

CS-UY 2413: Design & Analysis of Algorithms

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Homework 2

Due 11:59pm Monday, Sep 30, New York time. By handing in the homework you are agreeing to the Homework Rules; see EdStem.

Our Master Theorem: The version of the Master Theorem that we covered in class is on the last page of this homework. We won't be covering the version of the Master Theorem in the textbook and you're not responsible for knowing it. (But you may find it interesting!)

Reminder: For $r \neq 1$, $r^0 + r^1 + \dots + r^k = \frac{r^{k+1}-1}{r-1}$.

1. For each example, indicate whether $f = o(g)$ (little-oh), $f = \omega(g)$ (little-omega), or $f = \Theta(g)$ (big-Theta). No justification is necessary.
 - (a) $f(n) = n^2 \log n$, $g(n) = n^3$
 - (b) $f(n) = 2^n$, $g(n) = 3^n$
 - (c) $f(n) = n^2 + 5n$, $g(n) = n^2$
 - (d) $f(n) = 2 \sum_{i=0}^n i^2$, $g(n) = n^3$
 - (e) $f(n) = \log_2 n$, $g(n) = \log_{16} n$
2. Give a formal proof of the following statement: If $f(n) \geq 1$ for all $n \in \mathbb{N}$, $g(n) \geq 1$ for all $n \in \mathbb{N}$, $f(n) = \Omega(g(n))$, and $g(n)$ is unbounded (meaning $\lim_{n \rightarrow \infty} g(n) = \infty$) then $\log_2 f(n) = \Omega(\log g(n))$. Use the formal definition of big-Omega in your answer. In your proof, you can use the fact that the value of $\log_2 n$ increases as n increases.
3. For each of the following recurrences, determine whether Our Master Theorem (on the last page of this HW) can be applied to the recurrence. If it can, use it to give the solution to the recurrence in Θ notation; no need to give any details. If not, write "Our Master Theorem does not apply."
 - (a) $T(n) = 2T(n/2) + n \log n$
 - (b) $T(n) = 9T(n/3) + n^2$
 - (c) $T(n) = T(n/2) + 1$

4. Our Master Theorem can be applied to a recurrence of the form $T(n) = aT(n/b) + n^d$, where a, b, d are constants with $a > 0, b > 1, d > 0$. Consider instead a recurrence of the form $T_{\text{new}}(n) = aT_{\text{new}}(n/b) + n \log_d n$ where $a > 0, b > 1, d > 1$ (and $T(1) = 1$). For each of the following, state whether the given property of T_{new} is true. If so, explain why it is true. If not, explain why it is not true. (Even if you know the version of the Master Theorem in the textbook, don't use it in your explanation.)
- (a) $T_{\text{new}}(n) = O(n^2)$ if $\log_b a = 2$
 - (b) $T_{\text{new}}(n) = \Omega(n \log n)$
5. Consider the recurrence $T(n) = 2T(n/2) + 2n$ for $n > 1$, and $T(1) = 1$.
- (a) Compute the value of $T(4)$, using the recurrence. Show your work.
 - (b) Use a recursion tree to solve the recurrence and get a closed-form expression for $T(n)$, when n is a power of 2. (Check that your expression is correct by plugging in $n = 4$ and comparing with your answer to (a).)
 - (c) Suppose that the base case is $T(2) = 3$, instead of $T(1) = 1$. What is the solution to the recurrence in this case, for $n \geq 2$?
6. Consider a variation of mergesort that works as follows: If the array has size 1, return. Otherwise, divide the array into fourths, rather than in half. Recursively sort each fourth using this variation of mergesort. Then merge the first two fourths. Then merge the last two fourths. Finally, merge the two resulting halves.
- (a) Write a recurrence for the running time of this variation of mergesort. It should be similar to the recurrence for ordinary mergesort. Assume n is a power of 4.
 - (b) Apply Our Master Theorem to the recurrence to get the running time of the algorithm, in theta notation. Show your work.
7. Consider the following recursive algorithm. Assume n is a power of 2.
- If the array has only one element, return.
 - Recursively process the first half of the array.
 - Recursively process the second half of the array.
 - Recursively process the first half of the array again.
 - Print the whole array to verify it is sorted.
- (a) Write a recurrence expressing the running time of the algorithm. Assume the print operation takes $\Theta(n)$ time.
 - (b) Apply Our Master Theorem to your recurrence. What is the running time of the algorithm, in theta notation? Is the algorithm correct? Explain.

8. Let $T(n)$ be the running time of a recursive algorithm. The recurrence is given by $T(n) = 2T(n/4) + \sqrt{n}$. Assume n is a power of 4.
 - (a) Apply Our Master Theorem to find the asymptotic running time of the algorithm. Show your work.
9. Consider a divide-and-conquer algorithm that divides a problem of size n into 3 subproblems, each of size $n/3$, and then spends $\Theta(n)$ time to combine the results. Assume n is a power of 3.
 - (a) Write a recurrence relation for the running time $T(n)$ of this algorithm.
 - (b) Solve the recurrence using Our Master Theorem. Show your work.
10. Suppose we have a recurrence relation of the form $T(n) = 3T(n/2) + n \log n$, where n is a power of 2.
 - (a) Can Our Master Theorem be applied to this recurrence? If so, what is the solution? If not, why not?
11. Explain why the following statement is false: If $f(n) = O(g(n))$, then $2^{f(n)} = O(2^{g(n)})$. Provide a counterexample.

Theorem 0.1 (Our Master Theorem)

Let a, b, d, n_0 be constants such that $a > 0$, $b > 1$, $d \geq 0$ and $n_0 > 0$. Let $T(n) = aT(n/b) + \Theta(n^d)$ for when $n \geq n_0$, and $T(n) = \Theta(1)$ when $0 \leq n < n_0$. Then,

$$T(n) = \begin{cases} \Theta(n^d \log n) & \text{if } d = \log_b a \\ \Theta(n^{\log_b a}) & \text{if } d < \log_b a \\ \Theta(n^d) & \text{if } d > \log_b a \end{cases}$$

We assume here that $T(n)$ is a function defined on the natural numbers. We use $aT(n/b)$ to mean $a'T(\lfloor n/b \rfloor) + a''T(\lceil n/b \rceil)$ where $a', a'' > 0$ such that $a' + a'' = a$.