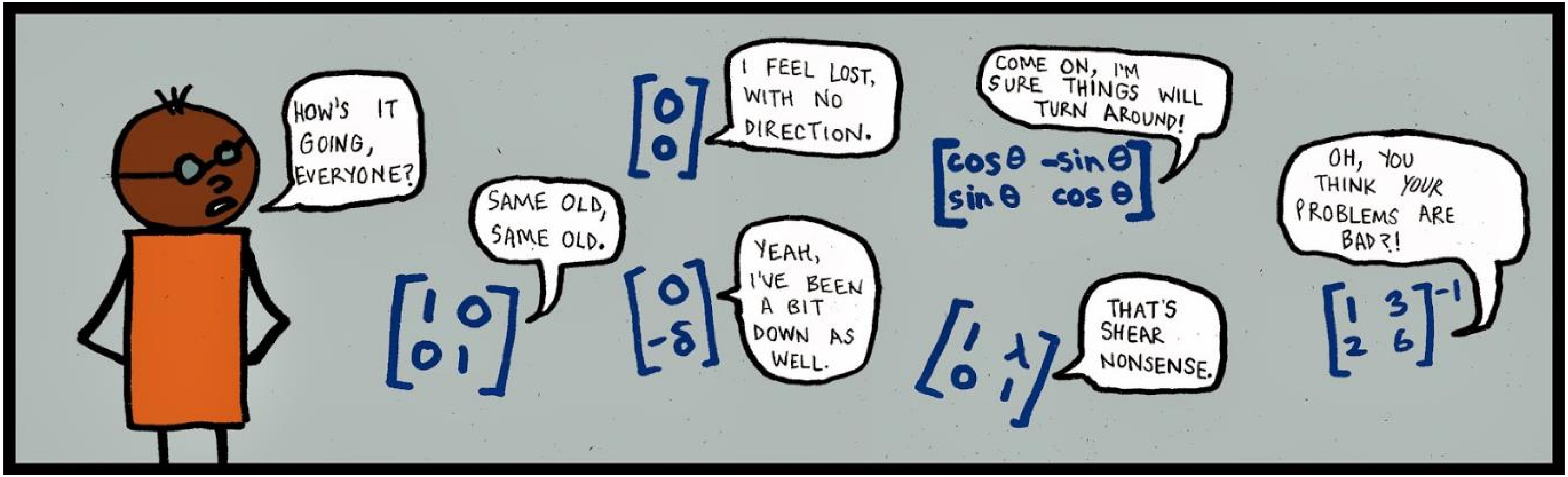


EE16A

Vector Spaces: Null-spaces and Column-spaces

Last time: Matrix inverses

$$\mathbf{A}\vec{x} = \vec{b} \longrightarrow \vec{x} = \mathbf{A}^{-1}\vec{b}$$

- If A is square and its columns are linearly independent, then A is invertible
- We can use Gaussian Elimination (Gauss-Jordan method) to find the inverse of a square matrix
- Once we have inverse, we can use it to solve system of equations

So matrix inverse is like division? Sort of, but matrix division doesn't technically exist

What if Ax=b has infinite solutions? No way to predict x from b, so A is not invertible

Equivalent Statements:

- Matrix A is **invertible**
- Ax=b has a unique solution
- A has linearly independent columns (A is **full rank**)
- A has a **trivial nullspace**
- The **determinant** of A is not zero

Jargon, old and new

The **range/span** of a set of vectors is the set of all possible linear combinations, i.e. the space of all outputs it can get to:

$$\underbrace{\text{span}\left\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_M\right\}}_{\text{the span of } \vec{a} \text{ vectors}} \underset{\text{is defined as}}{\triangleq} \left\{ \underbrace{\sum_{m=1}^{M} \alpha_m \vec{a}_m}_{\text{the set of all possible lin. combos. of the vectors}} \;\middle|\; \overbrace{\alpha_1, \alpha_2 \ldots \alpha_M \in \mathbb{R}}^{\text{coeffs. are scalars}} \right\}$$

The **dimensions** of a space tell you the degrees-of-freedom

The **rank** is the dimension of the span of the columns of A

-number of linearly independent columns of A

- must be equal or smaller than space

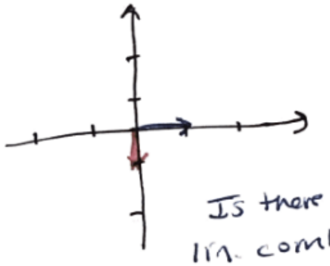
Today’s jargon

- **Nullspace** of a matrix A is the set of solutions to Ax=0
- A **vector space** is a set of vectors connected by two operators (+,x)
- A vector **subspace** is a subset of vectors that have nice properties
- A **basis** for a vector space is a minimum set of vectors needed to represent all vectors in the space
- **Dimension** of a vector space is the number of basis vectors
- **Column space** is the span(range) of the columns of a matrix
- **Row space** is the span of the rows of a matrix

$\boxed{A\vec{x}=\vec{0}}$ How many solutions does it have?

if A invertible, one unique sol'n.

Ex. $\underbrace{\begin{bmatrix}1&0\\0&-1\end{bmatrix}}_{\text{lin.indep. cols!}}\begin{bmatrix}\vec{x}\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$



what is sol'n? from graph $\vec{x}=\begin{bmatrix}0\\0\end{bmatrix}$

Is there any other lin. combo. that gives zero? No!

or do G.E. $\left[\begin{array}{cc|c}1&0&0\\0&-1&0\end{array}\right]$

$\left[\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right] \to \begin{array}{l}x_1=0\\x_2=0\end{array}$ ✓

Ex: $A=\underbrace{\begin{bmatrix}1&2\\0&0\end{bmatrix}}_{\text{cols lin. dep.}}$
→ not invertible!

$A\vec{x}=\vec{0}$ won't have unique sol'n

$\begin{bmatrix}1&2\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

→ G.E.

$\left[\begin{array}{cc|c}1&2&0\\0&0&0\end{array}\right] \to x_2=t$ ← free variable

$x_1+2x_2=0 \to x_1+2t=0$

$x_1=-2t$

So if i have a vector $\vec{v}=\begin{bmatrix}-2x_2\\x_2\end{bmatrix}=\begin{bmatrix}-2t\\t\end{bmatrix}$ ← a sol'n to the eqn for all t

$\begin{bmatrix}-2t\\t\end{bmatrix}=t\cdot\begin{bmatrix}-2\\1\end{bmatrix}$ ← $\underbrace{\text{span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\} \text{ are soln's to } \begin{bmatrix}1&2\\0&0\end{bmatrix}\vec{x}=\vec{0}}_{\text{aka } \boxed{\text{NULL SPACE}}}$

$\boxed{\text{Null Space}}$ of Matrix A: set of all solutions to $A\vec{x}=\vec{0}$

Null(A), N(A) → one example of a "vector space"

A has a trivial nullspace if $A\vec{x}=\vec{0}$ only has $\vec{0}$ as a sol'n (*zero is always a sol'n)

$\boxed{\text{How does this connect to invertibility?}}$

If $A\vec{x}=\vec{0}$ has unique sol'n (A invertible) → trivial nullspace

$\boxed{\text{Is null space always a span?}}$

will always be a vector space, which can be written as a span

---

Null space can help us characterize all the sol'ns to $A\vec{x}=\vec{b}$.

$A\vec{x}=\vec{b} \longrightarrow A\vec{x}_1=b$
↑some sol'n

If i know $\vec{v}_0 \in \text{Null}(A)$ ← some vector

$A\vec{v}_0=\vec{0}$

Then $\vec{x}_1+t\cdot\vec{v}_0 \Rightarrow$ also a sol'n ← tells set of sol'ns.

because $A(\vec{x}_1+t\cdot\vec{v}_0)=\vec{b}$

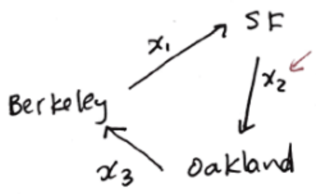
$A\vec{x}_1+A\cdot t\cdot\vec{v}_0=\vec{b}$ (scalar)

$A\vec{x}_1+t\underbrace{A\vec{v}_0}_{\vec{0}\text{ (in nullspace!)}}=\vec{b}$

$A\vec{x}_1=\vec{b}$
↑particular sol'n

Nullspace will be very useful in Module 2!

Traffic



cars going SF to OAK

measuring for better flow → Bayesian → Varaiya

"what goes in must come out?"
Assume no accumulation (no parking)

so
$$x_1 - x_2 = 0$$
$$x_2 - x_3 = 0$$
$$x_3 - x_1 = 0$$

What measurements do i need to find x_is?
At least one...

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

→ find null space x_1, x_2, x_3

$$\xrightarrow{R3+R1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R2+R3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Infinite sol'ns!

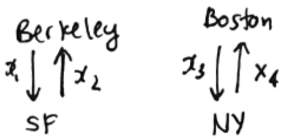
basic

free

How many meas. do you need to understand # of cars flowing?
One! x_1 predicts x_2, x_3

dim(Nullspace) → # free varia

What if it was!



Now need at least 2 meas.

What would be dim(Nullspace)?
2 free variables.

$\boxed{\text{Vector spaces}}$

Take a set of ~~vectors~~ elements (e.g. vectors) denoted by $\mathbb{V}$, a set of scalars $\mathbb{F}$ (e.g. $\alpha_1, \alpha_2 \ldots \alpha_N \in \mathbb{R}$), and two operators: $(+)$ plus? $(\cdot)$ multiply?

two operators ↳ operates on pairs of elements

so ~~vectors~~ elements can interact

↳ defined over some $\alpha \in \mathbb{R}$ (scalars) and element $\vec{v} \in \mathbb{V}$ (element is in vector space) $\Rightarrow \alpha \cdot \vec{v}$

We say that $\mathbb{V}$ (with the two operators) is a VECTORSPACE if the following properties (axioms) are satisfied:

axioms ↳ statement that is established/accepted

$\forall \vec{x}, \vec{y}, \vec{z} \in \mathbb{V}$ and $\forall \alpha, \beta \in \mathbb{R}$:

for all; 3 vectors that are in the set; for all; scalars that are real

"Axioms of Closure" (can't escape the set)

① $\alpha \vec{x} \in \mathbb{V}$ → scaling an element stays in the set

② $\vec{x} + \vec{y} \in \mathbb{V}$ → adding elements stays in the set

"Axioms of Addition" $(+)$

③ $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$ → Associative
↳ order of addition doesn't matter

④ $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ → Commutative

⑤ $\exists\, \vec{0} \in \mathbb{V}$ s.t. $\vec{x} + \vec{0} = \vec{x}$ → Additive Identity

there exists; some zero vector (origin); in the set; such that; if I add zero to an element it doesn't change

⑥ $\exists\, (-\vec{x}) \in \mathbb{V}$ s.t. $\vec{x} + (-\vec{x}) = \vec{0}$ → Additive Inverse

there exists; a minus x element; in the set; adding to element takes you back to zero

"Axioms of Scaling" $(\cdot)$

⑦ Distributivity: $\alpha \cdot (\vec{x} + \vec{y}) = \alpha \cdot \vec{x} + \alpha \cdot \vec{y}$ (vector addition)

⑧ Distributivity: (scalar addition) $(\alpha + \beta) \cdot \vec{x} = \alpha \cdot \vec{x} + \beta \cdot \vec{x}$

⑨ Associative: (multiplication) $\alpha \cdot (\beta \cdot \vec{x}) = (\alpha \cdot \beta) \vec{x}$ ← this operator b/w scalar & vector

⑩ Multiplicative Identity: $\exists\, 1 \in \mathbb{F}$ where $1 \cdot \vec{x} = \vec{x}$

This is called multiplicative identity

$\boxed{\text{Is } \mathbb{R} \text{ a vector space? Is } \mathbb{R}^n\text{?}}$ yes.

Example 1: $\underset{\text{the set}}{\mathbb{V}_1} = \left\{ \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{all 2x2 matrices}} \;\middle|\; \underbrace{a, b, c, d \in \mathbb{R}}_{\text{whose elements are real scalars}} \right\} = \mathbb{R}^{2\times 2}$

given

$\boxed{\text{Is it a vector space?}}$ Yes.

Let's verify from the definition:

✓① $\alpha \vec{x} \in \mathbb{V}_1$ $\quad \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$ ← still in set!

✓② $\vec{x} + \vec{y} \in \mathbb{V}_1$ $\quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$ ← still in set!

✓③ $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$ Yes, matrix addition for 2x2 is Associative!

✓④ $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ Also commutative

✓⑤ zeros element s.t. $\vec{x} + \vec{0} = \vec{x}$ yes! $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

✓⑥ minus $\vec{x}$ element s.t. $\vec{x} + (-\vec{x}) = \vec{0}$ yes! $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = -\vec{x}$

✓⑦ $\alpha \cdot (\vec{x} + \vec{y}) = \alpha \cdot \vec{x} + \alpha \cdot \vec{y}$
(distributive in vector addition)

✓⑧ $(\alpha + \beta) \cdot \vec{x} = \alpha \cdot \vec{x} + \beta \cdot \vec{x}$ distributive in scalar addition

e.g. $(\alpha + \beta) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (\alpha+\beta)a & (\alpha+\beta)b \\ (\alpha+\beta)c & (\alpha+\beta)d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix} + \begin{bmatrix} \beta a & \beta b \\ \beta c & \beta d \end{bmatrix}$

$= \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \beta \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

✓⑨ Associative (multiplication): $\alpha \cdot (\beta \cdot \vec{x}) = (\alpha \beta) \cdot \vec{x}$

✓⑩ Multiplicative Identity: $\exists\, 1 \in \mathbb{F}$ where $1 \cdot \vec{x} = \vec{x}$

All axioms are satisfied

$\therefore$ $\mathbb{V}_1 = \mathbb{R}^{2\times 2}$ with operators $(+)$ and $(\cdot)$ is a vector space!

Example 2: $\mathbb{V}_2 = \left\{ \underset{\text{scalars}}{\alpha} \;\middle|\; \underbrace{\alpha \geq 0}_{\text{all non-negative}},\ \alpha \in \mathbb{R} \right\} = \mathbb{R}^+$ → all non-negative real #s

& real

$\boxed{\text{Is it a vector space?}}$ No!

For example $\underbrace{3 \in \mathbb{V}_2, \text{ but } -3 \notin \mathbb{V}_2}$ ← is not in the set

axiom ⑥ is not satisfied! $\therefore$ Not a vector space

Example 3: $\mathbb{V}_3 = \{\underbrace{f : \mathbb{R} \to \mathbb{R}}_{\text{all the functions that map real to real}}\}$

e.g.
$f(x) = x$
$f(x) = 2x$
$f(x) = x^2$
$f(x) = e^x$

$\forall \underbrace{f, g, h}_{\text{elements in set}} \in \mathbb{V}_3$ and $\alpha, \beta \in \mathbb{R}$

$\boxed{\text{Is it a vector space?}}$

Let's check axioms:

✓ ① $\ell \triangleq \alpha f$ $\boxed{\text{Is it still real?}}$ yes!
some real function; is defined as

✓② $\ell \triangleq f + g$ $\boxed{\text{Is it still real?}}$ yes

✓③ $\ell \triangleq f + (g+h)$

is ~~output (for real input) real? yes~~

$\ell : \mathbb{R} \to \mathbb{R}$

some function that has real input & output.

1st evaluate $f(x), g(x), h(x)$
↳ add them

$\boxed{\text{does order matter?}}$ no!

$f + (g+h) = (f+g) + h$ ✓

✓④

✓⑤ zero function $\phi(x) \triangleq 0$ satisfies this & it is real (in set)!

✓⑥ minus element $\ell(x) \triangleq -f(x)$ $\boxed{\text{Is it real?}}$ yes

$f(x) + (-f(x)) = \phi(x)$ ✓

✓⑦ $\alpha(f+g) = \alpha f + \alpha g$

$\boxed{\text{What if I have : } f(x) = e^x?}$

$\ell(x) \neq e^{-x}$ No!

$\ell(x) = -e^x$

✓⑧ $(\alpha + \beta) \cdot f = \alpha f + \beta f$

✓⑨ $\alpha \cdot (\beta \cdot f) = (\alpha \cdot \beta) \cdot f$

✓⑩ $1 \cdot f = f$

$\therefore \mathbb{V}_3$ is a vector space!

$\boxed{\text{Subspaces}}$ Want to narrow down vector space to smaller set of possibilities, but still keep 'nice' properties.

A subspace $\mathbb{U}$ consists of a subset of the set $\mathbb{V}$ in the vector space $(\mathbb{V}, \mathbb{F})$ (including operations $(\cdot), (+)$) $\rightarrow$ can write as $\mathbb{U} \subset \mathbb{V}$

set of vectors — scalars — a linear vector subspace — is a subset

e.g. neighborhood $\subset$ city

Subspaces must have 3 properties:

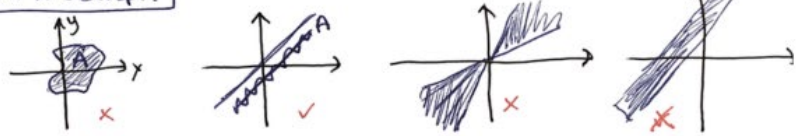
① Contains the zero vector $\vec{0} \in \mathbb{U}$

② Closed under vector addition: for any two vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{U}$, their sum is also in subspace $\rightarrow$ $\vec{v}_1 + \vec{v}_2 \in \mathbb{U}$

③ Closed under scalar multiplication: for any vector $\vec{v} \in \mathbb{U}$ and scalar $\alpha \in \mathbb{F}$ the product $\alpha \cdot \vec{v}$ must also be in $\mathbb{U}$ $\rightarrow$ $\alpha \cdot \vec{v} \in \mathbb{U}$

can't escape subset! (by $+, \cdot$)

$\boxed{\text{Is region A a subspace?}}$



"A subspace is a sub set of vectors in a vector space ~~that~~ where any linear combination of the vectors in the subset lies within the set."

Recall: Def'n RANGE of matrix $A \in \mathbb{R}^{n \times m}$ is the space of all outputs that the operator A can map to.

Range $(A) \subset \mathbb{R}^n$, but $\boxed{\text{is it a subspace?}}$

is a subset of

Let's check:

① $A\vec{0} = \vec{0}$, so $\vec{0}$ is in range(A) ✓

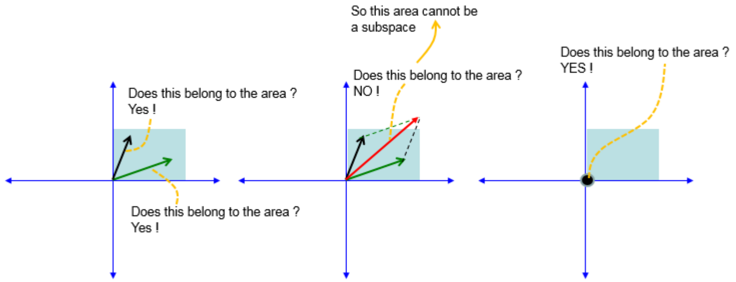
② if $\vec{v}_1, \vec{v}_2$ in range(A) then there exists $\vec{u}_1 \in \mathbb{R}^m$ s.t. $A\vec{u}_1 = \vec{v}_1$ and $\vec{u}_2 \in \mathbb{R}^m$ s.t. $A\vec{u}_2 = \vec{v}_2$

$\vec{v}_1 + \vec{v}_2 = A\vec{u}_1 + A\vec{u}_2 = A(\vec{u}_1 + \vec{u}_2)$ due to distributive

$\therefore \vec{v}_1 + \vec{v}_2$ in range ✓

③ if $\vec{v}$ is in range(A), then there exists $\vec{u} \in \mathbb{R}^m$ such that $A\vec{u} = \vec{v}$ $\rightarrow$ $\alpha \vec{v} = \alpha A \vec{u} = A(\alpha \vec{u})$ ✓ in range

$\therefore$ range(A) is a subspace.



Example 1 Consider $\mathbb{W}_1 = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| \, a,b,d \in \mathbb{R} \right\}$ set of all upper triangular 2x2 matrices

$\mathbb{W}_1 \subseteq \mathbb{V}_1$ so just need to check a few axioms (3 properties for subspace)

is a subset — all 2x2 matrices

① contains zero vector $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is in $\mathbb{W}_1$ ✓ still upper triangular

② $\vec{v}_1 + \vec{v}_2 \in \mathbb{W}_1$

$\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ 0 & c_1+c_2 \end{bmatrix}$ ← still upper triangular ✓

③ $\alpha \vec{v} \in \mathbb{W}_1$

$\alpha \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ 0 & \alpha d \end{bmatrix}$ ← stillllll upper triangular ✓

$\boxed{\text{Why do we need zero?}}$

I don't want to call empty set a subspace, so just ensures that it's not empty (convention)

$\therefore \mathbb{W}_1$ is a subspace.

Example 2 Consider $\mathbb{W}_3 = \{ f: \mathbb{R} \to \mathbb{R} \mid f(2) = 0 \}$

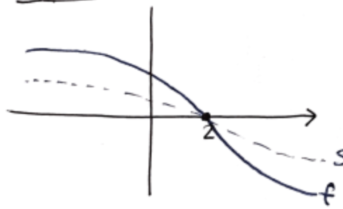
functions map real → real ↑ but all have $f(2)=0$

$\mathbb{W}_3 \subset \mathbb{V}_3$

subset — all real functions

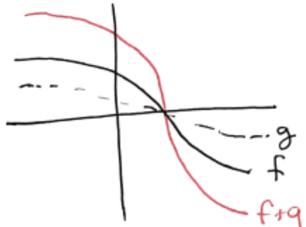
Check 3 axioms:

③



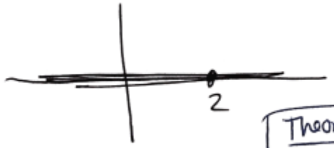
scaling ½f → $\alpha f \in \mathbb{W}_3$ ✓

②



f+g (add at each point) → still in set ✓

③



✓ all zeros

Theorem: Let $\mathbb{V}$ be a v.s. and $\{\vec{v}_1, \vec{v}_2 \ldots \vec{v}_N\} \subset \mathbb{V}$, then span $\{\vec{v}_1, \vec{v}_2 \ldots \vec{v}_N\}$ is ALWAYS a v.s. (and a subspace of $\mathbb{V}$)

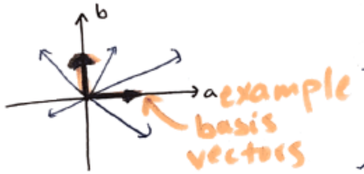
Bases → the minimum set of vectors needed to represent all vectors in the vector space

A set of vectors ~~defines~~ makes up a vector space. But maybe some vectors in the set are redundant (linearly dependent).

What is most efficient representation of vector space?

e.g. $\left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0.5\\-0.7\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}\pi\\e\end{bmatrix}, \begin{bmatrix}-6\\4\end{bmatrix}, \begin{bmatrix}4\\-6\end{bmatrix} \right\}$ → lots are lin. dep.!

Consider vector space is entire 2D plane: $\mathbb{V} = \left\{ \begin{bmatrix}a\\b\end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$



} any two vectors can span the vector space

→ Lots of choices of vectors span the space. But we don't need them all. Just pick two!

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* **Definition:** Given a vector space $(\mathbb{V}, \mathbb{F})$, a set of vectors $\{\vec{v}_1, \vec{v}_2 \dots \vec{v}_n\}$

($\mathbb{V}$: set of elements; $\mathbb{F}$: set of scalars)

is a BASIS of the vector space if:

- $\vec{v}_1, \vec{v}_2 \dots \vec{v}_n$ are linearly independent vectors,

AND
- for any vector $\vec{v} \in \mathbb{V}$, there exist scalars $\alpha_1, \alpha_2 \dots \alpha_n \in \mathbb{F}$ such that $\vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 \dots + \alpha_n \vec{v}_n$

any vector in the space can be written as a linear combo. of the basis vectors

"lin. ind. AND spans the vector space"

If I pick basis vectors $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$, is it unique?

No, for example could pick any two lin. ind. vectors in 2D space.

Dimension The dimension of a vector space is the number of basis vectors.

Does dimension change if I pick different basis vectors? No. Will always have same number of basis vectors! (Related to Rank of Matrix...)

→ If it could be represented w/ fewer, then not lin. ind. and ∴ not a basis

→ Any set with fewer vectors than 2 (from ex. above $\mathbb{R}^2$) will not span the vector space so is not a basis set. (can't describe

COLUMN SPACE: the span of the columns of a matrix ($\rightarrow$ = RANGE)

ROW SPACE: the span of the rows of a matrix

Are they both subspaces? Yes.

Do subspaces have bases? Yes. It is ~~the~~ a set of linearly independent vectors that span the subspace.
<u>Dimension</u> of subspace is # vectors in bases.

<u>Example:</u>

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{range}(A) = \left\{ \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

real scalars

3x3 $\rightarrow$ input space has dimension of 3

dim. output space?

this one is useless!

these two lin. ind

So $\boxed{\dim(\text{range}(A)) = 2}$

Can dim(range) be larger than dim(A)? No!

Where does extra dimension go? To the **NULLSPACE**

Nullspace of A consists of all vectors $\vec{x}$ in $\mathbb{R}^m$ such that $A\vec{x} = \vec{0}$

$$N(A) = \{ \vec{x} \mid A\vec{x} = \vec{0},\ \vec{x} \in \mathbb{R}^m \}$$

Null space; of matrix A; is the set of; vectors; input; output; that get mapped to zero by A (put $\vec{x}$ in, get $\vec{0}$ out) $\rightarrow$ means $\vec{x}$ not measurable

What is dimension of the nullspace? Should be $\leq m$ $\leftarrow$ # components in all input vectors.

However, unless A is zero matrix, then not all $\vec{x} \longrightarrow \vec{0}$, so should be $< m$

How many independent ways can we create the zero vector by taking lin. combos of the cols of A?

Recall $Ax = b$ $\leftarrow$ $\vec{0}$ means not measured $\rightarrow$ null space is things you can't measure with this system