EECS 16A   Designing Information Devices and Systems I
Fall 2020   Discussion 9A

1. Voltages Across Capacitors

For the circuits given below, determine the voltage across each capacitor and calculate the charge and energy stored on each capacitor (assume all capacitors start *uncharged*, and then we've let the system reach steady state). We are also given $C_1 = 1\,\mu F$, $C_2 = 3\,\mu F$, and $V_s = 1\,V$.

Recall charge has units of Coulombs (C), and capacitance is measured in Farads (F) $= \frac{Coulomb}{Volt}$.

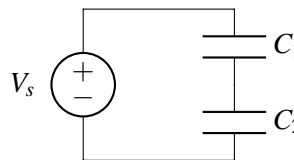It may also help to note metric prefix examples: $3\,\mu F = 3 \times 10^{-6} F$.

(a)
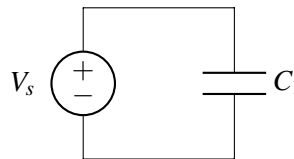
Answer:
In the steady state, this capacitor is charged up to match the voltage source V_s.
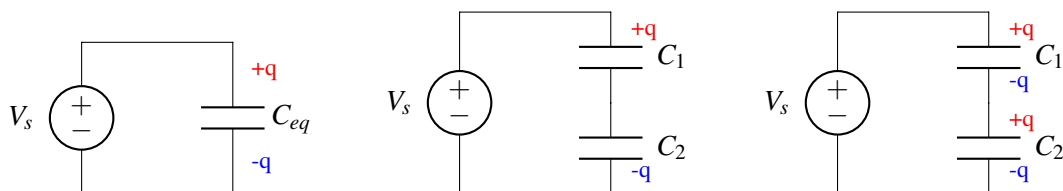The charge is $Q = C_1 V_s = 1\mu C$ ($+q$ accumulates on the top plate, $-q$ on the bottom plate).
The energy stored is $E = \frac{C_1 V_s^2}{2} = \frac{1}{2}\mu J$.

(b)

Helpful diagrams for considering the charges capacitors linked in series:
(without any initial charges)

Left: Our series capacitors may be modeled as one equivalent capacitor C_{eq}, which after some time is charged up by V_s to have $+q$ on the top plate and $-q$ on the bottom plate.

Middle: We return to the 2-capacitor picture, but carry this insight of equivalent charge with us. Now the charge $+q$ is on the top plate *of capacitor C_1*, and $-q$ is on the bottom plate *of capacitor C_2*.

Right: Since capacitor plates have opposite & equal charges, we attain this final right digram.

As another conceptual check, we notice that the node between C_1 and C_2 is isolated from any other connections and should always remain *charge neutral*. From the diagram right we see this is maintained since $(+q) + (-q) = 0$.

Answer:
In the spirit of 1(b)'s helpful diagram, we recognize both capacitors will have the same value of charge. Further we know this charge is equal to the expected charge from an equivalent capacitance model. So we first identify the equivalent capacitance:

$$C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2} \right)^{-1} = \left(\frac{3}{3\,\mu F} + \frac{1}{3\,\mu F} \right)^{-1} = \left(\frac{4}{3\,\mu F} \right)^{-1} = \frac{3}{4}\mu F$$

With C_{eq} known, we may use the capacitor definition $C = \frac{Q}{V}$ to compute the charge $Q_{eq} = C_{eq}V = \frac{3}{4}\mu C$. Note that this means $+\frac{3}{4}\mu C$ is on the top plate of C_1 and $-\frac{3}{4}\mu C$ is on the bottom plate of C_2.
Finally by using our reasoning above regarding equal charges on each capacitor matching the charge on this equivalent capacitor, we can again use $C = \frac{Q}{V}$ to identify the voltage across each capacitor.
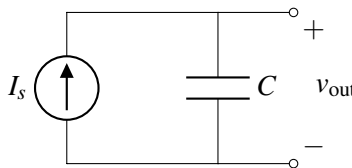
- The voltage V_1 across C_1 is $\frac{Q_{eq}}{C_1} = \frac{3}{4}V$.

- The voltage V_2 across C_2 is $\frac{Q_{eq}}{C_2} = \frac{1}{4}V$.

Again the energy stored can be calculated as $E = \frac{1}{2}CV^2$ for each capacitor

- $E_1 = 280\,nJ$
- $E_2 = 94\,nJ$

2. **Current Sources And Capacitors**

Given the circuit below, find an expression for $v_{out}(t)$ in terms of I_s, C, V_0, and t, where V_0 is the initial voltage across the capacitor at $t = 0$.

Then plot the function $v_{out}(t)$ over time on the graph below for the following conditions detailed below. Use the values $I_s = 1mA$ and $C = 2\,\mu F$.

(a) Capacitor is initially uncharged $V_0 = 0$ at $t = 0$.

(b) Capacitor has been charged with $V_0 = +1.5V$ at $t = 0$.

(c) **Practice:** Swap this capacitor for one with half the capacitance $C = 1\,\mu F$, which is initially uncharged $V_0 = 0$ at $t = 0$.

HINT: Recall the calculus identity $\int_a^b f'(x)dx = f(b) - f(a)$, where $f'(x) = \frac{df}{dx}$.

time (ms)
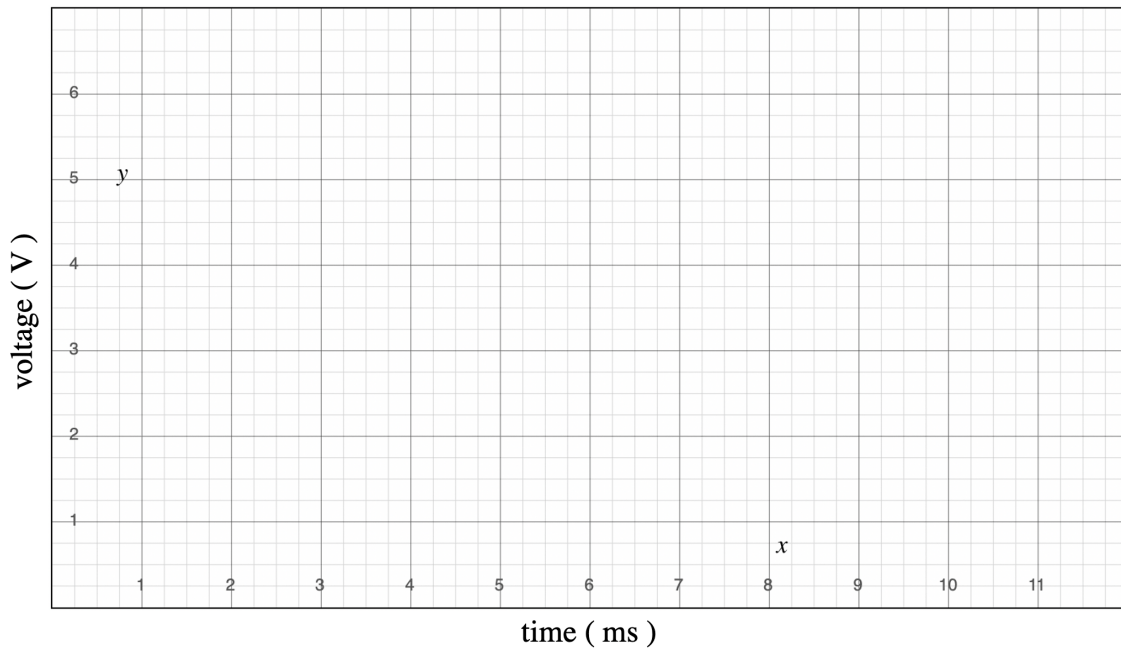
Answer:

The key here is to exploit the capacitor equation by taking its time-derivative

$$Q = C\,V_{out} \quad \longrightarrow \quad \frac{dQ}{dt} \equiv I_s = C\frac{dV_{out}}{dt}\,.$$

From here we can rearrange and show that $\frac{dV_{out}}{dt} = \frac{I_s}{C}$.

Thus the voltage has a constant slope!

Our solution is

$$V_{out}(t) = V_0 + \left(\frac{I_s}{C}\right)t \qquad \qquad \square$$

To be more mathematically formal, we are solving a differential equation that happens to return a linear function for $v_{out}(t)$:

$$\frac{dV_{out}}{dt} = \frac{I_s}{C} \quad \longrightarrow \quad \int_0^t \frac{dV_{out}}{dt}dt \equiv V_{out}(t) - V_{out}(0) = \int_0^t \frac{I_s}{C}dt \equiv \frac{I_s}{C}\int_0^t 1\,dt \equiv \frac{I_s}{C}t$$

Thus we arrive at the same statement as seen earlier $V_{out}(t) = V_{out}(0) + \left(\frac{I_s}{C}\right)t$.

From this stage we can compute the slope of $V_{out}(t)$ for parts (a) and (b) along with the slope for (c), which should be twice as large.

$$\frac{I_s}{C} = \frac{1\text{mA}}{2\mu\text{F}} = \frac{1000\frac{\mu\text{C}}{\text{s}}}{2\frac{\mu\text{C}}{\text{V}}} = 500\frac{\text{V}}{\text{s}} = \left(\frac{1}{2}\right)\frac{\text{V}}{\text{ms}}$$

For part (c):

$$\frac{I_s}{C} = \frac{1\text{mA}}{1\mu\text{F}} = \frac{1000\frac{\mu\text{C}}{\text{s}}}{1\frac{\mu\text{C}}{\text{V}}} = 1000\frac{\text{V}}{\text{s}} = 1\frac{\text{V}}{\text{ms}}$$

When plotting, make sure to recall (a) and (c) start at the origin, while (b) has initially charged plates by $V_0 = 1.5V$. Results are shown below