Online Appendix for Coexistence of Centralized and Decentralized Markets

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The results below has first been obtained in Idem (2021) for an environment with finitely many agents, divisible goods and arbitrary endowments. Here I restate them for the environment I study in the Coexistence of Centralized and Decentralized Markets, with the proofs adjusted accordingly.

1 Monagora Environment

I reproduce the setup and the initial statement of the mechanism design problem here for convenience.

- Good: There is a single, indivisible good in the market.
- Agents: There is a continuum of agents on $[0, 1]$.
- Endowments: Each agent has 1 unit of endowment of the good.
- Demands: Each agent demands up to 2 units of the good. Since the good is indivisible, this means, they can consume 0, 1, or 2 units, depending on whether they buy or sell, or neither buy nor sell.
- Valuations: Each agent has some valuation $\theta \in [0, 1]$ for a unit of the good. The valuations are drawn from some distribution F with support $[0, 1]$. Agents' valuations are their private information.

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- Marketplace: A mechanism designer wants to design a mechanism to maximize its profit. She knows the distribution of valuations, F.

By revelation principle, I focus on direct mechanisms. Moreover, as agents are symmetric other than their valuations, I focus on anonymous mechanisms, which is without loss. Then, the designer will choose a mechanism that allocates $q : \theta \to \mathbb{R}$ units of good to each agent with valuation θ and asks her to pay $t : \theta \to \mathbb{R}$. Hence, the net utility of the agent with the valuation θ from the monagorastic mechanism is

$$u(\theta) = \theta \min\{1, q(\theta)\} - t(\theta).$$

As agents have demands for two units, having more than 2 unit of the good is same as having 2 unit. As such, the expression for the utility above caps the maximum net trade that increases the utility at 1, since the agent already has 1 unit of endowment.

The profit of the marketplace is the net payments. Thus, the designer seeks to maximize total payment, given the incentive compatibility, individual rationality, and feasibility constraints.

$$\begin{aligned}
&\max_{(q,t)} \int_{[0,1]} t(\theta) f(\theta) d\theta \\
&\text{s. t.} \\
&\text{(IC)} && \theta \min\{1, q(\theta)\} - t(\theta) && \geq \theta \min\{1, q(\theta', \theta)\} - t(\theta') \\
&\text{(IR)} && \theta \min\{1, q(\theta)\} - t(\theta) && \geq 0 \\
&\text{(Individual Feasibility)} && q(\theta) && \geq -1 \\
&\text{(Aggregate Feasability)} && \int_{[0,1]} q(\theta) f(\theta) d\theta && \leq 0
\end{aligned}$$

1.1 Simplifying The Designer's Problem

We first develop a series of lemmata that help us state the maximization problem above as a concave program.

Lemma 1.1 (Monotonicity). *Suppose (q, t) is a direct, IC mechanism. Then,*

1. *If $q(\theta) < 1$ for some $\theta \in [0, 1]$, then $q(\theta)$ is increasing at (θ).*

2. *If $q(\theta) \geq 1$ for some $\theta \in [0, 1]$, then $q(\theta') \geq 1$ for each $\theta' \geq \theta$.*

The proof is standard, except for taking care of the capacities so it can be found in the Appendix A.

The next lemma presents the derivative of the utility of an agent in an IC mechanism.

Lemma 1.2 (Envelope Condition)**.** *If* (q, t) *is a direct, IC mechanism, then for each* $\theta \in [0, 1]$

$$\frac{\partial u(\theta)}{\partial \theta} = \begin{cases} q(\theta), & \text{if } q(\theta) < 1, \\ 1, & \text{otherwise.} \end{cases}$$

Again, the proof is similar to standard arguments and can be found in Appendix B.

Notation: For any direct mechanism (q, t), let

$$q^*(\theta) = \begin{cases} q(\theta), & \text{if } q(\theta) < 1, \\ 1, & \text{otherwise.} \end{cases}$$

Note that for a direct, IC mechanism, $q^*(\theta)$ is also weakly increasing.

The next lemma gives the representation of the utility of each type as the integral of the allocation rule, using the previous lemma.

Lemma 1.3 (Payoff Equivalence)**.** *If* (q, t) *is a direct, IC mechanism, then*

$$u(\theta) = u(0) + \int_0^\theta q^*(x)dx,$$

for each $\theta \in [0.1]$.

Proof. Since $u(\theta)$ is convex θ on both regions where $q(\theta) > 1$ and $q(\theta) \leq 1$ separately, it is absolutely continuous in θ. Then, it is the integral of its derivative. □

Next, we pin down the transfer rule in an IC mechanism.

Lemma 1.4 (Revenue Equivalence)**.** *If* (q, t) *is a direct, IC mechanism, then*

$$t(\theta) \quad = -u(0) + \theta q^*(\theta) - \int_0^\theta q^*(x)dx,$$

for each $\theta \in [0, 1]$.

Proof. From the definition of $u(\theta)$ and the previous lemma. $\square$

Now we show that the necessary conditions above for incentive compatibility of a mechanism are also sufficient to establish the incentive compatibility of a mechanism.

Proposition 1.1. *Let (q,t) be a direct mechanism. The mechanism is incentive compatible if and only if,*

1. *$q^*(\theta)$ is increasing at θ;*
2. *$t(\theta) = -u(0) + \theta q^*(\theta) - \int_0^\theta q^*(x)dx$.*

Proof can be found in Appendix C.

The next proposition provides the characterization of the IR mechanisms by establishing the types with the lowest utilities. The reason this is an issue in this model is that in an auction, the lowest allocation an agent could receive is 0. Hence, the utility is always increasing in agent's type, as can be seen from the envelope condition. Of course, this means the lowest type has the lowest utility. However, here, an agent with a relatively low type can be a seller, which means he would get a negative allocation. Therefore, the utility of the lowest type is not the lowest utility, which can again by seen from the envelope condition.

Proposition 1.2. *Let (q,t) be a direct IC mechanism. Then, it is IR if and only if,*

$$\theta^* q^*(\theta^*) \geq t(\theta^*),$$

where θ^* is defined as

1. *$\theta^* = 0$ if $q^*(0) \geq 0$,*
2. *$\theta^* = 1$ if $q^*(1) < 0$,*
3. *a solution to $q^*(\theta^*) = 0$ if such a type exists.*

Proof. Case 1: Suppose $q^*(0) \geq 0$. Then, by Lemma 1.3, incentive compatibility of a mechanism implies that the associated ex-post utilities $u(\theta)$ are increasing in θ. Hence, if $u(0) \geq 0$, we have $u(\theta) \geq 0$ for each $\theta \in [0,1]$.

Case 2: Suppose $q^*(1) < 0$. Then, by Lemma 1.3, $u(\theta)$ are decreasing and hence, $u(1)$ is the lowest payoff. Hence, if it is nonnegative, all other types' payoffs are nonnegative as above.

Case 3: Suppose there exists θ^* such that $q^*(\theta^*) = 0$. Then, by Lemma 1.3, $u(\theta)$ is decreasing up to θ^* and increasing after that point. Hence, type θ^* has the lowest payoff. So, if $u(\theta^*) \geq 0$, each type's IR condition must also hold.

□

Lemma 1.5. *If an IC and IR mechanism maximizes the expected revenue of the designer, then,*

$$t(\theta^*) = \theta^* q(\theta^*)$$

where θ^* is defined as

1. $\theta^* = 0$ *if* $q^*(0) \geq 0$,
2. $\theta^* = 1$ *if* $q^*(1) < 0$,
3. *the solution to* $q^*(\theta^*) = 0$ *if such a type exists.*

Proof. The previous proposition shows that IC and IR mechanisms must have $\theta^* q(\theta^*)$ greater than $t(\theta^*)$. However, if $\theta^* q(\theta^*) > t(\theta^*)$, then the seller can increase the expected revenue by increasing $t(0)$ and keeping the allocation rule the same. This would increase all types' payments and the revenue strictly, contradicting revenue maximization.

□

Using the condition about θ^* from Lemma 1.5 and the previous lemmata, we have

$$\begin{aligned}
\theta^* q^*(\theta^*) \quad &= t(\theta^*) \\
= \quad & -u(0) + \theta^* q^*(\theta^*) - \int_0^{\theta^*} q^*(x)dx \\
\iff \quad & u(0) = -\int_0^{\theta^*} q^*(x)dx \\
\iff \quad & t(\theta) = \int_0^{\theta^*} q^*(x)dx + \theta q^*(\theta) - \int_0^{\theta} q^*(x)dx
\end{aligned}$$

Now we are ready to show that the allocation rule in a revenue-maximizing mechanism is not 'wasteful'.

Proposition 1.3. *Let (q, t) be a direct mechanism that maximizes the revenue of the designer. Then, $q(\theta) \leq 1$ with probability 1 and the aggregate feasibility holds with equality: $\int\limits_{[0,1]} q(\theta)f(\theta)d\theta = 0$.*

Proof. First, suppose that in the optimal mechanism, there exists a set $\Theta \subset [0,1]$ with a positive measure such that for each $\theta \in \Theta$, $q(\theta) > 1$. Notice that decreasing the allocation to 1 unit has no effect on the agent's payoff. Hence, it doesn't effect any IC or IR constraints.

Next, let us examine the transfer rule in a direct, IC mechanism:

$$t(\theta) = \int_0^{\theta^*} q^*(x)dx + \theta q^*(\theta) - \int_0^{\theta} q^*(x)dx.$$

If we have $q(\theta) > 1$ for a positive measure of types, then we must have $q(\theta) < 0$ for a corresponding positive measure of types by the aggregate feasibility constraint. Hence, if we reduced $q(\theta) = 1$ for $\theta \in \Theta$, this wouldn't affect any constraints but instead strictly increase profit as it allows us to increase $q(\theta) < 0$ for a positive measure of types, contradicting the optimality of the mechanism.

By the same argument, having $\int\limits_{[0,1]} q(\theta)f(\theta)d\theta < 0$ cannot be optimal: Either buying less from types or selling more to some types would increase their payments, strictly increasing the profit. $\square$

Now, by fixing $t(\theta)$ to the characterization we have from above, we can restate the problem as follows.

$$\max_{q} \quad \int\limits_{[0,1]} \left[\int\limits_{\{y|q(y)\leq 0\}} q(x)dx + \left(\theta q(\theta) - \int\limits_0^{\theta} q(x)dx \right) \right] f(\theta)d\theta$$

s. t.

$q(\theta)$ is increasing

$q(\theta) \geq -1$

$\int\limits_{[0,1]} q(\theta)f(\theta)d\theta = 0$

After some transformations[1], the problem above can be rewritten as follows:

[1]The details can be followed in Appendix D.

$$\max_{q} \quad \left[\int_{[0,1]} q(\theta)\left[\frac{\mathbb{1}\{q(\theta) \leq 0\}}{f(\theta)} + \left(\theta - \frac{(1-F(\theta))}{f(\theta)}\right)\right] f(\theta)d\theta\right]$$

s. t.

$$q(\theta) \text{ is increasing}$$

$$q(\theta) \geq -1$$

$$\int_{[0,1]} q(\theta)f(\theta)d\theta = 0$$

References

A Proof of Lemma 1

Proof. Let $\theta, \theta' \in [0,1]$. Then, by incentive compatibility

$$\theta \min\{1, q(\theta)\} - t(\theta) \geq \theta \min\{1, q(\theta')\} - t(\theta')$$

and

$$\theta' \min\{1, q(\theta)\} - t(\theta) \leq \theta' \min\{1, q(\theta')\} - t(\theta').$$

Subtracting the second inequality from the first one leads to:

$$(\theta - \theta') \min\{1, q(\theta)\} \geq (\theta - \theta') \min\{1, q(\theta')\}$$

Suppose $q(\theta) < 1$ and $\theta > \theta'$. Then, we have

$$\begin{aligned} \min\{1, q(\theta)\} &\geq \min\{1, q(\theta')\} \iff \\ 1 > q(\theta) &\geq \min\{1, q(\theta')\} \iff \\ q(\theta) &\geq q(\theta') \end{aligned}$$

Now suppose $q(\theta) \geq 1$ and $\theta' \geq \theta$. Then,

$$\begin{aligned} \min\{1, q(\theta')\} &\geq \min\{1, q(\theta)\} \iff \\ \min\{1, q(\theta')\} &\geq 1 \iff \\ q(\theta') &\geq 1. \end{aligned}$$

$\square$

B Proof of Lemma 2

Proof. First, suppose $q(\theta) < 1$. Then, IC implies that for type θ agent:

$$\begin{aligned} u(\theta) &= \max_{\theta' \in [0,1]} \min\{1, q(\theta')\}\theta - t(\theta') \\ &= \max_{\theta' \in [0,1]} q(\theta')\theta - t(\theta'). \end{aligned}$$

Notice that the RHS is the maximum of affine functions of θ, so $u(\theta)$ is convex in θ on this region. Hence, $u(\theta)$ is differentiable almost everywhere in θ on this region. For any θ at which it is differentiable, for $\delta > 0$, IC implies that

$$\begin{aligned} & \lim_{\delta \to 0} \frac{u(\theta+\delta) - u(\theta)}{\delta} \\ \geq & \lim_{\delta \to 0} \frac{(q(\theta)(\theta+\delta) - t(\theta)) - (q(\theta)\theta - t(\theta))}{\delta} = q(\theta). \\ & \lim_{\delta \to 0} \frac{u(\theta) - u(\theta-\delta)}{\delta} \\ \leq & \lim_{\delta \to 0} \frac{(q(\theta)\theta - t(\theta)) - (q(\theta)(\theta-\delta) - t(\theta))}{\delta} = q(\theta). \end{aligned}$$

Then, two inequalities together imply that

$$\frac{\partial u(\theta)}{\partial \theta} = q(\theta).$$

Now suppose $q(\theta) \geq 1$. Then,

$$u(\theta) = \min\{1, q(\theta)\}\theta - t(\theta) = \theta - t(\theta).$$

Notice that $t(\theta)$ must be constant in θ on the region with $q(\theta) \geq 1$: Since agent's effective allocation is constant, otherwise, i would simply choose the type with the least cost. Then, of course,

$$\frac{\partial u(\theta)}{\partial \theta} = 1.$$

$\square$

C Proof of Proposition 1

Proof. We want to show that for each $\theta, \theta' \in [0,1]$, we have

$$
\begin{aligned}
& u(\theta) \geq \theta \min\{1, q(\theta')\} - t(\theta') \\
\iff \quad & u(\theta) \geq \theta \min\{1, q(\theta')\} + \theta' \min\{1, q(\theta')\} \\
& -\theta' \min\{1, q(\theta')\} - t(\theta') \\
\iff \quad & u(\theta) \geq \theta \min\{1, q(\theta')\} - \theta' \min\{1, q(\theta')\} \\
& +u(\theta') \\
\iff \quad & u(\theta) - u(\theta') \geq (\theta - \theta') \min\{1, q(\theta')\} \\
\iff \quad & \int_{\theta'}^{\theta} q^*(x) dx \geq \int_{\theta'}^{\theta} q^*(\theta') dx
\end{aligned}
$$

Suppose $\theta > \theta'$. Since $q^*(\cdot)$ is increasing, $q^*(x) \geq q^*(\theta')$ for each $x \in [\theta', \theta]$. Then, the last inequality above holds. Similar analysis holds for the case of $\theta < \theta'$.

□

D Transformations of the Designer's Problem

We start with the problem in Equation 1.1 and make the following transformation:

$$
\begin{aligned}
& \int_{[0,1]} \int_{0}^{\theta} q(x) dx f(\theta) d\theta \\
= \; & \int_{[0,1]} \int_{x}^{\overline{\theta}} f(\theta) d\theta q(x) dx \\
= \; & \int_{[0,1]} q(x)(1 - F(x)) dx \\
= \; & \int_{[0,1]} q(\theta) \left(\frac{(1 - F(\theta))}{f(\theta)} \right) f(\theta) d\theta
\end{aligned}
$$

So, the second part of the objective function becomes:

$$
\begin{aligned}
& \int_{[0,1]} \theta q(\theta) f(\theta) d\theta - \int_{[0,1]} q(\theta) \left(\frac{(1 - F(\theta))}{f(\theta)} \right) f(\theta) d\theta \\
= \; & \int_{[0,1]} \left(\theta q(\theta) - q(\theta) \frac{(1 - F(\theta))}{f(\theta)} \right) f(\theta) d\theta \\
= \; & \int_{[0,1]} q(\theta) \left(\theta - \frac{(1 - F(\theta))}{f(\theta)} \right) f(\theta) d\theta
\end{aligned}
$$

Next we look at the first summand in the objective function above. Notice that inside is actually a constant, so it can be expressed as below:

$$
\begin{aligned}
&\int_{[0,1]} \left[\int_{\{y|q(y)\leq 0\}} q(x)dx \right] f(\theta)d\theta \\
=& \int_{\{y|q(y)\leq 0\}} q(x)dx \\
=& \int_{[0,1]} q(x)\mathbb{1}\{q(x) \leq 0\}dx \\
=& \int_{[0,1]} q(\theta)\frac{\mathbb{1}\{q(\theta) \leq 0\}}{f(\theta)} f(\theta)d\theta
\end{aligned}
$$

Finally, the objective function can be written as:

$$
\begin{aligned}
&\int_{[0,1]} q(\theta)\frac{\mathbb{1}\{q(\theta) \leq 0\}}{f(\theta)} f(\theta)d\theta + \int_{[0,1]} q(\theta)\left(\theta - \frac{(1-F(\theta))}{f(\theta)}\right) f(\theta)d\theta \\
=& \int_{[0,1]} q(\theta)\left[\frac{\mathbb{1}\{q(\theta) \leq 0\}}{f(\theta)} + \left(\theta - \frac{(1-F(\theta))}{f(\theta)}\right)\right] f(\theta)d\theta
\end{aligned}
$$

Hence, the revenue maximization problem can be expressed as

$$
\max_{(q,t)} \quad \int_{\Theta} q(\theta)\left[\frac{\mathbb{1}\{q(\theta) \leq 0\}}{f(\theta)} + \left(\theta - \frac{(1-F(\theta))}{f(\theta)}\right)\right] f(\theta)d\theta
$$

s. t.

$q(\theta)$ is increasing in θ

$q(\theta) \geq -1$

$$0 \geq \int_{[0,1]} q(\theta)d\theta$$