EECS 16B Designing Information Devices and Systems II
Summer 2020 Note 4

1 Matrix Form

1.1 Introduction

Last time we saw how to solve simple scalar first order linear differential equations. Now, we are going to expand our tool set and learn how to tackle multivariable differential equations. (You will see in the homework that these exact same ideas will also equip us to deal with higher-order differential equations where we have second and third derivatives involved.) Let's motivate the need to understand two dimensional systems with the following circuit.

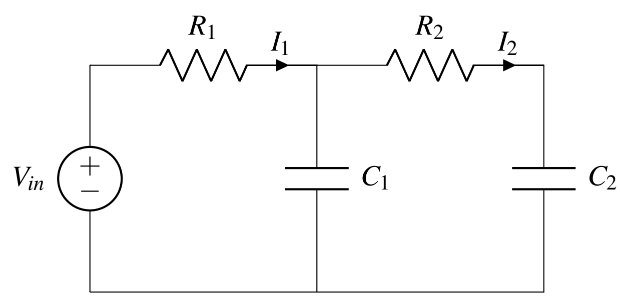


Figure 1: Two dimensional system

Let $C_1 = C_2 = 1\mu F$, $R_1 = \frac{1}{3}\text{M}\Omega$ and $R_2 = \frac{1}{2}\text{M}\Omega$.[1]

Concept Check: Before we begin, how would you solve for the above system if $R_2 = 0$?

Answer: If $R_2 = 0$, then the C_1 and C_2 are connected in parallel and can be combined into a single effective capacitance of $C_1 + C_2$. This is like what happened earlier when we had two gate capacitances in parallel.

As in Note 1, let us first consider the discharging case. In the above system, let $V_{in} = 1V$ for time $t < 0$, and $V_{in} = 0V$ for time $t \geq 0$. With this in mind, we have two steady state conditions:

(a) Initial condition $t = 0$: The voltage has been charging the capacitors for an infinite amount of time. Hence, both capacitors have voltage $v_{C_1} = v_{C_2} = 1V$ and the current $I_1 = I_2 = 0A$.

(b) As $t \to \infty$: After the capacitors have been allowed to discharge for a long period of time, they carry no charges on their plates, hence $v_{C_1} = v_{C_2} = 0V$.

Next, let us solve for the transients, i.e. how does our system go from (a) to (b)? First we need to set up the circuit equations.

[1] The SI prefixes 'M' and 'μ' stand for mega and micro and correspond to the decimal multiples of 10^6 and 10^{-6} respectively.

$$v_2 = v_1 - I_2 R_2 \tag{1}$$

$$I_2 = C_2 \frac{d}{dt} v_2 \tag{2}$$

$$0 - v_1 = I_1 R_1 \Rightarrow I_1 = -\frac{v_1}{R_1} \tag{3}$$

$$I_1 = I_2 + C_1 \frac{d}{dt} v_1 \tag{4}$$

For equations (1) and (2), we look at the current through Node 2, using Ohm's law for the former and voltage-current relation for the capacitor in the latter. Similarly, we find equation (3) and (4) by looking at Node 1. Similar to what we had seen in a single capacitor system, we have effectively introduced two new variables - $\frac{d}{dt}v_1$ and $\frac{d}{dt}v_2$. Next, to solve for the transients, we need to first define our system variables. **The standard approach that we will always take is to make anything that gets differentiated into a state variable.** Hence, we will need two state variables, v_1 and v_2, the voltages across C_1 and C_2 respectively. We will need to setup differential equations to solve for our system variables.

1.2 Systems of Differential Equations

We get a system of differential equations by isolating the derivative terms and solving for them in terms of their non-differentiated selves.

$$\frac{d}{dt} v_1(t) = -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) v_1(t) + \frac{v_2(t)}{R_2 C_1} \tag{5}$$

$$\frac{d}{dt} v_2(t) = \frac{v_1(t)}{R_2 C_2} - \frac{v_2(t)}{R_2 C_2} \tag{6}$$

To obtain equation (5), substitute for I_1 and I_2 from equations (2) and (4) into (3) and rearrange the terms. Similarly, substitute for I_2 (from equation (2)) into (1) to obtain (6).

Concept Check: Take a second to work out setting up the above differential equations. It is important to try to reduce all your branch equations from the circuit analysis to as few variables as possible.

As seen in EE16A, when encountering a system of equations, we try to put it into vector/matrix form to try to solve it. So, let's put equations (5) and (6) in the matrix differential form. We define $\vec{x}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$,

$$\frac{d}{dt}\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) v_1 + \frac{v_2}{R_2 C_1} \\ \frac{v_1}{R_2 C_2} - \frac{v_2}{R_2 C_2} \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) & \frac{1}{R_2 C_1} \\ \frac{1}{R_2 C_2} & -\frac{1}{R_2 C_2} \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \tag{7}$$

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \vec{x}(t) \tag{8}$$

In equation (8), we have substituted in for the component values defined above so that we get a matrix with concrete numbers in it.

Quick Aside: You may come across a dot over the variable of interest (e.g. $\dot{v}_1$) as short hand to represent the $\frac{d}{dt}$ operator. This is alternatively known as Newton's notation and is sometimes used for conciseness, especially in fields influenced by Physics notation. Similarly, we sometimes define $\ddot{V} \equiv \frac{d^2}{dt^2}V$ to make it faster to write second derivatives. However, we won't be using Newton's notation in this course.

In general, we want to set up our systems in the follwing generic form:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b} \tag{9}$$

In context of our above example, $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ and $\vec{b} = \vec{0}$ because there is no external voltage or current being applied for $t \geq 0$ in the above example.

So, why do we choose this form? Because it most closely resembles the first order scalar differential equation we studied in the previous note. Our goal in this note is to massage this equation to convert it to a collection of first order differential equations.

2 Towards a Diagonal System

2.1 A Diagonal System

The first thing we should try to do is to use the tools we developed for the scalar case. But, how can we do this, when our equations seem to be fundamentally dependent on two independent variables? Clearly, we wouldn't have this problem of 'coupling' if our A matrix were diagonal. In this section, we will develop the approach on how we can create a system where the matrix A has a diagonal representation so we can 'uncouple' our pair of differential equations.

Hence, we could proceed to solve the first order differential equations independently of each other, as seen in Note 1. Consider the following circuit:

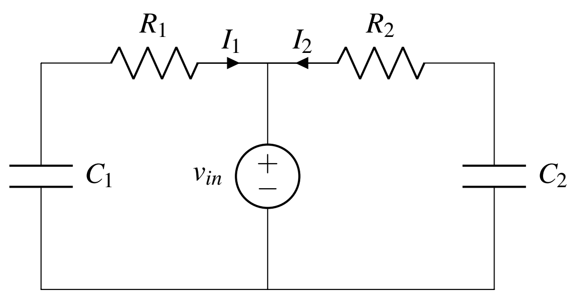


Figure 2: Diagonal System

Both the capacitors have been charged to v_{in} and at $t = 0$, we set $v_{in} = 0V$, and allow the capacitors to discharge. Hence our initial conditions are $v_1(0) = v_2(0) = V_{in}$. We get the following branch equations:

$$I_1 = -C_1 \frac{d}{dt} v_1 = \frac{v_1}{R_1} \tag{10}$$

$$I_2 = -C_2 \frac{d}{dt} v_2 = \frac{v_2}{R_2} \tag{11}$$

Hence from equations (10) and (11), we get the following uncoupled differential equation:

$$\frac{d}{dt}\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1C_1} & 0 \\ 0 & -\frac{1}{R_2C_2} \end{bmatrix}\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \tag{12}$$

Concept Check: We can easily solve the above system of equations by separately solving for v_1 and v_2. Review Note 1 if you are unsure about how to solve for the voltages.

2.2 Eigenvectors and Eigenvalues

What are **eigenvalues** and **eigenvectors**? When a matrix acts on an eigenvector we get the same eigenvector, except scaled by the relevant eigenvalue, i.e.

$$A\vec{v}_\lambda = \lambda\vec{v}_\lambda \tag{13}$$

Here, $\vec{v}_\lambda \neq \vec{0}$ is an eigenvector of A which corresponds to the scalar λ eigenvalue. If we look at all the eigenvectors of the matrix A corresponding to a single λ, these together form a subspace known as the λ-eigenspace. Each distinct eigenvalue is associated with its own nontrivial eigenspace.

We can see this subspace property by rearranging the above equation into:

$$(A-\lambda I_n)\vec{v} = \vec{0}. \tag{14}$$

Here, I_n is the $n \times n$ identity matrix. Since $A\vec{v}_\lambda = \lambda\vec{v}_\lambda$, we know that the matrix $A-\lambda I_n$ has a nullspace (namely the eigenspace corresponding to the eigenvalue λ of A). Consequently, this matrix $(A-\lambda I_n$ must be non-invertible and thus $\det(A-\lambda I_n) = 0$.

This gives us the **characteristic equation** and solving this equation will give us the eigenvalues λ. As a working example, let's compute the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$:

$$\det(A-\lambda I_2) = \det\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det(\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}) \tag{15}$$

Taking the determinant and setting it to zero, we get the characteristic equation:

$$0 = (1-\lambda)(3-\lambda)-(4)(2) = \lambda^2-4\lambda-5 = (\lambda-5)(\lambda+1) \tag{16}$$

Setting equation (16) to 0 and solving the quadratic equation, we get the eigenvalues as $\lambda_1 = 5$ and $\lambda_2 = -1$. Each eigenvalue will have it's own eigenspace, and it will define the nullspace of the matrix $A-\lambda I_n$. Hence to find the eigenspace, we can just find the relevant nullspace.

$$(A-\lambda_1 I_2)\vec{v}_1 = (A-5I_n)\vec{v}_1 = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \vec{0} \tag{17}$$

We see that both rows provide redundant information - $4v_{11}-2v_{12} = 0$, and hence an eigenvector is $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

2.2.1 Complex Eigenvalues

It can also be the case that when we solve $\det(A-\lambda I_n)=0$, there will be no real solutions to λ. Consider the rotation matrix we encountered in Note j:

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{18}$$

Concept Check: When does R_θ have real eigenvalues? Why? **Answer:** For $\theta = 0°$ or $\theta = 180°$.

However, when θ does not correspond to $0°$ or $180°$ rotation, there are no real vectors that are scaled versions of themselves after the transformation. This results in complex eigenvalues. For example, let's look at $45°$ rotation:

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\det(R-\lambda I) = \frac{1}{2}(1-\lambda)(1-\lambda)+\frac{1}{2}$$

Setting this determinant equal to zero and solving yields the complex eigenvalues, $\lambda = \frac{1}{\sqrt{2}}(1+j)$ and $\lambda = \frac{1}{\sqrt{2}}(1-j)$, which makes sense because there are no physical (real) eigenvectors for a rotation transformation in two dimensions.

2.3 Diagonalization

Now that we have identified our desire for a diaognal system and reviewed eigenvectors and eigenvalues, it is finally time to put all of our pieces into play. Let's suppose we had a system of differential equations of the form

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \tag{19}$$

For now we will assume that A is a 2×2 matrix and has a set of linearly independent eigenvector/eigenvalue pairs $(\lambda_1, \vec{v}_1)$ and $(\lambda_2, \vec{v}_2)$. [2]

Then by the definition of eigenvectors and eigenvalues, we know that

$$A\vec{v}_1 = \lambda_1\vec{v}_1 \tag{20}$$

$$A\vec{v}_2 = \lambda_2\vec{v}_2 \tag{21}$$

These two relationships can be expressed simultaneously using matrices that consolidate the eigenvectors (side by side) and eigenvalues (on a diagonal):

$$A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{22}$$

Calling the former two matrices V and the latter Λ,

$$AV = V\Lambda \tag{23}$$

[2]Note that this is not always the case and there are matrices that do not have a full set of eigenvectors that form a basis for $\mathbb{R}^n$. Such matrices are called defective matrices and we will explore them in the next note.

Because we chose two linearly independent eigenvectors to constitute V, V is invertible. Stating A in terms of its eigenvectors and eigenvalues is called the **eigendecomposition** or **diagonalization** of A :

$$A = V\Lambda V^{-1} \tag{24}$$

3 How to Solve?

Now that we've understood the motivation behind diagonal systems and on the **diagonalization** of a matrix A, we can go back to our original system of differential equations introduced in the first section.

Coming back to our original system, from equation (8),

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\vec{x}(t) \tag{25}$$

As discussed, let's use our diagonalization technique to solve this system of differential equations.

First, we must find it's eigenvalues, i.e. the roots of its characteristic equation:

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda+5 & -2 \\ -2 & \lambda+2 \end{bmatrix}\right) \tag{26}$$

$$= (\lambda+5)(\lambda+2) - 4 \tag{27}$$

$$= \lambda^2 + 7\lambda + 6 = (\lambda+6)(\lambda+1) \tag{28}$$

Solving the above characteristic equation, we get the eigenvalues as $\lambda_1 = -6$ and $\lambda_2 = -1$ and hence we can find a set of eigenvectors as $\vec{v}_1 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$. Defining the matrix $V = [\vec{v}_1, \vec{v}_2]$, and the diagonal matrix $\Lambda = \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix}$, we can write $A = V\Lambda V^{-1}$ to rewrite equation (8) as follows:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) = V\Lambda V^{-1}\vec{x}(t) \tag{29}$$

$$= \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}^{-1} \vec{x}(t) \tag{30}$$

We will now define a new variable $\vec{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = V^{-1}\vec{x}(t)$. By left multiplying V^{-1} on both sides of equation (29), we get the following:

$$V^{-1}\frac{d}{dt}\vec{x}(t) = V^{-1}A\vec{x}(t) \tag{31}$$

$$\Longrightarrow \frac{d}{dt}V^{-1}\vec{x}(t) = \Lambda\vec{z}(t) \tag{32}$$

$$\Longrightarrow \frac{d}{dt}\vec{z}(t) = \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix}\vec{z}(t) \tag{33}$$

Because differentiation is linear, we can go from (31) to (32). In equation (33), we have successfully uncoupled our equations and we can proceed to solve them independently as mentioned earlier:

$$\frac{d}{dt}z_1(t) = -6z_1(t) \implies z_1(t) = k_1 e^{-6t}$$
$$\frac{d}{dt}z_2(t) = -z_2(t) \implies z_2(t) = k_2 e^{-t}$$

Next, we need to solve for our constants k_1 and k_2. Recall our initial conditions, $v_1(0) = v_2(0) = 1V$. Hence, $z_1(0)$ and $z_2(0)$ are given by:

$$\vec{z}(0) = V^{-1}\begin{bmatrix} v_1(0) \\ v_2(0) \end{bmatrix} \implies \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{3}{\sqrt{5}} \end{bmatrix} \tag{34}$$

Hence, $k_1 = -\frac{1}{\sqrt{5}}$ and $k_2 = \frac{3}{\sqrt{5}}$. Now, we can transform back into our original variable $\vec{x}$ as follows to find $v_1(t)$ and $v_2(t)$:

$$\vec{x} = V\vec{z} \tag{35}$$
$$= \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{5}}e^{-6t} \\ \frac{3}{\sqrt{5}}e^{-t} \end{bmatrix} \tag{36}$$
$$= \begin{bmatrix} \frac{2}{5}e^{-6t} + \frac{3}{5}e^{-t} \\ -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t} \end{bmatrix} \tag{37}$$

For $t \geq 0$, we find that $v_1(t) = \frac{2}{5}e^{-6t} + \frac{3}{5}e^{-t}$ and $v_2(t) = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}$. Figure 3 is a plot of our solutions:

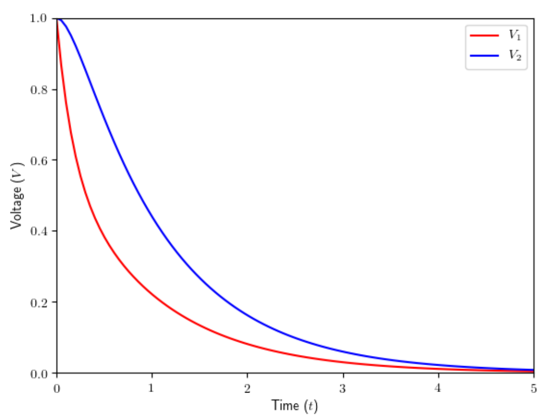
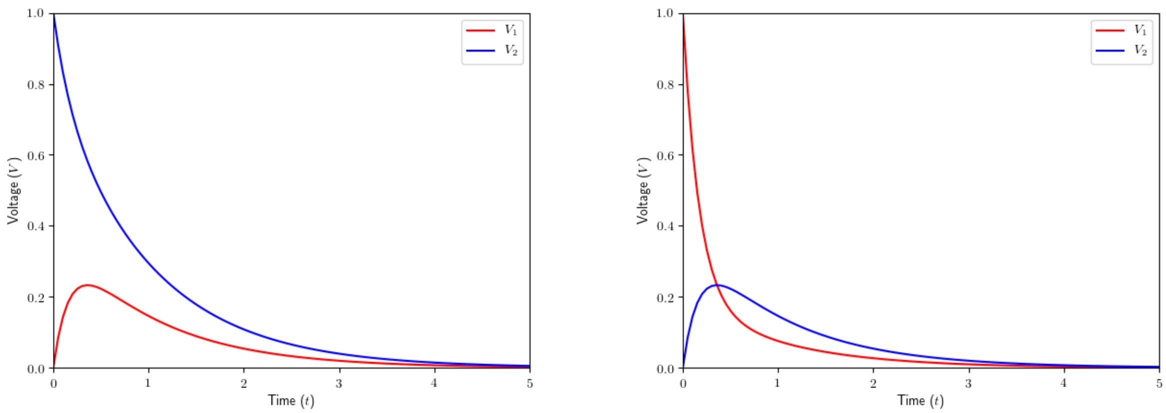


Figure 3: Initial Conditions: $v_1(0) = 1V$ and $v_2(0) = 1V$

Using the same argument, we can see how the voltage will vary with different initial conditions (as shown in figure 4):



(a) Initial Conditions: $v_1(0) = 0V$ and $v_2(0) = 1V$ **(b)** Initial Conditions: $v_1(0) = 1V$ and $v_2(0) = 0V$

Figure 4: Voltage transients for different initial conditions on the capacitors

Concept Check: Take a minute to qualitatively reason about the initial increase in voltages in Figure 4

4 Nonhomogenous Systems

Now that we have a good understanding of the homogenous case, let's look at the voltage transients of charging our two capacitor system. In this case, we have two uncharged capacitors, i.e. $v_1(0) = v_2(0) = 0V$, and we apply a voltage $V_{in} = 1V$ for time $t > 0$. We get the following branch equations:

$$v_2 = v_1 - I_2 R_2 \tag{38}$$

$$I_2 = C_2 \frac{d}{dt} v_2 \tag{39}$$

$$V_{in} - v_1 = I_1 R_1 \Rightarrow I_1 = \frac{V_{in} - v_1}{R_1} \tag{40}$$

$$I_1 = I_2 + C_1 \frac{d}{dt} v_1 \tag{41}$$

Hence, our matrix differential equation is:

$$\frac{d}{dt}\vec{x}(t) = \frac{d}{dt}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} \frac{V_{in}}{R_1C_1} \\ 0 \end{bmatrix} \tag{42}$$

$$= \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = A\vec{x} + \vec{b}. \tag{43}$$

Looking back at our diagonalization process, from (29) we can define a new variable $\vec{z}$ to get the differential equation

$$\frac{d}{dt}\vec{z}(t) = \Lambda\vec{z}(t) + \vec{c} \tag{44}$$

where $\vec{c} = V^{-1}\vec{b}$. We evaluate $\vec{c}$ and uncouple our system of differential equations and as a result, we get two first order scalar differential equations with a constant input.

Since the initial condition $\vec{x}(0)=\vec{0}$, the initial condition $\vec{z}(0)=V^{-1}\vec{x}(0)=\vec{0}$. Lastly, converting back to $\vec{x}$, we see that $\vec{c}=V^{-1}\vec{b}=\begin{bmatrix}-\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}\begin{bmatrix}3\\0\end{bmatrix}=\begin{bmatrix}-\frac{6}{\sqrt{5}}\\ \frac{3}{\sqrt{5}}\end{bmatrix}$

$$\frac{d}{dt}z_1(t)=-6z_1(t)-\frac{6}{\sqrt{5}} \implies z_1(t)=\frac{1}{\sqrt{5}}(e^{-6t}-1) \tag{45}$$

$$\frac{d}{dt}z_2(t)=-z_2(t)+\frac{3}{\sqrt{5}} \implies z_2(t)=-\frac{3}{\sqrt{5}}(e^{-t}-1) \tag{46}$$

$$\vec{x}(t)=V\vec{z}(t) \tag{47}$$

$$=\begin{bmatrix}-\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{5}}(e^{-6t}-1)\\ -\frac{3}{\sqrt{5}}(e^{-t}-1)\end{bmatrix} \tag{48}$$

$$=\begin{bmatrix}1-\frac{2}{5}e^{-6t}-\frac{3}{5}e^{-t}\\ 1+\frac{1}{5}e^{-6t}-\frac{6}{5}e^{-t}\end{bmatrix} \tag{49}$$

Finally, we have $v_1(t)=1-\frac{2}{5}e^{-6t}-\frac{3}{5}e^{-t}$ and $1+\frac{1}{5}e^{-6t}-\frac{6}{5}e^{-t}$. Figure 5 is a plot of our solutions.

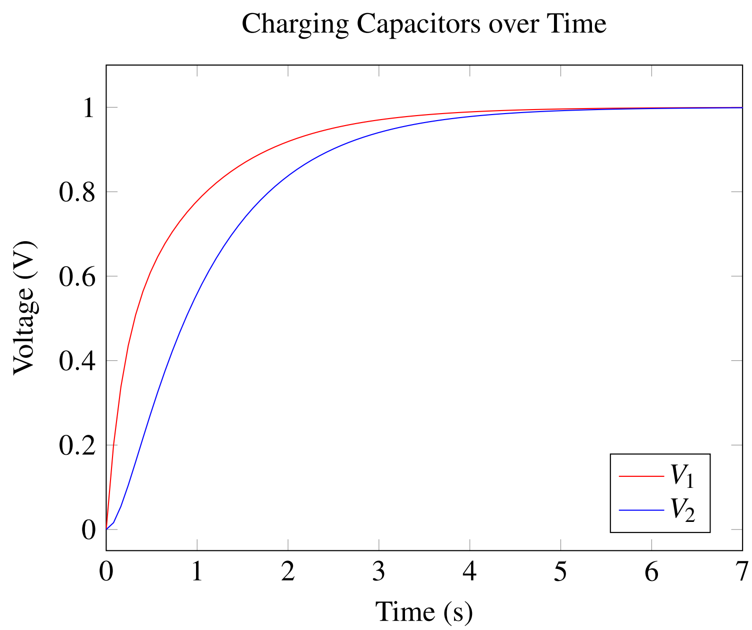


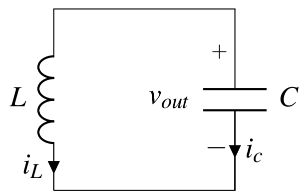
Figure 5: Voltage transients for charging capacitors

5 LC Tank

In our two capacitor circuit example, we found that our eigenvalues were real. But, we could also encounter a system whose eigenvalues are complex. In this section, we will explore a circuit, commonly known as an LC tank, whose matrix will have purely imaginary eigenvalues.

In the following circuit, we have an inductor $L = 10\,\text{nH}$ and capacitor $C = 10\,\text{pF}$ in parallel.

Let $I_L(0) = 50\,\text{mA}$ and $v_{out}(0) = 0\,\text{V}$:



Since the inductor and capacitor are in parallel:

$$v_L = v_c = v_{out} \tag{50}$$

KCL gives:

$$i_L = -i_c = -C\frac{dv_{out}}{dt} \implies \frac{dv_{out}}{dt} = -\frac{1}{C}i_L \tag{51}$$

$$v_L = v_{out} = L\frac{di_L}{dt} \implies \frac{di_L}{dt} = \frac{1}{L}v_{out} \tag{52}$$

Putting it into matrix form, as before:

$$\frac{d}{dt}\begin{bmatrix} v_{out} \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} v_{out} \\ i_L \end{bmatrix} \tag{53}$$

Finding the eigenvalues:

$$\det\left(\begin{bmatrix} -\lambda & -\frac{1}{C} \\ \frac{1}{L} & -\lambda \end{bmatrix}\right) = \lambda^2 + \frac{1}{LC} = 0 \tag{54}$$

$$\therefore \lambda_{1,2} = 0 \pm j\frac{1}{\sqrt{LC}} \tag{55}$$

Next, we can find the eigenvectors of the above matrix as $\vec{v}_1 = \begin{bmatrix} j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}$. We can use these vectors to transform our coordinates to one where the matrix becomes diagonal. More concretely,

$$\begin{bmatrix} v_{out} \\ i_L \end{bmatrix} = V\begin{bmatrix} \widetilde{v}_{out} \\ \widetilde{i}_L \end{bmatrix} \tag{56}$$

As discussed before, once in this new coordinates, our system becomes uncoupled, and we can solve for v_{out}

and i_L as follows:

$$\begin{bmatrix} \frac{d}{dt}\widetilde{v}_{out} \\ \frac{d}{dt}\widetilde{i}_L \end{bmatrix} = \begin{bmatrix} j\frac{1}{\sqrt{LC}} & 0 \\ 0 & -j\frac{1}{\sqrt{LC}} \end{bmatrix} \begin{bmatrix} \widetilde{v}_{out} \\ \widetilde{i}_L \end{bmatrix} \tag{57}$$

Next, we need to find initial conditions in this new coordinate system. Substituting the given values,

$$\begin{bmatrix} \widetilde{v}_{out}(0) \\ \widetilde{i}_L(0) \end{bmatrix} = \begin{bmatrix} j\sqrt{\frac{L}{C}} & -j\sqrt{\frac{L}{C}} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} v_{out}(0) \\ i_L(0) \end{bmatrix} \tag{58}$$

$$= \frac{1}{j20\sqrt{10}} \begin{bmatrix} 1 & j10\sqrt{10} \\ -1 & j10\sqrt{10} \end{bmatrix} \begin{bmatrix} 0 \\ 0.05 \end{bmatrix} \tag{59}$$

$$= \begin{bmatrix} 2.5 \times 10^2 \\ 2.5 \times 10^2 \end{bmatrix} \tag{60}$$

Hence, $\widetilde{v}_{out}(0) = 2.5 \times 10^{-2}$ and $\widetilde{i}_L(0) = 2.5 \times 10^{-2}$ so we can solve our pair of first order differential equations. Lastly, we can tranform back to our original coordinate system:

$$\begin{bmatrix} v_{out}(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} j10\sqrt{10} & -j10\sqrt{10} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2.5 \times 10^{-2} e^{j\sqrt{10}\times 10^9 t} \\ 2.5 \times 10^{-2} e^{-j\sqrt{10}\times 10^9 t} \end{bmatrix} \tag{61}$$

$$= \begin{bmatrix} j0.25\sqrt{10}e^{j\sqrt{10}\times 10^9 t} - j0.25\sqrt{10}e^{-j\sqrt{10}\times 10^9 t} \\ 2.5 \times 10^{-2} e^{j\sqrt{10}\times 10^9 t} + 2.5 \times 10^{-2} e^{-j\sqrt{10}\times 10^9 t} \end{bmatrix} \tag{62}$$

Concept Check: Write the above sum of exponentials as sine and cosine. *Hint: Use the Euler form of sin and cosine we encountered in the complex number note.*

Based on the intuition we have gained above, let's guess a solution with pure sines and cosines, as follows,

$$v_{out}(t) = c_1 \cos\left(\frac{1}{\sqrt{LC}}t\right) + c_2 \sin\left(\frac{1}{\sqrt{LC}}t\right) \tag{63}$$

Next, plugging in initial conditions to solve for the constants:

$$v_{out}(0) = 0 = c_1$$

$$i_c(0) = -i_L(0) = -50 \times 10^{-3}$$

$$\frac{dv_{out}(0)}{dt} = \frac{1}{C} i_c(0) = \frac{-50 \times 10^{-3}}{10^{-11}} = \frac{c_2}{\sqrt{10^{-8} \times 10^{-11}}}$$

$$c_1 = 0$$

$$c_2 = -\frac{5}{\sqrt{10}} = -0.5\sqrt{10}$$

$$v_{out}(t) = -0.5\sqrt{10}\sin\left(\sqrt{10} \times 10^9 t\right)$$

Notice that the amplitude of v_{out} is constant.

Concept Check: Follow the same steps above to find the current, $i_L(t)$. *Hint: The current will also be of the form in equation* (63)*, but with different constants.*

Solution:

$$i_L(t) = 50 \times 10^{-3} \cos\left(\sqrt{10} \times 10^9 t\right) \tag{64}$$

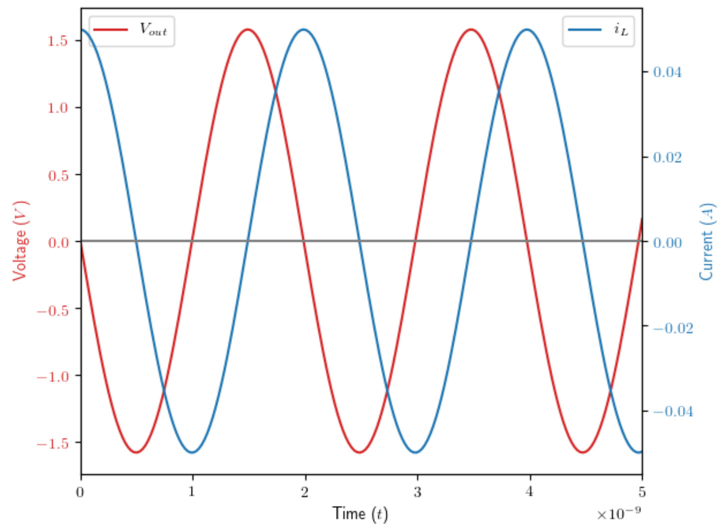


Figure 6: Voltage and Current response of LC Tank

Figure 6 plots the above solutions for the capacitor voltage and inductor current. This system is also called an oscillator because the circuit produces a repetitive voltage waveform under the right initial conditions.

From the above plots, we can see that the current and voltage are 90° out of phase, i.e. when the current is at its maximum or minimum, the voltage is at $0V$, and vice versa. What does this mean for the energy stored in these compoenets? We know that, energy in the capacitor, $E_C = \frac{1}{2}CV^2 = 1.25 \times 10^{-11} \sin^2\left(\sqrt{10} \times 10^9 t\right)$ and energy in the inductor, $E_L = 1.25 \times 10^{-11} \cos^2\left(\sqrt{10} \times 10^9 t\right)$. Figure 7 plots these energies. As it is clear, the total energy seems to be sloshing back and forth between the inductor and capacitor.

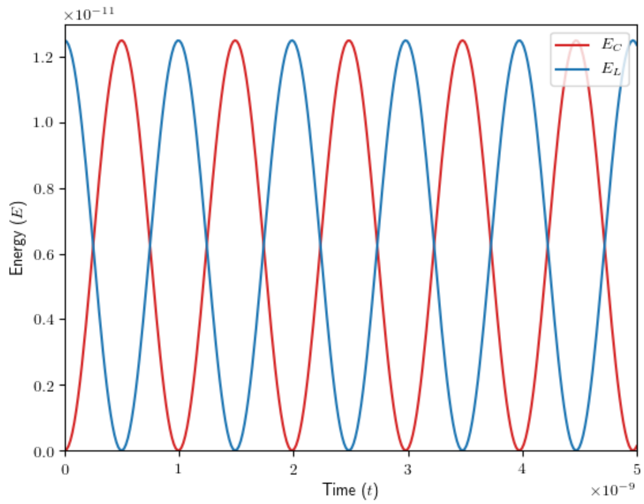


Figure 7: Energy stored in Inductor and Capacitor. Notice the sum is constant.

6 The Change of Basis Perspective

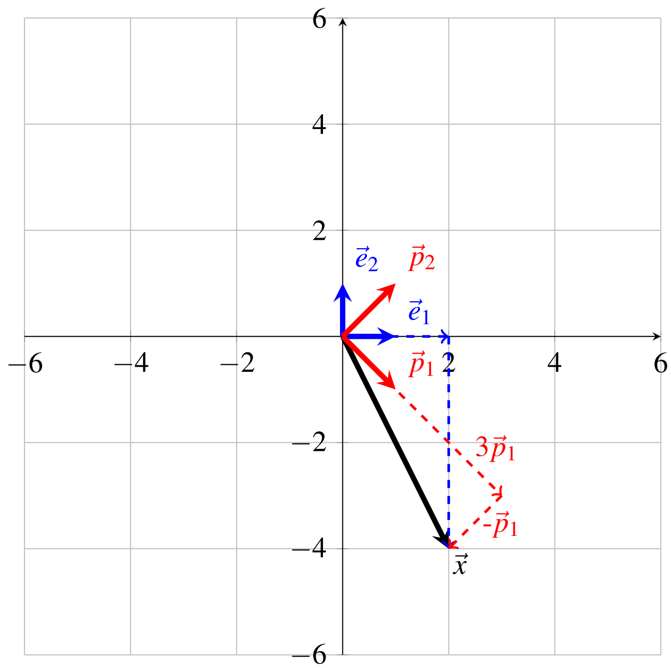
In this section of the note, we go back to our diagonalization process and draw the connection between eigendecomposition and coordinate systems. This provides a perspective where we are in fact transforming our coordinates into a basis in which the matrix A has a diagonal representation Λ.

Let's start with a vector $\vec{x} \in \mathbb{R}^n$. This vector represents a point in space. When you think about this vector written out using the coordinates, you are scaling the vectors in the standard basis (i.e. the columns of the identity matrix, I) by the components of $\vec{x}$ and then adding them up.

For example, we can write the vector $\vec{x} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ as the linear combination $\vec{x} = 2\vec{e}_1 - 4\vec{e}_2$ where $\vec{e}_1$ and $\vec{e}_2$ are the standard basis vectors. But, suppose that I think about this vector in terms of a different set of directions. More concretely, I define a new coordinate system:

$$\vec{p}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \vec{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{65}$$

Then to represent the vector $\vec{x} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, we would need an alternate name for this vector. A quick computation can show that $\vec{z} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ achieves this since $\vec{x} = 3 \cdot \vec{p}_1 - 1 \cdot \vec{p}_2$. [3]



The vectors that define this coordinate system form a basis, i.e. n linearly independent vectors $\vec{p}_1, \ldots \vec{p}_n$ defined with respect to the standard basis.

[3] We can find $\vec{z}$ by taking the inverse of P and multiply it by $\vec{x}$. We wil explore why this is the case in the next page.

6.1 Changing Coordinates

Now, let's say I have a vector $\vec{z}$ which I am representing as $\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ measuring with respect to my coordinate system. How can I translate this to the coordinates you are familiar with? Well, instead of scaling the vectors of the standard basis, we could scale the vectors defining my new basis.

Suppose that both of us were thinking of the same physical point in space and hence the vector $\vec{x}$ in your basis is:

$$\vec{x} = z_1\vec{p}_1 + \cdots + z_n\vec{p}_n \tag{66}$$

$$= \begin{bmatrix} | & & | \\ \vec{p}_1 & \cdots & \vec{p}_n \\ | & & | \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \tag{67}$$

$$= P\vec{z} \tag{68}$$

This also tells us that if we have a vector $\vec{x}$ written in the standard basis, we can transform it to its representation using P-basis vectors, $\vec{z}$, through the computation $\vec{z} = P^{-1}\vec{x}$.

6.2 Matrices in Different Bases

To transform a vector from my basis to your standard basis involves just a matrix multiplication. If this is the case, what would the matrix A look like in a different basis?

The matrix A performs a linear tranformation, $\vec{y} = A\vec{x}$. We would like to visualize this transformation in a different basis. In other words, we want to find the linear transformation D that performs the action $\vec{w} = D\vec{z}$ were $\vec{z}$ represents our coordinates in a new basis.

To do this, we must first change $\vec{x}$ into the basis P. This is done by left multiplying to get $\vec{z} = P^{-1}\vec{x}$. Then, we can apply the transformation D to get $\vec{w} = DP^{-1}\vec{z}$. Finally, since our vector $\vec{w}$ is in a different basis, we must convert it back to the standard basis by multiplying by P to get $\vec{y} = P\vec{w} = PDP^{-1}\vec{x}$. This shows that $A = PDP^{-1}$ or $D = P^{-1}AP$.

We summarize our results in the diagram given in Figure 8 using the up and down arrows.

$$\begin{tikzcd}
\vec{x} \arrow[r, mapsto, "A"] \arrow[d, shift right, "P^{-1}"'] & \vec{y} \arrow[d, shift left, "P^{-1}"] \\
\vec{z} \arrow[r, mapsto, "D"'] \arrow[u, shift right, "P"'] & \vec{w} \arrow[u, shift left, "P"]
\end{tikzcd}$$

Figure 8: Change of Basis Mapping. The top row has everything in the standard basis. The bottom row is in P-basis. The matrix D is supposed to represent the same linear transformation as A, except that it does so for vectors expressed in P-basis. It turns out $D = P^{-1}AP$ since matrix multiplication is done on the left.

6.3 Back to Diagonalization

Now that we've established what a matrix A looks like in a different basis, the question to ask is whether there is another basis within which this transformation is much simpler to understand.

What can we wish for? The transformation is clearly doing something nontrivial, and so it is not possible to find a basis in which the transformation is just the identity matrix. Nor is one going to be found where the matrix is just the zero matrix. So not all wishes can be fulfilled. What about wishing for a diagonal transformation? That would certainly be simpler to understand. But is that a reasonable wish? Let's see whether it is in this case.

Let's suppose we had our new basis V so that $\vec{x} = V\vec{z}$ and correspondingly, $\vec{z} = V^{-1}\vec{x}$. Similarly, $\vec{y} = V\vec{w}$ and correspondingly, $\vec{w} = V^{-1}\vec{y}$. Then:

$$A\vec{x} = AV\vec{z} = A(z_1\vec{v}_1 + z_2\vec{v}_2) \tag{69}$$

$$= z_1 A\vec{v}_1 + z_2 A\vec{v}_2 \tag{70}$$

Now, if chose our new basis to be the ones defined by the eigenvectors of A, then we can simplify:

$$= z_1\lambda_1\vec{v}_1 + z_2\lambda_2\vec{v}_2 \tag{71}$$

$$= \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{72}$$

$$= VD\vec{z} \tag{73}$$

$$= VDV^{-1}\vec{x} \tag{74}$$

where D is the diagonal matrix of eigenvalues and V is a matrix with the corresponding eigenvectors as its columns. Thus we have proved that $A = VDV^{-1}$. Furthermore, this also means that $D = V^{-1}AV$.

So, our wish can indeed be fulfilled — at least for this specific matrix. In general, the pattern we have used above will hold whenever we can find an eigenbasis — a full basis consisting of eigenvectors.

Why is this the case? Because $AV = [A\vec{v}_1 \cdots A\vec{v}_n] = [\lambda_1\vec{v}_1 \cdots \lambda_n\vec{v}_n]$ and then since $V^{-1}V = I$, we know that [4]

$$V^{-1}AV = \begin{bmatrix} \lambda_1 V^{-1}\vec{v}_1 & \cdots & \lambda_n V^{-1}\vec{v}_n \end{bmatrix} \tag{75}$$

$$= \begin{bmatrix} \lambda_1 & & \\ & \ddots & \vdots \\ & & \lambda_n \end{bmatrix} = D \tag{76}$$

6.3.1 Repeated Eigenvalues

For a 2×2 matrix, it's possible that the two eigenvalues that you end up with have the same value, leading to a phenomenon called a **repeated eigenvalues**. This repeated eigenvalue can have one or two dimensional eigenspace (unlike a single, unrepeated eigenvalue, which will only have a one dimensional eigenspace).

[4] If you aren't sure why $V^{-1}\vec{v}_i = \vec{e}_i$, try setting $\vec{x} = V^{-1}\vec{v}_1$. This would mean $V\vec{x} = \vec{v}_1$.

For example, the following matrix has a repeated eigenvalue of λ.

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

The λ-**eigenspace** of this matrix is all of $\mathbb{R}^2$ since for any vector $\vec{v} \in \mathbb{R}^2$, $A\vec{v} = \lambda\vec{v}$.

We can also have examples like

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

that have a single eigenvalue $\lambda = 0$. (Easy to see by looking at the characteristic equation $\lambda^2 = 0$.) In this case, the relevant eigenspace is one-dimensional — only $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and its multiples are eigenvectors here. (If there were any other linearly independent eigenvectors corresponding to the 0 eigenvalue, then everything would have to be an eigenvector and that would mean the matrix mapped everything to the zero vector. But the matrix is clearly not the zero matrix, so that isn't what is going on.)

7 Defective Matrices

In our approach to solve a system of differential equations, we developed a methodology in which we could turn a system of differential equations into n first-order scalar differential equations. This methodology involved a process called **diagonalization** in which we viewed a matrix A through its representation in a basis made up of eigenvectors. As a result, the matrix A had a diagonal representation Λ in our new basis.

However, note that each time we performed this process, we assumed that A was diagonalizable, or has a full basis consisting of eignevectors. Sadly, not every matrix has n linearly independent eigenvectors as we see above. In the next note, we will look at this case and a physical phenomena that arises from it. But for the time being, the best we can do is hope that our matrix A is diagonalizable.

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