CS-UY 2413: Design & Analysis of Algorithms
Homework 2

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Due 11:59pm Monday, Sep 30, New York time

By handing in the homework you are agreeing to the Homework Rules; see EdStem.

Our Master Theorem: The version of the Master Theorem that we covered in class is on the last page of this homework. We won't be covering the version of the Master Theorem in the textbook and you're not responsible for knowing it. (But you may find it interesting!)

Reminder: For $r \neq 1$, $r^0 + r^1 + ... + r^k = \frac{r^{k+1}-1}{r-1}$.

Problem 1

For each example, indicate whether $f = o(g)$ (little-oh), $f = \omega(g)$ (little-omega), or $f = \Theta(g)$ (big-Theta). No justification is necessary.

(a) $f(n) = 2n^2 + 5n$, $g(n) = n^2$ (b) $f(n) = n^3$, $g(n) = 2^n$ (c) $f(n) = 7n \log n$, $g(n) = n^2$ (d) $f(n) = \sum_{i=1}^{n} i^2$, $g(n) = n^3$ (e) $f(n) = \log_2 n$, $g(n) = \log_{16} n$

Problem 2

Give a formal proof of the following statement: If $f(n) \geq 1$ for all $n \in \mathbb{N}$, $g(n) \geq 1$ for all $n \in \mathbb{N}$, $f(n) = O(g(n))$, and $g(n)$ is unbounded (meaning $\lim_{n\to\infty} g(n) = \infty$) then $f(n)^2 = O(g(n)^2)$.

Use the formal definition of big-Oh in your answer. In your proof, you can use the fact that the value of n^2 increases as n increases.

Problem 3

For each of the following recurrences, determine whether Our Master Theorem (on the last page of this HW) can be applied to the recurrence. If it can, use it to give the solution to the recurrence in Θ notation; no need to give any details. If not, write "Our Master Theorem does not apply."

(a) $T(n) = 2T(n/2) + n \log n$ (b) $T(n) = 9T(n/3) + n^2$ (c) $T(n) = T(n/2) + 1$

Problem 4

Our Master Theorem can be applied to a recurrence of the form $T(n) = aT(n/b) + n^d$, where a, b, d are constants with $a > 0$, $b > 1$, $d > 0$. Consider instead a recurrence of the form $T_{new}(n) = aT_{new}(n/b) + n \log_d n$ where $a > 0$, $b > 1$, $d > 1$ (and $T(1) = 1$).

For each of the following, state whether the given property of T_{new} is true. If so, explain why it is true. If not, explain why it is not true. (Even if you know the version of the Master Theorem in the textbook, don't use it in your explanation.)

(a) $T_{new}(n) = O(n^2)$ if $\log_b a = 2$ (b) $T_{new}(n) = \Omega(n \log n)$

Problem 5

Consider the recurrence $T(n) = 2T(n/2) + n \log n$ for $n > 1$, and $T(1) = 1$.

(a) Compute the value of $T(4)$, using the recurrence. Show your work. (b) Use a recursion tree to get an upper bound for the recurrence and get a closed-form expression for $T(n)$, when n is a power of 2. (Check that your expression is correct by plugging in $n = 4$ and comparing with your answer to (a).) (c) Suppose that the base case is $T(2) = 3$, instead of $T(1) = 1$. What is an upper bound for the solution to the recurrence in this case, for $n \geq 2$?

Problem 6

Consider a variation of mergesort that works as follows: If the array has size 1, return. Otherwise, divide the array into fourths, rather than in half. Recursively sort each fourth using this variation of mergesort. Then merge the first two fourths. Finally, merge the result with the last two fourths.

(a) Write a recurrence for the running time of this variation of mergesort. It should be similar to the recurrence for ordinary mergesort. Assume n is a power of 4. (b) Apply Our Master Theorem to the recurrence to get the running time of the algorithm, in theta notation. Show your work.

Problem 7

Consider the following recursive sorting algorithm. Assume n is a power of 2. (Note: This is not a version of mergesort. No merges are performed.)

- If the array has only one element, return.
- Recursively sort the first half of the elements in the array.

- Recursively sort the second half of the elements in the array.
- Recursively sort the first half of the elements in the array again.
- Recursively sort the second half of the elements in the array again.

(a) Prove that the algorithm is correct by showing that the array will be sorted after the four recursive calls are performed, assuming the four recursive calls correctly sort their (sub)arrays. (b) Write a recurrence expressing the running time of the algorithm. (c) Apply Our Master Theorem to your recurrence. What is the running time of the algorithm, in theta notation?

Problem 8

Let $f(n) = n \log n$ and $g(n) = n^{1.1}$. Show that $f(n) = O(g(n))$.

Problem 9

Solve the recurrence relation $T(n) = 2T(n/2) + n$ using the Master Theorem.

Problem 10

Describe a divide-and-conquer algorithm to find the maximum and minimum elements in an unsorted array of n elements. Write a recurrence relation for the running time of your algorithm, and solve it using the Master Theorem.

Our Master Theorem

Let a, b, d, n_0 be constants such that $a > 0$, $b > 1$, $d \geq 0$ and $n_0 > 0$. Let $T(n) = aT(n/b) + \Theta(n^d)$ for when $n \geq n_0$, and $T(n) = \Theta(1)$ when $0 \leq n < n_0$. Then,

$$T(n) = \begin{cases} \Theta(n^d \log n) & \text{if } d = \log_b a \\ \Theta(n^{\log_b a}) & \text{if } d < \log_b a \\ \Theta(n^d) & \text{if } d > \log_b a \end{cases}$$

We assume here that $T(n)$ is a function defined on the natural numbers. We use $aT(n/b)$ to mean $a'T(\lfloor n/b \rfloor) + a''T(\lceil n/b \rceil)$ where $a', a'' > 0$ such that $a' + a'' = a$.