EECS 16A   Designing Information Devices and Systems I
Spring 2021
Homework 11

This homework is due April 16, 2020, at 23:59.

Self-grades are due April 19, 2020, at 23:59.

Midterm 2 re-do is due April 16, 2020, at 23:59. This homework is intentionally short to give you time to re-do and review the midterm.

Submission Format

Your homework submission should consist of **one** file.

- `hw11.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned)

Submit each file to its respective assignment on Gradescope.

1. **Reading Assignment**

   For this homework, please read Notes 21 and 22 to learn about inner products, norms, trilateration, and correlation. You are always encouraged to read beyond this as well.

   (a) What does it mean for two vectors $\vec{x}$ and $\vec{y}$ to be orthogonal, in terms of their inner product?

2. **Inner Product Properties**

   Learning Goal: *The objective of this problem is to exercise useful identities for inner products.*

   Our definition of the inner product in $\mathbb{R}^n$ is:

   $$\langle \vec{x}, \vec{y} \rangle \;=\; x_1 y_1 + x_2 y_2 + \ldots + x_n y_v \;=\; \vec{x}^\mathsf{T}\vec{y}, \quad \text{for any } \vec{x}, \vec{y} \in \mathbb{R}^n$$

   Prove the following identities in $\mathbb{R}^n$:

   (a) $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

   (b) $\langle \vec{x}, \vec{x} \rangle = \|\vec{x}\|^2$

   (c) $\langle -\vec{x}, \vec{y} \rangle = -\langle \vec{x}, \vec{y} \rangle.$

   (d) $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

   (e) $\langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle = \langle \vec{x}, \vec{x} \rangle + 2\langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{y} \rangle$

3. Cauchy-Schwarz Inequality

Learning Goal: *The objective of this problem is to understand and prove the Cauchy-Schwarz inequality for real-valued vectors.*

The Cauchy-Schwarz inequality states that for two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$:

$$|\langle \vec{v}, \vec{w} \rangle| = |\vec{v}^T \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\|$$

In this problem we will prove the Cauchy-Schwarz inequality for vectors in $\mathbb{R}^2$.

Take two vectors: $\vec{v} = r \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\vec{w} = t \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$, where $r > 0$, $t > 0$, θ, and ϕ are scalars. Make sure you understand why any vector in $\mathbb{R}^2$ can be expressed this way and why it is acceptable to restrict $r, t > 0$.

(a) In terms of some or all of the variables r, t, θ, and ϕ, what are $\|\vec{v}\|$ and $\|\vec{w}\|$?

(b) In terms of some or all of the variables r, t, θ, and ϕ, what is $\langle \vec{v}, \vec{w} \rangle$? *Hint: The trig identity* $\cos(a)\cos(b) + \sin(a)\sin(b) = \cos(a-b)$ *may be useful.*

(c) Show that the Cauchy-Schwarz inequality holds for any two vectors in $\mathbb{R}^2$. *Hint: consider your results from part (b). Also recall* $-1 \leq \cos x \leq 1$ *and use both inequalities.*

(d) Note that the inequality states that the inner product of two vectors must be less than *or equal to* the product of their magnitudes. What conditions must the vectors satisfy for the equality to hold? In other words, when is $\langle \vec{v}, \vec{w} \rangle = \|\vec{v}\| \cdot \|\vec{w}\|$?

4. Mechanical Linear Correlation

Learning Goal: *The objective of this problem is to understand how to compute the linear correlation between signals.*

We recall that the linear correlation of signal $\vec{y}$ with signal $\vec{x}$ is given as:

$$\text{corr}_{\vec{x}}(\vec{y})[k] = \sum_{n=-\infty}^{\infty} \vec{x}[n]\vec{y}[n-k]$$

Signal $\vec{s}_1$ and Signal $\vec{s}_2$
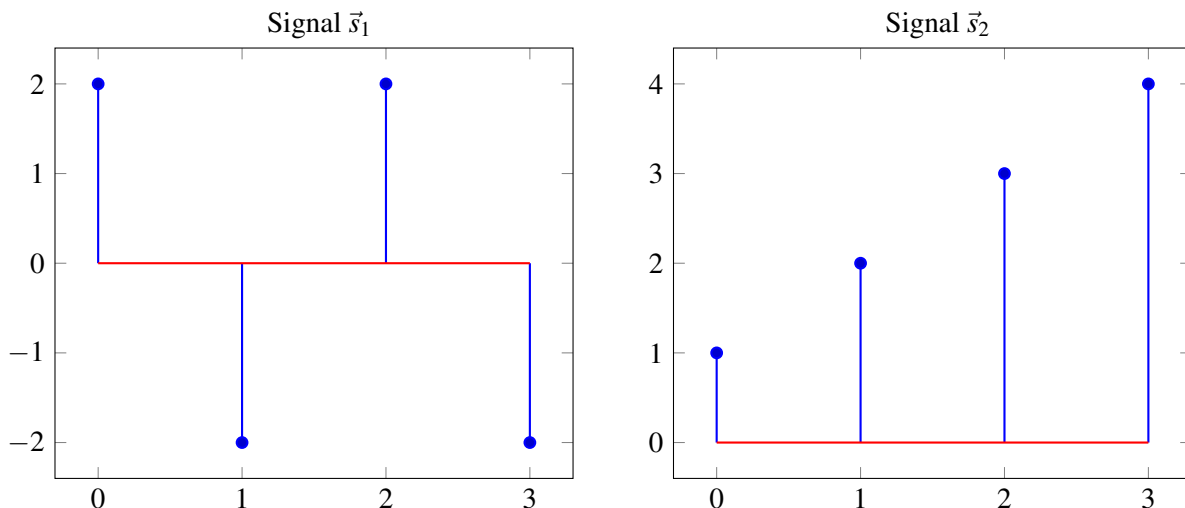
Assume that both of the above signals extend to $\pm\infty$, and are 0 everywhere outside of the region shown in the above graphs. First, we will demonstrate the procedure for linear correlation by computing the linear correlation between signal $\vec{s}_1$ with itself (*i.e.* $\text{corr}_{\vec{s}_1}(\vec{s}_1)[k]$). This is referred to as the linear autocorrelation. This can be computed by evaluating the inner product between the signal and the shifted version of the signal (outlined in the below tables). Here, we compute this quantity for shifts between -3 and 3. For all shifts outside this range, the inner product is zero. Finally, we plot the non-zero values of the linear autocorrelation.

$\vec{s}_1[n]$	0		0		0		2		-2		2		-2		0		0		0	
$\vec{s}_1[n+3]$	2		-2		2		-2		0		0		0		0		0		0	
$\langle\vec{s}_1[n],\vec{s}_1[n+3]\rangle$	0	+	0	+	0	+	-4	+	0	+	0	+	0	+	0	+	0	+	0	= -4

$\vec{s}_1[n]$	0		0		0		2		-2		2		-2		0		0		0	
$\vec{s}_1[n+2]$	0		2		-2		2		-2		0		0		0		0		0	
$\langle\vec{s}_1[n],\vec{s}_1[n+2]\rangle$	0	+	0	+	0	+	4	+	4	+	0	+	0	+	0	+	0	+	0	= 8

$\vec{s}_1[n]$	0		0		0		2		-2		2		-2		0		0		0	
$\vec{s}_1[n+1]$	0		0		2		-2		2		-2		0		0		0		0	
$\langle\vec{s}_1[n],\vec{s}_1[n+1]\rangle$	0	+	0	+	0	+	-4	+	-4	+	-4	+	0	+	0	+	0	+	0	= -12

$\vec{s}_1[n]$	0		0		0		2		-2		2		-2		0		0		0	
$\vec{s}_1[n+0]$	0		0		0		2		-2		2		-2		0		0		0	
$\langle\vec{s}_1[n],\vec{s}_1[n+0]\rangle$	0	+	0	+	0	+	4	+	4	+	4	+	4	+	0	+	0	+	0	= 16

$\vec{s}_1[n]$	0		0		0		2		-2		2		-2		0		0		0	
$\vec{s}_1[n-1]$	0		0		0		0		2		-2		2		-2		0		0	
$\langle\vec{s}_1[n],\vec{s}_1[n-1]\rangle$	0	+	0	+	0	+	0	+	-4	+	-4	+	-4	+	0	+	0	+	0	= -12

$\vec{s}_1[n]$	0		0		0		2		-2		2		-2		0		0		0	
$\vec{s}_1[n-2]$	0		0		0		0		0		2		-2		2		-2		0	
$\langle\vec{s}_1[n],\vec{s}_1[n-2]\rangle$	0	+	0	+	0	+	0	+	0	+	4	+	4	+	0	+	0	+	0	= 8

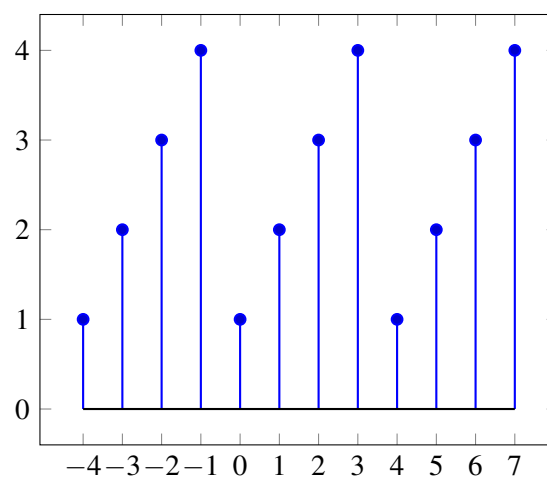| $\vec{s}_1[n]$	0		0		0		2		-2		2		-2		0		0		0	
$\vec{s}_1[n-3]$	0		0		0		0		0		0		2		-2		2		-2	
$\langle\vec{s}_1[n],\vec{s}_1[n-3]\rangle$	0	+	0	+	0	+	0	+	0	+	0	+	-4	+	0	+	0	+	0	= -4
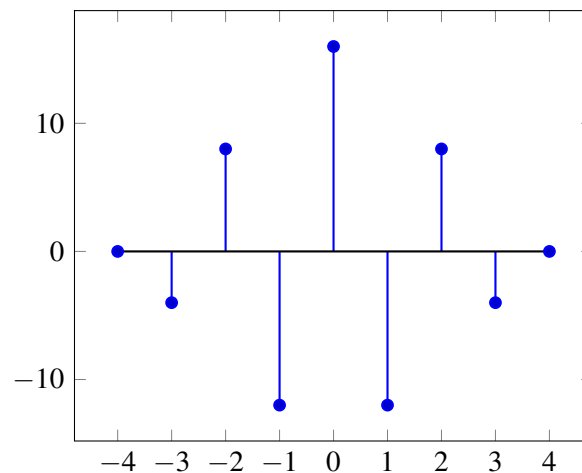
(a) Using the procedure demonstrated above, compute $\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$, the linear cross-correlation of $\vec{s}_2$ with $\vec{s}_1$. Like the example, use tables like the one given below for $k = -3$ and plot the resulting correlation.

$\vec{s}_1[n]$	0	0	0	2	-2	2	-2	0	0	0
$\vec{s}_2[n+3]$	1	2	3	4	0	0	0	0	0	0
$\langle \vec{s}_1[n], \vec{s}_2[n+3] \rangle$										

(b) Will the linear cross-correlation of $\vec{s}_2$ with $\vec{s}_1$ ($\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$) be the same as the cross-correlation of $\vec{s}_1$ with $\vec{s}_2$ ($\text{corr}_{\vec{s}_2}(\vec{s}_1)[k]$)? You can use the iPython notebook **prob11.ipynb** to figure this out. How are they related to each other?

Now, we will review the procedure to perform linear cross-correlation between one signal that is periodic with a period of 4 and another that is finite length and extended with zeros as in the previous parts. As an example, we will compute the linear correlation $\text{corr}_{\vec{p}_2}(\vec{s}_1)[k]$ between the periodic signal $\vec{p}_2$ (with period 4), formed by repeating $\vec{s}_2$, and the finite length signal $\vec{s}_1$ extended with zeros. The result will be a periodic signal with period 4.
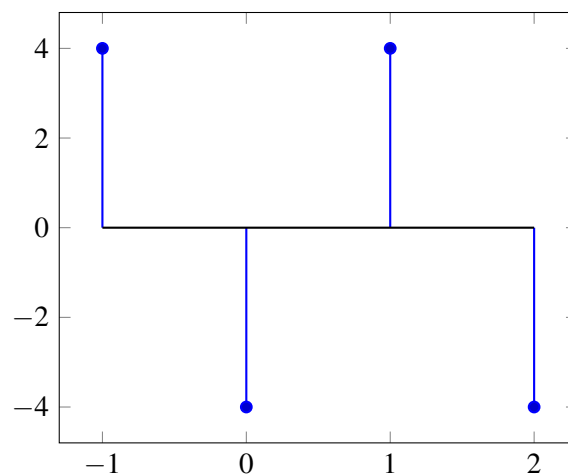
The periodic signal, $\vec{p}_2$, formed by repeating $\vec{s}_2$ is plotted below for indices -4 to 7. It is defined and non-zero for all indices from $-\infty$ to $+\infty$.


We compute one period of the result of the cross-correlation by starting at a shift of $k = -1$ and ending at a shift of $k = 2$.

$\vec{p}_2[n]$	2		3		4		1		2		3		4		1		2		3	
$\vec{s}_1[n+1]$	0		0		2		-2		2		-2		0		0		0		0	
$\langle \vec{p}_2[n], \vec{s}_1[n+1]\rangle$	0	+	0	+	8	+	-2	+	4	+	-6	+	0	+	0	+	0	+	0	= 4

$\vec{p}_2[n]$	2		3		4		1		2		3		4		1		2		3	
$\vec{s}_1[n+0]$	0		0		0		2		-2		2		-2		0		0		0	
$\langle \vec{p}_2[n], \vec{s}_1[n+0]\rangle$	0	+	0	+	0	+	2	+	-4	+	6	+	-8	+	0	+	0	+	0	= -4

$\vec{p}_2[n]$	2		3		4		1		2		3		4		1		2		3	
$\vec{s}_1[n-1]$	0		0		0		0		2		-2		2		-2		0		0	
$\langle \vec{p}_2[n], \vec{s}_1[n-1]\rangle$	0	+	0	+	0	+	0	+	4	+	-6	+	8	+	-2	+	0	+	0	= 4

$\vec{p}_2[n]$	2		3		4		1		2		3		4		1		2		3	
$\vec{s}_1[n-2]$	0		0		0		0		0		2		-2		2		-2		0	
$\langle \vec{p}_2[n], \vec{s}_1[n-2]\rangle$	0	+	0	+	0	+	0	+	0	+	6	+	-8	+	2	+	-4	+	0	= -4

The computed single period of the resulting linear cross correlation is plotted below.

The resulting linear cross correlation for shifts from $k = -4$ to $k = 7$ is plotted below.
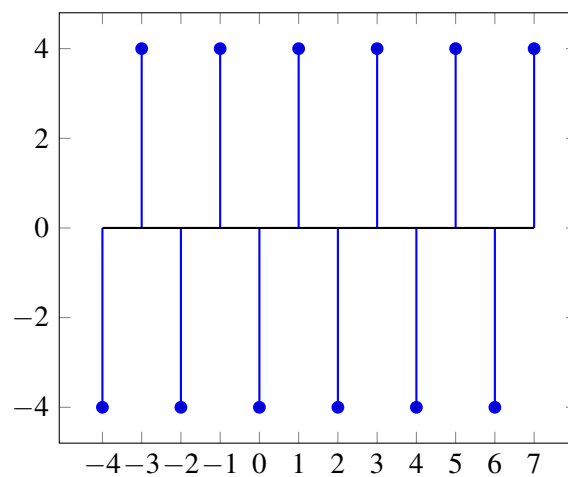
(c) Repeat the procedure described above to compute the correlation $\text{corr}_{\vec{p}_1}(\vec{s}_1)[k]$ between a periodic signal $\vec{p}_1$ (with period 4), formed by repeating $\vec{s}_1$, and the finite-length signal $\vec{s}_1$ extended with zeros. Like the example, evaluate tables like the one below for $k = -3$ for different shifts and plot a single period of the result.

$\vec{p}_1[n]$	-2	2	-2	2	-2	2	-2	2	-2	2
$\vec{s}_1[n+3]$	2	-2	2	-2	0	0	0	0	0	0
$\langle \vec{p}_1[n], \vec{s}_1[n+3] \rangle$										

5. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.