FINAL REPORT

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Project Title	Experimental Test of Ardehali Inequalities on a Five Qubit IBM Quantum Computer

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INTRODUCTION

The locality problem is aroused by the Einstein-Podolsky-Rosen (EPR) [1] argument, which suggests using a theory of hidden variables. Bell's Theorem can demonstrate the falsification of locality proposed by the Hidden Variable Theorem (HVT). There is a bound for expectation values on a multipartite system where quantum predictions violate this bound. Bell inequalities limit this bound. Clauser-Horne-Shimony-Holt (CHSH) [2] inequalities are derived for two-qubit systems, and their generalized N-qubit system form is called Mermin-Ardehali-Belinski-Klyshko (MABK) inequalities. Quantum mechanics violates Mermin inequalities by an exponential factor of $2^{(n-2)/2}$ for even n and $2^{(n-1)/2}$ for odd n, where n is the number of qubits. These factors increased in Ardehali inequalities by narrowing down the boundaries. The main hypothesis to be tested in this project is that quantum mechanics violates Ardehali inequalities by an exponential amount of $2^{(n-1)/2}$ for even n and $2^{(n-2)/2}$ for odd n. This type of inequality can be tested by using several techniques. One of them is to construct quantum circuits on superconducting qubits. These circuits allow us to construct corresponding quantum states which violate MABK inequalities. Several years ago, IBM [3] opened their quantum computers to the public for up to 5 qubits. We plan to construct such a test specifically for Ardehali inequalities on an IBM quantum computer for a maximum of 5 qubits.

A state is described by Greenberger-Horne-Zeilinger [4], and this state can be used to demonstrate the falsification of the EPR argument without using any inequalities. Since the EPR view gives a door to introduce an additional variable that provides a complete picture without randomness, a spin correlation experiment must always yield a certain outcome. When a quantum mechanical approach calculates this state, it cannot occur. Even though a simplistic approach does this refutation of the EPR argument, it cannot be implemented in an experimental environment due to non-ideal laboratory conditions. The non-perfect efficiency of any measurement alone is a problem to test the GHZ experiment. Therefore, one must use the help of softening effect of any inequality for locality problems to test it in a laboratory environment. According to Bell's Theorem [5], several inequalities are derived for a different number of qubits. The most known is CHSH inequality which is derived for two-qubit systems. The more generalized form of CHSH inequalities for a system of N-qubits is MABK inequalities. In Mermin's case [6], Bell's inequality derived for a state of n spin ½ particles using distribution and correlation functions. This n-particle state is simply:

$$|\phi\rangle = \frac{1}{\sqrt{2}}[|00..0\rangle + i|11..1\rangle] \tag{1}$$

where 0 and 1 refer to a component of the j^{th} particle along its z-axis, and an operator defined:

$$\hat{A} = \frac{1}{2i}\left[\prod_{j=1}^{n}\left(\sigma_x^j + i\sigma_y^j\right) - \prod_{j=1}^{n}\left(\sigma_x^j - i\sigma_y^j\right)\right] \tag{2}$$

with eigenvalue of 2^{n-1}. By taking the diagonal matrix element of A, one can expand this state as:

$$2^{n-1} = \langle \phi | \sigma_y^1 \sigma_x^2 \dots \sigma_x^n | \phi \rangle + \dots - \langle \phi | \sigma_y^1 \sigma_y^2 \sigma_y^3 \sigma_x^4 \dots \sigma_x^n | \phi \rangle - \dots + \langle \phi | \sigma_y^1 \dots \sigma_y^5 \sigma_x^6 \dots \sigma_x^n | \phi \rangle + \dots - \langle \phi | \sigma_y^1 \dots \sigma_y^7 \sigma_x^8 \dots \sigma_x^n | \phi \rangle - \dots + \dots \quad (3)$$

where j refers to the position of particle and n refers to the number of particles. Since it is investigated in a case where measurements are non-ideal and observed correlation function $E_{\mu 1 \dots \mu n}$ cannot attain extreme values $\langle \phi | \sigma_{\mu_1}^1 \dots \sigma_{\mu_n}^n | \phi \rangle$=±1, this conditionally independent form of representation of (3) is introduced where μ is x or y and m is 0 or 1:

$$P_{\mu_1 \dots \mu_n}(m_1 \dots m_n) \int d\lambda p(\lambda) \left[p_{\mu_1}^1(m_1, \lambda) \dots p_{\mu_n}^n(m_n, \lambda) \right] \quad (4)$$

Defined value λ is common to all n particles with distribution p(λ). It attributes property that the outcome of any one detector does not depend on the preference of components x and y, which made for other particles. In other words, information is already available in the particles even before the measurements are done and leave a common source. If such a representation exists, it can be said that the mean products of x and y components are given by:

$$E_{\mu_1 \dots \mu_n} = \int d\lambda p(\lambda) \left[E_{\mu_1}^1(\lambda) \dots E_{\mu_n}^n(\lambda) \right] \quad (5)$$

And the correlation function is given as

$$E_s^j(\lambda) = p_p^j(+, \lambda) - p_p^j(-, \lambda) \quad (6)$$

Now the correlation function combination is given as

$$F = \int d\lambda p(\lambda) \frac{1}{2i} \left[\prod_{j=1}^{n} \left(E_x^j + iE_y^j \right) - \prod_{j=1}^{n} \left(E_x^j - iE_y^j \right) \right] \quad (7)$$

The Quantum mechanical approach for this expression is:

$$F = \langle \phi | \hat{A} | \phi \rangle = 2^{n-1} \quad (8)$$

There is, however, a stricter bound can be written. Each E_x^j, E_y^j is constrained by a value of -1 and +1. Then imaginary part of F will have a magnitude of $\sqrt{2}$ and phases of $\pm \frac{\pi}{4}\ and \pm \frac{3\pi}{4}$ for each product of $\prod_{j=1}^{n} \left(E_x^j + iE_y^j \right)$. When this product lies on the imaginary axis, it is bounded by these inequalities:

$$F \leq 2^{\frac{n}{2}}, for\ n\ even$$
$$F \leq 2^{\frac{(n-1)}{2}}, for\ n\ odd \quad (9)$$

These are the boundaries formulated by Mermin. A much stricter bound was formulated by Ardehali [7].

A GHZ state defined as

$$|\phi\rangle = \frac{1}{\sqrt{2}}[|00..0\rangle - |11..1\rangle] \tag{10}$$

Like Mermin's case, every particle up to n-1 is measured in the x and y-axis, but the n^{th} particle is measured on the a and b axis, making 45° and 135° to the x-axis, respectively.

An operator is defined [8] as $A = A_1+A_2$ where

$$A_1 = \frac{1}{2}\left(\prod_{j=1}^{n-1}\left(\sigma_x^j + i\sigma_y^j\right) + \prod_{j=1}^{n-1}\left(\sigma_x^j - i\sigma_y^j\right)\right)(\sigma_a^n + \sigma_b^n) \tag{11}$$

$$A_2 = -\frac{1}{2i}\left(\prod_{j=1}^{n-1}\left(\sigma_x^j + i\sigma_y^j\right) - \prod_{j=1}^{n-1}\left(\sigma_x^j - i\sigma_y^j\right)\right)(\sigma_a^n - \sigma_b^n) \tag{12}$$

After assigning distinct values for every possible permutation and notate M_1 for A_1 and M_2 for A_2, a function F is defined:

$$F = \sum P_{x_1 y_1 \dots a_n b_n}(m_1, \dots, m_n)(M_1 + M_2) \tag{13}$$

Then F can be written in the form of,

$$F = \sum P_{x_1 y_1 \dots a_n b_n}(m_1, \dots, m_n)\begin{bmatrix} -Re\left(\prod_{j=1}^{n-1}\left(m_x^j + i m_y^j\right)\right)(m_a^n - m_b^n) + \\ Im\left(\prod_{j=1}^{n-1}\left(m_x^j + i m_y^j\right)\right)(m_a^n + m_b^n) \end{bmatrix} \tag{14}$$

To get the upper bound for F, CHSH lemma is used.

Lemma: If u, u', v, and U' are random variables having a probability distribution function P(u, v, u', v'), then the following elementary relation always holds:

$$\sum P(u,u',v,v')[u(v-v') + u'(v+v')] \leq 2\max\{|u|,|u'|\}\max\{|v|,|v'|\} \tag{15}$$

It is obvious to conclude that,

$$F \leq 2\max\{F_1, F_2\} \tag{16}$$

Maximum values of F_1 and F_2 are projections to their imaginary or real axis, and they attain similar values as Mermin's. Thus, by the assumption of local realism, F is bounded by:

$$F \leq \ 2^{\frac{n}{2}}, \text{for n even}$$
$$F \leq \ 2^{\frac{n+1}{2}}, for\ n\ odd \tag{17}$$

When the expected value of A calculated with a quantum mechanical approach, it gives:

$$\langle \phi | A | \phi \rangle = 2^{n-\left(\frac{1}{2}\right)} \tag{18}$$

These boundaries are violated by the quantum mechanical approach with a factor of $2^{(n-1)/2}$ for even n and $2^{(n-2)/2}$ for odd n. A quantum computer can test these inequalities.

METHOD

Mermin inequalities have already been tested for up to 5 qubits. The program used is "Qiskit", a python-based language designed for the usage of IBM quantum computers. In the Mermin case, state (1) is used, and to create such a state, CNOT and Hadamard gates, which are two-qubit gates, are used. Phases are indicated by S and T gates, which are one qubit gates. They multiply |1> terms with π/2 and π/4, respectively. σ_x and σ_y measurements can be simulated by using additional gates like H gate for σ_x measurement and an additional $S^\dagger$ gate for σ_y measurement, even though only σ_z basis measurement can be done. To switch from σ_y to σ_x, one must add or remove one $S^\dagger$ gate. Measurement for 45° on xy plane can be simulated by using a $T^\dagger$ gate and then H gate, similarly for 135° on xy plane can be simulated by using a T gate instead of a $T^\dagger$ gate then an H gate. All circuits needed can be obtained by using this technique. Using (11) and (12); we have 8, 16, and 32 measurements for 3, 4, and 5 qubits, respectively. We need to specify the setups to construct corresponding circuits for each qubit case.

Three qubits:

For three qubits, the setup is the following.

$$|\Phi_3\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle) \tag{19}$$

$$\begin{aligned}\langle M_3\rangle = \langle\sigma_x\sigma_x\sigma_a\rangle + \langle\sigma_x\sigma_x\sigma_b\rangle - \langle\sigma_y\sigma_y\sigma_a\rangle - \langle\sigma_y\sigma_y\sigma_b\rangle \\ -\langle\sigma_x\sigma_y\sigma_a\rangle + \langle\sigma_x\sigma_y\sigma_b\rangle - \langle\sigma_y\sigma_x\sigma_a\rangle + \langle\sigma_y\sigma_x\sigma_b\rangle\end{aligned} \tag{20}$$

$$\langle M_3\rangle_{LR} \leq 4, \langle M_3\rangle_{QM} \leq 5.66 \tag{21}$$

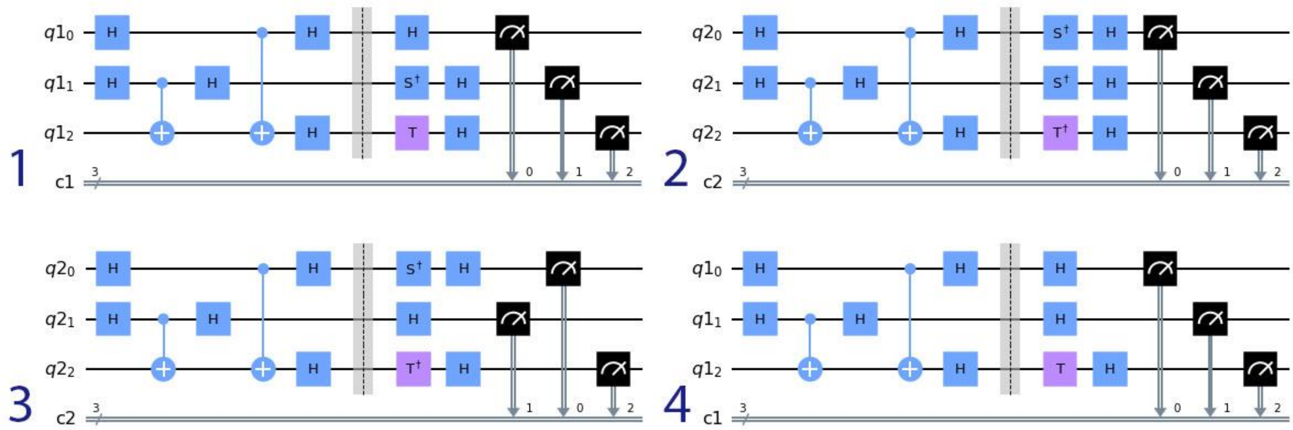


Figure.1: Four of the circuits needed to test (20). Left of the barrier is state (19) preparation; the rest is measurements of 1: $\sigma_x\sigma_y\sigma_b$, 2: $\sigma_y\sigma_y\sigma_a$, 3: $\sigma_y\sigma_x\sigma_a$ and 4: $\sigma_x\sigma_x\sigma_b$.

Each circuit is run for 4096 shots. The name of the IBM quantum computer used is "ibmq_manila". Every experiment was repeated by ten times.

Four qubits:

For three qubits, the setup is the following.

$$|\Phi_4\rangle = \frac{1}{\sqrt{2}}(|0,0,0,0\rangle + |1,1,1,1\rangle) \tag{22}$$

$$\begin{aligned}\langle M_4\rangle = &\langle\sigma_x\sigma_x\sigma_x\sigma_a\rangle + \langle\sigma_x\sigma_x\sigma_x\sigma_b\rangle - \langle\sigma_x\sigma_y\sigma_y\sigma_a\rangle - \langle\sigma_x\sigma_y\sigma_y\sigma_b\rangle\\ &-\langle\sigma_y\sigma_x\sigma_y\sigma_a\rangle - \langle\sigma_y\sigma_x\sigma_y\sigma_b\rangle - \langle\sigma_y\sigma_y\sigma_x\sigma_a\rangle - \langle\sigma_y\sigma_y\sigma_x\sigma_b\rangle\\ &-\langle\sigma_x\sigma_x\sigma_y\sigma_a\rangle + \langle\sigma_x\sigma_x\sigma_y\sigma_b\rangle + \langle\sigma_x\sigma_y\sigma_x\sigma_a\rangle + \langle\sigma_x\sigma_y\sigma_x\sigma_b\rangle\\ &-\langle\sigma_y\sigma_x\sigma_x\sigma_a\rangle + \langle\sigma_y\sigma_x\sigma_x\sigma_b\rangle + \langle\sigma_y\sigma_y\sigma_y\sigma_a\rangle + \langle\sigma_y\sigma_y\sigma_y\sigma_b\rangle\end{aligned} \tag{23}$$

$$\langle M_4\rangle_{LR} \leq 4, \langle M_4\rangle_{QM} \leq 11.31 \tag{24}$$

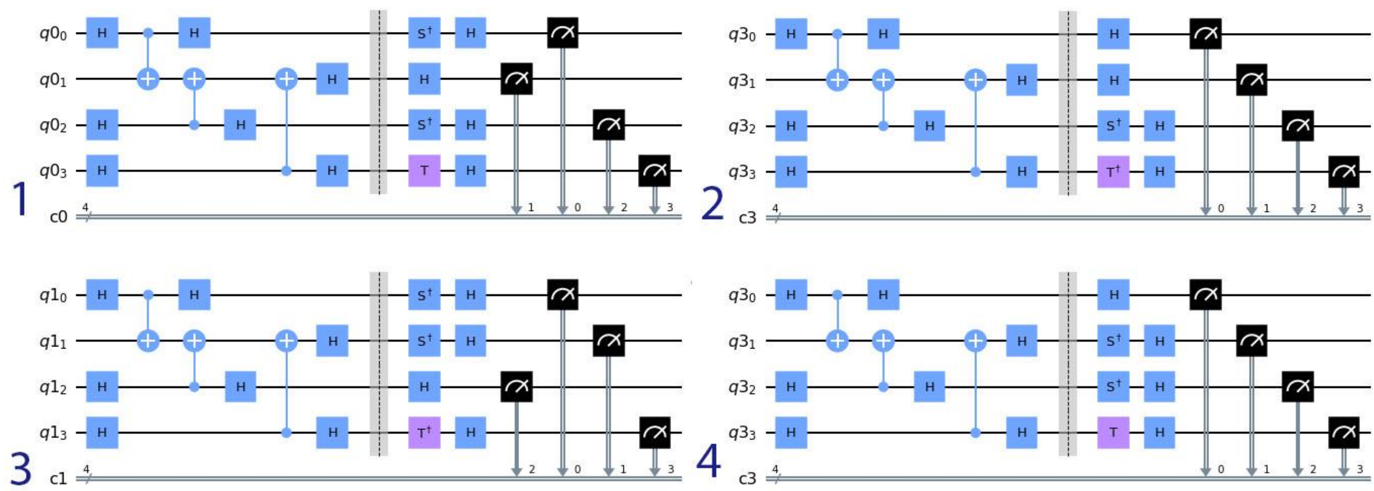


Figure.2: Four of the circuits needed to test (23). Left of the barrier is state (22) preparation; rest is measurements of 1: $\sigma_y\sigma_x\sigma_y\sigma_b$, 2: $\sigma_x\sigma_x\sigma_y\sigma_a$. 3: $\sigma_y\sigma_y\sigma_x\sigma_a$ and 4: $\sigma_x\sigma_y\sigma_y\sigma_b$.

Each circuit is run for 8192 shots. The name of the IBM quantum computer used is "ibmq_manila". Every experiment was repeated by ten times.

Five qubits:

$$|\Phi_5\rangle = \frac{1}{\sqrt{2}}(|0,0,0,0,0\rangle + |1,1,1,1,1\rangle) \tag{25}$$

$$\begin{aligned}\langle M_5\rangle = &\langle\sigma_x\sigma_x\sigma_x\sigma_x\sigma_a\rangle + \langle\sigma_x\sigma_x\sigma_x\sigma_x\sigma_b\rangle - \langle\sigma_x\sigma_y\sigma_y\sigma_x\sigma_a\rangle - \langle\sigma_x\sigma_y\sigma_y\sigma_x\sigma_b\rangle\\ &-\langle\sigma_y\sigma_x\sigma_y\sigma_x\sigma_a\rangle - \langle\sigma_y\sigma_x\sigma_y\sigma_x\sigma_b\rangle - \langle\sigma_y\sigma_y\sigma_x\sigma_x\sigma_a\rangle - \langle\sigma_y\sigma_y\sigma_x\sigma_x\sigma_b\rangle\\ &\langle\sigma_y\sigma_x\sigma_x\sigma_y\sigma_a\rangle - \langle\sigma_y\sigma_x\sigma_x\sigma_y\sigma_b\rangle + \langle\sigma_y\sigma_y\sigma_y\sigma_y\sigma_a\rangle + \langle\sigma_y\sigma_y\sigma_y\sigma_y\sigma_b\rangle\\ &- \cdots (up\ to\ 32)\end{aligned} \tag{26}$$

$$\langle M_5\rangle_{LR} \leq 8, \langle M_5\rangle_{QM} = 22.62 \tag{27}$$

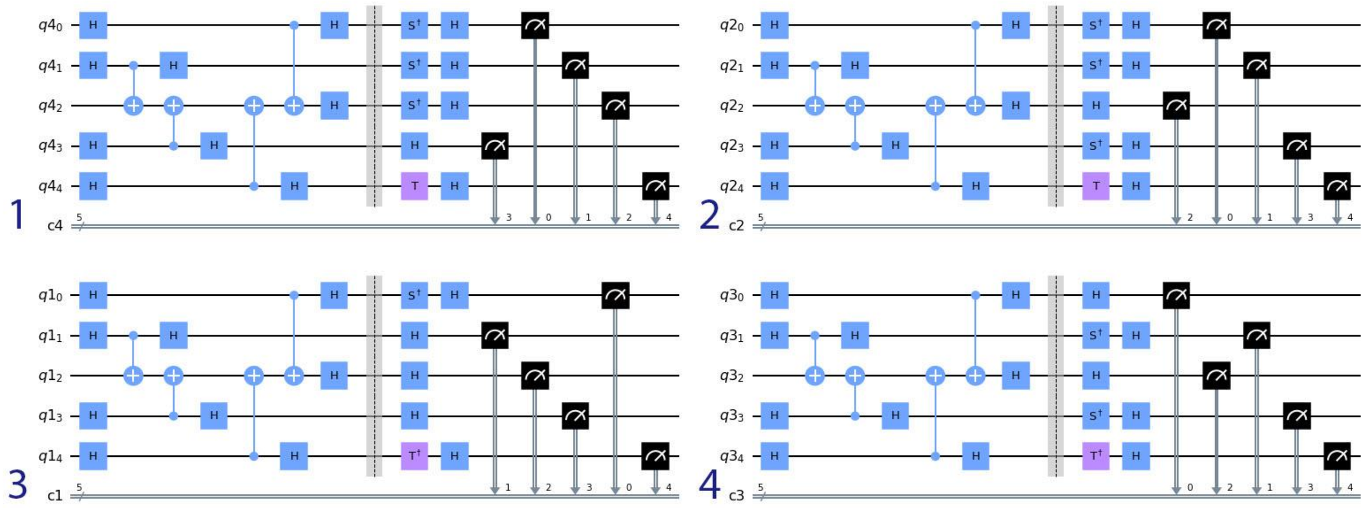


Figure.3: Four of the circuits needed to test (26). Left of the barrier is state (25) preparation; rest is measurements of 1: $\sigma_y\sigma_y\sigma_y\sigma_x\sigma_b$, 2: $\sigma_y\sigma_y\sigma_x\sigma_y\sigma_b$. 3: $\sigma_y\sigma_x\sigma_x\sigma_x\sigma_a$ and 4: $\sigma_x\sigma_y\sigma_x\sigma_y\sigma_a$.

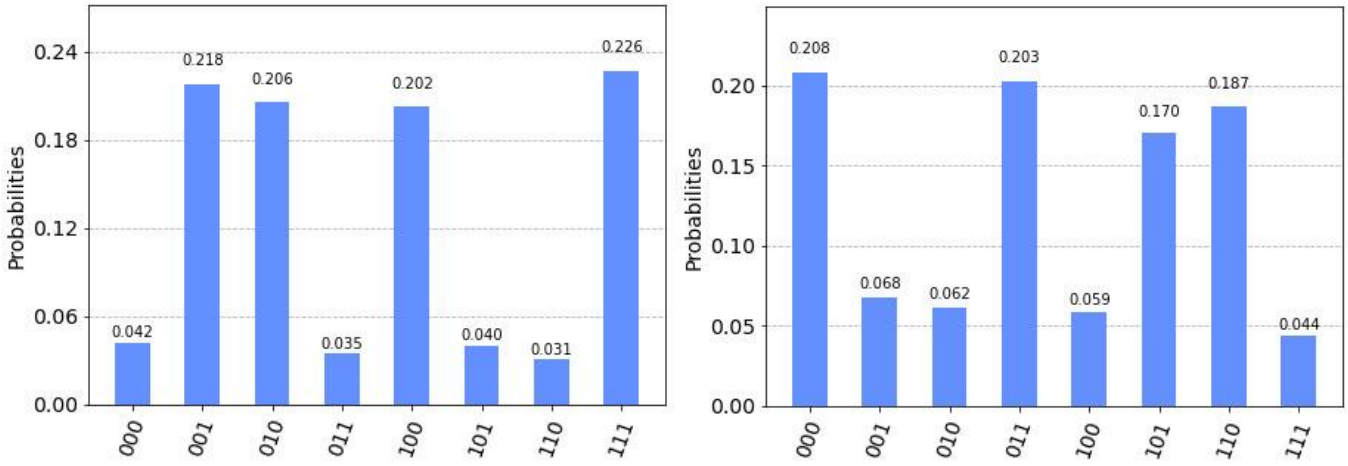
Each circuit is run for 8192 shots. The name of the IBM quantum computer used is "ibmq_manila". Every experiment was repeated by ten times.

These three cases result in a long time for every single circuit since IBM allows the usage of their computers in queues, which is five jobs maximum once a time, for the public. Therefore, setups coded in a way that they would repeat themselves for a long time in "qiskit".

RESULTS & DISCUSSION

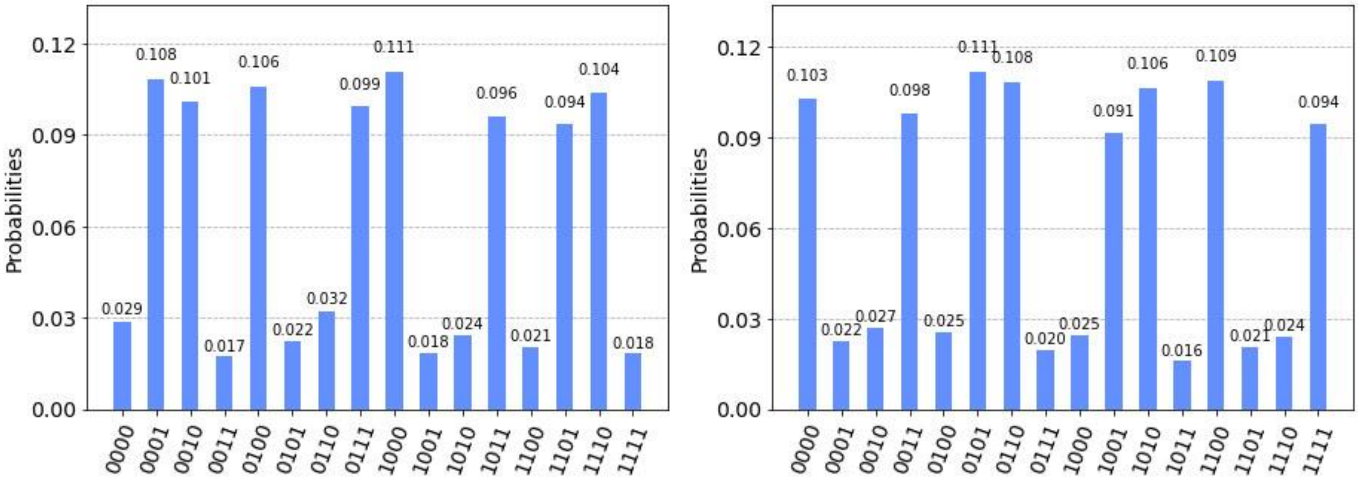
We obtained some results that do not show any correlations between particles when we use a different quantum computer with similar properties to "ibmq_manila". Also, there was some flawed data that shows a slight violation. But we excluded them and took the data that shows the highest violations.

Moreover, data taken was not in the form to study the individual circuit measurements in detail. We can only show what one-time measurements for each case and its probabilities with outcomes would look like.



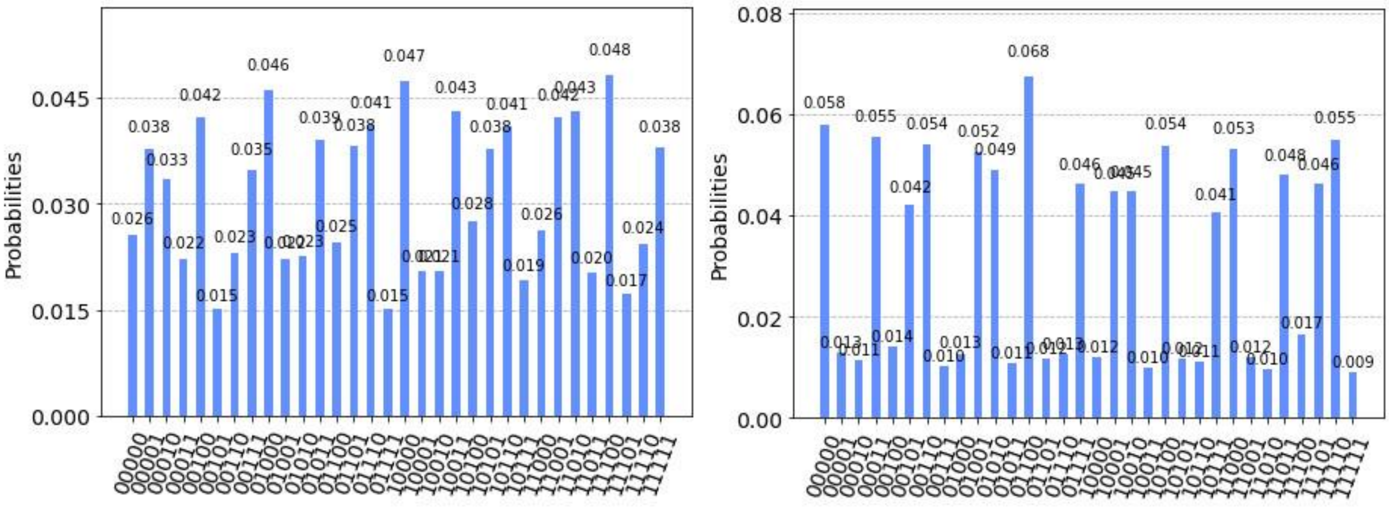
Graph.1: The histogram on the left shows the probabilities for the $\sigma_x\sigma_y\sigma_b$ measurement and, the right histogram is for the $\sigma_y\sigma_y\sigma_a$ measurement. Each term at the bottom of the graphs represents a possible outcome.

It can be seen from the histograms that there are some correlations between particles. And correlations are different for two measurements with the group of terms that contains an even or odd number of '1's in it.



Graph.2: Histogram on the left shows the probabilities for the $\sigma_x\sigma_y\sigma_y\sigma_b$ measurement and, the right histogram is for the $\sigma_x\sigma_y\sigma_y\sigma_a$ measurement. Each term at the bottom of the graphs represents a possible outcome.

These two measurements in the graph.2 only differs in the measurement at the last qubit. Also, it can be seen that their correlations are different, similarly to the first graph with their parity in their states.



Graph.2: Histogram on the left shows the probabilities for the $\sigma_y\sigma_y\sigma_y\sigma_x\sigma_b$ measurement and, the right histogram is for the $\sigma_y\sigma_y\sigma_y\sigma_x\sigma_a$ measurement. Each term at the bottom of the graphs represents a possible outcome.

Again, these two measurements in the graph.3 only differs in the measurement at the last qubit. Correlations switched with their parities. However, this time graph on the left shows a less uniform correlation from the other one, and it has more errors than usual.

The group with the terms consisting of an odd number of '1' must be subtracted from the group with the even numbers of '1' to calculate the expected value that appeared in the inequality. Then expected value for each possible permutation can be calculated.

Final products of the results shown as:

	Local Realism	Quantum Mechanics	Present Experiment
3 qubits	4	5.66	4.31 ± 0.02
4 qubits	4	11.31	7.71 ± 0.05
5 qubits	8	22.62	11.98 ± 0.12

Table.1: Results of the experiment. Upper bounds for the locally realistic case and quantum mechanical case are included for 3,4, and 5 qubits.

Measured expectation values exceed the boundaries established by the local realism hypothesis (17) with a sufficient statistical significance. Also, experimental results are in the range of the expectation values predicted by the quantum mechanical approach.

CONCLUSION

The primary motivation in this work was to apply a very similar approach that has been used in experiments on Mermin inequalities to Ardehali inequalities. Using a five-qubit IBM quantum computer with a python-based programming language "qiskit", we were able to construct every circuit and obtain the data needed. The experiment is done for 3, 4, and 5 qubits cases. On the negative side, most quantum computers were not operating sufficiently enough to conduct this experiment. The difference between the quantum mechanical approach and the measured values shows huge errors accumulated along with the experiment. Results clearly show that locally realistic expectation values exceeded by a significant statistical value in the experiment, and nature does not seem to be obeying this idea. Then we can say that this experiment allows us to reject the local realism hypothesis.

Moreover, the experiment shows that this method can be used to construct a benchmark for quantum computers and test their reliability considering the quantum mechanical limit.

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