EECS 16A   Designing Information Devices and Systems I
Spring 2022   Homework 8

This homework is due Friday, March 18, 2022 at 23:59. Self-grades are due Monday, March 28, 2022, at 23:59.

Submission Format

Your homework submission should consist of **one** file.

- `hw8.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit the file to the appropriate assignment on Gradescope.

1. **Mid-semester Class Survey**

   Please fill out the EECS16A Mid-semester Class Survey (click here). We appreciate your feedback towards improving this class — this really helps us improve future semesters as well as latter parts of this class. This survey is completely anonymous.

   An incentive to do this: If more than 70% of the class fills out this survey in full, we will add 1 point to everyone's MT1 score. If more than 80% of the class fills out this survey in full, we will add 2 poinst to everyone's MT1 score. Please note that EECS 47D students are not required to fill out the survey.

2. **Reading Assignment**

   For this homework, please read Note 15. Note 15 covers superposition and equivalence, two very helpful techniques to help simplify circuit analysis.

3. **Equivalent Resistance**

   Learning Goal: *The objective of this problem is to practice finding the equivalent to a series/parallel combination of resistors.*
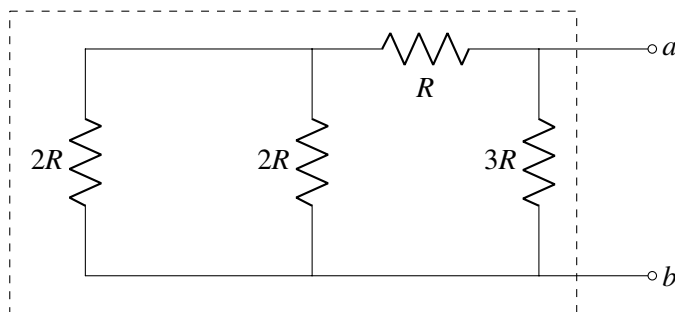
   (a) Find the equivalent resistance looking in from points a and b. In other words, express the resistive network in the dashed box as one resistor.
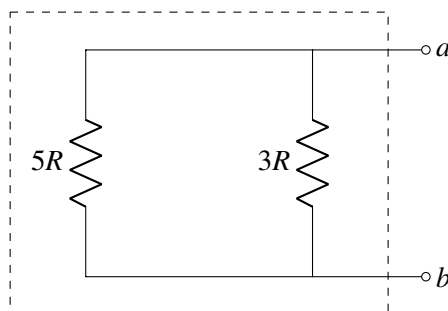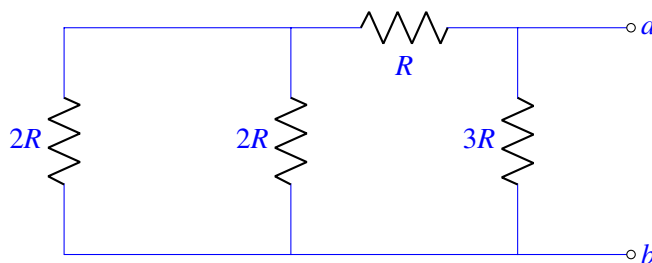

   Solution:

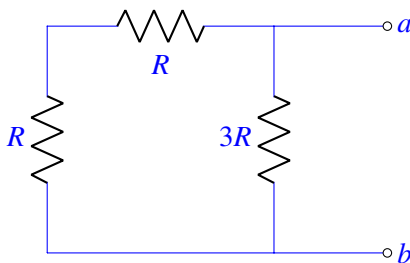 $$R_{eq} = 5R \parallel 3R = \frac{5R \cdot 3R}{5R + 3R} = \frac{15}{8}R$$

(b) Find the equivalent resistance looking in from points a and b. In other words, express the resistive network in the dashed box as one resistor.
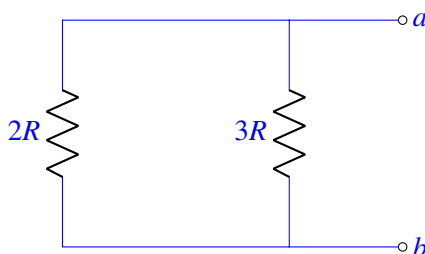

Solution: We find the equivalent resistance for the resistors from left to right. In general, it is easiest to start furthest away from the node voltages we wish to measure.

We can begin by combining the leftmost two resistors, which are in parallel and both have values $2R$. The equivalent resistor would have resistance $R_{eq} = 2R \parallel 2R = \frac{2R \cdot 2R}{2R + 2R} = R$, so we replace both resistors with a equivalent resistor of resistance R.


Now, the leftmost resistors are in series both with values R, so the equivalent resistor would have resistance $R_{eq} = R + R = 2R$.


Finally, we are left with a parallel combination of resistors with values $2R$ and $3R$ so the equivalent resistor would have resistance $R_{eq} = 2R \parallel 3R = \frac{6}{5}R$.
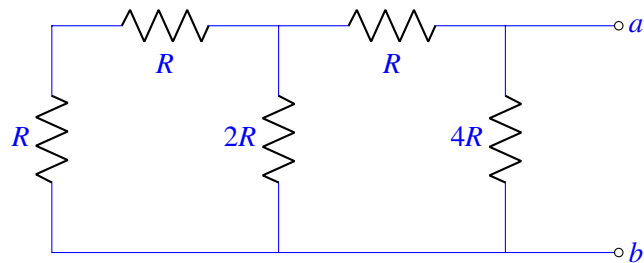
(c) Find the equivalent resistance looking in from points a and b. In other words, express the resistive network in the dashed box as one resistor.
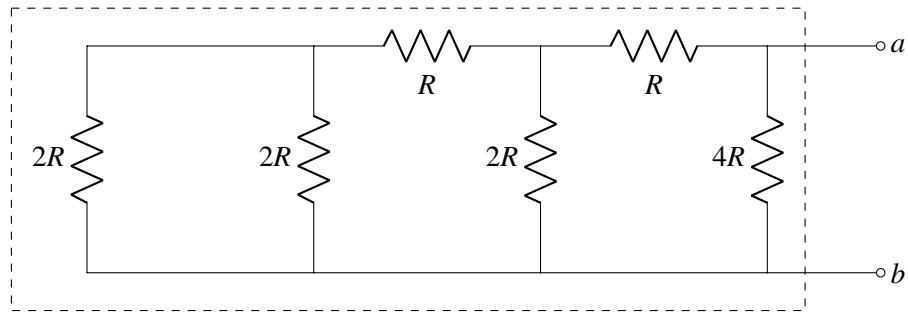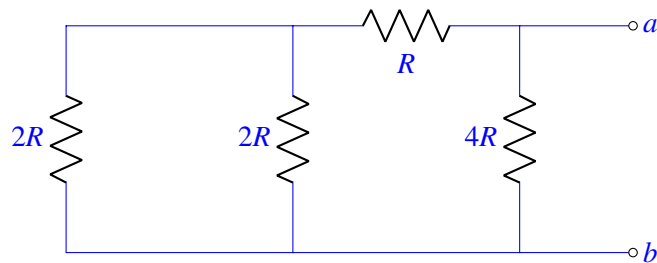

Solution: Again, we find the equivalent resistance for the resistors from left to right.
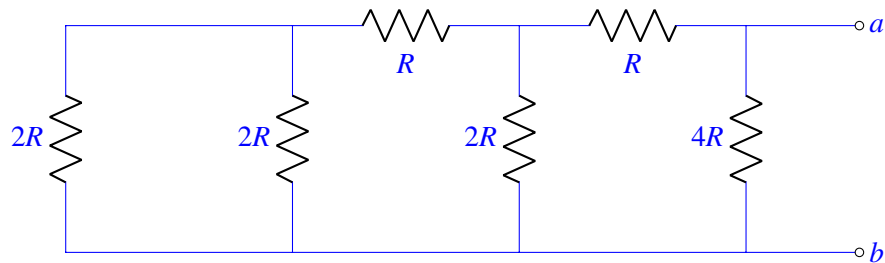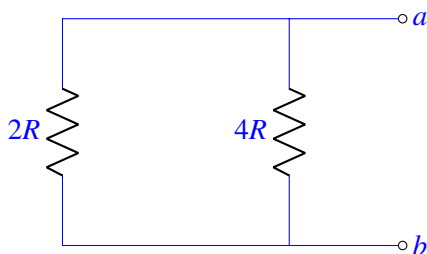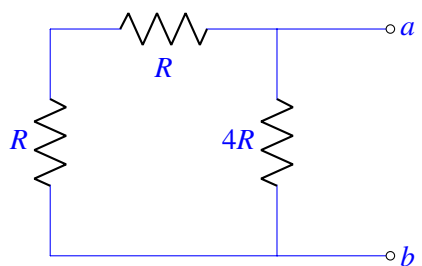

We first combine the two parallel resistors on the left:


Next, the two series resistors on the left:

Following the process, we can eventually simplify down to a circuit with a single equivalent resistor.

$$R_{eq} = 2R \parallel 4R = \frac{4}{3}R$$

(d) **(OPTIONAL, CHALLENGE)** Find the equivalent resistance for the infinite ladder looking in from points a and b. In other words, express the resistive network in the dashed region as one resistor. (*Hint: Let's call the resistance looking in from a and b as R_{in}, and the resistance looking to the left from points c and d as R_{in1}. Replace the entire circuit to the left of points c and d with a resistor whose value is given by R_{in1}. Find the relationship between R_{in} and R_{in1} using this circuit. Find another relationship between R_{in} and R_{in1} using the fact that the the ladder is infinite. For an infinite ladder, adding an another branch does not change the equivalent resistance. Think of this as a convergent infinite series.*)
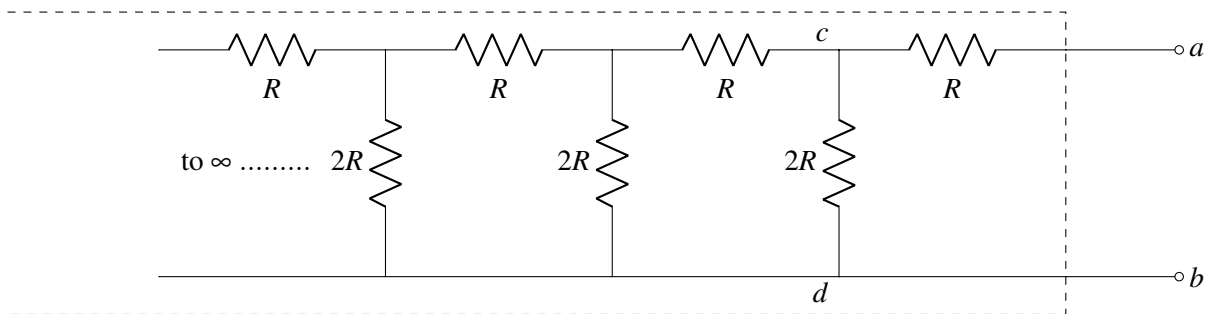
As a first step you can replace the circuit looking to the left from c and d by R_{in1}.
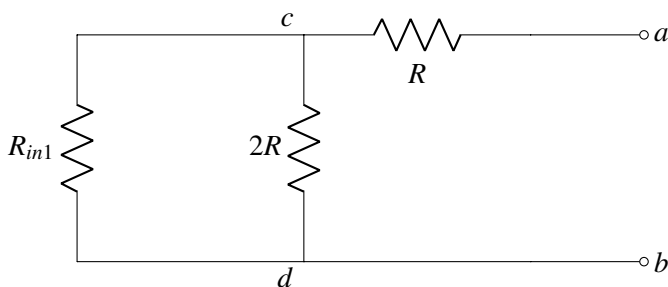
Solution: We wish to compute the equivalent resistance R_{in} looking to the left from nodes a and b. The equivalent resistance looking to the left from nodes c and d is given by R_{in1}. Clearly,

$$R_{in} = (R_{in1} \| 2R) + R$$

Additionally, since this is an infinite ladder, the equivalent resistance does not change by addition of an extra branch to the right, since having infinity + 1 steps is the same thing as having infinite steps. Therefore, $R_{in1} = R_{in}$. Using this result in the previous equation, we have,

$$R_{in} = (R_{in} \| 2R) + R$$
$$R_{in} = \frac{2R \cdot R_{in}}{2R + R_{in}} + R$$
$$R_{in}^2 - RR_{in} - 2R^2 = 0$$
$$(R_{in} - 2R)(R_{in} + R) = 0$$

Clearly, $R_{in} = -R$ is not a physically realizable solution. The equivalent resistance looking into this infinite ladder is given by $R_{in} = 2R$.

4. **Why Bother With Thévenin Anyway?**

   (a) Find a Thévenin eqiuvalent for the circuit shown below looking from the terminals a and b. (*Hint: That is, find the open circuit voltage V_R across the terminals a and b. Also, find the equivalent resistance looking from the terminals a and b when the input voltage source is zeroed.*)
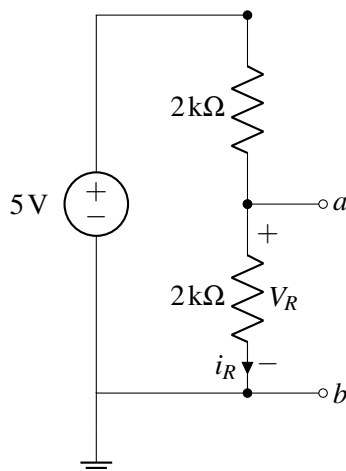
   Solution:

$$V_{Th} = \frac{2\,k\Omega}{2\,k\Omega + 2\,k\Omega} \cdot 5\,V = 2.5\,V$$
$$R_{Th} = 2\,k\Omega \,\|\, 2\,k\Omega = 1\,k\Omega$$
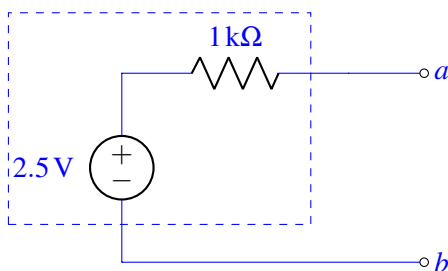
(b) Now consider the circuit shown below where a load resistor of resistance R_L is attached across the terminals a and b. Compute the voltage drop V_R across the terminals a and b in this new circuit with the attached load. Express your answer in terms of R_L. (*Hint: We have already computed the Thevenin equivalent of the unloaded circuit in part (a). To analyze the new circuit, attach R_L as the load resistance across the Thevenin equivalent computed in part (a), as shown in the figure below. One of the main advantages of using Thevenin (and Norton) equivalents is to avoid re-analyzing different circuits which differ only by the amount of loading.*)
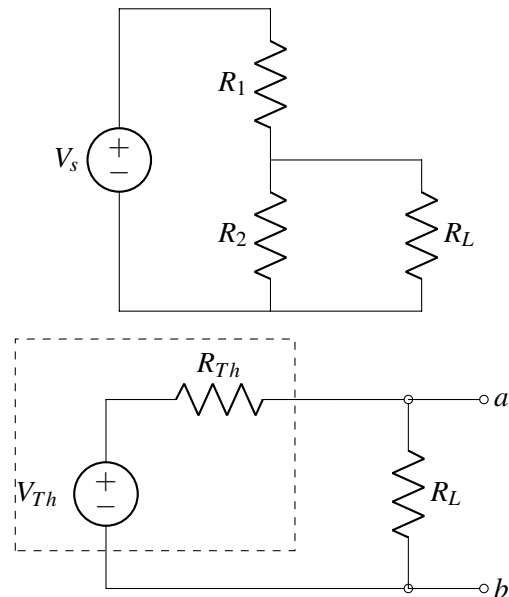
Solution:
We just attach the R_L resistor to our Thévenin equivalent circuit and calculate the voltage across it.
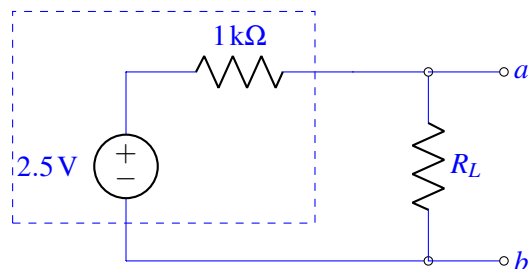
$$V_R = \frac{R_L}{1\,\text{k}\Omega + R_L} \cdot 2.5\,\text{V}$$

(c) Now compute the voltage drop V_R for three different values of R_L equal to $8/3\,\text{k}\Omega$, $8\,\text{k}\Omega$, $80\,\text{k}\Omega$? What can you comment on the value of R_L needed to ensure that the loading does not reduce the voltage drop V_R compared to the unloaded voltage V_R computed in part (a)? **Solution:**
$R_L = \frac{8}{3}\text{k}\Omega$:

$$V_R = \frac{\frac{8}{3}\text{k}\Omega}{1\,\text{k}\Omega + \frac{8}{3}\text{k}\Omega} \cdot 2.5\,\text{V} = 1.82\,\text{V}$$
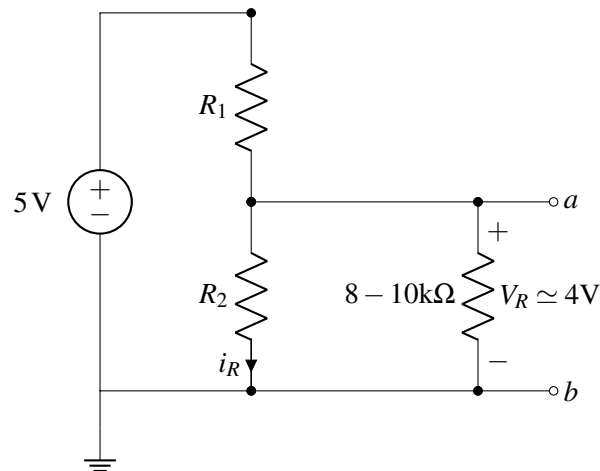
$R_L = 8\,\text{k}\Omega$:

$$V_R = \frac{8\,\text{k}\Omega}{1\,\text{k}\Omega + 8\,\text{k}\Omega} \cdot 2.5\,\text{V} = 2.22\,\text{V}$$

$R_L = 80\,\text{k}\Omega$:

$$V_R = \frac{80\,\text{k}\Omega}{1\,\text{k}\Omega + 80\,\text{k}\Omega} \cdot 2.5\,\text{V} = 2.46\,\text{V}$$

As the value of R_L is increased, the voltage drop V_R approaches the unloaded Thevenin voltage computed in part (a).
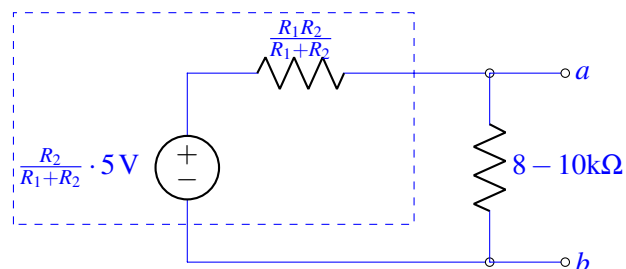
(d) Say that we want to support loads in the range of $8\,\text{k}\Omega$ to $10\,\text{k}\Omega$. We would like to maintain $4\,\text{V}$ across these loads. How can we approximately achieve this by setting R_1 and R_2 in the following circuit?

Solution:

$$V_{Th} = \frac{R_2}{R_1 + R_2} \cdot 5\,\text{V}$$

$$R_{Th} = R_1 \parallel R_2 = \frac{R_1 R_2}{R_1 + R_2}$$

$$V_R = \frac{R}{R + \frac{R_1 R_2}{R_1 + R_2}} \cdot \frac{R_2}{R_1 + R_2} \cdot 5\,\text{V} \simeq 4\,\text{V}$$

$$\frac{R R_2}{R(R_1 + R_2) + R_1 R_2} = \frac{4}{5}$$

If we set $R_1, R_2 \ll R$, then $\frac{R R_2}{R(R_1 + R_2) + R_1 R_2} \approx \frac{R R_2}{R(R_1 + R_2)} = \frac{R_2}{R_1 + R_2}$. Therefore, we can just choose two small resistors $R_1, R_2 \ll 8\,\text{k}\Omega$, such that $R_2 = 4R_1$.

(e) Thus far, we have seen how to use Thevenin equivalents to compute the voltage drop across a load without re-analyzing the entire circuit. We would like to see if we can use the Thevenin equivalent for power computations. Consider the case where the load resistance $R_L = 8\text{k}\Omega$, $V_S = 5V$, $R_1 = R_2 = 2\text{k}\Omega$. Compute the power dissipated across the load resistor R_L both using the original circuit and the Thevenin equivalent. Are they equal? Now, compute the power dissipated by the voltage source V_s in the original circuit. Also, compute the power dissipated by the Thevenin voltage source V_{Th} in the Thevenin equivalent circuit. Is the power dissipated by the two sources equal?
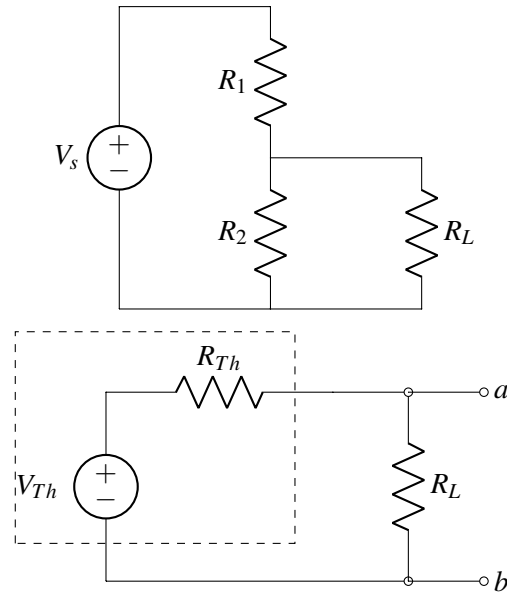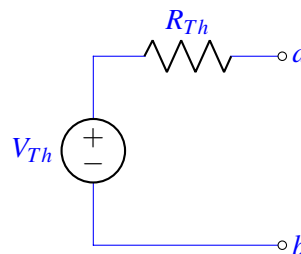

Solution:

We will compare the power dissipation in V_s vs. V_{Th} and R_L in either case. This could be done for the specific example above (with $R_L = 8\text{k}\Omega$), but it's more useful to go through this exercise generally. Thus, we will use the circuit shown below:
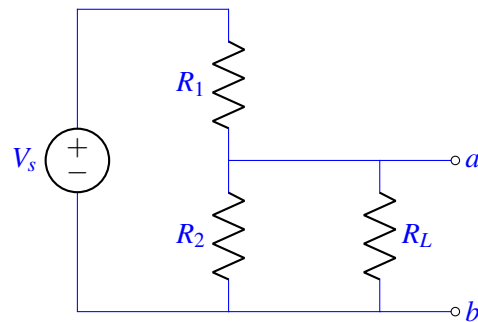
Recall that the Thévenin equivalent for the circuit above looks as follows:

where $R_{Th} = \frac{R_1 R_2}{R_1 + R_2}$ and $V_{Th} = \frac{R_2}{R_1 + R_2} V_s$.

Because we are going to end up writing a few expressions multiple times, we are going to define a new variable:

$$\beta = R_1 R_2 + R_{load} R_1 + R_{load} R_2$$

Let's start with our equivalent circuit. In the equivalent circuit, the current through the load resistor and equivalently every other element in the circuit is:

$$I = \frac{V_R}{R_L} = \frac{V_{Th}}{R_L + R_{Th}}$$

With this current, we find the power dissapated across the source and the load resistor.

$$P_{V_{Th}} = -IV = -\frac{V_{Th}^2}{R_L + R_{Th}} = -\frac{V_{Th}^2(R_1 + R_2)}{\beta} = -\frac{V_s^2 R_2^2}{\beta(R_1 + R_2)} = -0.694\,\text{mW}$$

$$P_{R_L} = I^2 R = \frac{V_{Th}^2}{(R_L + R_{Th})^2} \cdot R_L = \frac{V_{Th}^2(R_1 + R_2)^2}{\beta^2} \cdot R_L = \frac{V_s^2 R_2^2}{\beta^2} \cdot R_L = 0.617\,\text{mW}$$

Let's try to find the answer from the original circuit. We will begin by calculating the current through the source.

$$I_s = \frac{V_s}{R_{eq}} = \frac{V_s}{R_1 + R_2 \parallel R_L} = \frac{V_s(R_1 + R_2)}{\beta}$$

Now, we can calulate the power through the source.

$$P_{V_s} = -I_s V_s = -\frac{V_s^2(R_2 + R_L)}{\beta} = -6.94\,\text{mW}$$

The power dissipated by the source in the original circuit is not the same as the power dissipated in the new circuit. What about the load resistor? We will first calculate the voltage across the load resistor.

$$V_{load} = \frac{R_2 \parallel R_L}{R_1 + R_2 \parallel R_L} \cdot V_s = \frac{\frac{R_2 R_{load}}{R_2 + R_L}}{R_1 + \frac{R_2 R_{load}}{R_2 + R_L}} \cdot V_s = \frac{R_2 R_{load}}{\beta} \cdot V_s$$

$$P_{load} = \frac{V_{load}^2}{R_L} = \frac{V_s^2 R_2^2}{\beta^2} R_L = 0.617\,\text{mW}$$

The power through the load is the same! Thévenin equivalents can be used to calculate the power through elements that are not part of the circuit that was transformed.

5. Wheatstone Bridge

Thévenin equivalence is a powerful technique we can use to analyze the Wheatstone bridge circuit shown below. This circuit is used in many sensor applications where resistors R_1, R_2, R_3, R_4 are varying with respect to some external actuation. For example, it can be used to build a strain gauge or a scale (remember Fruity Fred from HW 7). In that case the resistors R_1, R_2, R_3, R_4 would vary with respect to a strain caused by a force, and the Wheatstone Bridge circuit would translate that variation into a voltage difference across the "bridge" terminals a and b. Assume that $R_1 = 2\,\text{k}\Omega$, $R_2 = 2\,\text{k}\Omega$, $R_3 = 1\,\text{k}\Omega$, $R_4 = 4\,\text{k}\Omega$
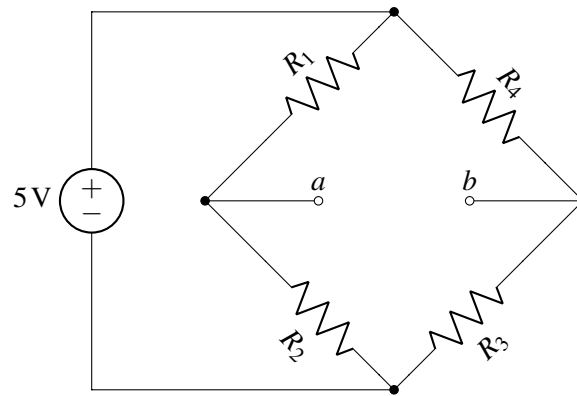
(a) Calculate the voltage V_{ab} between the two terminals a and b.

Solution:

Notice in the above circuit that there are two voltage dividers, so we can calculate v_a and v_b quickly.
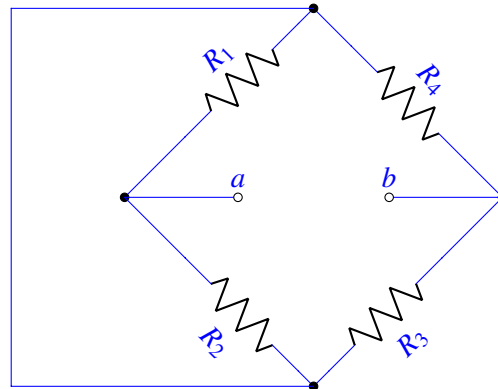
$$v_a = \frac{R_2}{R_1 + R_2} \cdot 5\,\text{V} = 2.5V$$

$$v_b = \frac{R_3}{R_3 + R_4} \cdot 5\,\text{V} = 1V$$

Thus, the voltage difference between the two terminals a and b is: $V_{ab} = v_a - v_b = 1.5V$.
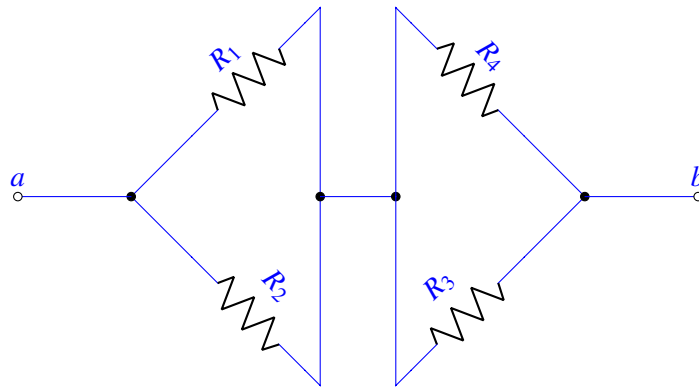
(b) Next, draw the Thévenin equivalent of the Wheatstone bridge circuit.

Solution:

We find the Thévenin resistance by replacing the voltage source with a short and calculating the resistance between the two terminals a and b. The circuit now looks like:
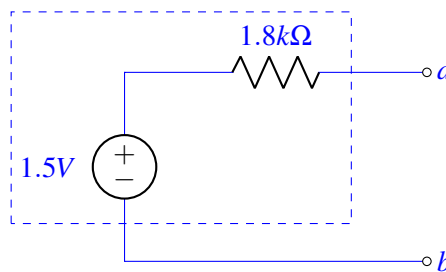

Notice that because the top and bottom node are shorted, we have $R_1 \parallel R_2$ in series with $R_3 \parallel R_4$ between nodes "a" and "b".
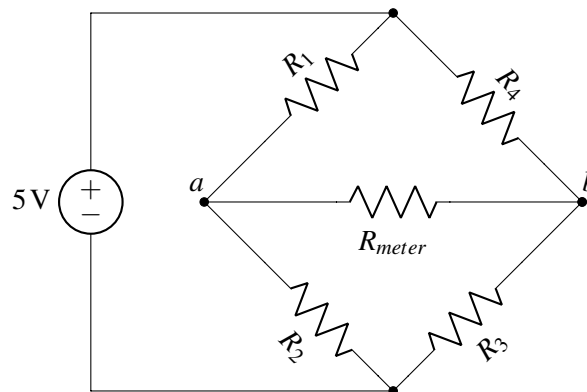
It follows that R_{th} is:

$$R_{\text{Th}} = (R_1 \parallel R_2) + (R_3 \parallel R_4), \text{ where } \parallel \text{ denotes the parallel operator.}$$
$$= \frac{R_1 R_2}{R_1 + R_2} + \frac{R_3 R_4}{R_3 + R_4}$$
$$= 1.8k\Omega$$

Using $V_{\text{Th}} = V_{ab} = 1.5V$ from part (a), we can construct the Thévenin equivalent circuit:

(c) Now assume that you are trying to measure the voltage V_{ab} using a voltmeter, whose resistance is R_{meter}, so you end up with the circuit below.


Unfortunately, you voltmeter is far from ideal, so $R_{meter} = 4k\Omega$. Is the voltage V_{ab} you found in part (a) equal to the new voltage $V_{R_{meter}}$ across the voltmeter resistor? Why or why not? Calculate the current $I_{R_{meter}}$ through the voltmeter resistor and the voltage $V_{R_{meter}}$ across the meter resistor.

Solution:

No, the Thévenin voltage we found in part (a) is the open-circuit voltage. If we add R_{meter} back into the original circuit, R_{meter} would load the other resistors (or, equivalently, the Thévenin resistance), so the Thévenin voltage is not equal to the actual voltage across the meter resistor.

Having derived the Thévenin equivalent circuit, we can now draw the following:
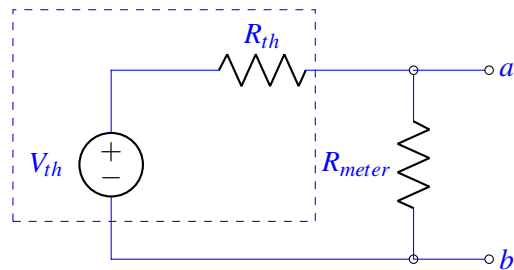
Using the facts that, $R_{meter} = 4k\Omega$, $R_{th} = 1.8k\Omega$, $V_{th} = 1.5V$ we can write:

$$I_{R_{meter}} = \frac{1.5V}{1.8k\Omega + 4k\Omega} \approx 0.26\text{mA}$$

$$V_{R_{meter}} = I_{R_{meter}}R_{meter} \approx 1.03V$$

Notice that we have an almost 20% error in our measurement because of the finite impedance of the voltmeter!

6. **Superposition**

Learning Goal: *The objective of this problem is to help you practice solving circuits using the principles of superposition.*

Find the current i indicated in the circuit diagram below using superposition.
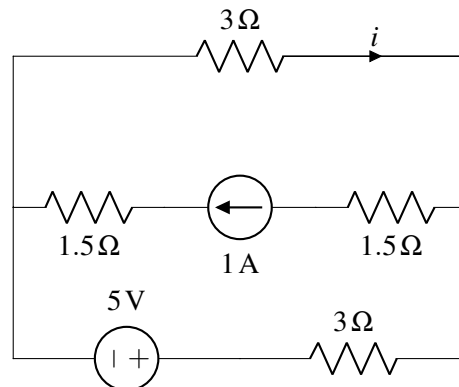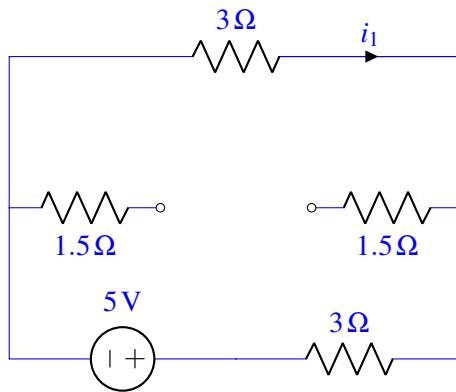

Solution:

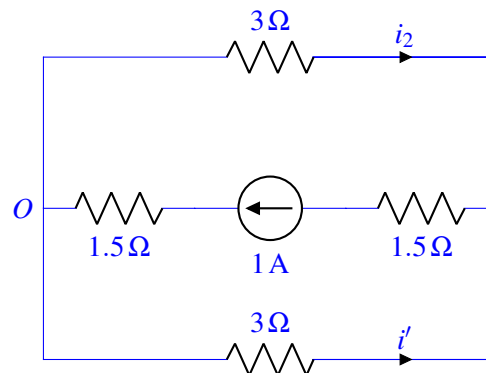$$i = -1/3\,\text{A}$$

Consider the circuits obtained by:

(a) Zeroing out the 1 A current source:

In the above circuit, no current flows through the middle branch, as it is an open circuit. Thus this is just a 5 V voltage source connected to two $3\,\Omega$ resistors in series so

$$i_1 = -5/6\,\text{A}$$

(b) Zeroing out the 5 V voltage source:

In the above circuit, notice that the $3\,\Omega$ resistors are in parallel and therefore form a current divider. Since the values of the resistances are equal, the current flowing through them will also be equal, that is $i_2 = i'$. Applying KCL to node O, we get

$$1 - i_2 - i' = 0$$

which gives us

$$i_2 = 1/2\,\text{A}$$

Now, applying the principle of superposition, we have $i = i_1 + i_2 = -5/6\,\text{A} + 1/2\,\text{A} = -1/3\,\text{A}$.

7. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.