EECS 16A   Designing Information Devices and Systems I
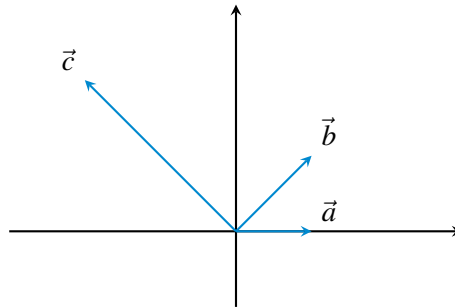Spring 2022
                                                                Discussion 2B

1. Visualizing Span

We are given a point $\vec{c}$ that we want to get to, but we can only move in two directions: $\vec{a}$ and $\vec{b}$. We know that to get to $\vec{c}$, we can travel along $\vec{a}$ for some amount α, then change direction, and travel along $\vec{b}$ for some amount β. We want to find these two scalars α and β, such that we reach point $\vec{c}$. That is, $\alpha\vec{a} + \beta\vec{b} = \vec{c}$.

(a) Formulate the system of equations as a matrix to find the unknowns, α, β, in terms of the vectors $\vec{a}, \vec{b}, \vec{c}$.

Answer:

$$\begin{cases} \alpha a_1 + \beta b_1 = c_1 \\ \alpha a_2 + \beta b_2 = c_2 \end{cases}$$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \vec{c} \tag{1}$$

(b) First, consider the case where $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\vec{c} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Draw these vectors on a sheet of paper. Now find the two scalars α and β, such that we reach point $\vec{c}$. What are these scalars if we use $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ instead?
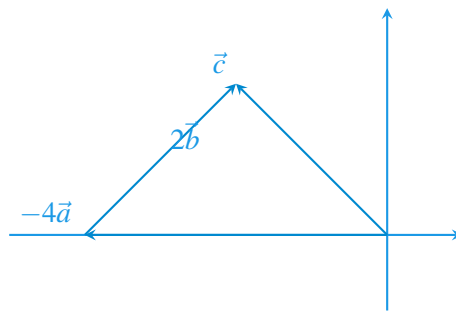
Answer: To solve for α and β, we can set up a system of equations

$$\begin{cases} \alpha + \beta = -2 \\ \beta = 2 \end{cases}$$

By substitution, we get $\beta = 2, \alpha = -4$. Therefore, the system can be written as

$$-4\vec{a} + 2\vec{b} = \vec{c}$$
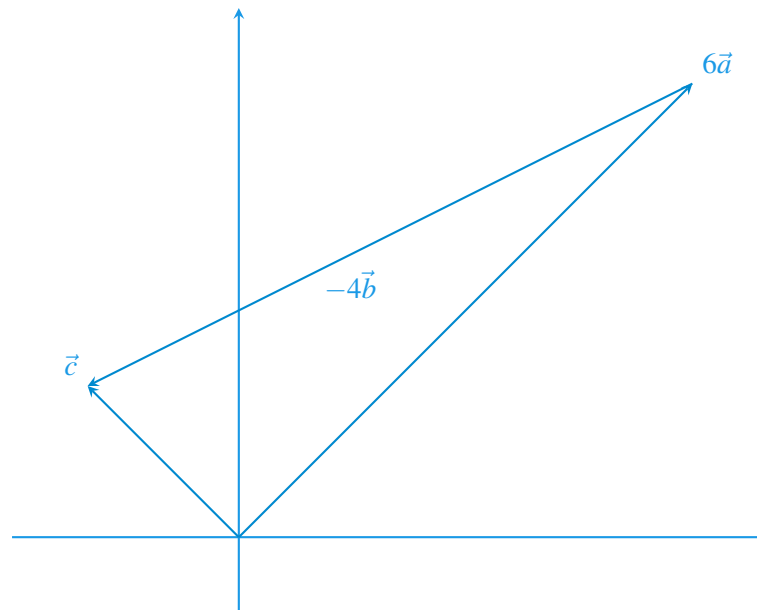
We can visualize this graphically

Similarly, we can write the next part as

$$\begin{cases} \alpha + 2\beta = -2 \\ \alpha + \beta = 2 \end{cases}$$

We can setup an augmented matrix to solve this

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 1 & 1 & 2 \end{array}\right]$$

Solving this, we get $\alpha = 6, \beta = -4$ Now we represent $6\vec{a} - 4\vec{b} = \vec{c}$

2. Span Basics

(a) What is span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

Answer: span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ contains any vector $\vec{v}$ that can be written as

$$\vec{v} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

We realize that any vector whose last component is 0 can be written in this form and any vector whose last component is nonzero cannot. Hence, the required span is the set of all vectors that can be written in the form $\begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$.

(b) Is $\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ in span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

Answer: From the definition of span, we know that if we can express $\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ as a linear combination

of $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, it is in the span. Assume such a linear combination exists, and the two vectors are respectively multiplied by α_1 and α_2, we can set up such an equation

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$$

. We can solve this by writing it into augmented matrix

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Solving this with Gaussian Elimination, we get

$$\begin{bmatrix} 1 & 0 & \frac{5}{3} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, we know $\alpha_1 = \frac{5}{3}$, $\alpha_2 = \frac{5}{3}$, so we can conclude the statement is true.

(c) What is a possible choice for $\vec{v}$ that would make span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \vec{v} \right\} = \mathbb{R}^3$?

Answer: From part (a), we realize that any vector whose last component is 0 can be written as a linear combination of the two vectors already in the set. Hence, if we include, for example, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ into the set, then we should be able to reach any vector in $\mathbb{R}^3$. Any vector whose last component is non-zero is a valid addition to the set to achieve the desired span.

(d) For what values of b_1, b_2, b_3 is the following system of linear equations consistent? ("Consistent" means there is at least one solution.)
$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Answer: For the system of linear equations to be consistent, there must exist some x such that the equality above holds. Performing matrix vector multiplication, we can rewrite the above equality as

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \vec{b}$$

The question now becomes: which vectors $\vec{b}$ can be written in the above form, i.e as a linear combination of the columns of A? This is exactly the definition of span, and the answer must be the same as that from part (a).

3. **Span Proofs**

   Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

   (a)
   $$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}, \text{ where } \alpha \text{ is a non-zero scalar}$$

   In other words, we can scale our spanning vectors and not change their span.

   (b) (for practice)
   $$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_2, \vec{v}_1, \ldots, \vec{v}_n\}$$

   In other words, we can swap the order of our spanning vectors and not change their span.

   Answer:

   (a) Suppose we have some arbitrary $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars a_i:

   $$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \left(\frac{a_1}{\alpha}\right)\alpha\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n.$$

   Scalar multiplication cancels out. Thus, we have shown that $\vec{q} \in \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we have $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Now, we must show the other direction. Suppose we have some arbitrary $\vec{r} \in \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars b_i:

   $$\vec{r} = b_1(\alpha\vec{v}_1) + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n = (b_1\alpha)\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n.$$

   Thus, we have shown that $\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we now have $\text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Combining this with the earlier result, the spans are thus the same.

   (b) Suppose $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars a_i:

   $$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = a_2\vec{v}_2 + a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$$

   Swapping the order in addition does not affect the sum, so $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_2, \vec{v}_1, \ldots, \vec{v}_n\}$. Similarly, starting with some $\vec{r} \in \text{span}\{\vec{v}_2, \vec{v}_1, \ldots, \vec{v}_n\}$, again swapping the order does not affect the sum, so putting both together, the spans are thus the same.