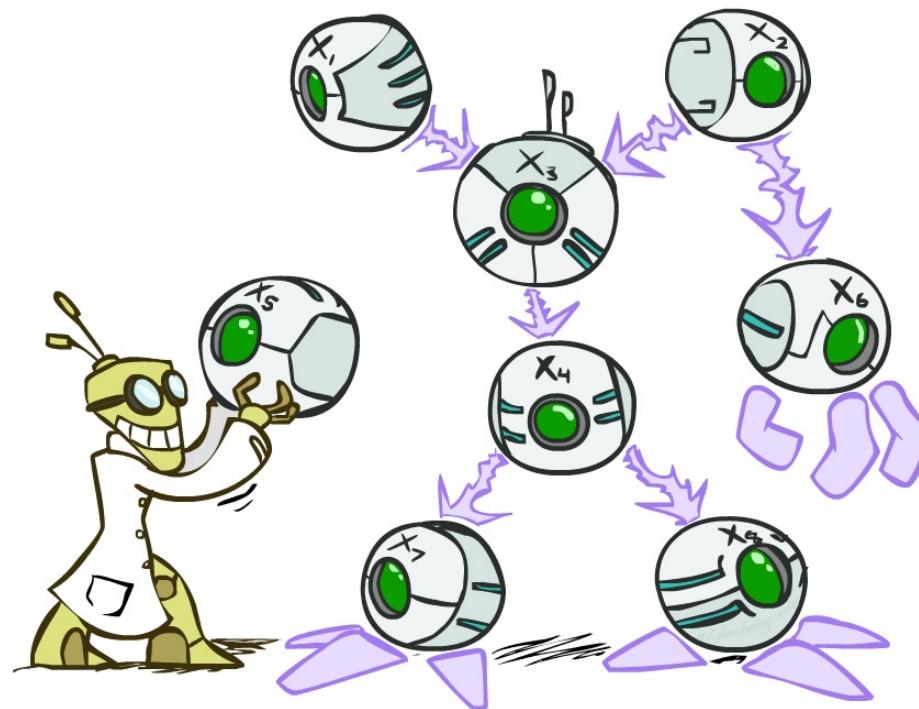


# CS 188: Artificial Intelligence

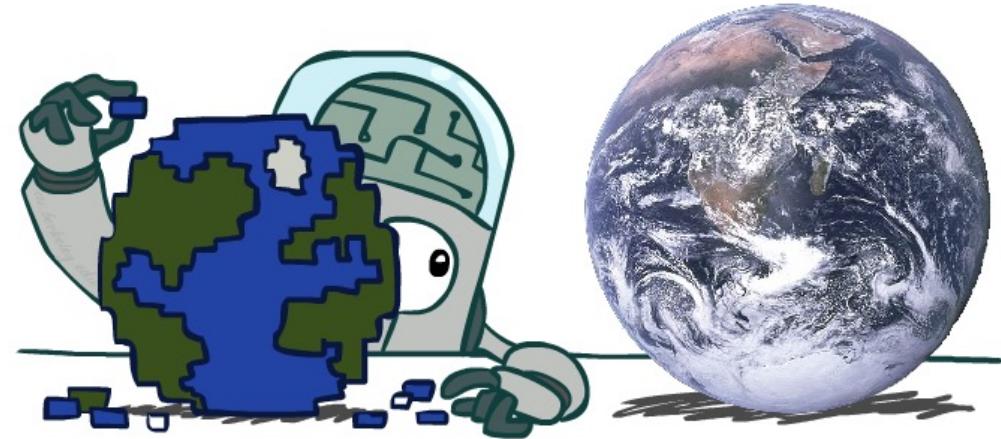
## Bayes' Nets



Instructor: Anca Dragan --- University of California, Berkeley

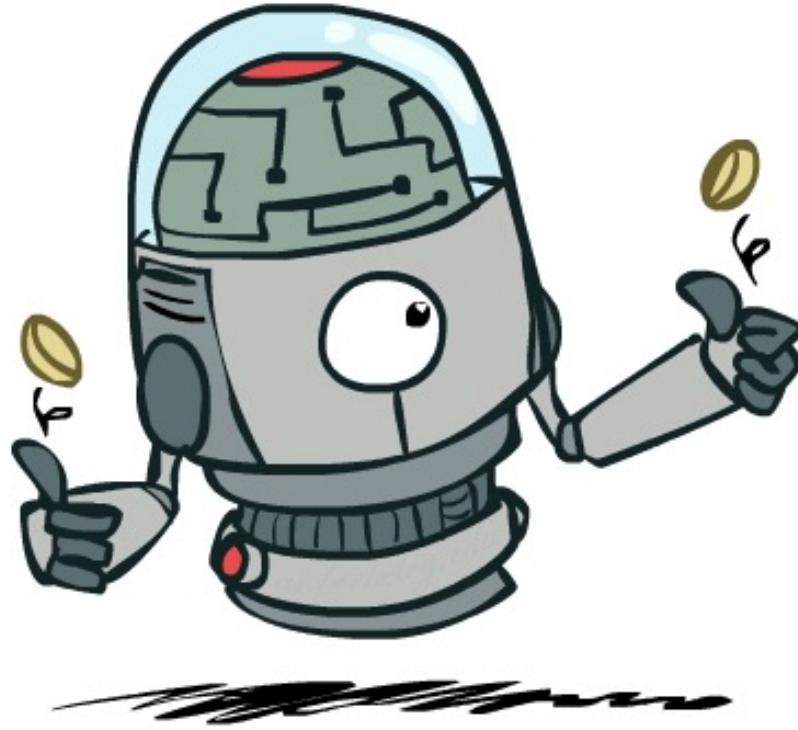
# Probabilistic Models

- Models describe how (a portion of) the world works
- **Models are always simplifications**
  - May not account for every variable
  - May not account for all interactions between variables
  - “All models are wrong; but some are useful.”
    - George E. P. Box
- What do we do with probabilistic models?
  - We (or our agents) need to reason about unknown variables, given evidence
  - Example: explanation (diagnostic reasoning)
  - Example: prediction (causal reasoning)
  - Example: value of information



# Independence

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# Independence

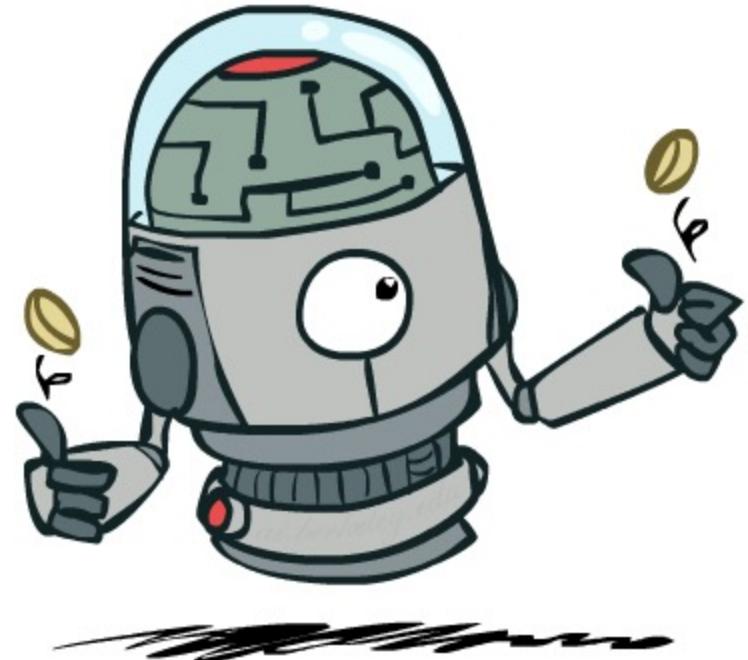
- Two variables are *independent* if:

$$\forall x, y : P(x, y) = P(x)P(y)$$

- This says that their joint distribution *factors* into a product two simpler distributions
- Another form:

$$\forall x, y : P(x|y) = P(x)$$

- We write:  $X \perp\!\!\!\perp Y$
- Independence is a simplifying *modeling assumption*



- *Empirical* joint distributions: at best “close” to independent
- What could we assume for {Weather, Traffic, Cavity, Toothache}?

# Example: Independence?

$P_1(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$P(T)$

T	P
hot	0.5
cold	0.5

$P_2(T, W)$

T	W	P
hot	sun	0.3
hot	rain	0.2
cold	sun	0.3
cold	rain	0.2

$P(W)$

W	P
sun	0.6
rain	0.4

# Example: Independence

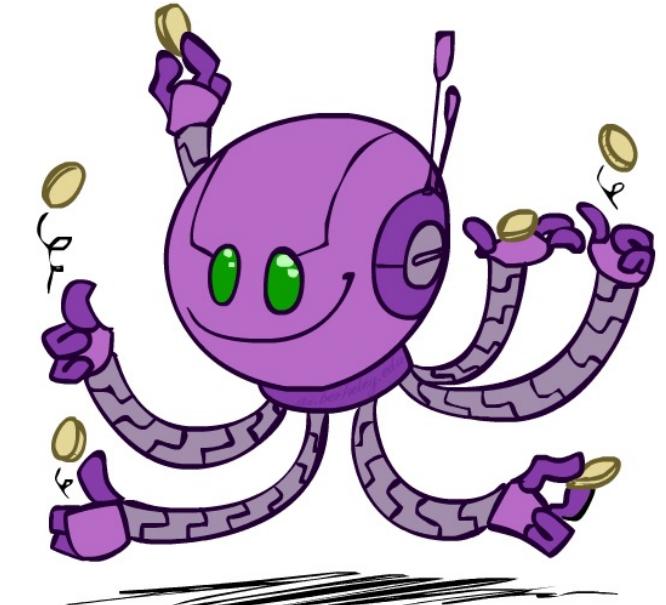
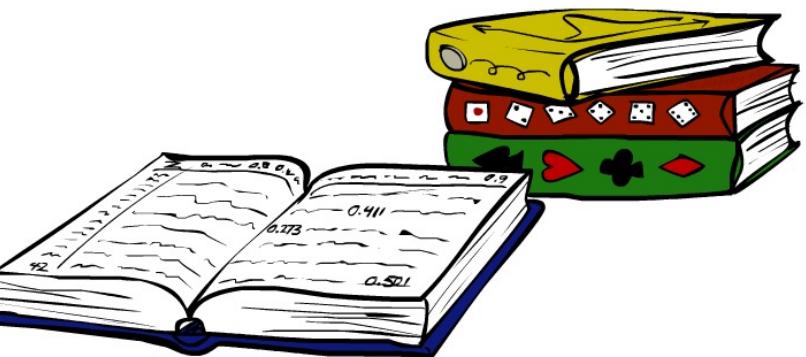
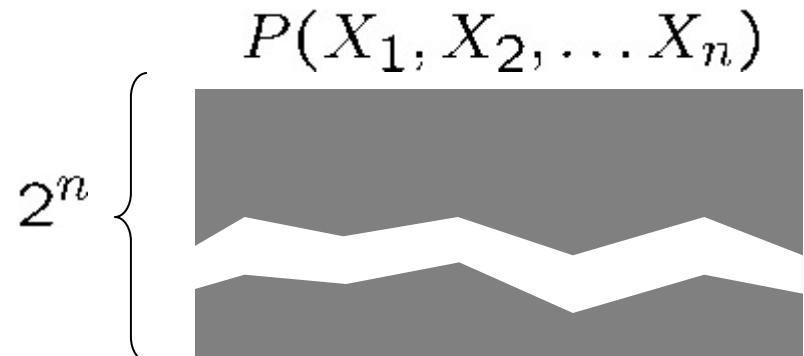
- N fair, independent coin flips:

$P(X_1)$	
H	0.5
T	0.5

$P(X_2)$	
H	0.5
T	0.5

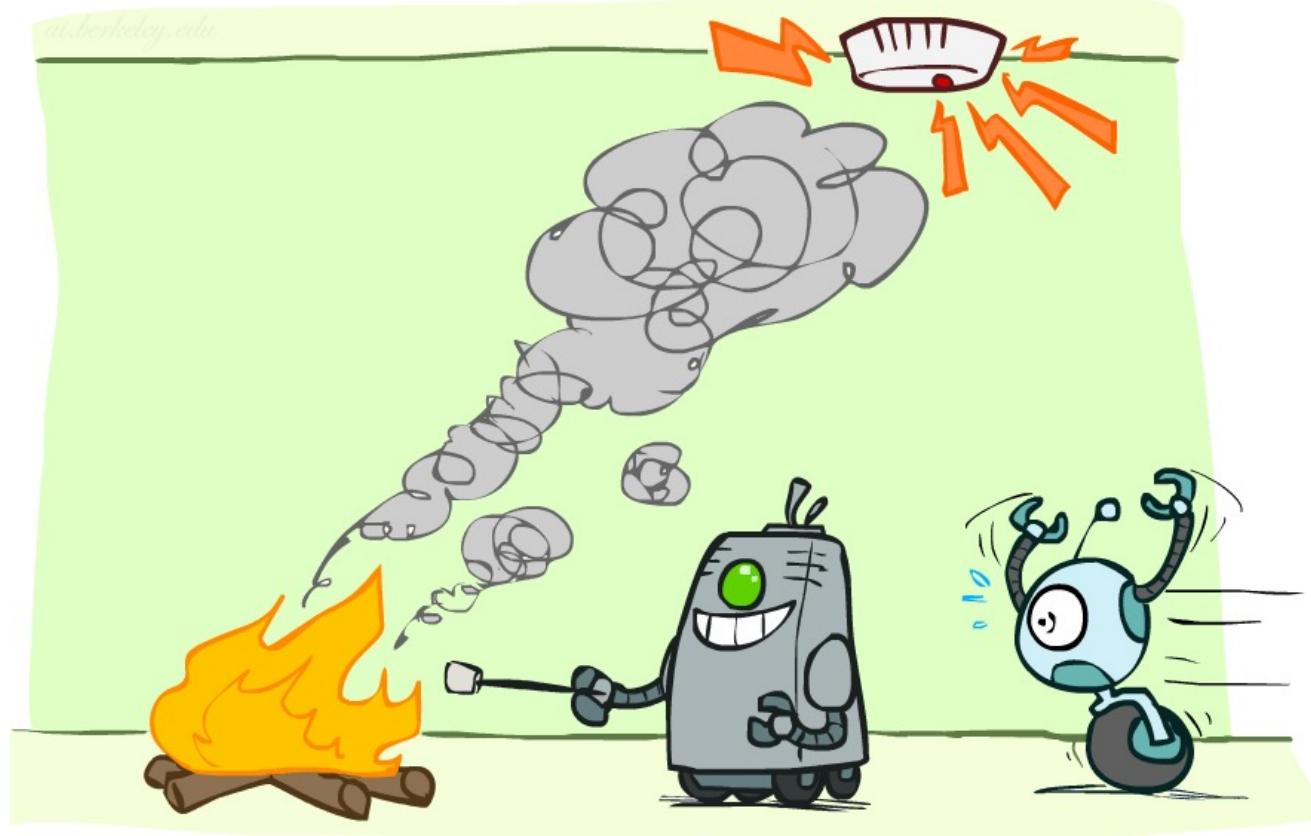
...

$P(X_n)$	
H	0.5
T	0.5



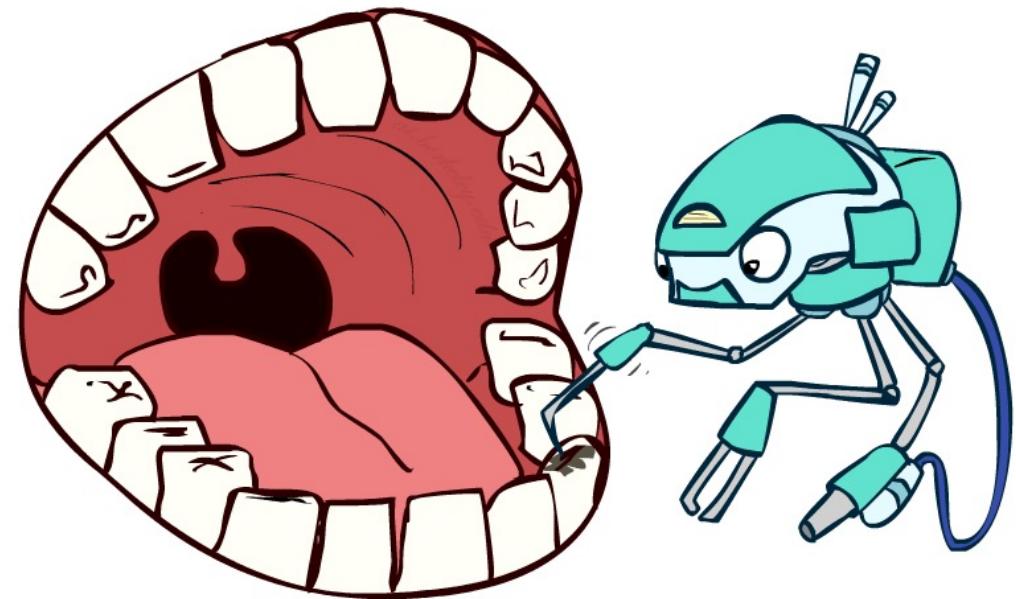
# Conditional Independence

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# Conditional Independence

- $P(\text{Toothache}, \text{Cavity}, \text{Catch})$
- If I have a cavity, the probability that the probe catches it doesn't depend on whether I have a toothache:
  - $P(+\text{catch} | +\text{toothache}, +\text{cavity}) = P(+\text{catch} | +\text{cavity})$
- The same independence holds if I don't have a cavity:
  - $P(+\text{catch} | +\text{toothache}, -\text{cavity}) = P(+\text{catch} | -\text{cavity})$
- Catch is *conditionally independent* of Toothache given Cavity:
  - $P(\text{Catch} | \text{Toothache}, \text{Cavity}) = P(\text{Catch} | \text{Cavity})$
- Equivalent statements:
  - $P(\text{Toothache} | \text{Catch}, \text{Cavity}) = P(\text{Toothache} | \text{Cavity})$
  - $P(\text{Toothache}, \text{Catch} | \text{Cavity}) = P(\text{Toothache} | \text{Cavity}) P(\text{Catch} | \text{Cavity})$
  - One can be derived from the other easily



# Conditional Independence

---

- Unconditional (absolute) independence very rare (why?)
- *Conditional independence* is our most basic and robust form of knowledge about uncertain environments.
- X is conditionally independent of Y given Z  $X \perp\!\!\!\perp Y | Z$

if and only if:

$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$

or, equivalently, if and only if

$$\forall x, y, z : P(x|z, y) = P(x|z)$$

# Conditional Independence

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or, equivalently, if and only if

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$$\begin{aligned} P(x|z, y) &= \frac{P(x, z, y)}{P(z, y)} \\ &= \frac{P(x, y|z)P(z)}{P(y|z)P(z)} \\ &= \frac{P(x|z)P(y|z)P(z)}{P(y|z)P(z)} \end{aligned}$$

# Conditional Independence

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- What about this domain:

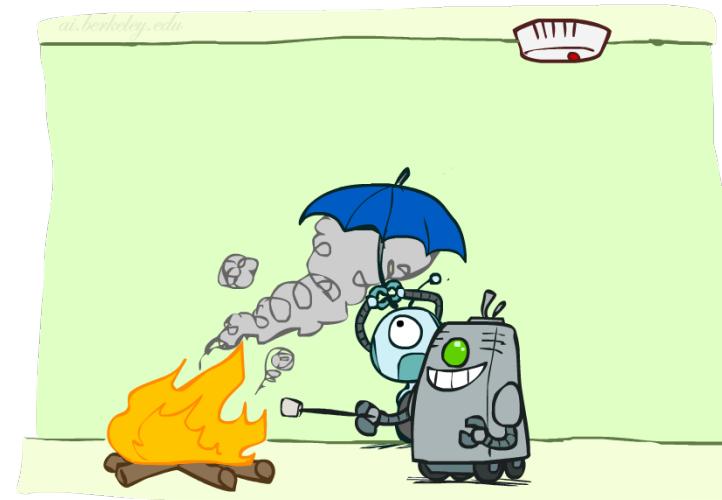
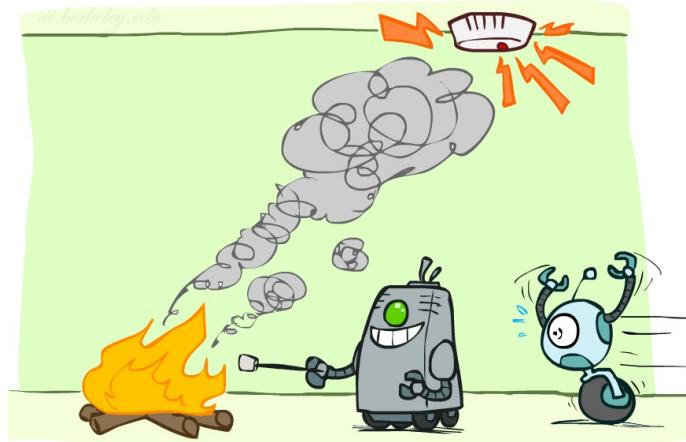
- Traffic
- Umbrella
- Raining



# Conditional Independence

- What about this domain:

- Fire
- Smoke
- Alarm



# Conditional Independence and the Chain Rule

- Chain rule:

$$P(X_1, X_2, \dots, X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2) \dots$$

- Trivial decomposition:

$$P(\text{Traffic, Rain, Umbrella}) =$$

$$P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain, Traffic})$$

- With assumption of conditional independence:

$$P(\text{Traffic, Rain, Umbrella}) =$$

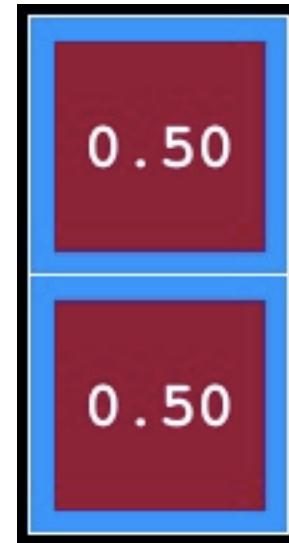
$$P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain})$$



- Bayes' nets / graphical models help us express conditional independence assumptions

# Ghostbusters Chain Rule

- Each sensor depends only on where the ghost is
- That means, the two sensors are conditionally independent, given the ghost position
- T: Top square is red  
B: Bottom square is red  
G: Ghost is in the top
- Givens:  
 $P(+g) = 0.5$   
 $P(-g) = 0.5$   
 $P(+t | +g) = 0.8$   
 $P(+t | -g) = 0.4$   
 $P(+b | +g) = 0.4$   
 $P(+b | -g) = 0.8$



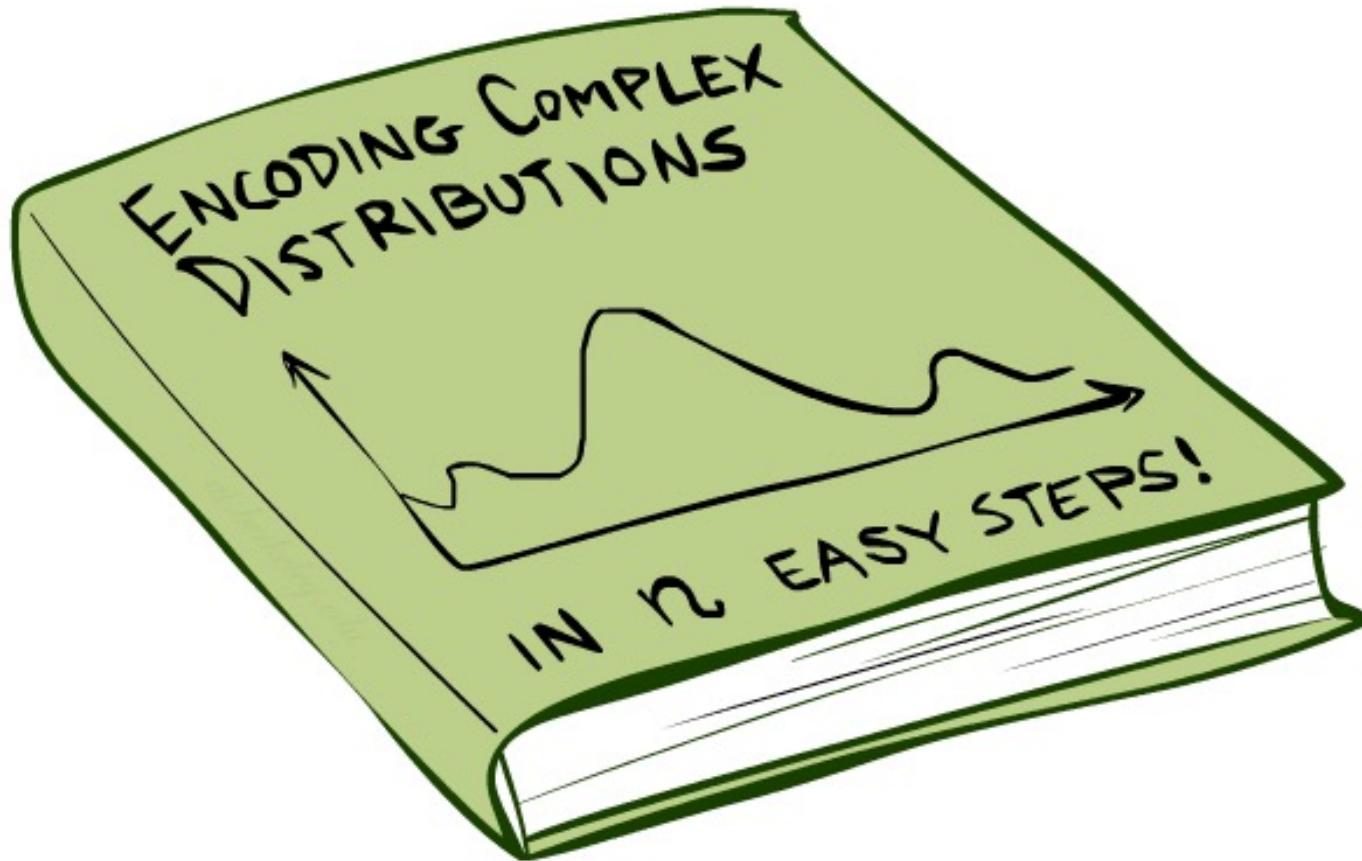
$$P(T, B, G) = P(G) P(T|G) P(B|G)$$

T	B	G	$P(T, B, G)$
+t	+b	+g	0.16
+t	+b	-g	0.16
+t	-b	+g	0.24
+t	-b	-g	0.04
-t	+b	+g	0.04
-t	+b	-g	0.24
-t	-b	+g	0.06
-t	-b	-g	0.06



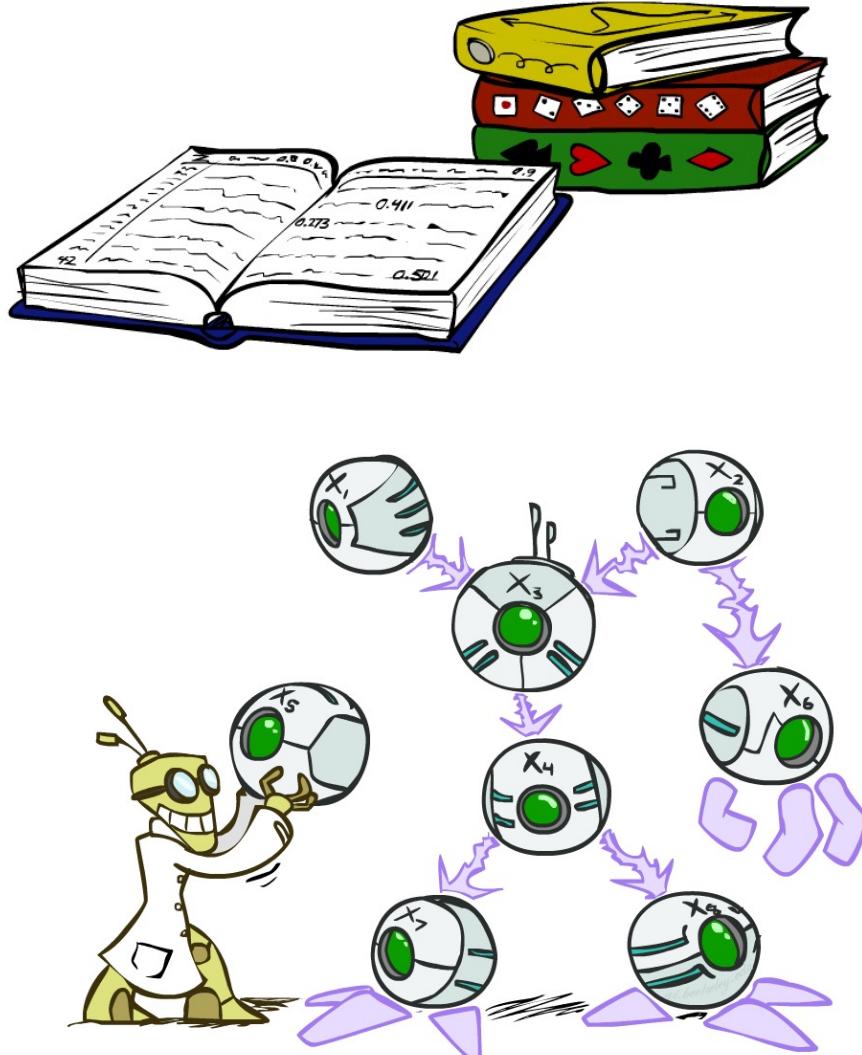
# Bayes'Nets: Big Picture

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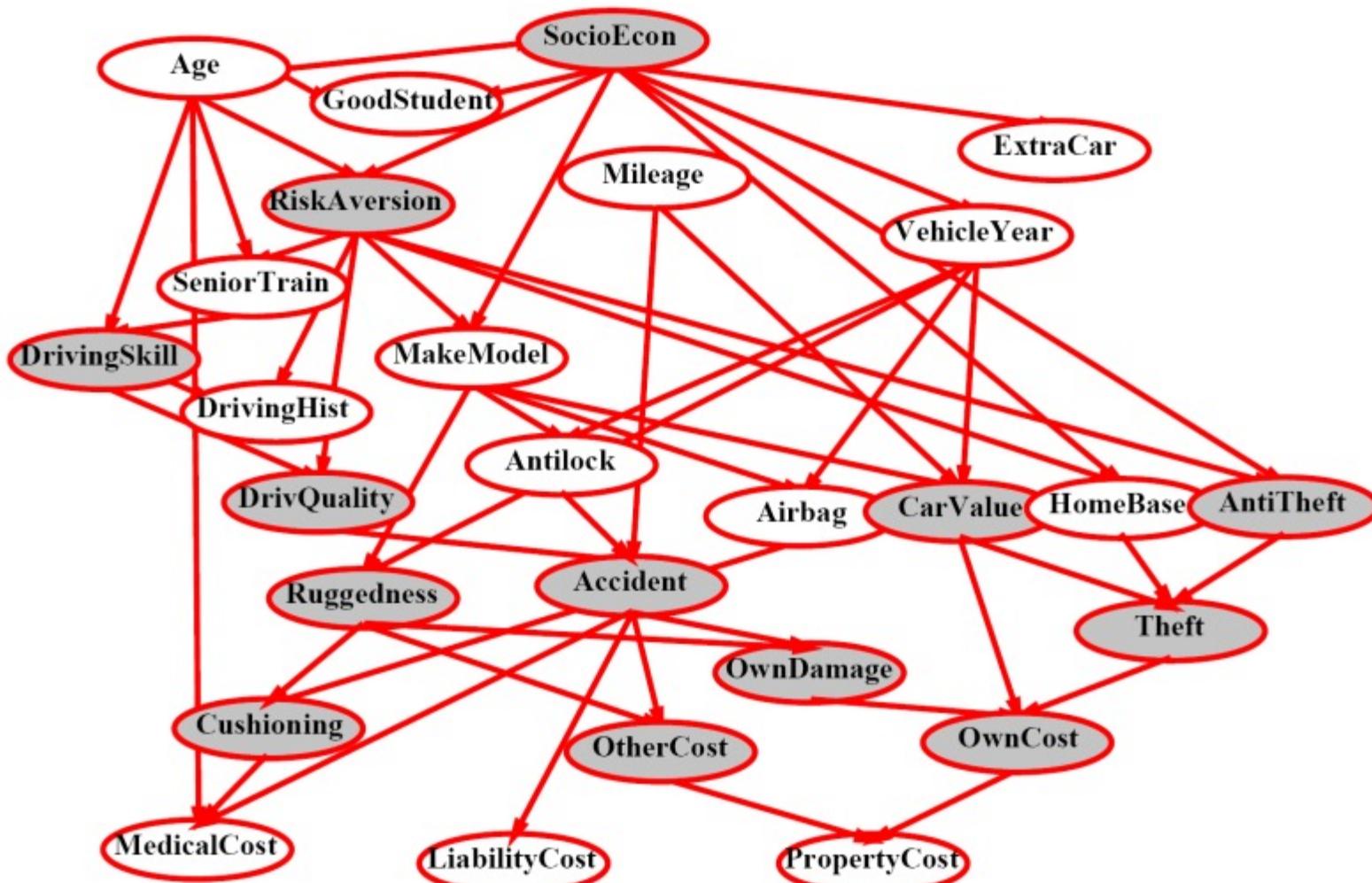


# Bayes' Nets: Big Picture

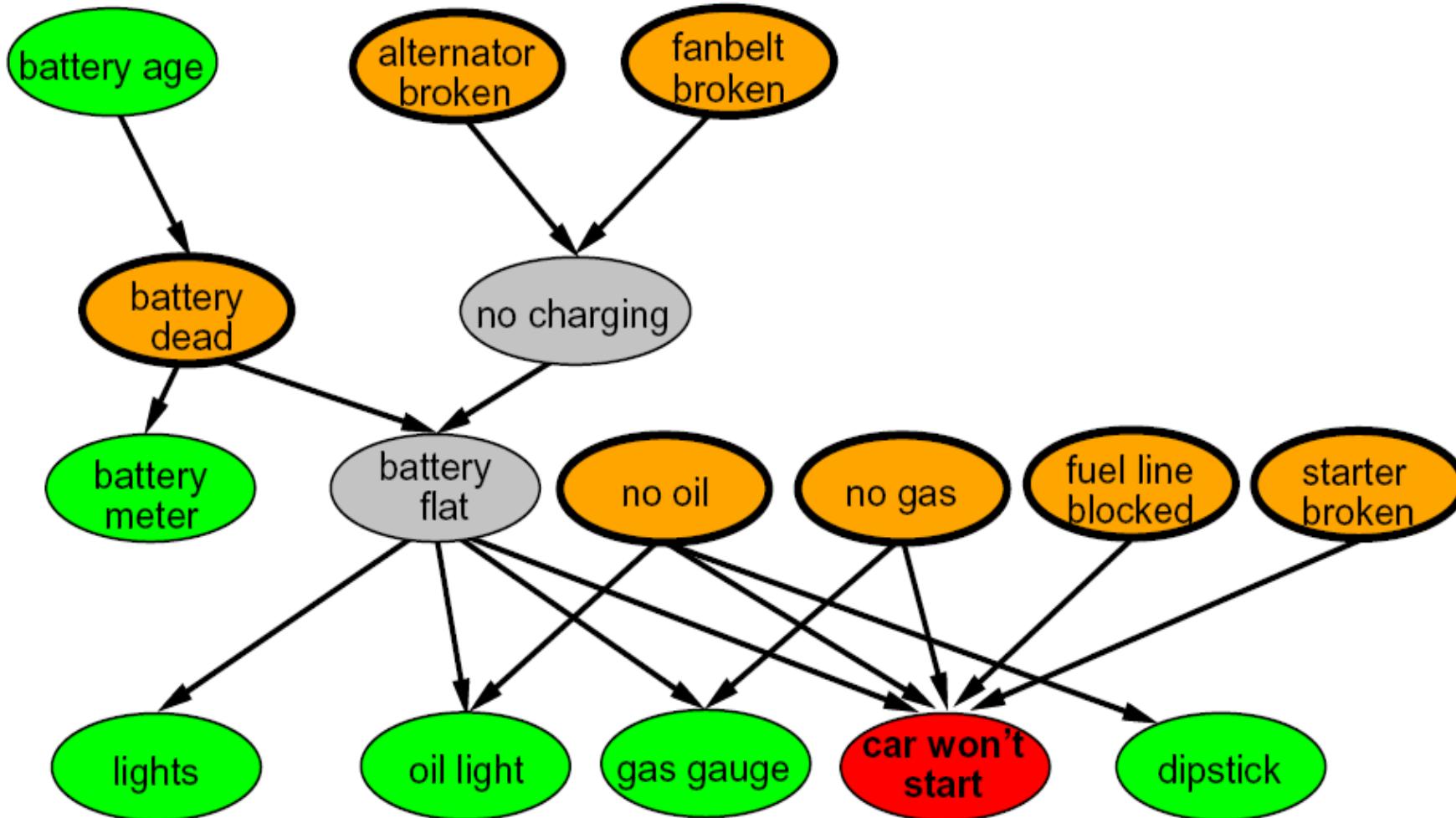
- Two problems with using full joint distribution tables as our probabilistic models:
  - Unless there are only a few variables, the joint is WAY too big to represent explicitly
  - Hard to learn (estimate) anything empirically about more than a few variables at a time
- Bayes' nets: a technique for describing complex joint distributions (models) using simple, local distributions (conditional probabilities)
  - More properly called graphical models
  - We describe how variables locally interact
  - Local interactions chain together to give global, indirect interactions
  - For about 10 min, we'll be vague about how these interactions are specified



# Example Bayes' Net: Insurance

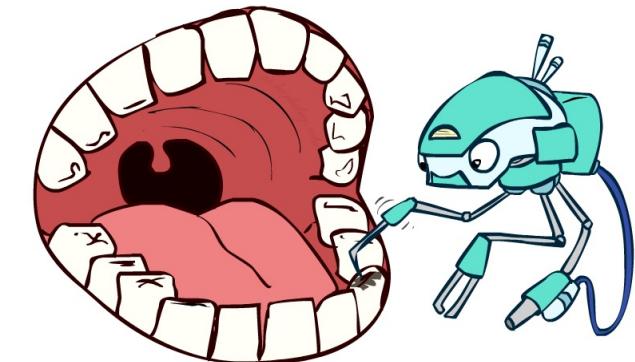
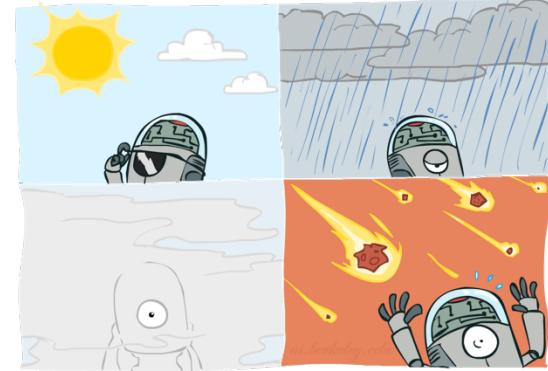
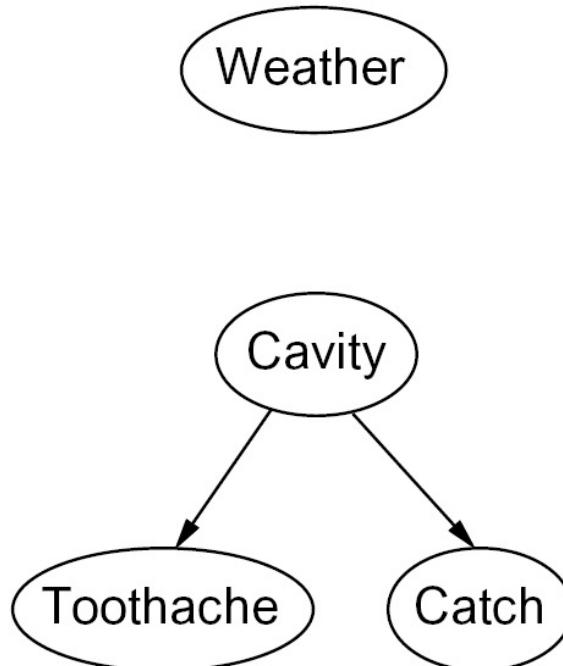


# Example Bayes' Net: Car



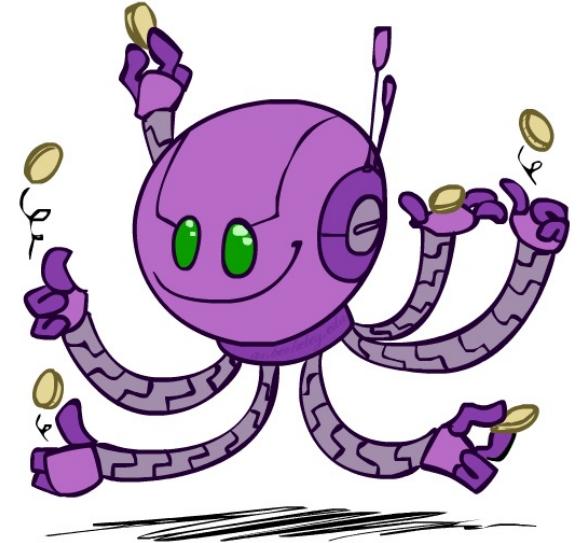
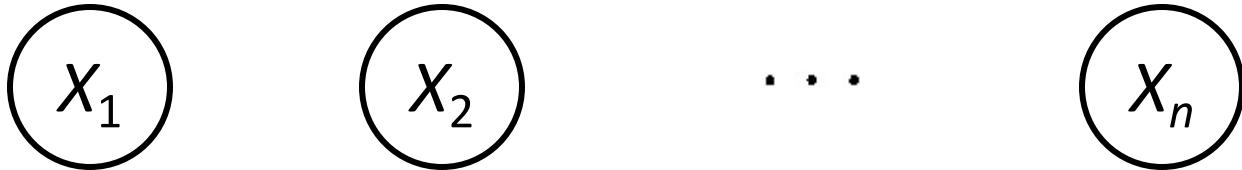
# Graphical Model Notation

- Nodes: variables (with domains)
  - Can be assigned (observed) or unassigned (unobserved)
- Arcs: interactions
  - Similar to CSP constraints
  - Indicate “direct influence” between variables
  - Formally: encode conditional independence (more later)
- For now: imagine that arrows mean direct causation (in general, they don’t!)



# Example: Coin Flips

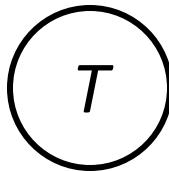
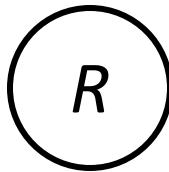
- N independent coin flips



- No interactions between variables: **absolute independence**

# Example: Traffic

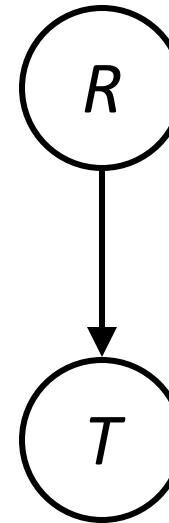
- Variables:
  - $R$ : It rains
  - $T$ : There is traffic
- Model 1: independence



- Model 2: rain causes traffic



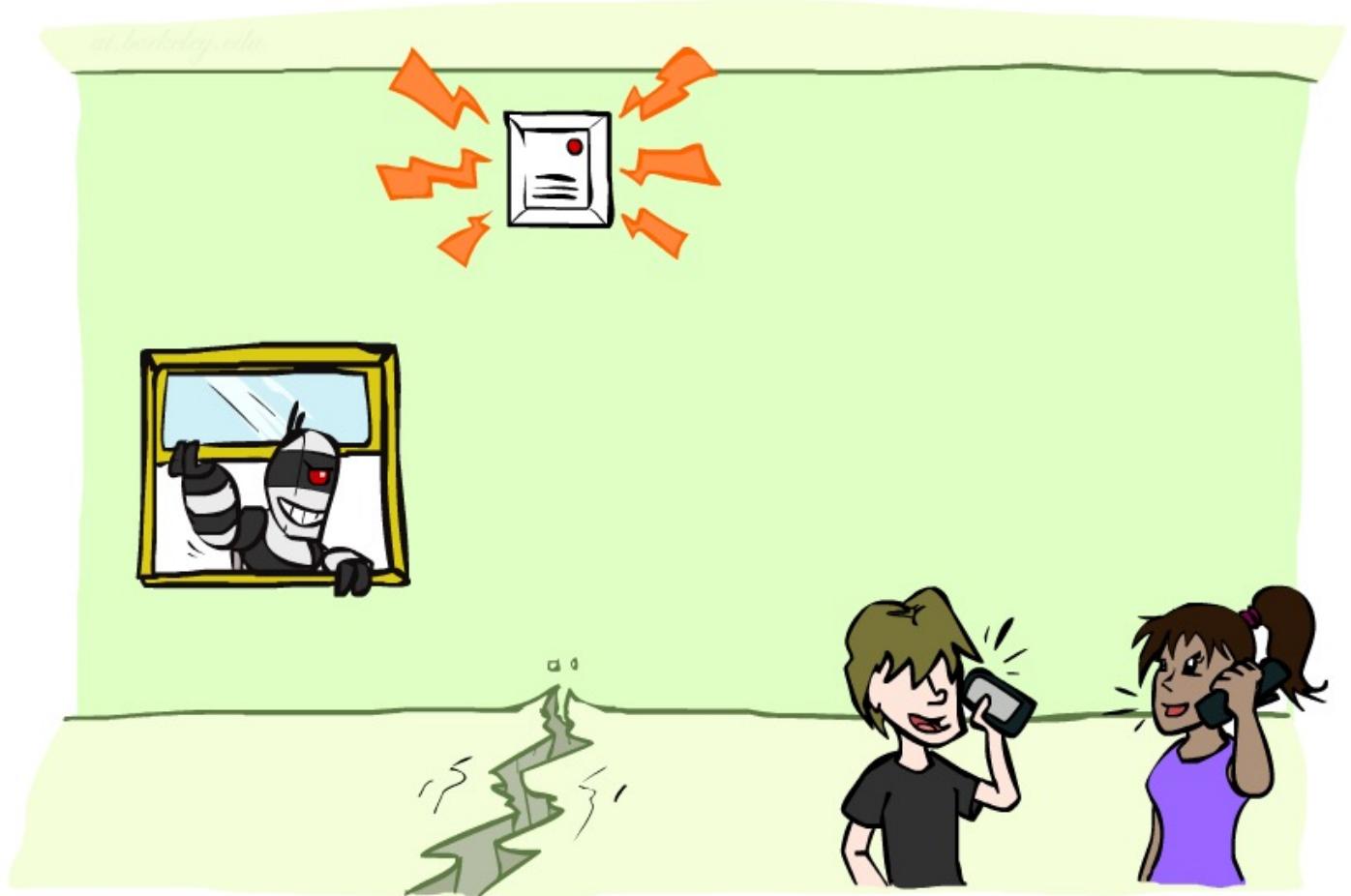
- Why is an agent using model 2 better?



# Example: Alarm Network

## ■ Variables

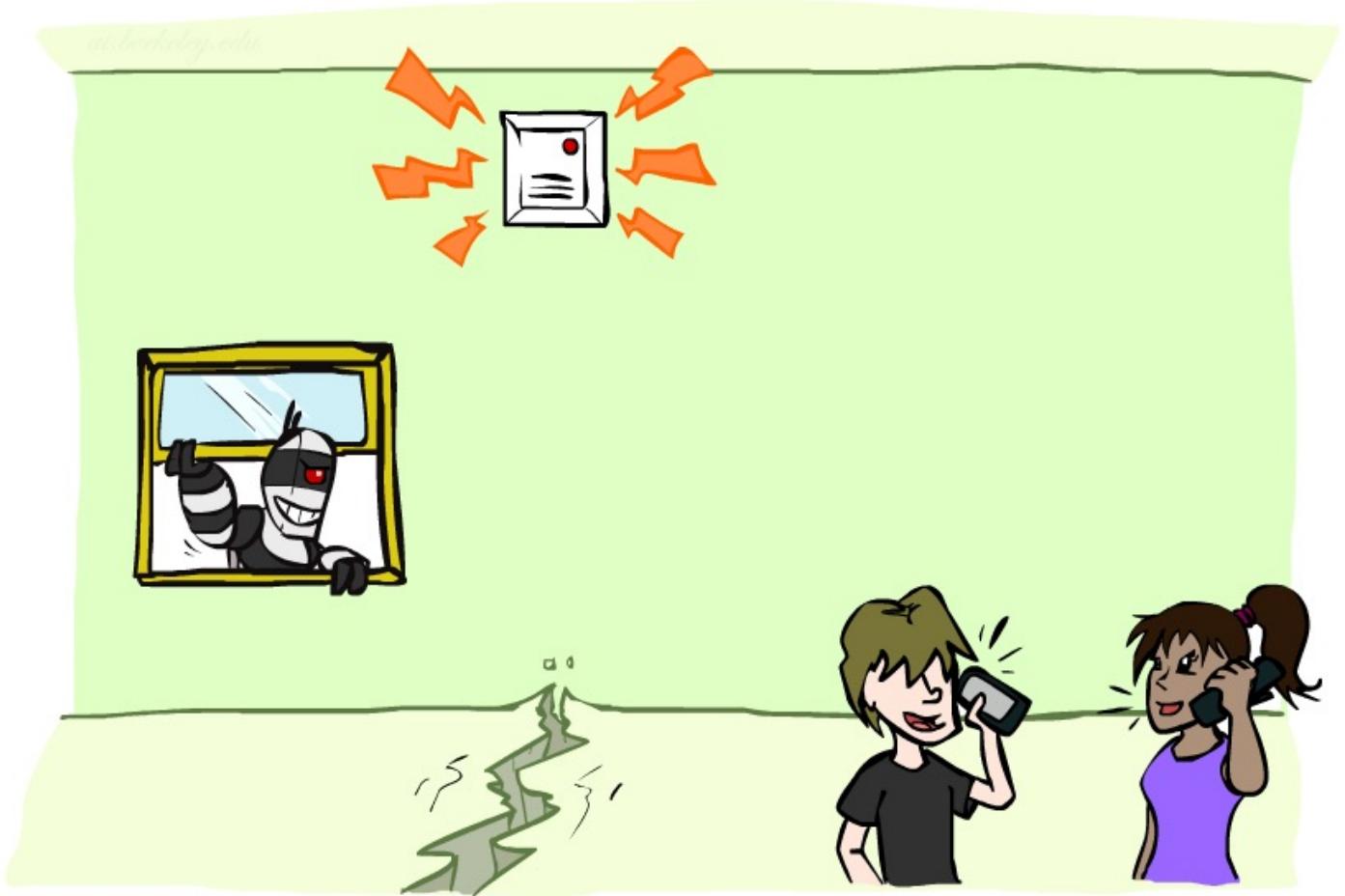
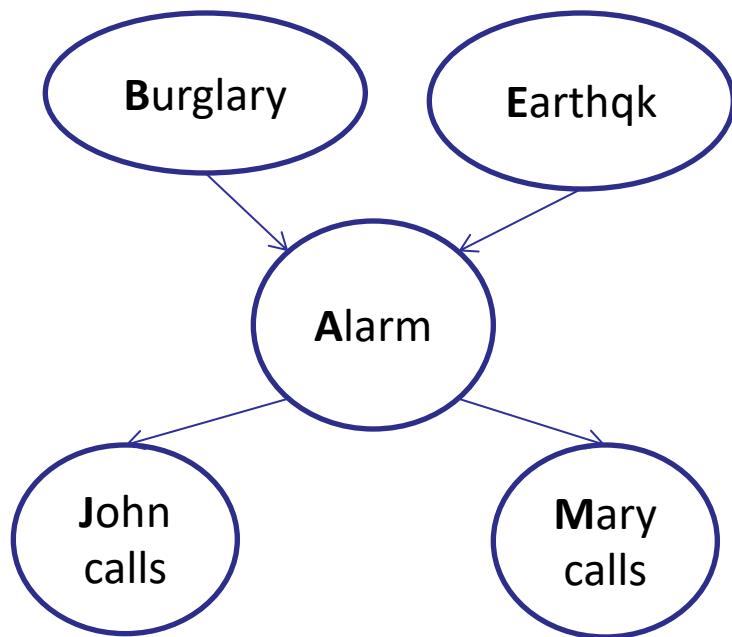
- B: Burglary
- A: Alarm goes off
- M: Mary calls
- J: John calls
- E: Earthquake!



# Example: Alarm Network

## Variables

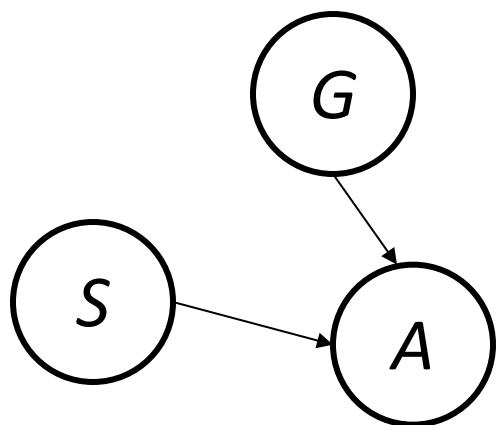
- B: Burglary
- A: Alarm goes off
- M: Mary calls
- J: John calls
- E: Earthquake!



# Example: Humans

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- G: human's goal / human's reward parameters
- S: state of the physical world
- A: human's action



# Example: Traffic II

- Variables

- T: Traffic
- R: It rains
- L: Low pressure
- D: Roof drips
- B: Ballgame
- C: Cavity



# Bayes' Net Semantics

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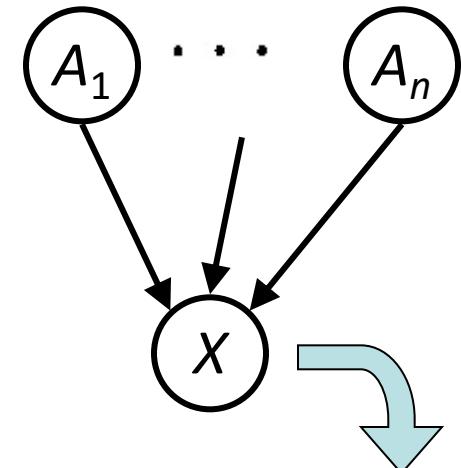


# Bayes' Net Semantics

- A set of nodes, one per variable  $X$
- A directed, acyclic graph
- A conditional distribution for each node
  - A collection of distributions over  $X$ , one for each combination of parents' values

$$P(X|a_1 \dots a_n)$$

- CPT: conditional probability table
- Description of a noisy “causal” process



$$P(X|A_1 \dots A_n)$$

*A Bayes net = Topology (graph) + Local Conditional Probabilities*

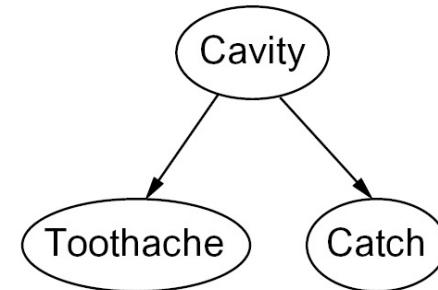
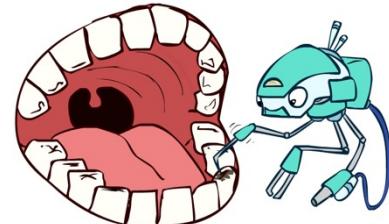
# Probabilities in BNs



- Bayes' nets **implicitly** encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$

- Example:



$P(+\text{cavity}, +\text{catch}, -\text{toothache})$

$= P(-\text{toothache}|\text{+cavity})P(\text{+catch}|\text{+cavity})P(\text{+cavity})$

# Probabilities in BNs



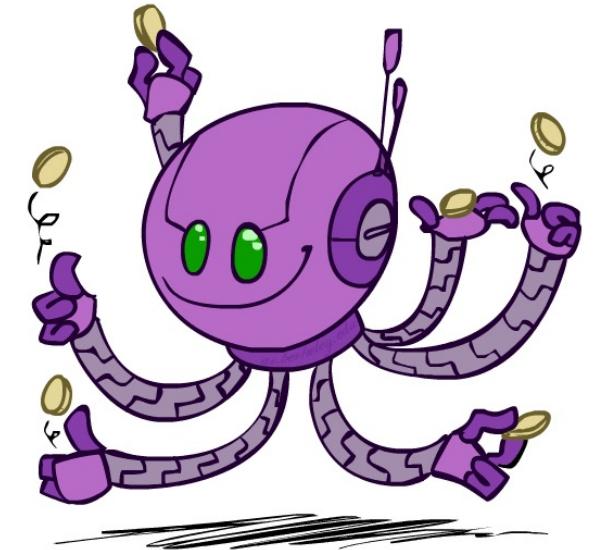
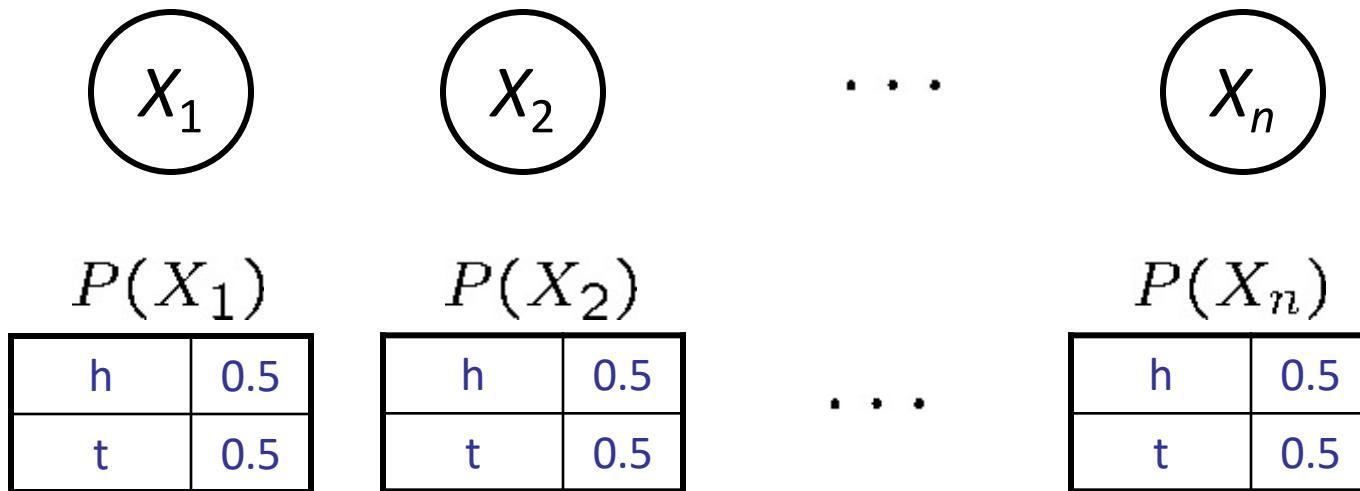
- Why are we guaranteed that setting

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$

results in a proper joint distribution?

- Chain rule (valid for all distributions):  $P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | x_1 \dots x_{i-1})$
- Assume conditional independences:  $P(x_i | x_1, \dots, x_{i-1}) = P(x_i | \text{parents}(X_i))$   
→ Consequence:  $P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$
- Not every BN can represent every joint distribution
  - The topology enforces certain conditional independencies

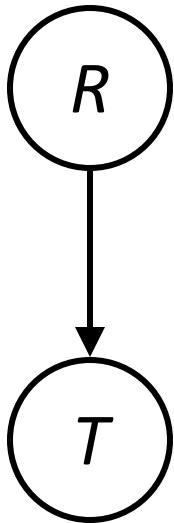
# Example: Coin Flips



$$P(h, h, t, h) = P(h)P(h)P(t)P(h)$$

*Only distributions whose variables are absolutely independent can be represented by a Bayes' net with no arcs.*

# Example: Traffic



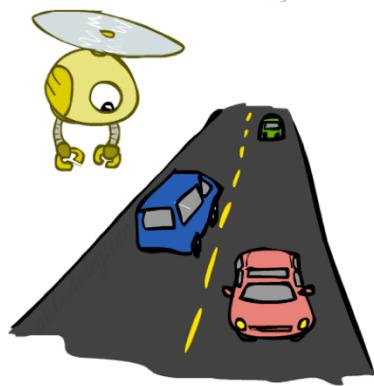
$P(R)$

+r	1/4
-r	3/4

$$P(+r, -t) = P(+r)P(-t|r) = \frac{1}{4} * \frac{1}{4}$$

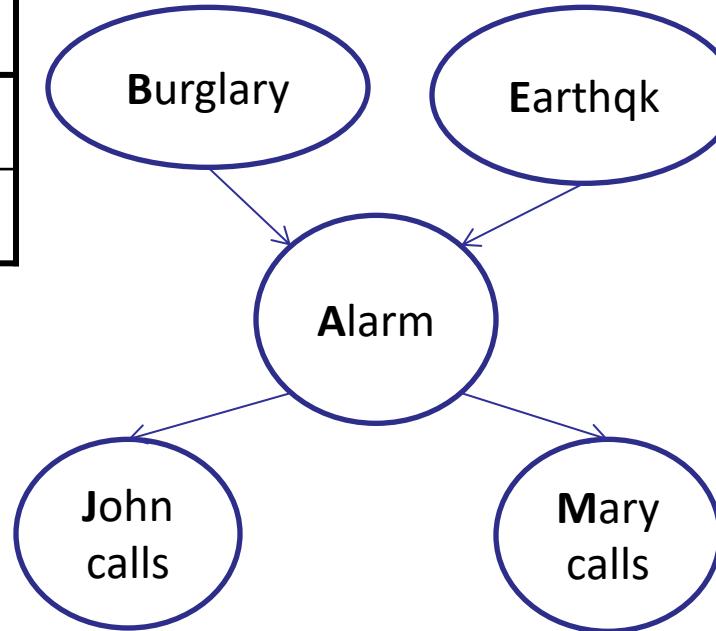
$P(T|R)$

+r	+t	3/4
	-t	1/4
-r	+t	1/2
	-t	1/2



# Example: Alarm Network

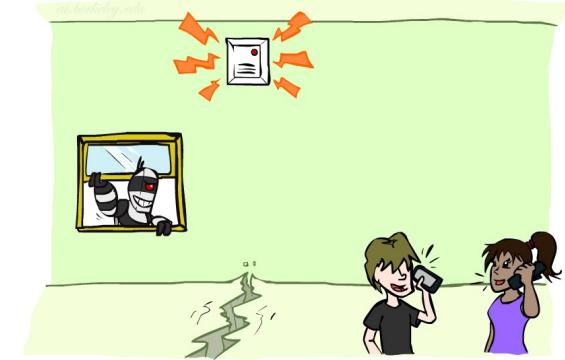
B	P(B)
+b	0.001
-b	0.999



A	J	P(J A)
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

A	M	P(M A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99

E	P(E)
+e	0.002
-e	0.998

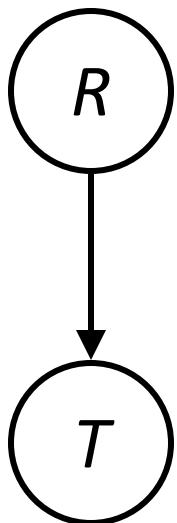


B	E	A	P(A B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

$P(M|A)P(J|A)$   
 $P(A|B,E)P(E)$   
 $P(B)$

# Example: Traffic

- Causal direction

 $P(R)$ 

+r	1/4
-r	3/4

 $P(T|R)$ 

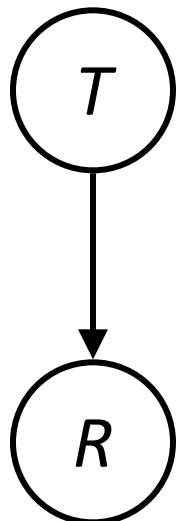
+r	+t	3/4
	-t	1/4
-r	+t	1/2
	-t	1/2

 $P(T, R)$ 

+r	+t	3/16
+r	-t	1/16
-r	+t	6/16
-r	-t	6/16

# Example: Reverse Traffic

- Reverse causality?



$P(T)$

+t	9/16
-t	7/16

$P(R|T)$

+t	+r	1/3
	-r	2/3
-t	+r	1/7
	-r	6/7



$P(T, R)$

+r	+t	3/16
+r	-t	1/16
-r	+t	6/16
-r	-t	6/16

# Causality?

- When Bayes' nets reflect the true causal patterns:
  - Often simpler (nodes have fewer parents)
  - Often easier to think about
  - Often easier to elicit from experts
- BNs need not actually be causal
  - Sometimes no causal net exists over the domain (especially if variables are missing)
  - E.g. consider the variables *Traffic* and *Drips*
  - End up with arrows that reflect correlation, not causation
- What do the arrows really mean?
  - Topology may happen to encode causal structure
  - **Topology really encodes conditional independence**

$$P(x_i|x_1, \dots x_{i-1}) = P(x_i|\text{parents}(X_i))$$

