# Coexistence of Centralized and Decentralized Markets

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**Abstract**

Centralized marketplaces that provide a venue to trade goods and services, such as Airbnb, Amazon, and Uber, have seen a rapid growth in the last decade. When these marketplaces decide the rules of trade, their goal is to maximize their own profit. To understand them better, I employ a mechanism design approach to characterize the optimal rules of a profit-maximizing marketplace in an environment where endowments are known but values are not. If the mechanism is the only trading venue, I characterize the optimal mechanism which ranks agents according to their virtual values and costs. Building on this characterization, I analyze equilibria of a market choice game where agents choose between the centralized marketplace and decentralized bilateral trade. I show that there is an equilibrium in which both the centralized marketplace and decentralized trade coexist. Agents with low and high values join the centralized marketplace while the agents with intermediate values choose to decentralized trade. Moreover, this is the unique equilibrium of the market choice game I study. I establish that the equilibrium profit of the marketplace is at least half of what it would be if there were no decentralized trade. The ratio of the profits with and without competition from the decentralized market is independent of the distribution of the values. I provide conditions under which this equilibrium results in higher welfare than either institution on its own. Finally, I consider an alternative trading protocol for the decentralized market and show that under uniform distribution of valuations, all of the main results extend to this environment as well.

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# 1 Introduction

Centralized marketplaces have seen massive growth in the last decade. For instance, a recent report predicts that Amazon generates half of all e-commerce sales in the US (Congress Majority Staff, 2020). Another analysis estimates that usage of ride-share apps beat taxis in NYC as early as 2017 (Schneider, 2021). Airbnb and Vrbo's market share in vacation rentals reached half of the market (Hinote, 2021). All these suggest that the centralized marketplaces have a significant role in their respective markets.

Indeed, a congressional panel investigating competition in the digital markets asserted that Amazon has monopoly power in the US e-commerce market (Congress Majority Staff, 2020). Although this is another indication of the major role Amazon plays in the e-commerce market, their discussion makes it clear that they are actually referring to the status of Amazon as a centralized marketplace that facilitates trade between third-party sellers and consumers. This usage does not fit the traditional meaning of the word monopoly; Amazon is in an intermediary role here. Yet they are not alone in using the word in this way.[1] This points to a gap in the economic knowledge on how a profit-maximizing marketplace differs from a profit-maximizing producer as well as a lack of terminology.

A marketplace was traditionally 'an open space in a town where a market is or was formerly held' (OED, 2021) or 'the system of buying and selling in competitive conditions' (CED, 2021). These notions of a marketplace are compatible with economics under perfect competition; they would not allow a marketplace to have its own agenda. However, as the reports cited above show, today the marketplaces that maximize their own profit have significant shares in many markets. Thus, there is no perfect competition in or among the marketplaces, and some marketplaces in fact have their own agendas. Although monopoly, oligopoly, and monopsony have been studied extensively by economists, our understanding of the centralized marketplaces that maximize their profit is still lacking. Moreover, we know even less about the impact of a centralized marketplace that competes with numerous other trading venues. Would the centralized marketplace monopolize all trade or is there room for some forms of decentralized trade to coexist? Moreover, if coexistence is possible, what are the welfare and profit consequences of multiple trading modes?

[1] A Google search of "Amazon is a monopoly", with quotes, returns close to 100000 results as of June 2021.

To illustrate the point, consider the example of someone who needs a ride. She can check Uber's price and if she is happy with the price, she can simply take this deal. However, if she believes she can get a better price by asking a friend for a ride and negotiating a mutually beneficial agreement, she may choose to do so. Even though Uber may have a large market share in this sector, there are still endless possibilities for trading outside the formal market, not to mention simply calling a taxi. When Uber chooses what price to offer for each ride, ignoring these possibilities would harm its profit. Moreover, it is hard to guess the impact of Uber's response to these other options.

Financial markets provide another example. Many assets can be traded at the stock exchanges as well as over-the-counter. In the stock exchanges, there is no uncertainty; there are essentially posted prices that an agent can take advantage of. However, in the over-the-counter markets, trade is not as transparent. Dealers often don't post prices in a public manner. Instead, they provide quotes when someone is interested in trading with them. The agents who trade with these dealers only observe the prices offered by a limited number of dealers before they trade. For this reason, parts of the over-the-counter trade are referred to as the dark market. This paper provides a framework to think about the problems faced by a stock exchange that competes with a dark market.

In this paper, I develop a model of a centralized marketplace which competes for agents who also have option of trading among themselves in a decentralized manner. I consider a setup with a single, indivisible good where agents have some endowments and a valuation for at most two of the good. The endowments are common knowledge whereas the valuations are the private information for agents. A designer chooses an ex-post individually rational mechanism to maximize revenue of the marketplace– say, the commissions charged for intermediation. By revelation principle, without loss, I focus on direct mechanisms that are dominant strategy incentive compatible.

I first restrict my attention to the case where the centralized marketplace is the only trading venue, or a *monagora*.[2] I allow each agent to be both buyer and seller depending on their valuation, and the market conditions. I show that the revenue-maximizing allocation rule ranks the agents according to their virtual values and virtual costs. The optimal allocation allows a transaction from an agent to another if and only if one has a higher virtual value than the virtual cost of the other.

[2] I use this term as a parallel of monopoly (the only seller) and monopsony (the only buyer). It is derived by combining 'mono' and 'agora' to mean 'the only market'.

The transfers are pinned down by an interplay of incentive compatibility, individual rationality, and profit maximization.

After studying the marketplace on its own, I introduce the possibility that agents can trade bilaterally among themselves in a decentralized manner– agents are randomly matched among those who decide not to participate in the centralized market. I initially model the trade procedure in the decentralized market as a Nash Bargaining, then later I establish robustness by showing that the results extend to trading with double auction. I consider a market choice game where (i) the marketplace designer announces a mechanism, (ii) agents choose whether to join the mechanism or to search for a trading partner, (iii) outcomes are realized in both markets. I first establish that there is an equilibrium in which the centralized market and decentralized trade coexist. In this equilibrium, the agents with low and high values join the marketplace while the agents with intermediate values choose to search. Agents who join the marketplace are offered utilities higher than their expected utilities from the search market while the agents who join the search market are offered lower utilities. Moreover, I show that the coexistence is the unique equilibrium. Thus, when the agents can choose between these two modes of trade, it is never an equilibrium for all agents to join the same market.

One might expect that competition from the search market will significantly decrease the profits of the marketplace. This is not the case. I show that the profit of the marketplace in the coexistence equilibrium is at least half of the profit that the marketplace would make if it had been a monagora. Moreover, I show that the ratio of the reduction in profit of the marketplace as result of competition from the decentralized market is independent of the distribution of agents valuations. This is remarkable since every aspect of the model heavily depends on the distribution. This ratio is only a function of the efficiency parameter of the decentralized market. As the decentralized market gets more efficient, the profit of the marketplace decreases. However, even in the case of most efficient decentralized market, the profit of the marketplace is half of its profit when it operates on its own.

Finally, I provide conditions under which the coexistence generates higher total welfare than either the marketplace or the search market generates alone. Essentially, the search market extends the extensive margin of trade (more agents trades in the coexistence equilibrium than in the monagorastic marketplace) while the marketplace extends the intensive margin (some agents trade with a higher probability in the coexistence equilibrium than in the monagorastic search market).

Thus, a combination of them leads to increased efficiency.

### 1.1 Literature Review

There is a vast literature on designing optimal auctions from the perspective of the seller. Most notably, Myerson (1981) established many of the standard tools in that literature. Although my focus is different, I follow a similar methodology in that I invoke the revelation principle, establish a new version of payoff equivalence and pin down the transfers using IC and IR constraints together with profit-maximization condition.[3]

Myerson and Satterthwaite (1983) studied the problem of choosing a trade mechanism to maximize the total welfare in the economy. Their main result shows that it is generically impossible to have an efficient trade mechanism -that allocates the good always to the agent who values it the most- without outside resources to finance it. In our model without a decentralized market, unsurprisingly, the welfare achieved is even less than what Myerson and Satterthwaite (1983) provides. However, introducing the option to search improves the efficiency of the market as a whole.

Peivandi and Vohra (2021) study a model where agents are allowed to deviate from a market mechanism to trade among themselves according to a different mechanism. Their main result states that almost every market mechanism is inherently unstable in the sense that there is always a positive measure of agents who would like to deviate from it. Our findings provide a partial counterpart to their result in that by restricting the possible deviations from the market mechanism, I am able to find a stable market structure.

We have also seen rapid developments in the literature on two-sided markets and platforms Rochet and Tirole (2003); Armstrong (2006). In this literature, the questions mainly focus on the competition among platforms under numerous configurations of fee and price structures that could be employed by the platforms. More recently Hartline and Roughgarden (2014) bridges the gap between mechanism design and two-sided markets with a model where sellers can choose to join a platform that sets a menu of selling procedures or to develop their own selling venue. This paper complements these studies by providing a benchmark for the profit that can be obtained by a single platform.

Lu and Robert (2001) considers the problem of a broker maximizing a mixture of profit and

[3] For excellent surveys of the literature, see Krishna (2009); Vohra (2011); Börgers (2015).

welfare while facing a group of traders who can be buyers and sellers, which is similar to the model where the marketplace doesn't face any competition here. They provide some properties of the optimal Bayesian incentive compatible mechanism. However, focusing on Bayesian IC causes a new kind of ironing or bunching in the mechanism, which makes it very difficult for them to obtain a closed-form solution. Under some conditions, (see Gershkov et al. (2013)) our solution does not lose any revenue by focusing on dominant strategy incentive compatibility but this approach allows us to pin down the exact solution and study further properties of these markets.

Miao (2006) also studies the coexistence of centralized and decentralized market in a different setup. Instead of designing mechanisms, the centralized marketplaces there are limited to choosing prices in a dynamic search environment. The paper shows that when the search and transactions costs vanish, the coexistence equilibrium converges to an equilibrium where everyone joins the decentralized market and obtain the Walrasian equilibrium outcomes. However, in this paper, I obtain coexistence even when the search cost vanishes.

## 2 Monagora

In this part of the paper, I restrict all trade to the centralized marketplace; hence there is a unique market or monagora. I use monagora as a parallel of monopoly (only seller) and monopsony (only buyer). It is derived by combining 'mono' and 'agora' to mean 'only market'.

### 2.1 Setup

- Good: There is a single, indivisible good in the market.
- Agents: There is a continuum of agents on $[0, 1]$.
- Endowments: Each agent has 1 unit of endowment of the good.
- Demands: Each agent demands up to 2 units of the good. Since the good is indivisible, this means, they can consume 0, 1, or 2 units, depending on whether they buy or sell, or neither buy nor sell.
- Valuations: Each agent has some valuation $\theta \in [0, 1]$ for a unit of the good. The valuations are drawn from some distribution $F$ with support $[0, 1]$. Agents' valuations are their private

information.

- Marketplace: A mechanism designer wants to design a mechanism to maximize its profit. She knows the distribution of valuations, $F$.

By revelation principle, I focus on direct mechanisms. Moreover, as agents are symmetric other than their valuations, I focus on anonymous mechanisms, which is without loss. Then, the designer will choose a mechanism that allocates $q : \theta \to \mathbb{R}$ units of good to each agent with valuation $\theta$ and asks her to pay $t : \theta \to \mathbb{R}$. Hence, the net utility of the agent with the valuation $\theta$ from the monagorastic mechanism is

$$u(\theta) = \theta \min\{1, q(\theta)\} - t(\theta).$$

As agents have demands for two units, having more than 2 unit of the good is same as having 2 unit. As such, the expression for the utility above caps the maximum net trade that increases the utility at 1, since the agent already has 1 unit of endowment.

The profit of the marketplace is the expected net payments. Thus, the designer seeks to maximize total payment, given the incentive compatibility, individual rationality, and feasibility constraints.

$$\begin{array}{lll}
\max\limits_{(q,t)} & \mathbb{E}_\theta\left[t(\theta)\right] & \\
\text{s. t.} & & \\
\text{(IC)} & \theta \min\{1, q(\theta)\} - t(\theta) & \geq \theta \min\{1, q(\theta', \theta)\} - t(\theta') \\
\text{(IR)} & \theta \min\{1, q(\theta)\} - t(\theta) & \geq 0 \\
\text{(Individual Feasibility)} & q(\theta) & \geq -1 \\
\text{(Aggregate Feasability)} & \mathbb{E}_\theta\left[q(\theta)\right] & \leq 0
\end{array}$$

This problem is very similar to the one studied in Idem (2021) and many of the results obtained there are true for this environment as well. I restate and prove them in the Online Appendix[4] for this environment. These results imply that the above problem can in fact be reduced to the following, simpler problem.

[4]Online Appendix is available at this link.

$$\max_q \quad \left[\int_{[0,1]} q(\theta)\left[\frac{\mathbb{1}\{q(\theta)\le 0\}}{f(\theta)} + \left(\theta - \frac{(1-F(\theta))}{f(\theta)}\right)\right] f(\theta)d\theta\right]$$

s. t.

$q(\theta)$ is increasing

$q(\theta) \ge -1$

$\mathbb{E}_\theta\left[q(\theta)\right] = 0$

Let the *virtual value* and *virtual cost* of an agent with value $\theta$ be $\mathcal{V}(\theta) = \left(\theta - \frac{(1-F(\theta))}{f(\theta)}\right)$ and $C(\theta) = \left(\theta + \frac{(F(\theta)}{f(\theta)}\right)$, respectively. Virtual value and virtual cost can be thought of as the marginal revenue and marginal cost. When an agent is a seller, that is, when an agent has a negative allocation, her deduction from the profit of the marketplace is the virtual cost as can be seen from the objective function above. Similarly, when an agent is a buyer, her contribution to the profit is the virtual value, since the indicator in the objective function takes the value of 0 in that case.

**Definition 2.1.** The distribution of an agent $i$'s type, $F$ is **regular** if both $\mathcal{V}$ and $C$ are increasing.

Since the allocation needs to be increasing, if $q(\theta) < 0$ for some $\theta$, we would have $q(\theta') < 0$ for each $\theta' \le \theta$. Similarly, if $q(\theta) > 0$ for some $\theta$, we would have $q(\theta') > 0$ for each $\theta' \ge \theta$. So, let $0 \le \underline{\theta} \le \overline{\theta} \le 1$ such that $\underline{\theta}$ is the supremum of values with negative allocation and $\overline{\theta}$ is the infimum of the values with positive allocation. Then, we can write the objective function as follows:

$$\begin{aligned}\Pi^M &= \mathbb{P}[\theta \in [0,\underline{\theta}]]\mathbb{E}[C(\theta)q(\theta)|\theta \in [0,\underline{\theta}]] + \mathbb{P}[\theta \in [\overline{\theta},1]]\mathbb{E}[\mathcal{V}(\theta)q(\theta)|\theta \in [\overline{\theta},1]] \\ &= \int_0^{\underline{\theta}} C(x)q(x)f(x)dx + \int_{\overline{\theta}}^1 \mathcal{V}(x)q(x)f(x)dx\end{aligned}$$

Note that for $\theta \le \underline{\theta}$, we must have $q(\theta) = -1$ as $-1$ is the only possible negative allocation with the indivisible good and similarly, $q(\theta) = 1$ for $\theta \ge \overline{\theta}$. Thus, the optimal allocation will have the following form and the next step is to choose the cutoffs, $\underline{\theta}$ and $\overline{\theta}$ optimally.

$$q(\theta) = \begin{cases} -1 & \text{if } \theta \le \underline{\theta} \\ 0 & \text{if } \underline{\theta} \le \theta \le \overline{\theta} \\ 1 & \text{if } \theta \ge \overline{\theta} \end{cases}$$

Then, the problem can be restated as follows:

$$\max_{\underline{\theta},\overline{\theta}} \quad \left[ -\int_0^{\underline{\theta}} C(x) f(x) dx + \int_{\overline{\theta}}^1 \mathcal{V}(x) f(x) dx \right]$$

s. t.

$$F(\underline{\theta}) = 1 - F(\overline{\theta})$$

$$0 \le \underline{\theta} \le \overline{\theta} \le 1$$

Integrating out the total virtual value and the virtual cost using integration by parts[5] shows that the objective function is equal to:

$$-\underline{\theta} F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta})) = F(\underline{\theta})(\overline{\theta} - \underline{\theta})$$

where the equality is obtained by using the feasibility condition $F(\underline{\theta}) = 1 - F(\overline{\theta})$. Then, by Weierstrass Theorem, there exist a solution to this problem. Moreover, the solution is interior: If $\underline{\theta} = 0$ or $\underline{\theta} = \overline{\theta}$, the profit is 0. However, positive profit is feasible by any feasible interior solution as is clear from the objective function.

Moreover, by using the formula for the transfer rule and the optimal allocation from above, we can compute the transfers in the mechanism to be

$$t(\theta) = \begin{cases} -\underline{\theta} & \text{if } \theta \le \underline{\theta} \\ 0 & \text{if } \underline{\theta} \le \theta \le \overline{\theta} \\ \overline{\theta} & \text{if } \theta \ge \overline{\theta} \end{cases}$$

[5] See Appendix A for details.

Notice that each 'seller' gets the same payment while each 'buyer' pays the same amount. Thus, this mechanism is equivalent to offering bid-ask prices that the transfer rule above suggest, that is a price for buying and a price for selling, and letting agents choose whether they want to buy or sell or not trade.

Finally, notice that the solution must have $\mathcal{C}(\underline{\theta}) = \mathcal{V}(\overline{\theta})$: If we had $\mathcal{C}(\underline{\theta}) > \mathcal{V}(\overline{\theta})$, decreasing $\underline{\theta}$ and adjusting $\overline{\theta}$ accordingly for feasibility would increase the profit since buying from $\underline{\theta}$ is costlier than what selling to $\overline{\theta}$ pays off. Similarly, if we had $\mathcal{C}(\underline{\theta}) < \mathcal{V}(\overline{\theta})$, then increasing $\underline{\theta}$ and adjusting $\overline{\theta}$ so that the feasibility binds would again increase the profit since there are more agents whose trade is profitable.

We summarize our findings in the next theorem.

**Theorem 2.1.** *Suppose the distribution $F$ is regular. Then, the optimal mechanism has the allocation rule*

$$q(\theta) = \begin{cases} -1 & \text{if } \theta \leq \underline{\theta} \\ 0 & \text{if } \underline{\theta} \leq \theta \leq \overline{\theta} \\ 1 & \text{if } \theta \geq \overline{\theta} \end{cases}$$

*and the transfer rule*

$$t(\theta) = \begin{cases} -\underline{\theta} & \text{if } \theta \leq \underline{\theta} \\ 0 & \text{if } \underline{\theta} \leq \theta \leq \overline{\theta} \\ \overline{\theta} & \text{if } \theta \geq \overline{\theta} \end{cases}$$

*where $\underline{\theta}$ and $\overline{\theta}$ satisfies $\mathcal{C}(\underline{\theta}) = \mathcal{V}(\overline{\theta})$ and solves the problem*

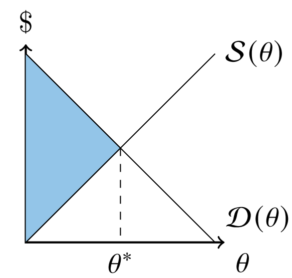


Figure 1: Profit in case there was no decentralized market.

$$\max_{\underline{\theta},\overline{\theta}} \quad \left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta}))\right]$$

$$s.\ t.$$

$$F(\underline{\theta}) = 1 - F(\overline{\theta})$$

$$0 \leq \underline{\theta} \leq \overline{\theta} \leq 1.$$

## 2.2 The Simple Economics of Optimal Marketplaces

Bulow and Roberts (1989) has shown that the optimal auction design problem can be understood using elementary tools we are familiar from the profit maximization problem of a monopolist, as is taught in an undergraduate level course. Here, I provide an analysis of the optimal profit-maximizing marketplaces in the same spirit.

We have already seen that the objective function of the designer is the expected payments of the agents and an equivalent way of thinking about the expected payments is the expected virtual surplus. Thus, I will now use virtual values and costs to define the (inverse) demand and supply functions. Let $\mathcal{S}(\theta) = \mathcal{C}(\theta)$ and $\mathcal{D}(\theta) = \mathcal{V}(F^{-1}(1 - F(\theta)))$. Here, the virtual cost is the marginal cost of the marketplace so it represents the effective supply in the marketplace. Similarly, the virtual value is the marginal revenue curve so it represents demand. However, I define the demand so that at each level of $\theta$, we have the marginal revenue of the 'buyer' that corresponds to the 'seller' with the value $\theta$. This is parallel to the way the demand and supply are first studied in an 'introduction to economics' course. However, instead of agents' willingness to pays (or willingness to get paid), I use their virtual values and costs, since these are the effective marginal revenue and cost curves.

Figure 2.2 shows $\mathcal{D}$ and $\mathcal{S}$ together. As the marketplace is maximizing the profit, the optimal level of trade is given by $\theta^*$ such that $\mathcal{D}$ and $\mathcal{S}$ equals each other. This simply corresponds to the point where the marginal revenue equals the marginal cost. Moreover, the area of the region between $\mathcal{D}$ and $\mathcal{S}$ is equal to the profit of the marketplace.

### 2.3 Illustrative Example with Uniform Distribution

Suppose $\theta$ is distributed uniformly over $[0, 1]$ with c.d.f. $F(\theta) = \theta$. Then, the virtual values and costs are given by

$$\mathcal{V}(\theta) = 2\theta - 1 \text{ and } \mathcal{C}(\theta) = 2\theta\,.$$

It is easy to show that the objective function becomes $\underline{\theta}(1 - 2\underline{\theta})$ after substituting for $\overline{\theta} = 1 - \underline{\theta}$ (feasibility). So, the optimal cutoffs are $\underline{\theta} = \frac{1}{4}$ and $\overline{\theta} = \frac{3}{4}$.

Let us consider two agents with valuations $\theta_1$ and $\theta_2$. Then, 1 will buy from 2 in the monagorastic marketplace if and only if $\theta_1 \geq 0.75$ and $\theta_2 \leq 0.75$, and 2 will buy from 1 if and only if $\theta_2 \geq 0.75$ and $\theta_1 \leq 0.75$. The Figure 2 depicts the space of $(\theta_1, \theta_2)$ where the shaded areas represent these trading regions.

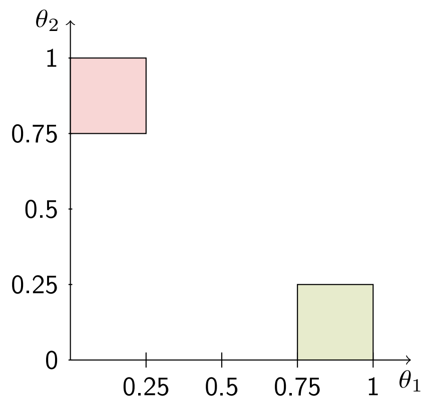


Figure 2: $x$-axis represents $\theta_1 \in [0, 1]$ and $y$-axis represents $\theta_2 \in [0, 1]$. Green and red areas show the type profiles at which agent 1 and 2 is the buyer, respectively.

Using the payment formula from the theorem above, we can compute the payments as below:

$$t(\theta) = \begin{cases} -\dfrac{1}{4}, & \text{if } \theta \leq \dfrac{1}{4}\,, \\ 0, & \text{if } \dfrac{1}{4} \leq \theta \leq \dfrac{3}{4}\,, \\ \dfrac{1}{4}, & \text{if } \theta \geq \dfrac{3}{4}\,. \end{cases}$$

The large area of the type space where there is no trade motivates our consideration for the decentralized market. The marketplace excludes these type profiles from trade because their is not profitable; this is akin to a monopolist or a monopsonist excluding some agents from trade. Indeed, the marketplace acts as both a monopolist and a monopsonist. However, unlike the buyers a monopolist excludes, the agents the marketplace excludes actually have a surplus that they can create, if they were allowed to trade. Thus, assumption that they will remain in the marketplace even though they are not trading is not very realistic. Next, I am going to allow them to choose between the marketplace and a decentralized market.

## 3 Decentralized Market

Now, I introduce a search market and give the agents a choice between joining the mechanism or the search market. Timeline is as follows: A mechanism designer announces a mechanism through which agents can trade the good and invite some of the agents to join it. Upon observing the mechanism, agents can either join the mechanism or they can join a search market, or not join either market.

Agents who join the search market are randomly matched to each other. This process is governed by a matching function, adapted from the search theory. Once matched, they play Nash Bargaining, that is, the agents observe each other's valuations and then split the surplus that would be created by trade. Although most of the text focuses on Nash bargaining, I later extend the model to a double auction where each agent makes a bid and the agent with the higher bid buys the other agent's endowment at the mid-point of their bids. I show that with uniform distribution, all main results obtained for the Nash Bargaining also applies to the environment with the double auction.

Agents who join the mechanism get what the announced mechanism promises. However, the mechanism can be conditional on the set of agents who join it. The simplest example of such a mechanism would be one that operates if agents who join it are exactly those who were invited and shuts down otherwise. Many of the results I obtain here would be valid under this extreme mechanism. However, I obtain an obedience principle that shows this is not necessary: With proper handling of the off-path payoffs, there would be no interlopers in the marketplace in an equilibrium. I expand on this in the Section 3.3.

Here, as a solution concept I use SPE with no bilateral deviations, meaning no pair of agents would prefer to change markets together. Notice that the continuum of agents implies there is no aggregate uncertainty for the designer. Thus, I restrict attention of the designer to deterministic mechanism. Moreover, for technical reasons, I assume that the designer invites a closed set of agents to join the marketplace.

I first study these markets under the following assumption that restricts the set of mechanisms the designer can choose. After gaining a good understanding of these equilibria, I will consider the general case without this assumption. It will turn out that this restriction is without loss of profit, since every equilibrium with coexistence will have to satisfy it.

**Assumption 3.1.** Suppose a convex set of types, $(\underline{\theta}, \overline{\theta})$ joins the search market in the equilibrium.

So, the designer chooses $\underline{\theta}$ and $\overline{\theta}$ optimally, anticipating the agents' best responses to the announced mechanism. We will call the equilibria that satisfy this assumption *simple equilibria* and the mechanisms that induce these equilibria *simple mechanisms*. We will later show that the simple equilibria will indeed have a very simple structure with bid and ask prices. This structure will make sure that the mechanism has the agents with the high virtual values and the low virtual costs. We will also observe that this assumption is without loss of profit: This is the structure we would have under the optimal marketplace without any simplicity restriction.

## 3.1 Matching Functions

I assume there is a matching function that determines the measure of meetings in the search market as a function of the measure of agents in the market, $M(\mu(\Theta^d)$ where $\Theta^d$ is the set of agents who join the search market and $\mu$ is the measure with respect to the distribution $F$. Given the market structure under a strategy profile, each agent gets a meeting with probability $p(\Theta^d)$. In search theory, matching functions are commonly assumed to have constant returns to scale (CRS), meaning doubling the size of the market also doubles the number of meetings. Since we focus on a market where every agent has the same endowment, this is a one-sided market. Thus, CRS matching functions are simply linear in the size of the market: $M(\mu(\Theta^d) = m \times \mu(\Theta^d)$. Then, each agent in this market would have a meeting with probability $2m$ since the total measure of meetings is $M(\mu(\Theta^d)$ and there will be two agents in each meeting.

When there is no cause for confusion, I will simply denote the probability that an agent gets a meeting by $p$.

### 3.2 Payoffs from Search

Let's fix values of $\underline{\theta}$ and $\overline{\theta}$ for now and consider agents' outside options given these values.

If an agent with valuation $\theta$ joins the search market, her expected payoff is

$$u^d(\theta) = p\mathbb{P}[x > \theta | x \in [\underline{\theta}, \overline{\theta}]]\mathbb{E}\left[\frac{x-\theta}{2} | x \in [\underline{\theta}, \overline{\theta}], x > \theta\right] + p\mathbb{P}[x < \theta | x \in [\underline{\theta}, \overline{\theta}]]\mathbb{E}\left[\frac{\theta - x}{2} | x \in [\underline{\theta}, \overline{\theta}], x < \theta\right]$$

where $p$ is the probability of matching someone, $\mathbb{P}[x > \theta | x \in [\underline{\theta}, \overline{\theta}]]$ and $\mathbb{P}[x < \theta | x \in [\underline{\theta}, \overline{\theta}]]$ are the probabilities that the match has a higher or lower values, respectively, and $\mathbb{E}\left[\frac{x-\theta}{2} | x \in [\underline{\theta}, \overline{\theta}], x > \theta\right]$ and $\mathbb{E}\left[\frac{\theta - x}{2} | x \in [\underline{\theta}, \overline{\theta}], x < \theta\right]$ are the expected shares of the surplus given the match has a higher or a lower value, respectively. The denominator 2 in the expression for the surplus comes from the fact that each agent gets half of the surplus from the trade.

Suppose $\theta \in [\underline{\theta}, \overline{\theta}]$. Then,

$$\begin{aligned}
u^d(\theta) &= p\frac{F(\overline{\theta}) - F(\theta)}{2(F(\overline{\theta}) - F(\underline{\theta}))}\left[\int_{\theta}^{\overline{\theta}} \frac{xf(x)dx}{F(\overline{\theta}) - F(\theta)} - \theta\right] + p\frac{F(\theta) - F(\underline{\theta})}{2(F(\overline{\theta}) - F(\underline{\theta}))}\left[\theta - \int_{\underline{\theta}}^{\theta} \frac{xf(x)dx}{F(\theta) - F(\underline{\theta})}\right] \\
&= \frac{p}{2(F(\overline{\theta}) - F(\underline{\theta}))}\left[\int_{\theta}^{\overline{\theta}} xf(x)dx - \theta[F(\overline{\theta}) - F(\theta)] + \theta[F(\theta) - F(\underline{\theta})] - \int_{\underline{\theta}}^{\theta} xf(x)dx\right] \\
&= \frac{p}{2(F(\overline{\theta}) - F(\underline{\theta}))}\left[\int_{\theta}^{\overline{\theta}} xf(x)dx - \int_{\underline{\theta}}^{\theta} xf(x)dx + \theta[2F(\theta) - F(\overline{\theta}) - F(\underline{\theta})]\right]
\end{aligned}$$

Suppose $\theta \leq \underline{\theta}$. Then,

$$
\begin{aligned}
u^d(\theta) &= \frac{p}{2}\left[\int_{\underline{\theta}}^{\overline{\theta}} \frac{xf(x)dx}{F(\overline{\theta}) - F(\underline{\theta})} - \theta\right] \\
&= \frac{p}{2(F(\overline{\theta}) - F(\underline{\theta}))}\left[\int_{\underline{\theta}}^{\overline{\theta}} xf(x)dx - \theta[F(\overline{\theta}) - F(\underline{\theta})]\right]
\end{aligned}
$$

Similarly, if $\theta \geq \overline{\theta}$ then,

$$
\begin{aligned}
u^d(\theta) &= \frac{p}{2}\left[\theta - \int_{\underline{\theta}}^{\overline{\theta}} \frac{xf(x)dx}{F(\overline{\theta}) - F(\underline{\theta})}\right] \\
&= \frac{p}{2(F(\overline{\theta}) - F(\underline{\theta}))}\left[\theta[F(\overline{\theta}) - F(\underline{\theta})] - \int_{\underline{\theta}}^{\overline{\theta}} xf(x)dx\right]
\end{aligned}
$$

Notice that we write the net utilities from trade, not the total utilities. The total utility would also include the utility they get from consuming their endowment. To be consistent, we will also compute the net utilities mechanism promises to the agents. Hence, we ignore the utility they get from consuming their own endowment and simply focus on net utilities from each market.

We differentiate the utilities to understand their shape:

$$
\frac{\partial u^d(\theta)}{\partial \theta} = \begin{cases} -\dfrac{p}{2} & \text{if } \theta \leq \underline{\theta}, \\ \dfrac{p(2F(\theta) - F(\underline{\theta}) - F(\overline{\theta}))}{2(F(\overline{\theta}) - F(\underline{\theta}))} & \text{if } \in [\underline{\theta}, \overline{\theta}], \\ \dfrac{p}{2} & \text{if } \theta \geq \overline{\theta} \end{cases}
$$

For $\theta \in [\underline{\theta}, \overline{\theta}]$, we have:

$$
\begin{aligned}
& 2F(\overline{\theta}) \geq 2F(\theta) \geq 2F(\underline{\theta}) \\
\iff & F(\overline{\theta}) - F(\underline{\theta}) \geq 2F(\theta) - F(\underline{\theta}) - F(\overline{\theta}) \geq F(\underline{\theta}) - F(\overline{\theta}) \\
\iff & \frac{1}{2} \geq \frac{2F(\theta) - F(\underline{\theta}) - F(\overline{\theta})}{2(F(\overline{\theta}) - F(\underline{\theta}))} \geq -\frac{1}{2}
\end{aligned}
$$

Thus, the slope of the expected utility from the search is always in the interval $[-\frac{p}{2}, \frac{p}{2}]$. This will be helpful when we want to show where the utilities the mechanism offers and the agents expect from search can cross each other.

The Figure 3 shows the utilities from the search market for all agents given that agents in $[\underline{\theta}, \overline{\theta}]$ join the search market.

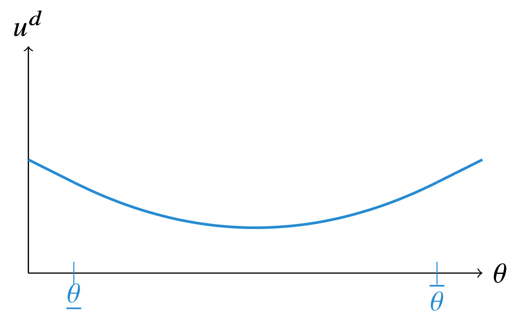


Figure 3: Agents' utilities from their outside options, given $(\underline{\theta}, \overline{\theta})$ join the decentralized market.

### 3.3 An Obedience Principle

Here I show that in a Simple Equilibrium, the agents who join the centralized marketplace are those who had been invited.

I assume that the designer announces the cutoffs $\underline{\theta}, \overline{\theta}$ as well as the mechanism and commits to operating the announced mechanism as long as (i) the agents who join the mechanism is a subset of the invited agents, (ii) the set of agents who join the mechanism is union of two intervals $[0, a] \cap [b, 1]$ where $a < \underline{\theta}$ and $\overline{\theta} < b$ and finally (iii) $F(a) = 1 - F(b)$. Otherwise, the designer shuts down the marketplace.

**Proposition 3.1.** *In a simple equilibrium, set of agents who join the marketplace are exactly those who were invited.*

*Proof.* First, note that if some agents join the marketplace but the set of agents who join the marketplace does not satisfy the requirements for the operation of the announced mechanism, then, the marketplace shuts down and agents there gets 0 utility. Then, they would strictly prefer to be in the decentralized market since the expected payoff there is always positive. Thus, a different set of agents than the invited ones can only be in the marketplace if the marketplace is active.

Suppose the designer announced $\underline{\theta}, \overline{\theta}$ (meaning she invited $[0, \underline{\theta}] \cap [\overline{\theta}, 1]$) and a strict subset of invited agents, $[0, a] \cap [b, 1]$ that satisfy the above requirements joined the mechanism. For this to be an equilibrium, agents with valuations $a$ and $b$ must be indifferent between the two market. To see this, suppose the utilities $a$ or $b$ get from the mechanism is strictly higher than their expected payoffs from search. Then, the designer can increase the transfers to strictly increase the profit. Now suppose the expected payoff from search is strictly lower than the utility from the mechanism for $a$ ($b$). Note that both the expected payoff from search and the utilities the mechanism offer are continuous in agents' own valuations. Then, a positive measure of agents with valuations just above $a$ (below $b$) must also have been offered a higher utility in the mechanism than what they expect from search, which contradicts their joining the search market in an equilibrium. Thus, we conclude that in the equilibrium, the realized cutoffs in this potential equilibrium, $a$ and $b$ must be indifferent between two markets. We now write the utilities these agents expect from both markets.

$$
\begin{aligned}
u^d(a) &= \frac{p}{2}\left[\mathbb{E}[\theta|\theta \in [a,b]] - a\right] && = \frac{p}{2}\left[\mathbb{E}[\theta|\theta \in [\underline{\theta}, \overline{\theta}]] - \underline{\theta}\right] + \underline{\theta} - a = u^m(a) \\
u^d(b) &= \frac{p}{2}\left[b - \mathbb{E}[\theta|\theta \in [a,b]]\right] && = \frac{p}{2}\left[\overline{\theta} - \mathbb{E}[\theta|\theta \in [\underline{\theta}, \overline{\theta}]]\right] + b - \overline{\theta} = u^m(b)
\end{aligned}
$$

where we calculate the expected payoff from the mechanism using the payoff equivalence: The mechanism would make $\underline{\theta}$ and $\overline{\theta}$ indifferent between two markets and then have a slope of $-0.5$ for sellers and of $0.5$ for buyers.

By adding these equations up, we end up with

$$
\begin{aligned}
\frac{p}{2}\left[b-a\right] =& \frac{p}{2}\left[\overline{\theta}-\underline{\theta}\right]+b+\underline{\theta}-a-\overline{\theta} \\
\frac{p}{2}\left[b-a-\overline{\theta}+\underline{\theta}\right] =& b+\underline{\theta}-a-\overline{\theta} \\
p =& 2,
\end{aligned}
$$

which is of course impossible since $p$ is a probability. Thus, we conclude that there is no equilibrium where the set of agents in the mechanism is nonempty and different from the set of invited agents. □

## 3.4 Designing the Mechanism

Under Assumption 1, the agents who join the mechanism will have values in $[0,\underline{\theta}]$ and $[\overline{\theta},1]$. We can write their utilities as follows, using payoff equivalence from the Online Appendix.

$$
u^m(\theta)=\begin{cases} u^m(\overline{\theta})+\int_{\overline{\theta}}^{\theta} q(x)dx & \text{if } \theta \geq \overline{\theta} \\ u^m(0)+\int_{0}^{\theta} q(x)dx & \text{if } \theta \leq \underline{\theta} \end{cases}
$$

Similarly, we can write the transfers:

$$
t(\theta)=\begin{cases} \theta q(\theta)-u^m(\overline{\theta})-\int_{\overline{\theta}}^{\theta} q(x)dx & \text{if } \theta \geq \overline{\theta} \\ \theta q(\theta)-u^m(0)-\int_{0}^{\theta} q(x)dx. & \text{if } \theta \leq \underline{\theta} \end{cases}
$$

### 3.4.1 The Profit Function

Now, let us compute the profit from the optimal allocation given the cutoffs. The step-by-step derivation can be followed in the Appendix B but here is the end-result:

$$
\begin{aligned}
\Pi_{\underline{\theta},\overline{\theta}} &= \mathbb{P}[\theta \in [0,\underline{\theta}]]\mathbb{E}[t(\theta)|\theta \in [0,\underline{\theta}]] + \mathbb{P}[\theta \in [\overline{\theta},1]]\mathbb{E}[t(\theta)|\theta \in [\overline{\theta},1]] \\
&= \int_0^{\underline{\theta}} t(\theta)f(\theta)d\theta + \int_{\overline{\theta}}^{1} t(\theta)f(\theta)d\theta \\
&= -F(\underline{\theta})u^d(\underline{\theta}) - (1-F(\overline{\theta}))u^d(\overline{\theta}) + \int_0^{\underline{\theta}} \left[\left(x + \frac{F(x)}{f(x)}\right)q(x)\right]f(x)dx + \int_{\overline{\theta}}^{1} \left[\left(x - \frac{1-F(x)}{f(x)}\right)q(x)\right]f(x)dx
\end{aligned}
$$

Earlier, I defined the virtual cost, $C$ and the virtual value $\mathcal{V}$ as:

$$
C(x) = \left(x + \frac{F(x)}{f(x)}\right) \text{ and } \mathcal{V}(x) = \left(x - \frac{1-F(x)}{f(x)}\right).
$$

I continue to assume both of them are increasing and call such distributions regular.

### 3.4.2 The Constraints

First of all, we are assuming that $0 \le \underline{\theta} \le \overline{\theta} \le 1$, since the support of the distribution of values is $[0,1]$.

Second, the feasibility requires that

$$
\int_0^{\underline{\theta}} q(x)f(x)dx + \int_{\overline{\theta}}^{1} q(x)f(x)dx \le 0.
$$

Third, we need $-1 \le q(\theta) \le 1$ for all agents respectively as they have a unit of endowment and up to two units of unit demand, in total, by assumption. In fact, since the good is indivisible, this means each agent can have either $-1$, 0 or 1 as their allocation.

Fourth, we know that for incentive compatibility in the mechanism, we need the allocation to be increasing.

Finally, we need to consider the implications of the individual rationality.

**Individual Rationality** Notice that in any profit maximizing mechanism, individual rationality (IR) constraint for at least one type of agent in each segment who joins the mechanism ($[0,\underline{\theta}]$ and $[\overline{\theta},1]$) must bind. If not, then uniformly increasing the payment of all agents in the particular segment without a binding IR until there is a binding constraint increases the profit. Moreover, the binding IR constraints in the optimal mechanism must be $\underline{\theta}$ and $\overline{\theta}$. To see this, notice that under the Assumption 1, we want to construct an equilibrium such that agents in $[0, \underline{\theta}]$ and $[\overline{\theta}, 1]$ join the mechanism while the rest of the agents join the search market. Since agents will choose the market that offers them a higher utility, this means, $\underline{\theta}$ and $\overline{\theta}$ must be the points where $u^d$ and $u^m$ cross each other such that on $[0, \underline{\theta}]$ and $[\overline{\theta}, 1]$, $u^m$ should be higher while on $(\underline{\theta}, \overline{\theta})$, $u^d$ must be higher. Then, we must have: $u^d(\underline{\theta}) = u^m(\underline{\theta}) = u^m(0) + \int_0^{\underline{\theta}} q(x)dx$ so that $u^m(0) = u^d(\underline{\theta}) - \int_0^{\underline{\theta}} q(x)dx$ and $u^m(\overline{\theta}) = u^d(\overline{\theta})$.

Moreover, binding IR constraints for $\underline{\theta}$, $\overline{\theta}$, together with the monotonicity of the allocations significantly simplifies the problem since it can be used to obtain the optimal allocation in a simple equilibrium. Notice that around $\underline{\theta}$, the utility from the mechanism should have a left derivative below $-0.5p$:

$$
\begin{aligned}
u^m(0) + \int_0^{\theta} q(x)dx \geq u^d(\theta) &\iff \\
u^d(\underline{\theta}) - \int_0^{\underline{\theta}} q(x)dx + \int_0^{\theta} q(x)dx \geq u^d(\theta) &\iff \\
u^d(\underline{\theta}) - \int_{\theta}^{\underline{\theta}} q(x)dx \geq u^d(\theta) &\iff \\
p\left[\frac{\int_{\underline{\theta}}^{\overline{\theta}} xf(x)dx}{2\left(F(\overline{\theta}) - F(\underline{\theta})\right)} - \frac{\underline{\theta}}{2}\right] - \int_{\theta}^{\underline{\theta}} q(x)dx \geq p\left[\frac{\int_{\underline{\theta}}^{\overline{\theta}} xf(x)dx}{2\left(F(\overline{\theta}) - F(\underline{\theta})\right)} - \frac{\theta}{2}\right] &\iff \\
p\frac{\theta - \underline{\theta}}{2} \geq \int_{\theta}^{\underline{\theta}} q(x)dx &\iff \\
p\frac{\theta - \underline{\theta}}{2} \geq u^m(\underline{\theta}) - u^m(\theta).
\end{aligned}
$$

Since only possible allocations are $-1$, $0$, and $1$, this means we must have $q(\underline{\theta}) = -1$. Moreover, due to monotonicity of the allocation for incentive compatibility, this would only be true if for each $\theta \in [0, \underline{\theta}]$, $q(\theta) = -1$. Following the same steps around $\overline{\theta}$ also shows that $q(\theta) = 1$ for each $\theta \in [\overline{\theta}, 1]$. We note the observations we have made here in the following lemma.

**Lemma 3.1.** *In a simple equilibrium with cutoffs $\underline{\theta}$, $\overline{\theta}$, we have:*

$$q(\theta) = \begin{cases} -1 \ \textit{if} \ \theta \leq \underline{\theta} \\ 1 \ \textit{if} \ \theta \geq \overline{\theta} \end{cases}$$

### 3.5 Simple Equilibrium

Now we are going to show that a simple equilibrium exists.

Using the observations from the previous subsection, the profit from any such pairs of thresholds can be written as follows.[6]

$$\Pi = \frac{1}{2}\left([2-p]\left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1-F(\overline{\theta}))\right] - p\left[F(\underline{\theta}) - (1-F(\overline{\theta}))\right] E[\theta|\underline{\theta} \leq \theta \leq \overline{\theta}]\right)$$

Our constraints are $0 \leq \underline{\theta} \leq \overline{\theta} \leq 1$ and $F(\underline{\theta}) \geq 1 - F(\overline{\theta})$. First, we can't have $\overline{\theta} < m(F)$ where $m(F)$ is the median of $F$ since that would require $0.5 > F(\overline{\theta}) \geq F(\underline{\theta}) \geq 1-F(\overline{\theta}) \geq 0.5$. Second, given that $\Pi$ is strictly decreasing in $\underline{\theta}$, $F(\underline{\theta}) \geq 1 - F(\overline{\theta})$ should bind[7]. Third, $\underline{\theta} \leq m(F) \leq \overline{\theta}$ as a result of previous two observations. Thus, for a strictly increasing $F$, this is essentially a single parameter problem with a continuous objective and a compact domain. Hence, it has a solution by Weierstrass Theorem. Moreover, the solution is interior in the sense that the constraints $\underline{\theta} \leq m(F) \leq \overline{\theta}$ don't bind. If they did, then the profit would be 0 while it is possible to achieve a positive profit when they don't bind.[8] The Appendix C shows this in more details but it is easy to see that when the feasibility condition binds, the expectation term disappears from the profit. Then, the profit in this equilibrium is equal to the profit when there was no search market, times a constant, $\frac{2-p}{2}$. Thus, the solution must still have $\mathcal{C}(\underline{\theta}) = \mathcal{V}(\overline{\theta})$, as shown in the Theorem 2.1.

**Theorem 3.1.** *Suppose $F$ is regular. Then, there exists $\underline{\theta}, \overline{\theta}$ such that in the simple equilibrium,*

- *agents in $[0, \underline{\theta}]$ and $[\overline{\theta}, 1]$ join the mechanism,*
- *agents in $(\underline{\theta}, \overline{\theta})$ join the search market,*
- $\mathcal{C}(\underline{\theta}) = \mathcal{V}(\overline{\theta})$ *and* $F(\underline{\theta}) = 1 - F(\overline{\theta})$.

[6]The details can be followed in Appendix C.
[7]We show this in the Appendix C.
[8]Again, Appendix C shows this in more detail.

To illustrate the utilities agents are offered in the mechanism and expect from the search market, suppose $p = 1$ and $F(\theta) = \theta$ so that everyone gets a meeting and agents are uniformly distributed over the unit interval. Then, the optimal cutoffs are $\underline{\theta} = 0.25$ and $\overline{\theta} = 0.75$. Then, Figure 3.5 show the utilities agents can expect from either market as well as the profit of the marketplace, and the compensations of the agents.

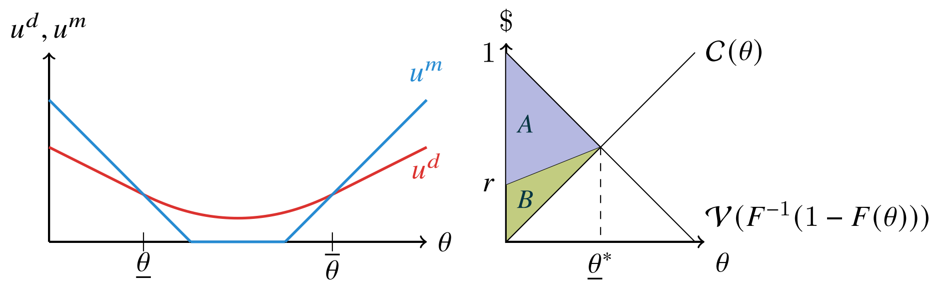


Figure 4: On the left, we see the utilities from the search market and the optimal mechanism under the simple equilibrium. On the right, we see the profit $(A)$, compensations 'paid' to agents because of their outside option of search $(B)$, and the profit in case there was no search market $(A + B)$.

The preceding theorem describes the structure of the simple equilibrium. However, it doesn't completely describe the mechanism that induces this equilibrium. Although the allocations and transfers for the agents who join the mechanism are exactly pinned down by the discussions from the previous sections, especially in Lemma 3.1, we haven't described what is offered to agents in $[\underline{\theta}, \overline{\theta}]$ except for saying that they are offered lower utilities than $u^d$, since it is the only thing that matters for the equilibrium behavior. There are in fact several mechanisms that would induce the same equilibrium that only differ in the off-path payoffs: As long as what the mechanism promises to agents in $[\underline{\theta}, \overline{\theta}]$ is less than $u^d$, the outcomes would remain the same. The next proposition describes one such mechanism. As described before, it essentially offers ask and bid prices, i.e., prices for selling and buying as well as the option to not trade.

**Proposition 3.2.** *In the simple equilibrium, the mechanism the designer offers has the following allocation and transfer rules:*

$$
q(\theta) = \begin{cases} -1 & \textit{if } \theta \leq \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\underline{\theta}}{2} \\ 0 & \textit{if } \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\underline{\theta}}{2} \leq \theta \leq \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\overline{\theta}}{2} \\ 1 & \textit{if } \theta \geq \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\overline{\theta}}{2} \end{cases}
$$

$$
t(\theta) = \begin{cases} -\dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\underline{\theta}}{2} & \textit{if } \theta \leq \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\underline{\theta}}{2} \\ 0 & \textit{if } \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\underline{\theta}}{2} \leq \theta \leq \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\overline{\theta}}{2} \\ \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\overline{\theta}}{2} & \textit{if } \theta \geq \dfrac{p\mathbb{E}[x|x \in [\underline{\theta}, \overline{\theta}]] + (2-p)\overline{\theta}}{2} \end{cases}
$$

Proof is discussed in the Appendix C.1. Essentially, it computes the transfers for the agents who join the mechanism and then extends the allocation and the transfer rules to the rest of the agents in a way that the allocations are increasing and we have $u^m(\theta) < u^d(\theta)$ for each $\theta \in (\underline{\theta}, \overline{\theta})$.

## 3.6 Profit of the Marketplace

Next, we are going to obtain the relationship between the profit in this mechanism and the profit of the monagorastic marketplace.

**Proposition 3.3.** *Suppose $F$ is regular. Then, the profit of the marketplace in coexistence (described in Theorem 3.1) is $1 - \frac{p}{2} = 1 - m$ times the profit the marketplace would make if there were no search market.*

Mathematically, this proposition is a direct consequence of the expression for the profit we obtain in the Appendix C.

It is remarkable that the ratio of the profits is completely independent of the distribution, $F$. The ratio only depends on the probability that an agents finds a match in the search market, $p$. Clearly, the profit is decreasing in $p$: As the matching process becomes more efficient, the decentralized market becomes more attractive and the profit of the marketplace decreases. However, even when $p = 1$, which means when every agent finds a match in the decentralized market, the profit of the marketplace is half of its profit when it operate on its own. This is because even when each agent in the decentralized market gets a match with certainty, the matches themselves may not be very efficient. An agent might meet someone whose valuation is very close to her own. However, marketplace solves this problem and makes profit by creating efficient matches. At the other extreme, $p = 0$, when there is no match in the decentralized market, the marketplace makes the profit it makes on its own, since agents cannot trade in the decentralized market anymore.

### 3.7 Efficiency of the Coexistence

Now, we focus on a concrete distribution for valuations to obtain some comparisons of the profit. In particular, we assume that $F(\theta) = \theta$ so that agents' valuations are drawn from the uniform distribution.

**Proposition 3.4.** *Suppose $F = U[0, 1]$. Then, the total welfare under the simple equilibrium is greater than when either market operates on its own.*

It is clear that the coexistence improves efficiency over the monagorastic marketplace: Since the profit function in the coexistence case is just a scaled version of the profit of the monagorastic marketplace, the cutoffs for trading in the marketplace are the same in these two cases, as we have noted before. However, in the coexistence equilibrium, agents who don't get to trade in the marketplace join the search market and create some surplus. Thus, coexistence equilibrium improves the efficiency over the monagorastic marketplace. What remains to be shown is that it also improves over the pure search market. We show this in the Appendix D.1.

Intuition behind this result is that the search market extends the extensive margin of trade while the centralized marketplace extends the intensive margin. Thus, when they operate together, everyone gets to trade with some probability (which causes this to create more surplus than the monagorastic marketplace, since it doesn't allow some agents to trade) and some agents trade for certain (which is why the coexistence equilibrium provides a higher total welfare than the pure search market).

Although the proposition only mentions the uniform distribution, the result is true for many other distribution. Next, we provide a condition on the distribution of the valuations which guarantees that the coexistence is more efficient than either the pure search market or the monagorastic marketplace. This condition is satisfied by most commonly used distributions.

**Assumption 3.2.** Under $F$, the following inequality holds:

$$2\mathbb{E}[\theta F(\theta)|\theta \in [\underline{\theta}, \overline{\theta}]] + \mathbb{E}[\theta|\theta > \overline{\theta}] \geq \mathbb{E}[\theta|\theta \in [\underline{\theta}, \overline{\theta}]] + \mathbb{E}[\theta F(\theta)|\theta \leq \underline{\theta}] + \mathbb{E}[\theta F(\theta)|\theta \geq \overline{\theta}],$$

where $\underline{\theta}$ and $\overline{\theta}$ are the optimal cutoffs for a simple equilibrium, given the distribution $F$.

As this condition depends on the optimal cutoffs, it is difficult to show that it is satisfied by any regular distribution. However, it is in fact hard to find a regular distribution that doesn't satisfy it. Standard Normal Distribution, Logistic Distribution, Exponential Distribution, Standard Beta Distribution are among the distributions that has this property[9].

**Proposition 3.5.** *If F satisfies Assumption 3.2, then the total welfare under the simple equilibrium is greater than when either market operates on its own.*

I prove this in the Appendix D.2.

### 3.8 Unrestricted Equilibrium

In this section, we remove the restriction to the 'simple' market structure that required an interval of buyers and sellers to join the search market. Here, we allow the designer to invite any closed set of agents. As the following proposition shows, the restriction of the simple equilibrium is without loss of generality. Thus, it shows that the Assumption 3.1 was without loss of generality. Therefore, everything we have studied so far holds for all equilibria where both markets are active.

**Theorem 3.2.** *If both markets are active in an equilibrium, then the sets of agents in the search market must each be an interval.*

The proof is insightful yet lengthy and involved. Thus, it is covered in the Appendix F.

### 3.9 Uniqueness of the Equilibrium

So far, we have seen that the simple equilibrium is the unique coexistence equilibrium. This leaves the question of whether there can be equilibria where only one of the markets is active. First, it is relatively easy to argue that there cannot be any equilibrium where all agents join

[9]Mathematica codes for computations that show these distributions satisfy this assumption are available upon request.

the centralized marketplace. This is because the marketplace will exclude some agents from trade even if they are in the marketplace. But then, these agents would have a profitable bilateral deviation to the decentralized market where they would get a positive payoff, rather than getting 0 in the marketplace. Thus, the only other possibility is an equilibrium where all agents join the decentralized market. Next, we show that, everyone joining the decentralized market is not an equilibrium either.

**Proposition 3.6.**

1. *There is no equilibrium where all agents join the centralized marketplace.*

2. *There is no equilibrium where all agents join the decentralized market.*

3. *Thus, the simple equilibrium is the unique equilibrium.*

The proposition is proved in Appendix G.

The uniqueness of the equilibrium increases the robustness of the predictions about the markets I study. If coexistence was only one of the several equilibria, then my predictions would be based on the assumption that the particular equilibrium I study is the one that is actually played. Without this kind of assumptions, it would be hard to make any prediction, since everything depends on the equilibrium being played. However, since coexistence is the unique equilibrium, there is no need to worry about these issues. Simply focusing on equilibrium behavior is sufficient to have robust predictions.

### 3.10 Simple Equilibrium with Double Auction

Suppose the agents who join the decentralized market are randomly matched to each other and then they participate in a double auction. The random matching process operates in the same manner as before. The double auction works as follows. Each agent makes some bids, $b_i$. Agent with the higher bid buys the other's endowment and pays $\frac{b_i+b_j}{2}$, when bids are given by $b_i$ and $b_j$; so they are trading a unit of the good at the midpoint of the bids. This is a special case of the 'simple trading rule' studied by Cramton et al. (1987). They show that this game has a symmetric equilibrium where the bids are increasing in agents' valuations. Thus, the equilibrium is ex-post efficient in the sense that, the agent who values the good more ends up with the whole quantity.

Moreover, Kittsteiner (2003) has shown that the symmetric equilibrium characterized by Cramton et al. (1987) is indeed the unique equilibrium. Now, I will describe the bidding functions in this equilibrium as well as agents' expected payoffs from the game.

Suppose the distribution of agents who participates in this double auction is given by some CDF, $G$. Suppose $G$ is strictly increasing on its support, $[\underline{\theta}, \overline{\theta}]$ and is differentiable. Then, agents' bids in the unique equilibrium are given by:

$$b(\theta) = \theta - \frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2}$$

It follows from the Proposition 5 of Cramton et al. (1987) that following this bidding strategy constitutes an equilibrium. Theorem 1 in Kittsteiner (2003) further shows that there is no other equilibrium.

As before, we will focus on simple mechanisms that exclude an interval of agents: We suppose agents in $[\underline{\theta}, \overline{\theta}]$ join the decentralized market and the rest of the agents join the centralized marketplace. Then, $G(\theta) = \dfrac{F(x) - F(\underline{\theta})}{F(\overline{\theta}) - F(\underline{\theta})}$ on its support $[\underline{\theta}, \overline{\theta}]$.

To compute all agents' expected payoffs from the decentralized market, we need to know the optimal bids of agents whose valuations lie outside $[\underline{\theta}, \overline{\theta}]$. For agents with values below $\theta \leq \underline{\theta}$, if they join the decentralized market and get matched to someone, the best response is to bid $b(\underline{\theta})$ and for agents with values above $\overline{\theta}$, the best response is to bid $b(\overline{\theta})$. I show this in two steps. First, I show that for any agent, the optimal bid must lie in $[b(\underline{\theta}), b(\overline{\theta})]$. Then, I show that for agents with values less than $\underline{\theta}$ the optimal bid is $b(\underline{\theta})$ while for agents with values above $\overline{\theta}$, the optimal bid is $b(\overline{\theta})$. This is stated in the next lemma and proved in the Appendix H.

**Lemma 3.2.** *For each $\theta \in [0, \underline{\theta}]$, $b(\theta) = b(\underline{\theta})$ and for each $\theta \in [\overline{\theta}, 1]$, $b(\theta) = b(\overline{\theta})$.*

Agents' utilities from the decentralized market:

Suppose $\theta \in [\underline{\theta}, \overline{\theta}]$. If the agent has the lower value, she has the lower bid, since the bidding function is monotone. Then, the agent gives up her endowment but gets paid. If the agent has the higher value, then, she has the higher bid so she gets the other's endowment and pays for it. Thus, expected payoff is

$$u^{da}(\theta) = pG(\theta)\left[\theta - \int_{\underline{\theta}}^{\theta} \frac{\frac{1}{2}[b(x)+b(\theta)]g(x)dx}{G(\theta)}\right] + p(1-G(\theta))\left[\int_{\theta}^{\overline{\theta}} \frac{\frac{1}{2}[b(x)+b(\theta)]g(x)dx}{1-G(\theta)} - \theta\right]$$

$$= p\theta[2G(\theta)-1] + \frac{p}{2}\int_{\theta}^{\overline{\theta}} [b(x)+b(\theta)]g(x)dx - \frac{p}{2}\int_{\underline{\theta}}^{\theta} [b(x)+b(\theta)]g(x)dx$$

$$= p\theta[2G(\theta)-1] + \frac{p}{2}(1-2G(\theta))b(\theta) + \frac{p}{2}\int_{\theta}^{\overline{\theta}} b(x)g(x)dx - \frac{p}{2}\int_{\underline{\theta}}^{\theta} b(x)g(x)dx$$

If $\theta \leq \underline{\theta}$, then, as above argument shows, the optimal bid is $b(\underline{\theta})$ and the expected payoff from the decentralized market is:

$$u^{da}(\theta) = p\left[\int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{2}[b(\underline{\theta})+b(x)]g(x)dx - \theta\right]$$

Similarly, agents with $\theta \geq \overline{\theta}$ bid $b(\overline{\theta})$ and get the expected payoff:

$$u^{da}(\theta) = p\left[\theta - \int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{2}[b(\overline{\theta})+b(x)]g(x)dx\right]$$

Next, we compute $u^{da}(\underline{\theta})$ and $u^{da}(\overline{\theta})$ by using the formulas for $b(\cdot)$, as the IR constraints of $\underline{\theta}$ and $\overline{\theta}$ will again play an important role in the equilibrium. The details can be found in the Appendix I but here is the end result:

$$u^{da}(\underline{\theta}) = \frac{p}{2}\left[-\underline{\theta} + 4\int_{\underline{\theta}}^{G^{-1}(\frac{1}{2})} \left[G(x)-\frac{1}{2}\right]^2 dx + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]$$

$$u^{da}(\overline{\theta}) = \frac{p}{2}\left[\overline{\theta} + 4\int_{G^{-1}(\frac{1}{2})}^{\overline{\theta}} \left[G(x)-\frac{1}{2}\right]^2 dx - \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]$$

In the simple equilibrium, we are going to make the IR constraints of $\underline{\theta}$ and $\overline{\theta}$ bind, as otherwise decreasing the payment until they bind increases the profit. For these cutoffs to work, we need the slope of the utility from the decentralized market, $\frac{\partial u^{da}(\theta)}{\partial \theta}$ for types below $\underline{\theta}$ to be greater than $-1$, since $-1$ is the allocation they will be offered in the mechanism and we want $u^{da}$ to be less than $u^m$ on this region. Moreover, we need the slope of $u^{da}$ to be greater than $-1$ around $\underline{\theta}$ and the slope should be increasing (thus the utility function should be convex). Finally, we need the slope of the utility from the decentralized market to be less than 1 for agents with values above $\overline{\theta}$.

**Lemma 3.3.**

$$\frac{\partial u^{da}(\theta)}{\partial \theta} = \begin{cases} -p & \textit{if } \theta \leq \underline{\theta} \\ p(G(\theta) - 1) & \textit{if } \underline{\theta} \leq \theta \leq \overline{\theta} \\ p & \textit{if } \theta \geq \overline{\theta} \end{cases}$$

Proof can be found in the Appendix J.

Since this is between $-1$ and 1, for agents with values in $[\underline{\theta}, \overline{\theta}]$, the designer can indeed offer lower utilities to these agents. One way of doing this would be offering the allocation $-1$ for agents between $\underline{\theta}$ and $G^{-1}(\frac{1}{2})$ and allocation 1 for agents between $G^{-1}(\frac{1}{2})$ and $\overline{\theta}$. This would make sure these agents are offered utilities lower than their expected payoff from the decentralized market since $u^m(\underline{\theta}) = u^{da}(\underline{\theta})$ and $u^m(\overline{\theta}) = u^{da}(\overline{\theta})$. (This may offer utilities below zero for some agents. The designer may not be concerned about this, since these agents are not wanted anyway. However, if the designer wishes the mechanism to offer nonnegative utilities, this can be achieved by flattening the utility when it reaches zero, as the Proposition 3.2 does.)

Now, we look at the profit function. As before, it is equal to

$$\begin{aligned}
\Pi_{\underline{\theta},\overline{\theta}} &= \mathbb{P}[\theta \in [0, \underline{\theta}]]\mathbb{E}[t(\theta)|\theta \in [0, \underline{\theta}]] + \mathbb{P}[\theta \in [\overline{\theta}, 1]]\mathbb{E}[t(\theta)|\theta \in [\overline{\theta}, 1]] \\
&= \int_0^{\underline{\theta}} t(\theta)f(\theta)d\theta + \int_{\overline{\theta}}^1 t(\theta)f(\theta)d\theta \\
&= -F(\underline{\theta})u^d(\underline{\theta}) - (1 - F(\overline{\theta}))u^d(\overline{\theta}) + \int_0^{\underline{\theta}} C(x)q(x)f(x)dx + \int_{\overline{\theta}}^1 \mathcal{V}(x)q(x)f(x)dx \\
&= -F(\underline{\theta})u^d(\underline{\theta}) - (1 - F(\overline{\theta}))u^d(\overline{\theta}) - \int_0^{\underline{\theta}} C(x)f(x)dx + \int_{\overline{\theta}}^1 \mathcal{V}(x)f(x)dx
\end{aligned}$$

First two terms are the compensations for agents to join the centralized marketplace, while the last two terms are the total virtual surplus. We know expression for the total surplus from before because that part is unchanged:

$$-\int_{0}^{\underline{\theta}} \mathcal{C}(x)f(x)dx + \int_{\overline{\theta}}^{1} \mathcal{V}(x)f(x)dx = -\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta})).$$

Next, we study the compensations, since now they will be different from what we had for the Nash bargaining.

$$
\begin{aligned}
&F(\underline{\theta})u^d(\underline{\theta}) + (1 - F(\overline{\theta}))u^d(\overline{\theta}) \\
=&F(\underline{\theta})p\left[\frac{1}{2}\left[b(\underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right] - \underline{\theta}\right] + (1 - F(\overline{\theta}))p\left[\overline{\theta} - \frac{1}{2}\left[b(\overline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]\right] \\
=&\frac{p}{2}\left[F(\underline{\theta})\left[b(\underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx - 2\underline{\theta}\right] + (1 - F(\overline{\theta}))\left[2\overline{\theta} - b(\overline{\theta}) - \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]\right] \\
=&p\left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta}))\right] \\
+&\frac{p}{2}\left[\left[F(\underline{\theta}) + F(\overline{\theta}) - 1\right]\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx + F(\underline{\theta})b(\underline{\theta}) - (1 - F(\overline{\theta}))b(\overline{\theta})\right]
\end{aligned}
$$

Remember that the optimal bidding strategy is given by

$$b(\theta) = \theta - \frac{\int_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2}$$

with $G(\theta) = \dfrac{F(x) - F(\underline{\theta})}{F(\overline{\theta}) - F(\underline{\theta})}$ on $[\underline{\theta}, \overline{\theta}]$ and $G^{-1}(\frac{1}{2}) = F^{-1}\left(\frac{F(\underline{\theta})+F(\overline{\theta})}{2}\right)$. Although it is relatively easy to show that a simple equilibrium exists with arbitrary distributions, the general solution to the profit maximization problem is too complicated to provide some useful comparative statics. Hence, I focus on the uniform distribution, $U[0,1]$ from here on to be able to find a closed form solution to the problem above.

Using the uniform distribution, with some algebra (see Appendix K), we can show that

$$b(\theta) = \frac{\overline{\theta} + \underline{\theta} + 4\theta}{6},$$
$$\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx = \frac{\overline{\theta} + \underline{\theta}}{2}.$$

Now, we can plug these back into the expression for the profit to see that it is decreasing in $\underline{\theta}$. Thus, the feasibility must bind, which means $\underline{\theta} = 1 - \overline{\theta}$. Using this, we simplify the profit further and obtain the following simple expression for the profit. (The derivations can be followed in Appendix L.)

$$\Pi_{\underline{\theta},\overline{\theta}} = \frac{6-5p}{6}\left[\underline{\theta}(\overline{\theta} - \underline{\theta})\right] = \frac{6-5p}{6}\Pi^M$$

Clearly, this problem has an interior solution, which is the same as the solution of the problem of the marketplace when it operated on its own: The profits in two cases are equal up to a constant multiplier. Thus, almost everything we have seen under the Nash bargaining hold here with the uniform distribution. I state them for this environment as well for completeness.

**Theorem 3.3.** *There exists $\underline{\theta}, \overline{\theta}$ such that in the simple equilibrium,*

- *agents in $[0, \underline{\theta}]$ and $[\overline{\theta}, 1]$ join the mechanism,*
- *agents in $(\underline{\theta}, \overline{\theta})$ join the decentralized market with double auction,*
- $\mathcal{C}(\underline{\theta}) = \mathcal{V}(\overline{\theta})$ *and* $F(\underline{\theta}) = 1 - F(\overline{\theta})$.

**Proposition 3.7.** *The profit of the marketplace in coexistence is $1 - \frac{5p}{6} = 1 - \frac{5m}{3}$ times the profit the marketplace would make if there were no decentralized market with double auction.*

**Proposition 3.8.** *The total welfare under the simple equilibrium with the double auction is greater than when either market operates on its own.*

**Proposition 3.9.** *Same agents trade in the marketplace, with or without the decentralized market.*

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# A Integrating the Virtual Surplus

$$
\begin{aligned}
& -\int_0^{\underline{\theta}} C(x) f(x) dx + \int_{\overline{\theta}}^1 V(y) f(y) dy \\
&= -\int_0^{\underline{\theta}} C(x) f(x) dx + \int_{\overline{\theta}}^1 V(y) f(y) dy \\
&= -\int_0^{\underline{\theta}} \left[ x + \frac{F(x)}{f(x)} \right] f(x) dx + \int_{\overline{\theta}}^1 \left[ y - \frac{1-F(y)}{f(y)} \right] f(y) dy \\
&= -\int_0^{\underline{\theta}} [x]\, f(x) dx - \int_0^{\underline{\theta}} [F(x)]\, dx + \int_{\overline{\theta}}^1 [y]\, f(y) dy - \int_{\overline{\theta}}^1 [1 - F(y)]\, dy \\
&= -\left( [xF(x)]_0^{\underline{\theta}} - \int_0^{\underline{\theta}} [F(x)]\, dx \right) - \int_0^{\underline{\theta}} [F(x)]\, dx \\
&+ \left( [yF(y)]_{\overline{\theta}}^1 - \int_{\overline{\theta}}^1 [F(y)]\, dy \right) - \int_{\overline{\theta}}^1 [1 - F(y)]\, dy \\
&= -\underline{\theta} F(\underline{\theta}) + 1 - \overline{\theta} F(\overline{\theta}) - \int_{\overline{\theta}}^1 [F(y)]\, dy - (1 - \overline{\theta}) + \int_{\overline{\theta}}^1 [F(y)]\, dy \\
&= -\underline{\theta} F(\underline{\theta}) + 1 - \overline{\theta} F(\overline{\theta}) - 1 + \overline{\theta} \\
&= -\underline{\theta} F(\underline{\theta}) - \overline{\theta} F(\overline{\theta}) + \overline{\theta}
\end{aligned}
$$

# B Simplifying the Profit Function

Here we simplify the profit function.

$$
\Pi_{\underline{\theta},\overline{\theta}} = \int_0^{\underline{\theta}} t(\theta) f(\theta) d\theta + \int_{\overline{\theta}}^1 t(\theta) f(\theta) d\theta.
$$

We will study each integral separately. We start with the first one.

$$\int_0^{\underline{\theta}} t(\theta) f(\theta) d\theta \tag{1}$$

$$= \int_0^{\underline{\theta}} \left[ \theta q(\theta) - u^m(0) - \int_0^{\theta} q(x)dx \right] f(\theta)d\theta \tag{2}$$

$$= \int_0^{\underline{\theta}} \left[ \theta q(\theta) - u^m(0) \right] f(\theta)d\theta - \int_0^{\underline{\theta}} \int_0^{\theta} q(x) f(\theta) dx d\theta \tag{3}$$

$$= \int_0^{\underline{\theta}} \left[ \theta q(\theta) - u^m(0) \right] f(\theta)d\theta - \int_0^{\underline{\theta}} \int_x^{\underline{\theta}} q(x) f(\theta) d\theta dx \tag{4}$$

$$= \int_0^{\underline{\theta}} \left[ \theta q(\theta) - u^m(0) \right] f(\theta)d\theta - \int_0^{\underline{\theta}} q(x) \int_x^{\underline{\theta}} f(\theta) d\theta dx \tag{5}$$

$$= \int_0^{\underline{\theta}} \left[ \theta q(\theta) - u^m(0) \right] f(\theta)d\theta - \int_0^{\underline{\theta}} q(x)(F(\underline{\theta}) - F(x))dx \tag{6}$$

$$= \int_0^{\underline{\theta}} \left[ -u^m(0) + \left( x - \frac{F(\underline{\theta}) - F(x)}{f(x)} \right) q(x) \right] f(x)dx \tag{7}$$

$$= \int_0^{\underline{\theta}} \left[ -u^d(\underline{\theta}) + \int_0^{\underline{\theta}} q(y)dy + \left( x - \frac{F(\underline{\theta}) - F(x)}{f(x)} \right) q(x) \right] f(x)dx \tag{8}$$

$$= \int_0^{\underline{\theta}} F(\underline{\theta}) q(y) \frac{f(y)}{f(y)} dy + \int_0^{\underline{\theta}} \left[ -u^d(\underline{\theta}) + \left( x - \frac{F(\underline{\theta}) - F(x)}{f(x)} \right) q(x) \right] f(x)dx \tag{9}$$

$$= \int_0^{\underline{\theta}} \left[ -u^d(\underline{\theta}) + \left( x + \frac{F(x)}{f(x)} \right) q(x) \right] f(x)dx \tag{10}$$

$$= -F(\underline{\theta}) u^d(\underline{\theta}) + \int_0^{\underline{\theta}} \left[ \left( x + \frac{F(x)}{f(x)} \right) q(x) \right] f(x)dx \tag{11}$$

In line 4, we change the order of integration; in line 5, we isolate the inner integral by extracting the allocations out; in line 6, we replace the value of the inner integral; in line 7, we merge the sum back; in line 8, we replace the value of the utility of the lowest type; in line 9, we integrate out the information rent for these types; in line 10, we cancel the new double integral with the $-\underline{\theta} q(x)$

as that integral turns out to be just the integral of $\underline{\theta}q(x)$ by changing the order of integration as above. We follow the similar steps for the transfers from $[\overline{\theta}, 1]$.

$$\int_{\overline{\theta}}^{1} t(\theta)f(\theta)d\theta$$
$$= \int_{\overline{\theta}}^{1} \left[\theta q(\theta) - u^m(\overline{\theta}) - \int_{\overline{\theta}}^{\theta} q(x)dx\right] f(\theta)d\theta$$
$$= \int_{\overline{\theta}}^{1} \left[\theta q(\theta) - u^m(\overline{\theta})\right] f(\theta)d\theta - \int_{\overline{\theta}}^{1}\int_{\overline{\theta}}^{\theta} q(x)f(\theta)dxd\theta$$
$$= \int_{\overline{\theta}}^{1} \left[\theta q(\theta) - u^m(\overline{\theta})\right] f(\theta)d\theta - \int_{\overline{\theta}}^{1}\int_{x}^{1} q(x)f(\theta)d\theta dx$$
$$= \int_{\overline{\theta}}^{1} \left[\theta q(\theta) - u^m(\overline{\theta})\right] f(\theta)d\theta - \int_{\overline{\theta}}^{1} q(x)\int_{x}^{1} f(\theta)d\theta dx$$
$$= \int_{\overline{\theta}}^{1} \left[\theta q(\theta) - u^m(\overline{\theta})\right] f(\theta)d\theta - \int_{\overline{\theta}}^{1} q(x)(1 - F(x))dx$$
$$= \int_{\overline{\theta}}^{1} \left[-u^m(\overline{\theta}) + \left(x - \frac{1 - F(x)}{f(x)}\right) q(x)\right] f(x)dx$$
$$= -(1 - F(\overline{\theta}))u^d(\overline{\theta}) + \int_{\overline{\theta}}^{1} \left[\left(x - \frac{1 - F(x)}{f(x)}\right) q(x)\right] f(x)dx$$

# C Profit from the simple equilibrium

From the Appendix A, we have

$$-\int_{0}^{\underline{\theta}} C(x)f(x)dx + \int_{\overline{\theta}}^{1} V(y)f(y)dy = -\underline{\theta}F(\underline{\theta}) - \overline{\theta}F(\overline{\theta}) + \overline{\theta}$$

Moreover, under a CRS matching function $M$ for the search market, $p$ is independent of the segmentation of the market. Then, the compensations paid to the agents will be given by:

$$
\begin{aligned}
&F(\underline{\theta})u^d(\underline{\theta}) + (1 - F(\overline{\theta}))u^d(\overline{\theta}) \\
&= p\frac{F(\underline{\theta})}{2}\left[\frac{\int_{\underline{\theta}}^{\overline{\theta}} \theta f(\theta)d\theta}{F(\overline{\theta}) - F(\underline{\theta})} - \underline{\theta}\right] + p\frac{1 - F(\overline{\theta})}{2}\left[\overline{\theta} - \frac{\int_{\underline{\theta}}^{\overline{\theta}} \theta f(\theta)d\theta}{F(\overline{\theta}) - F(\underline{\theta})}\right] \\
&= p\frac{1}{2}\left(F(\underline{\theta})\left[\frac{\int_{\underline{\theta}}^{\overline{\theta}} \theta f(\theta)d\theta}{F(\overline{\theta}) - F(\underline{\theta})} - \underline{\theta}\right] + (1 - F(\overline{\theta}))\left[\overline{\theta} - \frac{\int_{\underline{\theta}}^{\overline{\theta}} \theta f(\theta)d\theta}{F(\overline{\theta}) - F(\underline{\theta})}\right]\right) \\
&= \frac{p}{2}\left(F(\underline{\theta})E[\theta|\underline{\theta} \le \theta \le \overline{\theta}] - \underline{\theta}F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta})) - (1 - F(\overline{\theta}))E[\theta|\underline{\theta} \le \theta \le \overline{\theta}]\right) \\
&= \frac{p}{2}\left(\left(F(\underline{\theta}) - (1 - F(\overline{\theta}))\right)E[\theta|\underline{\theta} \le \theta \le \overline{\theta}] - \underline{\theta}F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta}))\right)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\Pi \\
&= -\frac{p}{2}\left(\left(F(\underline{\theta}) - (1 - F(\overline{\theta}))\right)E[\theta|\underline{\theta} \le \theta \le \overline{\theta}] - \underline{\theta}F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta}))\right) + \left[-\underline{\theta}F(\underline{\theta}) - \overline{\theta}F(\overline{\theta}) + \overline{\theta}\right] \\
&= \frac{1}{2}\left((2 - p)\left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta}))\right] - p\left[F(\underline{\theta}) - (1 - F(\overline{\theta}))\right]E[\theta|\underline{\theta} \le \theta \le \overline{\theta}]\right)
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\mathbb{E}[\theta|\theta \in [\underline{\theta}, \overline{\theta}]] &= \frac{\int_{\underline{\theta}}^{\overline{\theta}} x f(x) dx}{F(\overline{\theta}) - F(\underline{\theta})} \\
\frac{\partial \mathbb{E}[\theta|\theta \in [\underline{\theta}, \overline{\theta}]]}{\partial \underline{\theta}} &= \frac{-\underline{\theta} f(\underline{\theta})[F(\overline{\theta}) - F(\underline{\theta})] + f(\underline{\theta}) \left[\int_{\underline{\theta}}^{\overline{\theta}} x f(x) dx\right]}{[F(\overline{\theta}) - F(\underline{\theta})]^2} \\
&= f(\underline{\theta}) \frac{\int_{\underline{\theta}}^{\overline{\theta}} x f(x) dx - \underline{\theta}[F(\overline{\theta}) - F(\underline{\theta})]}{[F(\overline{\theta}) - F(\underline{\theta})]^2} \\
&= f(\underline{\theta}) \frac{\mathbb{E}[\theta|\theta \in [\underline{\theta}, \overline{\theta}]] - \underline{\theta}}{[F(\overline{\theta}) - F(\underline{\theta})]} > 0.
\end{aligned}
$$

Then, using this,

$$
\begin{aligned}
&2\frac{\partial \Pi}{\partial \underline{\theta}} \\
&= -(2-p)F(\underline{\theta}) - (2-p)\underline{\theta} f(\underline{\theta}) - p[f(\underline{\theta})]E[\theta|\underline{\theta} \leq \theta \leq \overline{\theta}] - p\left[F(\underline{\theta}) - (1 - F(\overline{\theta}))\right] \frac{\partial \mathbb{E}[\theta|\theta \in [\underline{\theta}, \overline{\theta}]]}{\partial \underline{\theta}}
\end{aligned}
$$

For a feasible mechanism, we need $F(\underline{\theta}) - (1 - F(\overline{\theta})) \geq 0$. Thus, each term above is negative, and $\Pi$ is decreasing in $\underline{\theta}$. Given that $\Pi$ is decreasing in $\underline{\theta}$, $F(\underline{\theta}) - (1 - F(\overline{\theta})) \geq 0$ will bind in any equilibrium. Therefore,

$$
\begin{aligned}
\Pi &= \frac{1}{2}(2-p)\left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1 - F(\overline{\theta}))\right] \\
&= \frac{2-p}{2}\left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(F(\underline{\theta}))\right] \\
&= \frac{2-p}{2}\left[F(\underline{\theta})[\overline{\theta} - \underline{\theta}]\right]
\end{aligned}
$$

Notice that for interior values with $\overline{\theta} > \underline{\theta}$, $\Pi > 0$. Thus, positive profit is feasible and will be achieved in the equilibrium.

## C.1 The Mechanism That Induces The Simple Equilibrium

*Proof of Proposition 3.2.* First, we compute the transfers for the agents who join the mechanism using the binding IR constraints for $\underline{\theta}$ and $\overline{\theta}$.

In the simple equilibrium we construct, for an agent with $\theta \in [0, \underline{\theta}]$,

$$\begin{aligned} t(\theta) &= \theta q(\theta) - u^m(0) - \int_0^\theta q(x)dx \\ &= \theta(-1) - u^m(0) - \theta(-1) \\ &= -u^m(0) = -u^d(\underline{\theta}) + \int_0^{\underline{\theta}} q(x)dx = -u^d(\underline{\theta}) + \int_0^{\underline{\theta}} (-1)dx \\ &= -\frac{p}{2}\left[E[x|x \in [\underline{\theta}, \overline{\theta}]] - \underline{\theta}\right] - \underline{\theta} \\ &= -\frac{p}{2}E[x|x \in [\underline{\theta}, \overline{\theta}]] - \underline{\theta}\frac{2-p}{2} \end{aligned}$$

Similarly, we can compute the transfer of agents with $\theta \in [\overline{\theta}, 1]$,

$$\begin{aligned} t(\theta) &= \theta q(\theta) - u^m(\overline{\theta}) - \int_{\overline{\theta}}^\theta q(x)dx \\ &= \theta(1) - u^m(\overline{\theta}) - 1(\theta - \overline{\theta}) \\ &= \overline{\theta} - u^m(\overline{\theta}) = \overline{\theta} - \frac{p}{2}\left[\overline{\theta} - E[x|x \in [\underline{\theta}, \overline{\theta}]]\right] \\ &= \frac{p}{2}E[x|x \in [\underline{\theta}, \overline{\theta}]] + \overline{\theta}\frac{2-p}{2} \end{aligned}$$

Knowing these, the designer can offer $-1$ allocation to all agents whose valuations are below $\frac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\underline{\theta}}{2}$. Similarly, for agents with valuations above $\frac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\overline{\theta}}{2}$, 1 unit of allocation can be offered. In between, they aren't offered any trade. This allocation is clearly increasing. Moreover, accompanied by the transfers $-\frac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\underline{\theta}}{2}$ for agents with negative allocations and $\frac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\overline{\theta}}{2}$ for agents with positive allocations, agents with values in $(\underline{\theta}, \overline{\theta})$ would strictly prefer the search market. To see this, note that these agents have utilities with slopes

$-1$ until $u^m$ hits 0, then it is constant at 0 and then it has the slope 1, after $\frac{p\mathbb{E}[x|x\in[\underline{\theta},\overline{\theta}]]+(2-p)\overline{\theta}}{2}$. Moreover, $u^m$ and $u^d$ are equal at $\underline{\theta}$ and $\overline{\theta}$. Since the slope of $u^d$ is bounded between $-\frac{p}{2}$ and $\frac{p}{2}$, and $u^d$ is positive, $u^m$ and $u^d$ cannot cross each other at any point other than $\underline{\theta}$ and $\overline{\theta}$. Thus, $u^d$ remains below $u^m$ for values in $(\underline{\theta},\overline{\theta})$.

□

# D Efficiency of Coexistence

## D.1 Under Uniform Distribution

*Proof of Proposition 3.4.* Suppose everyone is in the search market. Then, the total welfare can be computed as follows:

$$\begin{aligned}\mathbb{E}[u^d(\theta)] &= p\int_0^1 [\theta\theta-(1-\theta)\theta]d\theta \\ &= p\int_0^1 [2\theta^2-\theta]d\theta \\ &= p\left[\frac{2\theta^3}{3}-\frac{\theta^2}{2}\right]_0^1 = \frac{p}{6}.\end{aligned}$$

Next, we compute the welfare created by the marketplace alone in the coexistence equilibrium. The welfare marketplace generates will be more than enough to exceed the total welfare of the pure search market, so we don't need to compute the welfare created by the search market in the coexistence.

The profit function from the simple equilibrium under the uniform distribution is a constant times $\underline{\theta}(\overline{\theta}-\underline{\theta}) = \underline{\theta}(1-2\underline{\theta})$ using the fact that the feasibility binds so that $\underline{\theta} = 1-\overline{\theta}$. This is maximized at $\underline{\theta} = \frac{1}{4}$ and $\overline{\theta} = \frac{3}{4}$. Then, the welfare generated by the marketplace is

$$\int_{0.75}^1 \theta_b d\theta_b - \int_0^{0.25} \theta_s d\theta_s = \left[\frac{\theta^2}{2}\right]_{0.75}^1 - \left[\frac{\theta^2}{2}\right]_0^{0.25} = \frac{3}{16}.$$

The total welfare from the search market is $\frac{p}{6} \le \frac{1}{6}$ for any matching function since the probability a meeting will be less than or equal to 1. Moreover, $\frac{3}{16} > \frac{1}{6}$. Thus, for any matching function, the coexistence equilibrium creates a welfare higher than the pure search market. □

## D.2 Under General Distribution

*Proof of Proposition 3.5.* For the pure search market, the total welfare created is given by

$$\begin{aligned}&\int_0^1 \left[pF(\theta)\theta - p(1-F(\theta))\theta\right] f(\theta)d\theta \\ =&p\int_0^1 \theta\left[2F(\theta)-1\right] f(\theta)d\theta\end{aligned}$$

In the first line above, $pF(\theta)$ is the probability that the agent with the value $\theta$ meets with an agent with a value less than $\theta$, so she gets $\theta$ in the trade and $p(1-F(\theta))$ is the probability that she meets with an agent whose value is higher so she loses $\theta$. As with the uniform case, we ignore the transfers in the utilities as the transfers will cancel in the search market.

In the coexistence equilibrium, the total welfare created in the search market is

$$\begin{aligned}&\int_0^1 \left[p\left[\frac{F(\theta)-F(\underline{\theta})}{F(\overline{\theta})-F(\underline{\theta})}\right]\theta - p\left[\frac{F(\overline{\theta})-F(\theta)}{F(\overline{\theta})-F(\underline{\theta})}\right]\theta\right] f(\theta)d\theta \\ =&p\int_{\underline{\theta}}^{\overline{\theta}} \theta\left[\frac{2F(\theta)-F(\underline{\theta})-F(\overline{\theta})}{F(\overline{\theta})-F(\underline{\theta})}\right] f(\theta)d\theta \\ =&p\int_{\underline{\theta}}^{\overline{\theta}} \theta\left[\frac{2F(\theta)-1}{F(\overline{\theta})-F(\underline{\theta})}\right] f(\theta)d\theta\end{aligned}$$

The welfare generated by the marketplace in the coexistence is

$$\int_{\overline{\theta}}^1 xf(x)dx - \int_0^{\underline{\theta}} xf(x)dx$$

$$
\begin{aligned}
& p\int_{\underline{\theta}}^{\overline{\theta}} \theta\left[\frac{2F(\theta)-1}{F(\overline{\theta})-F(\underline{\theta})}\right] f(\theta)d\theta + \int_{\overline{\theta}}^{1} xf(x)dx - \int_{0}^{\underline{\theta}} xf(x)dx \geq p\int_{0}^{1}\theta\left[2F(\theta)-1\right]f(\theta)d\theta \\
\Longleftrightarrow\ & p\int_{\underline{\theta}}^{\overline{\theta}} \theta\left[\frac{2F(\theta)}{F(\overline{\theta})-F(\underline{\theta})}\right] f(\theta)d\theta + \int_{\overline{\theta}}^{1} xf(x)dx + p\int_{0}^{1}\theta f(\theta)d\theta \\
& \geq p\int_{\underline{\theta}}^{\overline{\theta}}\left[\frac{\theta f(\theta)d\theta}{F(\overline{\theta})-F(\underline{\theta})}\right] + p\int_{0}^{1}\theta\left[2F(\theta)\right]f(\theta)d\theta + \int_{0}^{\underline{\theta}} xf(x)dx \\
\Longleftrightarrow\ & p\int_{\underline{\theta}}^{\overline{\theta}} \theta\left[\frac{2F(\theta)}{F(\overline{\theta})-F(\underline{\theta})}\right] f(\theta)d\theta + (1+p)\int_{\overline{\theta}}^{1}\theta f(\theta)d\theta \\
& \geq p\left[\frac{2F(\underline{\theta})}{1-2F(\underline{\theta})}\right]\int_{\underline{\theta}}^{\overline{\theta}}\theta f(\theta)d\theta + 2p\int_{0}^{1}\theta\left[F(\theta)\right]f(\theta)d\theta + (1-p)\int_{0}^{\underline{\theta}} xf(x)dx
\end{aligned}
$$

In the first line above, either we have

$$
\int_{\underline{\theta}}^{\overline{\theta}} \theta\left[\frac{2F(\theta)-1}{F(\overline{\theta})-F(\underline{\theta})}\right] f(\theta)d\theta \geq \int_{0}^{1}\theta\left[2F(\theta)-1\right]f(\theta)d\theta
$$

in which case the inequality is satisfied for any $p$, since $\int_{\overline{\theta}}^{1} xf(x)dx - \int_{0}^{\underline{\theta}} xf(x)dx \geq 0$ when $F(\underline{\theta}) = 1 - F(\overline{\theta})$ or

$$
\int_{\underline{\theta}}^{\overline{\theta}} \theta\left[\frac{2F(\theta)-1}{F(\overline{\theta})-F(\underline{\theta})}\right] f(\theta)d\theta < \int_{0}^{1}\theta\left[2F(\theta)-1\right]f(\theta)d\theta.
$$

Thus, to show that for any $p$, the coexistence is more efficient, it is enough to show it with $p = 1$.

□

# E Slope of Utilities from Search Market

**Lemma E.1.** *Under any equilibrium,* $-\frac{p}{2} \leq \frac{\partial u^d(\theta)}{\partial\theta} \leq \frac{p}{2}$ *for each agent.*

*Proof.* Suppose $\Theta^d$ is the set of agents who join the decentralized market, $\mu(\Theta^d) = \mathbb{P}[x \in \Theta^d]$ their measure, and let $\theta \in Cov(\Theta^d)$. Then,

$$
\begin{aligned}
u^d(\theta) &= \frac{p}{2}\mathbb{P}[x > \theta | x \in \Theta^d] \left[\mathbb{E}[x|x > \theta, x \in \Theta^d] - \theta\right] \\
&+ \frac{p}{2}\mathbb{P}[x < \theta | x \in \Theta^d] \left[\theta - \mathbb{E}[x|x < \theta, x \in \Theta^d]\right] \\
&= \frac{p}{2}\frac{\mathbb{P}[x > \theta, x \in \Theta^d]}{\mu(\Theta^d)} \left[\mathbb{E}[x|x > \theta, x \in \Theta^d] - \theta\right] \\
&+ \frac{p}{2}\frac{\mathbb{P}[x < \theta, x \in \Theta^d]}{\mu(\Theta^d)} \left[\theta - \mathbb{E}[x|x < \theta, x \in \Theta^d]\right] \\
&= \frac{p}{2\mu(\Theta^d)}\left[\int\limits_{\{x\in\Theta^d : x>\theta\}} xf(x)dx - \theta\mathbb{P}[x > \theta, x \in \Theta^d]\right] \\
&+ \frac{p}{2\mu(\Theta^d)}\left[\theta\mathbb{P}[x < \theta, x \in \Theta^d] - \int\limits_{\{x\in\Theta^d : x<\theta\}} xf(x)dx\right] \\
\frac{\partial u^d(\theta)}{\partial\theta} &= \frac{p}{2\mu(\Theta^d)}\left[\mathbb{P}[x < \theta, x \in \Theta^d] - \mathbb{P}[x > \theta, x \in \Theta^d]\right].
\end{aligned}
$$

If $\theta \leq \theta'$ for each $\theta' \in \Theta^d$, then

$$
\begin{aligned}
u^d(\theta) &= \frac{p}{2}\mathbb{P}[x > \theta | x \in \Theta^d] \left[\mathbb{E}[x|x \in \Theta^d] - \theta\right] \\
&= \frac{p}{2}\left[\int\limits_{\{x\in\Theta^d\}} \frac{xf(x)dx}{\mu(\Theta^d)} - \theta]\right] \\
\frac{\partial u^d(\theta)}{\partial\theta} &= -\frac{p}{2}.
\end{aligned}
$$

Finally, if $\theta \geq \theta'$ for each $\theta' \in \Theta^d$, then

$$
\begin{aligned}
u^d(\theta) &= \frac{p}{2}\mathbb{P}[x < \theta | x \in \Theta^d] \left[\theta - \mathbb{E}[x|x \in \Theta^d]\right] \\
&= \frac{p}{2}\left[\theta - \int\limits_{\{x\in\Theta^d\}} \frac{xf(x)dx}{\mu(\Theta^d)}]\right] \\
\frac{\partial u^d(\theta)}{\partial\theta} &= \frac{p}{2}.
\end{aligned}
$$

□

## F Unrestricted Mechanisms

*Proof of Theorem 3.2.* We start with a simple observation: For the mechanism to make a positive profit, there has to be both agents who buy and sell at the marketplace. This means for a positive measure of agents, $u^m(\theta) \geq u^d(\theta)$ on both regions with $\frac{\partial u^m(\theta)}{\partial \theta} = 1$ and $\frac{\partial u^m(\theta)}{\partial \theta} = -1$.

We have shown in Appendix E that for an arbitrary segmentation of agents, the expected utility from search has a slope between $-0.5$ and $0.5$.

Notice that if for an agent $\frac{\partial u^m(\theta)}{\partial \theta} = 1$, then for each $\theta' > \theta$, $q(\theta') = 1$ by the envelope condition and monotonicity of the allocation for an IC mechanism. Similarly, if $\frac{\partial u^m(\theta)}{\partial \theta} = -1$, then for each $\theta' < \theta$, $q(\theta') = -1$.

Given this, if for $\theta$, $u^m(\theta) \geq u^d(\theta)$ and $\frac{\partial u^m(\theta)}{\partial \theta} = 1$, then for each $\theta' > \theta$, $u^m(\theta') \geq u^d(\theta')$ and similarly for the sellers. Thus, let $\underline{\theta}$ be the highest value such that $u^m(\theta) \geq u^d(\theta)$ and $q(\theta) = -1$ in the equilibrium. Similarly, let $\overline{\theta}$ be the lowest value such that $u^m(\theta) \geq u^d(\theta)$ and $q(\theta) = 1$.

There must be at least one type such that $u^m(\theta) = u^d(\theta)$. If not, either the utilities from search are above the utilities from the mechanism everywhere so that no one comes to the marketplace and the profit of the marketplace is zero or the utilities from the mechanism is strictly higher everywhere so everyone is in the mechanism and the mechanism can reduce the utilities until some IR constraint binds to strictly increase the profit.

Next, we argue that it cannot be the case that $u^m$ and $u^d$ are only tangent at $\underline{\theta}$ and $\overline{\theta}$, the utilities have to cross each other at these cutoffs: If they were only tangent but didn't cross each other, then $\underline{\theta}$ or $\overline{\theta}$ would have to be the point of a kink on $u^m$. Then, there are two cases: Either (i) every agent joins the mechanism or (ii) only one of the cutoffs is at a kink, and an interval of agents near the other cutoff join the search market. (i) In the former case, the outside option is zero for every agent so the mechanism doesn't compensate them. But then, they bilaterally deviate for a positive payoff. (ii) In the latter case, suppose there is a kink at $\overline{\theta}$. Then, if the search market is active, $u^d$ has to be increasing at $\overline{\theta}$ since everyone above it is in the mechanism so that an agent with the value $\overline{\theta}$ can only be a buyer in the search market. For there to be agents in the search market, $u^d$ should cross $u^m$ at a point $\theta$ such that $u^d$ is decreasing since it cannot cross $u^m$ on the part it is constant or has a slope of 1 below $\overline{\theta}$. But if there is such a point, then $u^d$ would be increasing at $\theta$, since all agents in the search market will be below it as well, which shows this case is impossible as well. Thus, $u^m$ and $u^d$ cannot be tangent at $\underline{\theta}$ and $\overline{\theta}$, they have to cross each other at these points.

Then, due to the shape of the feasible utility functions ($u^m$ can have slopes $-1$, 0, and 1 in this order and $u^d$ is first decreasing and then increasing -with a potentially constant 0 slope in the middle- with a slope that remains between $-\frac{1}{2}$ and $\frac{1}{2}$), either all agents with values in $[\underline{\theta}, \overline{\theta}]$ join the search market or the flat part of the $u^m$ crosses $u^d$ twice again, in which case agents with values in $[\underline{\theta}, a]$ and $[b, \overline{\theta}]$ join the search market for some $\underline{\theta} < a < b < \overline{\theta}$ and agents with values in $a, b$ join the mechanism as well. Moreover, in the latter case, we need $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$; otherwise the $u^d$ would be either strictly decreasing or strictly increasing for agents with values in $[a, b]$, in which case $u^d$ and $u^m$ wouldn't cross at both $a$ and $b$, as this case requires. When $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$, $u^d$ would be flat, as it can be seen from the slope we computed above. So, we can write the profit as follows where the case with $a = b$ corresponds to the situation where the flat part of $u^m$ doesn't cross $u^d$.

$$\Pi = -F(\underline{\theta})u^d(\underline{\theta}) - (1 - F(\overline{\theta}))u^d(\overline{\theta}) - (F(b) - F(a))u^d(a) - \int_0^{\underline{\theta}} C(x)f(x)dx + \int_{\overline{\theta}}^1 V(x)f(x)dx$$

Moreover, the constraints are $0 \le \underline{\theta} \le a \le b \le \overline{\theta} \le 1$, $F(\underline{\theta}) \ge 1 - F(\overline{\theta})$ and $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$.

Let $\Theta^d = [\underline{\theta}, a] \cup [b, \overline{\theta}]$. Then,

$$\Pi = -F(\underline{\theta})u^d(\underline{\theta}) - (1-F(\overline{\theta}))u^d(\overline{\theta}) - (F(b)-F(a))u^d(a) - \int_0^{\underline{\theta}} C(x)f(x)dx + \int_{\overline{\theta}}^1 V(x)f(x)dx$$

$$= -\frac{pF(\underline{\theta})}{2}\left[\int_{\{x\in\Theta^d\}} \frac{xf(x)dx}{\left[F(\overline{\theta}) - F(b) + F(a) - F(\underline{\theta})\right]} - \underline{\theta}\right]$$

$$-\frac{p(1-F(\overline{\theta}))}{2}\left[\overline{\theta} - \int_{\{x\in\Theta^d\}} \frac{xf(x)dx}{\left[F(\overline{\theta}) - F(b) + F(a) - F(\underline{\theta})\right]}\right]$$

$$-\frac{p(F(b)-F(a))}{2\left[F(\overline{\theta}) - F(b) + F(a) - F(\underline{\theta})\right]}\left[\int_b^{\overline{\theta}} xf(x)dx - \int_{\underline{\theta}}^a xf(x)dx\right] - \left[-\underline{\theta}F(\underline{\theta}) - \overline{\theta}F(\overline{\theta}) + \overline{\theta}\right]$$

$$= \frac{p\left[(1-F(\overline{\theta}) - F(\underline{\theta}))\right]}{2}\left[\int_{\{x\in\Theta^d\}} \frac{xf(x)dx}{\left[F(\overline{\theta}) - F(b) + F(a) - F(\underline{\theta})\right]}\right] + \frac{p(F(\underline{\theta})\underline{\theta} - (1-F(\overline{\theta}))\overline{\theta})}{4}$$

$$-\frac{p(F(b)-F(a))}{2\left[F(\overline{\theta}) - F(b) + F(a) - F(\underline{\theta})\right]}\left[\int_b^{\overline{\theta}} xf(x)dx - \int_{\underline{\theta}}^a xf(x)dx\right] + \frac{1}{2}\left[-\underline{\theta}F(\underline{\theta}) - \overline{\theta}F(\overline{\theta}) + \overline{\theta}\right]$$

$$\frac{\partial \Pi}{\partial \underline{\theta}} = \frac{p\left[(1-F(\overline{\theta}) - F(\underline{\theta}))\right]}{2}\left[\frac{-\underline{\theta}f(\underline{\theta})\left[F(\overline{\theta}) - F(b) + F(a) - F(\underline{\theta})\right] + f(\underline{\theta})\int_{\{x\in\Theta^d\}} xf(x)dx}{\left[F(\overline{\theta}) - F(b) + F(a) - F(\underline{\theta})\right]^2}\right]$$

$$-\frac{pf(\underline{\theta})}{2}\left[\int_{\{x\in\Theta^d\}} \frac{xf(x)dx}{\left[F(\overline{\theta}) - F(b) + F(a) - F(\underline{\theta})\right]}\right] + \frac{p(\underline{\theta}f(\underline{\theta}) + F(\underline{\theta}))}{2}$$

$$-\frac{p(F(b)-F(a))}{2\left[F(\overline{\theta}) - F(b) + F(a) - F(\underline{\theta})\right]}\left[\underline{\theta}f(\underline{\theta})\right]$$

$$-\frac{pf(\underline{\theta})(F(b)-F(a))}{2\left[F(\overline{\theta}) - F(b) + F(a) - F(\underline{\theta})\right]^2}\left[\int_b^{\overline{\theta}} xf(x)dx - \int_{\underline{\theta}}^a xf(x)dx\right] - (\underline{\theta}f(\underline{\theta}) + F(\underline{\theta}))$$

$$= \frac{pf(\underline{\theta})\left[(1-F(\overline{\theta}) - F(\underline{\theta}))\right]}{2}\left[\frac{\mathbb{E}[x|x\in\Theta^d] - \underline{\theta}}{\left[F(\overline{\theta}) - F(b) + F(a) - F(\underline{\theta})\right]}\right]$$

$$-\frac{pf(\underline{\theta})}{2}\left[\int_{\{x\in\Theta^d\}} \frac{xf(x)dx}{\left[F(\overline{\theta}) - F(b) + F(a) - F(\underline{\theta})\right]}\right] - \frac{p(F(b)-F(a))}{2\left[F(\overline{\theta}) - F(b) + F(a) - F(\underline{\theta})\right]}\left[\underline{\theta}f(\underline{\theta})\right]$$

$$-\frac{pf(\underline{\theta})(F(b)-F(a))}{2\left[F(\overline{\theta}) - F(b) + F(a) - F(\underline{\theta})\right]^2}\left[\int_b^{\overline{\theta}} xf(x)dx - \int_{\underline{\theta}}^a xf(x)dx\right] - \left[\underline{\theta}f(\underline{\theta}) + F(\underline{\theta})\right]\left[1 - \frac{p}{2}\right]$$

By noting that $(1 - F(\overline{\theta}) - F(\underline{\theta})) \leq 0$ by feasibility, each term in the above sum is negative and hence $\Pi$ is decreasing in $\underline{\theta}$. We will use this to show that the feasibility binds.

Next, we consider the Lagrangian problem to study the KKT conditions. Here, we will initially relax the problem by relaxing the equality constraint $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$ to $F(a) + F(b) \geq F(\underline{\theta}) + F(\overline{\theta})$ but focus on solutions where it binds. From here, we are going to learn that the feasibility constraint must bind. We will use this to observe that $a = b$ should hold in the equilibrium, which will reduce the unrestricted equilibrium to a simple equilibrium.

$$\mathcal{L}(\underline{\theta}, a, b, \overline{\theta}, \lambda) = \Pi + \lambda_1(F(\underline{\theta}) + F(\overline{\theta}) - 1) + \lambda_2(1 - \overline{\theta}) + \lambda_3(\overline{\theta} - b) + \lambda_4(b - a) + \lambda_5(a - \underline{\theta}) + \lambda_6\underline{\theta}$$
$$+\lambda_7(F(a) + F(b) - F(\underline{\theta}) - F(\overline{\theta}))$$
$$\frac{\partial \mathcal{L}}{\partial \underline{\theta}} = \frac{\partial \Pi}{\partial \underline{\theta}} + \lambda_1 f(\underline{\theta}) - \lambda_5 + \lambda_6 - \lambda_7 f(\underline{\theta}) = 0$$
$$\frac{\partial \mathcal{L}}{\partial a} = \frac{\partial \Pi}{\partial a} - \lambda_4 + \lambda_5 + \lambda_7 f(a) = 0$$
$$\frac{\partial \mathcal{L}}{\partial b} = \frac{\partial \Pi}{\partial b} - \lambda_3 + \lambda_4 + \lambda_7 f(b) = 0$$
$$\frac{\partial \mathcal{L}}{\partial \overline{\theta}} = \frac{\partial \Pi}{\partial \overline{\theta}} + \lambda_1 f(\overline{\theta}) - \lambda_3 + \lambda_4 - \lambda_7 f(\overline{\theta}) = 0$$
$$\lambda_i \geq 0$$
$$\lambda_1(F(\underline{\theta}) + F(\overline{\theta}) - 1) = 0$$
$$\lambda_2(1 - \overline{\theta}) = 0$$
$$\lambda_3(\overline{\theta} - b) = 0$$
$$\lambda_4(b - a) = 0$$
$$\lambda_5(a - \underline{\theta}) = 0$$
$$\lambda_6\underline{\theta} = 0$$
$$\lambda_7(F(a) + F(b) - F(\underline{\theta}) - F(\overline{\theta})) = 0.$$

First, we note that for $\Pi > 0$, we need $1 > \overline{\theta}$. Moreover, for $1 > \overline{\theta}$, we need $\underline{\theta} > 0$ by feasibility. Then, we have $\lambda_2 = \lambda_6 = 0$ by complementary slackness conditions.

Remember that for $\underline{\theta} > 0$, we have $\dfrac{\partial \Pi}{\partial \underline{\theta}} < 0$. Then, since $\lambda_6 = 0$ and $\lambda_5, \lambda_7 \geq 0$, for $\dfrac{\partial \mathcal{L}}{\partial \underline{\theta}} = 0$, we need $\lambda_1 > 0$. By complementary slackness, this implies the feasibility constraint must bind.

Then, the profit function becomes:

$$\Pi = -\frac{p(F(b)-F(a))}{2\left[F(\overline{\theta})-F(b)+F(a)-F(\underline{\theta})\right]}\left[\int_b^{\overline{\theta}} xf(x)dx - \int_{\underline{\theta}}^{a} xf(x)dx\right] + \frac{2-p}{2}\left[-\underline{\theta}F(\underline{\theta}) - \overline{\theta}F(\overline{\theta}) + \overline{\theta}\right].$$

Next we are going to argue that in any solution to the above problem with a positive profit, we must have $a = b$. Suppose $(\underline{\theta}, a, b, \overline{\theta})$ maximizes $\Pi$ and $b > a$. Remember that we reject any solution that doesn't satisfy $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$, since this is an equilibrium requirement. Then, the differences of integrals in the above equation is nonnegative. Moreover, it is strictly positive if $\Pi > 0$:

Notice that for $\Pi > 0$, we need $\overline{\theta} > \underline{\theta}$. If $\overline{\theta} = \underline{\theta}$, then it must be the case that $\overline{\theta} = a = b = \underline{\theta} = F^{-1}(0.5)$ and then we can verify that $\Pi = 0$. $\overline{\theta} > \underline{\theta}$ implies $\overline{\theta} > b$ and $a > \underline{\theta}$ because (i) we need $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$ in the equilibrium and (ii) $\overline{\theta} = b > a = \underline{\theta}$ cannot happen in the equilibrium as shown before stating the Lagrangian problem. But when we have $\overline{\theta} > b \geq a > \underline{\theta}$ and $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$, we have:

$$\begin{aligned}\left[\int_b^{\overline{\theta}} xf(x)dx - \int_{\underline{\theta}}^{a} xf(x)dx\right] &= \left[(F(\overline{\theta})-F(b))\int_b^{\overline{\theta}} \frac{xf(x)dx}{(F(\overline{\theta})-F(b))} - (F(a)-F(\underline{\theta}))\int_{\underline{\theta}}^{a} \frac{xf(x)dx}{(F(a)-F(\underline{\theta}))}\right] \\ &= (F(\overline{\theta})-F(b))\mathbb{E}[x|x\in[b,\overline{\theta}]] - (F(a)-F(\underline{\theta}))\mathbb{E}[x|x\in[\underline{\theta},a]] \\ &= (F(\overline{\theta})-F(b))\left[\mathbb{E}[x|x\in[b,\overline{\theta}]] - \mathbb{E}[x|x\in[\underline{\theta},a]]\right] > 0.\end{aligned}$$

Then, we must have $a = b$, since this doesn't effect the virtual surplus but minimizes the cost. Moreover, it must be the case that $F(a) = F(b) = F^{-1}(\frac{1}{2})$ since we need $F(a) - F(\underline{\theta}) = F(\overline{\theta}) - F(b)$ and the feasibility binds. Thus, we have reduced the unrestricted equilibrium to a simple equilibrium. □

# G Uniqueness of the Equilibrium

*Proof.* 3.6

1. Suppose all agents join the centralized market. Then, everyone's outside option is 0, since

no agent trades in the decentralized market. Thus, the marketplace must be operating the monagorastic mechanism to maximize the profit. Then, agents who don't get to trade get 0 utility. However, any two agent can profitably deviate to the decentralized market: They almost surely have different valuations and as long as they have different valuations, they have a positive surplus to share. Thus, it isn't a best response for them to stay in the marketplace.

2. Suppose all agents joined the decentralized market. We are going to show that there exist $a \leq \underline{\theta}$ and $b \geq \overline{\theta}$ such that agents in $[0, a) \cup (b, 1]$ strictly prefer joining the centralized marketplace instead.

   When all agents join the decentralized market,

$$u^d(0) = \frac{p}{2}\mathbb{E}\left[\theta | \theta \in [0, 1]\right]$$
$$u^d(1) = \frac{p}{2}\left[1 - \mathbb{E}\left[\theta | \theta \in [0, 1]\right]\right]$$

   Next, we compute the utilities the centralized marketplace promises them.

$$u^m(0) = \frac{p}{2}\left[\mathbb{E}\left[\theta | \theta \in [\underline{\theta}, \overline{\theta}]\right] - \underline{\theta}\right] + \underline{\theta}$$
$$u^m(1) = \frac{p}{2}\left[\overline{\theta} - \mathbb{E}\left[\theta | \theta \in [\underline{\theta}, \overline{\theta}]\right]\right] + 1 - \overline{\theta}$$

   Now, we check whether these two types can profitably deviate from the decentralized market to the marketplace:

$$\begin{aligned} & u^m(0) + u^m(1) > u^d(0) + u^d(1) \\ \iff \frac{p}{2}\left[-\underline{\theta}\right] + \underline{\theta} + \frac{p}{2}\left[\overline{\theta}\right] + 1 - \overline{\theta} &> \frac{p}{2}[1] \\ \iff \underline{\theta} + 1 - \overline{\theta} &> \frac{p}{2}\left[1 - \overline{\theta} + \underline{\theta}\right] \\ \iff 1 &> \frac{p}{2} \end{aligned}$$

In fact, since both $u^m$ and $u^d$ are continuous functions, the above inequality show that there are indeed $a, b$ such that pairs of agents with values in $[0, a)$ and in $(b, 1]$ strictly prefer the mechanism to the decentralized market. Thus, all agents joining the decentralized market cannot be an equilibrium.

3. Earlier, I have shown that the only possible coexistence equilibrium is the simple equilibrium, and that a simple equilibrium exists. Now, I have further shown that there is no equilibrium without coexistence. Thus, the unique equilibrium is the simple equilibrium.

□

# H Optimal Bids for Agents in the Marketplace

*Proof of Lemma 3.2.* Step 1: Bidding below $b(\underline{\theta})$ can never be optimal: For any bid $b \leq b(\underline{\theta})$, the agent sells her endowment with certainty but get paid less than she would get if she bid $b(\underline{\theta})$. Mathematically, the expected utility of an agent who bids $b \leq b(\underline{\theta})$ is given by:

$$\frac{1}{2}\int_{\underline{\theta}}^{\overline{\theta}} [b + b(x)]g(x)dx - \theta$$

Thus, bids strictly below $b(\underline{\theta})$ cannot be optimal.

Similarly, bids strictly above $b(\overline{\theta})$ cannot be optimal either. In that case, the agent's expected payoff would be

$$\theta - \frac{1}{2}\int_{\underline{\theta}}^{\overline{\theta}} [b + b(x)]g(x)dx$$

Hence, for each $\theta \in [0, \underline{\theta}] \cup [\overline{\theta}, 1]$, the optimal bid must be the bid of some type joins the decentralized market, that is: $b(\theta) \in [b(\underline{\theta}), b(\overline{\theta})]$.

Step 2: Let us first define the following notation: If an agent bids $b(\theta')$, then the expected price for selling is $p_s(\theta') = \frac{1}{2}\int_{\underline{\theta}}^{\theta'} \frac{[b(\theta')+b(x)]g(x)dx}{1-G(\theta')}$ and the expected price for buying is $p_b(\theta') = \frac{1}{2}\int_{\theta'}^{\overline{\theta}} \frac{[b(\theta')+b(x)]g(x)dx}{G(\theta')}$.

Since the best response of agent with value $\underline{\theta}$ is $b(\underline{\theta})$, her expected payoff from this bid should be higher than any other $b(\theta')$ by revealed preference. Then,

$$
\begin{aligned}
p_s(\underline{\theta}) - \underline{\theta} &\geq (1 - G(\theta')) \left[p_s(\theta') - \underline{\theta}\right] + G(\theta') \left[\underline{\theta} - p_b(\theta')\right] \\
&= \underline{\theta}[2G(\theta') - 1] + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \\
\iff p_s(\underline{\theta}) - (1 - G(\theta'))p_s(\theta') + G(\theta')p_b(\theta') &\geq 2\underline{\theta}G(\theta')
\end{aligned}
$$

Suppose $\theta \leq \underline{\theta}$. We want to show that bidding $b(\underline{\theta})$ gives a higher payoff than any other type's bid $b(\theta')$:

$$
\begin{aligned}
p_s(\underline{\theta}) - \theta &\geq (1 - G(\theta')) \left[p_s(\theta') - \theta\right] + G(\theta') \left[\underline{\theta} - p_b(\theta')\right] \\
&= \theta[2G(\theta') - 1] + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \\
p_s(\underline{\theta}) - (1 - G(\theta'))p_s(\theta') + G(\theta')p_b(\theta') &\geq 2\theta G(\theta')
\end{aligned}
$$

But this is true since $p_s(\underline{\theta}) - (1 - G(\theta'))p_s(\theta') + G(\theta')p_b(\theta') \geq 2\underline{\theta}G(\theta') \geq 2\theta G(\theta')$ where the first inequality follows from the revealed preference argument above and the second one follows from $\underline{\theta} \geq \theta$ and $G(\theta') \geq 0$.

Similarly, the best response of an agent with value $\underline{\theta}$ is $b(\overline{\theta})$. Thus,

$$
\begin{aligned}
\overline{\theta} - p_b(\overline{\theta}) &\geq (1 - G(\theta')) \left[p_s(\theta') - \overline{\theta}\right] + G(\theta') \left[\overline{\theta} - p_b(\theta')\right] \\
&= \overline{\theta}[2G(\theta') - 1] + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \\
\iff 2\overline{\theta}(1 - G(\theta')) &\geq p_b(\overline{\theta}) + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta')
\end{aligned}
$$

Suppose $\theta \geq \overline{\theta}$. In this case, we want to show that bidding $b(\overline{\theta})$ gives a higher payoff than any other type's bid $b(\theta')$:

$$
\begin{aligned}
\theta - p_b(\overline{\theta}) &\geq (1 - G(\theta'))\,[p_s(\theta') - \theta] + G(\theta')\,[\theta - p_b(\theta')] \\
&= \theta[2G(\theta') - 1] + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \\
\iff 2\theta(1 - G(\theta')) &\geq p_b(\overline{\theta}) + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta')
\end{aligned}
$$

Again, this is true since $2\theta(1-G(\theta')) \geq 2\overline{\theta}(1-G(\theta')) \geq p_b(\overline{\theta}) + (1-G(\theta'))p_s(\theta') - G(\theta')p_b(\theta')$ where the first inequality again follows from the revealed preference argument above and the second one follows from $\theta \geq \overline{\theta}$ and $1 - G(\theta') \geq 0$. □

# I Binding IR constraints

$$
\begin{aligned}
u^{da}(\underline{\theta}) &= p\left[\int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{2}[b(\underline{\theta}) + b(x)]g(x)dx - \underline{\theta}\right] \\
&= p\left[\frac{1}{2}\int_{\underline{\theta}}^{\overline{\theta}} [b(\underline{\theta}) + b(x)]g(x)dx - \underline{\theta}\right] \\
&= p\left[\frac{1}{2}\left[b(\underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right] - \underline{\theta}\right] \\
&= p\left[\frac{1}{2}\left[\underline{\theta} - \frac{\int_{G^{-1}(\frac{1}{2})}^{\underline{\theta}} \left[G(x) - \frac{1}{2}\right]^2 dx}{\left[G(\underline{\theta}) - \frac{1}{2}\right]^2} + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right] - \underline{\theta}\right] \\
&= \frac{p}{2}\left[-\underline{\theta} + 4\int_{\underline{\theta}}^{G^{-1}(\frac{1}{2})} \left[G(x) - \frac{1}{2}\right]^2 dx + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]
\end{aligned}
$$

Similarly, we obtain,

$$
\begin{aligned}
u^{da}(\overline{\theta}) &= p\left[\overline{\theta} - \int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{2}[b(\overline{\theta}) + b(x)]g(x)dx\right] \\
&= p\left[\overline{\theta} - \frac{1}{2}\left[b(\overline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]\right] \\
&= p\left[\overline{\theta} - \frac{1}{2}\left[\overline{\theta} - \frac{\int_{G^{-1}(\frac{1}{2})}^{\overline{\theta}} \left[G(x) - \frac{1}{2}\right]^2 dx}{\left[G(\overline{\theta}) - \frac{1}{2}\right]^2} + \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]\right] \\
&= \frac{p}{2}\left[\overline{\theta} + 4\int_{G^{-1}(\frac{1}{2})}^{\overline{\theta}} \left[G(x) - \frac{1}{2}\right]^2 dx - \int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx\right]
\end{aligned}
$$

# J Slope of Utilities from the Double Auction

*Proof of Lemma 3.3.* It is easy to verify that for agents with values less than $\underline{\theta}$, $\frac{\partial u^{da}(\theta)}{\partial \theta} = -p \geq -1$ and for agents with values above $\overline{\theta}$, $\frac{\partial u^{da}(\theta)}{\partial \theta} = p \leq 1$. Next we show that $\frac{\partial u^{da}(\theta)}{\partial \theta}$ is greater than $-1$ for $\underline{\theta}$ and less than 1 for $\overline{\theta}$.

First, we need the derivative of the bidding function:

$$b(\theta) = \theta - \frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2}$$

$$b'(\theta) = 1 - \frac{[G(\theta) - \frac{1}{2}]^4 - 2g(\theta)(G(\theta) - \frac{1}{2}) \int\limits_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^4}$$

$$= \frac{2g(\theta)(G(\theta) - \frac{1}{2}) \int\limits_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^4}$$

$$= 2g(\theta) \frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^3}$$

$$u^{da}(\theta) = p\theta[2G(\theta) - 1] + \frac{p}{2}(1 - 2G(\theta))b(\theta) + \frac{p}{2} \int\limits_{\theta}^{\overline{\theta}} b(x)g(x)dx - \frac{p}{2} \int\limits_{\underline{\theta}}^{\theta} b(x)g(x)dx$$

$$
\begin{aligned}
\frac{\partial u^{da}(\theta)}{\partial\theta} &= p\left[(2G(\theta)-1)+2\theta g(\theta)\right]+\frac{p}{2}\left[-2g(\theta)b(\theta)+(1-2G(\theta))b'(\theta)\right]-\frac{p}{2}2b(\theta)g(\theta) \\
&= p(2G(\theta)-1)+2pg(\theta)(\theta-b(\theta))+\frac{p}{2}\left[(1-2G(\theta))b'(\theta)\right] \\
&= p\left[G(\theta)-1+2g(\theta)(\theta-b(\theta))+\frac{1}{2}\left[(1-2G(\theta))b'(\theta)\right]\right] \\
&= p\left[G(\theta)-1+2g(\theta)\left[\theta-\theta+\frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta}[G(x)-\frac{1}{2}]^2dx}{[G(\theta)-\frac{1}{2}]^2}\right]+\frac{1}{2}(1-2G(\theta))b'(\theta)\right] \\
&= p\left[G(\theta)-1+2g(\theta)\frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta}[G(x)-\frac{1}{2}]^2dx}{[G(\theta)-\frac{1}{2}]^2}+\frac{1}{2}(1-2G(\theta))\left[2g(\theta)\frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta}[G(x)-\frac{1}{2}]^2dx}{[G(\theta)-\frac{1}{2}]^3}\right]\right] \\
&= p\left[G(\theta)-1+2g(\theta)\frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta}[G(x)-\frac{1}{2}]^2dx}{[G(\theta)-\frac{1}{2}]^2}-2g(\theta)\frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta}[G(x)-\frac{1}{2}]^2dx}{[G(\theta)-\frac{1}{2}]^2}\right] \\
&= p\left[G(\theta)-1\right]
\end{aligned}
$$

□

# K Bids with Uniform Distribution

$$b(\theta) = \theta - \frac{\int\limits_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2} = \theta - \frac{\int\limits_{\frac{\underline{\theta}+\overline{\theta}}{2}}^{\theta} \left[\frac{x - \underline{\theta}}{\overline{\theta} - \underline{\theta}} - \frac{1}{2}\right]^2 dx}{\left[\frac{\theta - \underline{\theta}}{\overline{\theta} - \underline{\theta}} - \frac{1}{2}\right]^2}$$

$$= \theta - \frac{\int\limits_{\frac{\underline{\theta}+\overline{\theta}}{2}}^{\theta} \left[\frac{2x - \underline{\theta} - \overline{\theta}}{2(\overline{\theta} - \underline{\theta})}\right]^2 dx}{\left[\frac{2\theta - \underline{\theta} - \overline{\theta}}{2(\overline{\theta} - \underline{\theta})}\right]^2} = \theta - \frac{\int\limits_{\frac{\underline{\theta}+\overline{\theta}}{2}}^{\theta} \left[2x - \underline{\theta} - \overline{\theta}\right]^2 dx}{\left[2\theta - \underline{\theta} - \overline{\theta}\right]^2}$$

$$= \theta - \frac{\left[\frac{1}{2\times 3}\left[2x - \underline{\theta} - \overline{\theta}\right]^3\right]_{\frac{\underline{\theta}+\overline{\theta}}{2}}^{\theta}}{\left[2\theta - \underline{\theta} - \overline{\theta}\right]^2} = \theta + \frac{1}{6}\frac{\left[2\theta - \underline{\theta} - \overline{\theta}\right]^3}{\left[2\theta - \underline{\theta} - \overline{\theta}\right]^2}$$

$$= \theta - \frac{2\theta - \underline{\theta} - \overline{\theta}}{6} = \frac{4\theta + \underline{\theta} + \overline{\theta}}{6}$$

Next, we compute another expression from the profit function:

$$\int\limits_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx = \int\limits_{\underline{\theta}}^{\overline{\theta}} \frac{4\theta + \underline{\theta} + \overline{\theta}}{6} \frac{1}{\overline{\theta} - \underline{\theta}} dx = \frac{1}{6(\overline{\theta} - \underline{\theta})} \int\limits_{\underline{\theta}}^{\overline{\theta}} (4\theta + \underline{\theta} + \overline{\theta}) dx$$

$$= \frac{1}{6(\overline{\theta} - \underline{\theta})}\left[2\overline{\theta}^2 - 2\underline{\theta}^2 + (\overline{\theta} + \underline{\theta})(\overline{\theta} - \underline{\theta})\right] = \frac{1}{6(\overline{\theta} - \underline{\theta})}\left[3(\overline{\theta} + \underline{\theta})(\overline{\theta} - \underline{\theta})\right] = \frac{\overline{\theta} + \underline{\theta}}{2}$$

# L Profit from Simple Equilibrium under Double Auction

$$
\begin{aligned}
\Pi_{\underline{\theta},\overline{\theta}} &= -\int_0^{\underline{\theta}} C(x)f(x)dx + \int_{\overline{\theta}}^{1} \mathcal{V}(x)f(x)dx - F(\underline{\theta})u^d(\underline{\theta}) - (1-F(\overline{\theta}))u^d(\overline{\theta}) \\
&= \left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1-F(\overline{\theta}))\right] - p\left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1-F(\overline{\theta}))\right] \\
&-\frac{p}{2}\left[\left[F(\underline{\theta}) + F(\overline{\theta}) - 1\right]\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx + F(\underline{\theta})b(\underline{\theta}) - (1-F(\overline{\theta}))b(\overline{\theta})\right] \\
&= (1-p)\left[-\underline{\theta}F(\underline{\theta}) + \overline{\theta}(1-F(\overline{\theta}))\right] \\
&-\frac{p}{2}\left[\left[F(\underline{\theta}) + F(\overline{\theta}) - 1\right]\int_{\underline{\theta}}^{\overline{\theta}} b(x)g(x)dx + F(\underline{\theta})b(\underline{\theta}) - (1-F(\overline{\theta}))b(\overline{\theta})\right] \\
&= (1-p)\left[-\underline{\theta}^2 + \overline{\theta}(1-\overline{\theta})\right] \\
&-\frac{p}{2}\left[\left[\underline{\theta} + \overline{\theta} - 1\right]\frac{\overline{\theta}+\underline{\theta}}{2} + \underline{\theta}\frac{5\underline{\theta}+\overline{\theta}}{6} - (1-\overline{\theta})\frac{5\overline{\theta}+\underline{\theta}}{6}\right]
\end{aligned}
$$

The profit is decreasing in $\underline{\theta}$:

$$
\begin{aligned}
\frac{\partial \Pi_{\underline{\theta},\overline{\theta}}}{\partial \underline{\theta}} &= (1-p)\left[-2\underline{\theta}\right] - \frac{p}{2}\left[\frac{\overline{\theta}+\underline{\theta}}{2} + \frac{1}{2}\left[\underline{\theta} + \overline{\theta} - 1\right] + \frac{5\underline{\theta}+\overline{\theta}}{6} + \underline{\theta}\frac{5}{6} - (1-\overline{\theta})\frac{1}{6}\right] \\
&= (1-p)\left[-2\underline{\theta}\right] - \frac{p}{2}\left[\frac{2}{3}(4\underline{\theta} + 2\overline{\theta} - 1)\right] \leq 0
\end{aligned}
$$

Notice that the first summand is negative and inside the brackets of the second summand is positive since $\underline{\theta} \geq 1 - \overline{\theta}$ by feasibility. Thus, the profit is decreasing in $\underline{\theta}$. Hence, the feasibility binds and we have $\underline{\theta} = 1 - \overline{\theta}$, otherwise decreasing $\underline{\theta}$ until the feasibility binds strictly increases the profit. Then, we have

$$
\begin{aligned}
\Pi_{\underline{\theta},\overline{\theta}} &= (1-p)\left[-\underline{\theta}^2 + \overline{\theta}(1-\overline{\theta})\right] - \frac{p}{2}\left[\underline{\theta}\frac{5\underline{\theta}+\overline{\theta}}{6} - (1-\overline{\theta})\frac{5\overline{\theta}+\underline{\theta}}{6}\right] \\
&= (1-p)\left[-\underline{\theta}^2 + \overline{\theta}(1-\overline{\theta})\right] - \frac{p}{2}\left[\underline{\theta}\frac{5\underline{\theta}+\overline{\theta}-5\overline{\theta}-\underline{\theta}}{6}\right] \\
&= (1-p)\left[-\underline{\theta}^2 + (1-\underline{\theta})\underline{\theta}\right] - \frac{p}{2}\left[\underline{\theta}\frac{5\underline{\theta}+(1-\underline{\theta})-5(1-\underline{\theta})-\underline{\theta}}{6}\right] \\
&= (1-p)\left[\underline{\theta}(1-2\underline{\theta})\right] - \frac{p}{2}\left[\underline{\theta}\frac{8\underline{\theta}-4}{6}\right] = (1-p)\left[\underline{\theta}(1-2\underline{\theta})\right] + \frac{p}{6}\left[\underline{\theta}(1-2\underline{\theta})\right] \\
&\frac{6-5p}{6}\left[\underline{\theta}(1-2\underline{\theta})\right] = \frac{6-5p}{6}\Pi^M
\end{aligned}
$$