Optimal Marketplace Design

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Abstract

Online marketplaces that provide a venue to trade goods and services conveniently, such as Airbnb, Ebay, Etsy and Uber, have seen a rapid growth in the last decade. When these marketplaces decide the rules of trade, their goal is to maximize their own profit; not the profit of the sellers or welfare of the consumers. We employ mechanism design approach to characterize the optimal rules of a marketplace that maximizes profit in an environment where agents arrive with endowments of a good. In the optimal marketplace, agents are ranked according to their virtual values from top to bottom and according to their virtual costs from bottom to top. Then, we use an algorithm to find the optimal allocation rule. The agents' virtual values and costs also provide a simple interpretation as virtual demand and supply. We study the welfare effects by comparing the welfare created by the platform to that of a decentralized market, and conclude that the marketplaces do not necessarily decrease the welfare. Lastly, we study a model where agents can choose between a platform and a decentralized market. In this model, we construct some equilibria where agents are divided between the platform and the decentralized market as well as equilibria where all agents join either the platform or the decentralized market.

1 Introduction

Stock exchange is an age-old example of a market where each participant can be a buyer or a seller, depending on the state of the market. However, as the internet became an essential part of life, online marketplaces such as Airbnb, Ebay, Etsy and Uber have also grown rapidly. Noticeably, all of these platforms are designed to maximize their own profits, which is different from how the markets have been studied traditionally. The common current approaches in the market design literature are to consider models where either a seller can choose the rules of trade [Myerson, 1981] to maximize their own profit or a social planner can choose the rules of trade to maximize the social surplus [Myerson and Satterthwaite, 1983]. However, these models are insufficient to understand platforms that set the terms of the trade to maximize their own profits.

In this study, we develop models to gain a better understanding of trading venues where each participant can be a buyer or a seller, depending on the state of the market. In the main model, we consider a setup with a single good where each agent has some endowment, which is common knowledge, and a valuation for at most one unit of the good; valuation is the private information of the agent. A mechanism designer chooses an ex-post individually rational mechanism to maximize profit. By revelation principle, without loss, we focus on direct mechanisms that are dominant strategy incentive compatible. In this environment, we show that the revenue maximizing allocation rule ranks the agents according to their virtual values and virtual costs. The optimal allocation only allows a transaction from an agent to another if one of their virtual values higher than the other's virtual cost. We describe an algorithm that would determine the optimal allocation for profit maximization according to this principle, and the transfers are pinned down by an interplay of incentive compatibility, individual rationality and profit maximization. One interesting remark is that the utility of an agent is not increasing in her type, unlike the common observation in the literature. This also leads to a distinct revenue equivalence result.

Next, by example, we show that this mechanism can increase or decrease the social welfare compared to a market where a form of double auction takes place between randomly matched agents. Finally, we extend our model by giving the agents an option to choose between a decentralized market or to join the platform after the terms of trade in the platform is announced. We construct equilibria where some types of agents choose the decentralized market and others choose the platform as well as equilibria where all agents join either the decentralized market or the platform.

1.1 The Simple Economics of Optimal Marketplaces

Here we illustrate the revenue-maximizing allocation rule using an analysis similar to [Bulow and Roberts, 1989]. We have an environment with n agents and each agent i draws a valuation θ_i for the good from some distribution F_i. For simplicity, suppose each agent has the endowment $e_i = 0.5$ so that each of them either wants to sell 0.5 unit of the good or buy the same amount. The principal chooses a mechanism knowing the distributions and the endowments. Invoking the revelation principle, we can restrict the attention to direct, dominant strategy incentive compatible and ex-post individually rational mechanisms.

When we study the model more formally later on in the paper, we will use a series of lemmata and propositions to simplify the principal's problem. In its most simple form, the objective function becomes the expected sum of each agent's virtual value or virtual cost, depending on whether the agent is ultimately assigned the role of a buyer or a seller. Thus, we will define these two functions here without motivating them further; why they emerge this way will become clear later. Let the virtual values and costs be

$$B_i(\theta_i) = \left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}\right) \text{ and } S_i(\theta_i) = \left(\theta_i + \frac{F_i(\theta_i)}{f_i(\theta_i)}\right).$$

We assume that they are both increasing functions for each agent. We will now use these functions to define the virtual (inverse) demand and supply functions. We rank the agents' virtual values highest to lowest and denote them $b_1, b_2, \ldots, b_n$. Similarly, we rank them according to the virtual costs but from the lowest to the highest this time and denote those numbers as $s_1, s_2, \ldots, s_n$. We define the virtual demand D and supply S as

$$D(q) = b_i \text{ if } \frac{i-1}{2} < q \leq \frac{i}{2} \text{ and } S(q) = s_i \text{ if } \frac{i-1}{2} < q \leq \frac{i}{2}.$$

This is in line with the way inverse demand and supply functions are taught as nonincreasing and nondecreasing functions of quantity in introductory level courses. However, here they are step functions with steps of length 0.5 since there are finitely many agents with 0.5 unit of demand and supply each.

We can now express a simple version of our main theorem (2.1) using the virtual demand and supply functions.

Theorem 1.1. *The allocation rule associated with the mechanism that maximizes the profit under incentive compatibility and individual rationality is as follows:*

If $B(0.5) \leq S(0.5)$*, no trade takes place.*

If $B(0.5) > S(0.5)$*, then the total volume of trade is given by*

$$q^* = \max_{q \in [0, 0.5n]} \{q | B(q) > S(q)\}.$$

Moreover, each agent with a virtual value greater than $B(q^*)$ *buys* 0.5 *unit; each agent with a virtual cost less than* $S(q^*)$ *sells* 0.5 *unit; the rest do not trade.*

Our Theorem 2.1 provides more details about the case where the agents have heterogeneous endowments as well as providing the transfers they would pay and receive.

We will now provide a concrete example to see how the allocation rule would work. Suppose there are 4 agents. For the sake of simplicity, we assume that each agent i's valuation θ_i is drawn from the uniform distribution on $[0, 1]$. In this case, virtual values and costs are given by $2\theta_i - 1$ and $2\theta_i$ respectively. Suppose the principal receives the reports $0.2, 0.4, 0.6, 0.8$. This would translate to the virtual values $-0.6, -0.2, 0.2, 0.6$ and virtual costs $0.4, 0.8, 1.2, 1.6$. If we plot the supply and demand, we would get Figure 1 (left). The theorem above implies that in the profit-maximizing mechanism, total quantity traded is 0.5 and agent whose value is 0.8 buys 0.5 unit, agent whose value is 0.2 sells 0.5 and the other agents do not participate in any trade. For instance, if the agent whose valuation is 1.6 valued the good at 1.2, then the corresponding plot is provided in Figure 1 (right). Since the highest value of D is lower than the lowest value of S, in the optimal mechanism, there would not be any trade with these realizations.

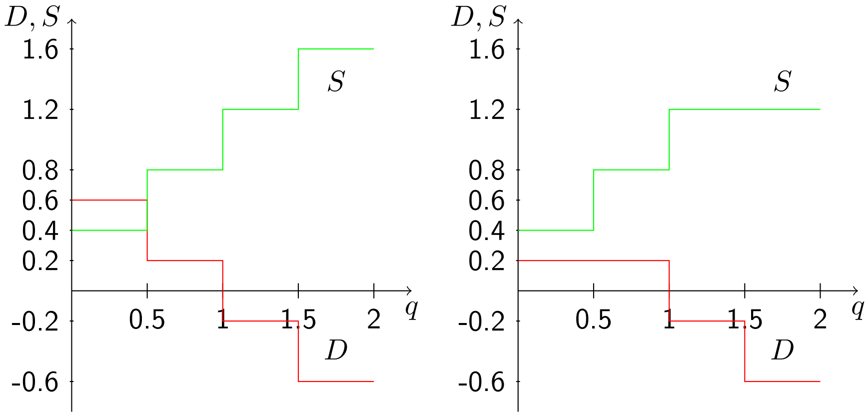

Figure 1: Demand and Supply curves given realization $\{0.2, 0.4, 0.6, 0.8\}$ (left) and $\{0.2, 0.4, 0.6, 0.8\}$ (right). In the right, D is always below S, hence the theorem tells that there would not be any trade in this instance. However, in the left, D is above S up to $q = 0.5$. Hence, according to the theorem, exactly 0.5 unit of good should be traded in the platform.

1.2 Literature Review

There is a vast literature on designing optimal auctions from the perspective of the seller. Most notably, [Myerson, 1981] established many of the standard tools in that literature. Although our focus is different, we follow a similar methodology in that we invoke the revelation principle, establish a new version of payoff equivalence and pin down the transfers using IC and IR constraints together with profit-maximization condition.[1]

[Myerson and Satterthwaite, 1983] studied the problem of choosing a trade mechanism to maximize the total welfare in the economy. Their main result shows that it is generically impossible to have an efficient trade mechanism -that allocates the good always to the agent who values it the most- without outside resources to finance it. In our main model, unsurprisingly, the welfare achieved even less than what [Myerson and Satterthwaite, 1983] provides. However, we provide an example that shows that this platform may improve the welfare compared to a decentralized market. Moreover, we show that when agents can choose between the platform and a decentralized market, the platform performs the efficient trade to incentivize joining the platform.

We have also seen rapid developments in the literature on two-sided markets and platforms [Rochet and Tirole, 2003, Armstrong, 2006]. In this literature, the questions mainly focus on the competition among platforms under numerous configurations of fee and price structures that could be employed by the platforms. More recently [Hartline and Roughgarden, 2014] bridges the gap between mechanism design two-sided markets with a model where sellers can choose to join a platform that sets a menu of selling procedures or to develop their own selling venue. We complement these studies by providing a benchmark for the profit that can be obtained by a single platform.

[Lu and Robert, 2001] considers the problem of a broker maximizing a mixture of profit

[1] For excellent surveys of the literature, see [Krishna, 2009, Vohra, 2011, Börgers, 2015].

and welfare while facing a group of traders who can be buyers and sellers. They provide some properties of the optimal Bayesian incentive compatible mechanism. However, focusing on Bayesian IC causes a new kind of ironing or bunching in the mechanism, which makes it very difficult to obtain a closed-form solution. Our solution does not lose any revenue by focusing on dominant strategy incentive compatibility (see [Gershkov et al., 2013]) but this approach allows us to pin down the exact solution and study further properties of these markets.

Finally, [Peivandi and Vohra, 2017] study a model where agents are allowed to deviate from a market mechanism to trade among themselves according to a different mechanism. Their main result states that almost every market mechanism is inherently unstable in the sense that there is always a positive measure of agents who would like to deviate from it. Our findings in the last section provide a partial counterpart to their result in that by restricting the possible deviations from the market mechanism, we are able to find a stable market structure.

2 Monagora

We call this model monagora to emphasize the fact that in this part of the paper, we restrict all trade to the platform; hence there is a unique market or monagora.[2]

2.1 Setup

- Agents: $n > 1$ agents; $N = \{1, \ldots, n\}$.
- Types: Each agent i has a value for a single-unit of a single good, $\theta_i \in \Theta_i$ which is private information. (We let $\Theta = \prod_{i=1}^{n} \Theta_i$ and $\theta \in \Theta$). We assume they have quasi-linear preferences and that each agent is interested in at most 1 unit of the good.
- Endowments: Each agent i has $e_i \in [0, 1]$ units of good which we and all agents take as given. All endowments are common knowledge among the agents and a mechanism designer.
- A mechanism designer wants to design a mechanism to maximize its profit. We start at a stage where the endowments are common knowledge and agents valuations are private information.

By revelation principle, we focus on direct mechanisms that allocates $q_i : \theta \times [0, 1]^n \to \mathcal{R}$ units of good to each agent i and asks her to pay $t_i : \theta \times [0, 1]^n \to \mathcal{R}$. Hence, the utility of the agent i from the mechanism with the valuation θ_i and endowment e_i is

$$u_i(\theta, e) = \theta_i \min\{1, q_i(\theta_i, \theta_{-i}, e) + e_i\} - t_i(\theta_i, \theta_{-i}, e) - \theta_i e_i.$$

[2]We use this term as a parallel of monopoly (only seller) and monopsony (only buyer).

$$\max_{(q_i,t_i)_{i\in N}} \int_{\Theta} \sum_{i=1}^{n} t_i(\theta,e) f(\theta) d\theta$$

s. t.

(IC)	$\theta_i \min\{1, q_i(\theta_i,\theta_{-i},e)+e_i\} - t_i(\theta_i,\theta_{-i},e)$
	$\geq \theta_i \min\{1, q_i(\theta_i',\theta_{-i},e)+e_i\} - t_i(\theta_i',\theta_{-i},e)$
(IR)	$\theta_i \min\{1, q_i(\theta_i,\theta_{-i},e)+e_i\} - t_i(\theta_i,\theta_{-i},e) \geq e_i\theta_i$
(Individual Feasibility)	$q_i(\theta,e) \geq -e_i$
(Aggregate Feasability)	$0 = \sum_{i=1}^{n} q_i(\theta,e)$

2.2 Simplifying The Principal's Problem

We first develop a series of lemma's that help us state the maximization problem above as a concave program.

Lemma 2.1 (Monotonicity). *Suppose $(q_i,t_i)_{i\in N}$ is a direct, IC mechanism. Then,*

1. *If $q_i(\theta,e)+e_i < 1$ for some $\theta \in \Theta$, $e \in [0,1]^n$ $i \in N$, then $q_i(\theta_i,\theta_{-i},e)$ is increasing in θ_i at (θ,e).*

2. *If $q_i(\theta,e)+e_i \geq 1$ for some $\theta \in \Theta$, $e \in [0,1]^n$ $i \in N$, then $q_i(\theta_i',\theta_{-i},e)+e_i \geq 1$ for each $\theta_i' \geq \theta_i$.*

The proof is standard, except for taking care of the capacities so it can be found in the Appendix A.

The next lemma presents the derivative of the utility of an agent in an IC mechanism.

Lemma 2.2 (Envelope Condition). *If $(q_i,t_i)_{i\in N}$ is a direct, IC mechanism, then for each $\theta \in \Theta$*

$$\frac{\partial u(\theta,e)}{\partial \theta_i} = \begin{cases} q_i(\theta,e), & \text{if } q_i(\theta,e)+e_i < 1, \\ 1-e_i, & \text{otherwise.} \end{cases}$$

Again, the proof is similar to standard arguments and can be found in Appendix B.

The next lemma gives the representation of the utility of each type as the integral of the allocation rule, using the previous lemma.

Lemma 2.3 (Payoff Equivalence). *If $(q_i,t_i)_{i\in N}$ is a direct, IC mechanism, then*

$$u_i(\theta,e) = u_i(\underline{\theta_i},\theta_{-i},e) + \int_{\underline{\theta_i}}^{\min\{\theta_i,\theta_i^*\}} q_i(x,\theta_{-i},e)dx + (\theta_i - \min\{\theta_i,\theta_i^*\})(1-e_i),$$

for each $\theta \in \Theta$ where θ_i^* is such that $q_i(\theta_i,e)+e_i = 1$ if such a solution exists, $\theta_i^* = \theta_i$ otherwise.

Proof. Since $u_i(\theta,e)$ is convex in θ_i restricted to regions where $q_i(\theta,e)+e_i > 1$ and $q_i(\theta,e)+e_i \leq 1$ separately, it is absolutely continuous in θ_i. Then, it is the integral of its derivative. □

Notation: For any direct mechanism $(q_i, t_i)_{i\in N}$, let

$$q_i^*(\theta_i, e) = \begin{cases} q_i(\theta, e), & \text{if } q_i(\theta, e) + e_i < 1, \\ 1 - e_i, & \text{otherwise.} \end{cases}$$

Note that for a direct, IC mechanism, $q_i^*(\theta_i, e)$ is also weakly increasing.

Next, we pin down the transfer rule in an IC mechanism.

Lemma 2.4 (Revenue Equivalence)**.** *If $(q_i, t_i)_{i\in N}$ is a direct, IC mechanism, then*

$$t_i(\theta, e) \;\; = -u_i(\underline{\theta_i}, \theta_{-i}, e) + \theta_i q_i^*(\theta, e) - \int_{\underline{\theta_i}}^{\theta_i} q_i^*(x, \theta_{-i}, e)dx,$$

for each $\theta \in \Theta$.

Proof. From the definition of $u_i(\theta, e)$ and the previous lemma. □

Now we provide a sufficient condition for incentive compatibility of a mechanism.

Proposition 2.1. *Let $(q_i, t_i)_{i\in N}$ be a direct mechanism. The mechanism is incentive compatible if and only if,*

1. *If $q_i(\theta, e) + e_i < 1$, then $q_i(\theta, e)$ is increasing in θ_i at (θ, e);*
2. *If $q_i(\theta, e) + e_i \geq 1$, then $q_i(\theta_i', \theta_{-i}, e) + e_i \geq 1$ for each $\theta_i' \geq \theta_i$;*
3. *$t_i(\theta_i, \theta_{-i}, e) = -u_i(\underline{\theta_i}, \theta_{-i}, e) + \theta_i q_i^*(\theta, e) - \int_{\underline{\theta_i}}^{\theta_i} q_i^*(x, \theta_{-i}, e)dx$.*

Proof can be found in Appendix C.

The next proposition provides the characterization of the IR mechanisms by establishing the types with the lowest utilities. The reason this is an issue in this model, unlike in the auction theory is that in an auction, the lowest allocation an agent could receive is 0. Hence, the utility is always increasing in agent's type, as can be seen from the envelope condition. However, here, an agent with a relatively low type can be a seller, which means he would get a negative allocation. Therefore, the utility of the lowest type is not the lowest utility in this case, which can again by seen from the envelope condition.

Proposition 2.2. *Let $(q_i, t_i)_{i\in N}$ be a direct IC mechanism. Then, it is IR if and only if for each $e \in [0, 1]^n$, $\theta_{-i} \in \Theta_i$, for each agent $i \in N$,*

$$\theta_i^* q_i^*(\theta_i^*, \theta_{-i}, e) \geq t_i(\theta_i^*, \theta_{-i}, e),$$

where θ_i^* is defined as

1. *$\theta_i^* = 0$ if $q_i^*(\underline{\theta_i}, \theta_{-i}, e) \geq 0$,*
2. *$\theta_i^* = \overline{\theta_i}$ if $q_i^*(\overline{\theta_i}, \theta_{-i}, e) < 0$,*
3. *the solution to $q_i^*(\theta_i^*, \theta_{-i}, e) = 0$ if such a type exists.*

Proof. *Case 1:* Suppose $q_i^*(\underline{\theta_i}, \theta_{-i}, e) \geq 0$. Then, by Lemma 2.3, incentive compatibility of a mechanism implies that the associated ex-post utilities $u_i(\theta, e)$ are increasing in θ_i. Hence, if $u_i(\underline{\theta_i}, \theta_{-i}, e) \geq 0$, we have $u_i(\theta_i, \theta_{-i}, e) \geq 0$ for each $\theta_i \in \Theta_i$. Of course, $u_i(\underline{\theta_i}, \theta_{-i}, e) \geq 0$ means $\underline{\theta_i} q_i^*(\underline{\theta_i}, \theta_{-i}, e) \geq t_i(\underline{\theta_i}, \theta_{-i}, e)$

Case 2: Suppose $q_i^*(\overline{\theta_i}, \theta_{-i}, e) < 0$. Then, by Lemma 2.3, $u_i(\theta_i, \theta_{-i}, e)$ are decreasing and hence, $u_i(\overline{\theta_i}, \theta_{-i}, e)$ is the lowest payoff. Hence, if it is nonnegative, all other types' payoffs are nonnegative as above.

Case 3: Suppose there exists θ_i^* such that $q_i^*(\theta_i^*, \theta_{-i}, e) = 0$. Then, by Lemma 2.3, $u_i(\theta_i, \theta_{-i}, e)$ is decreasing up to θ_i^* and increasing after that point. Hence, type θ_i^* has the lowest payoff. So, if $u_i(\theta_i^*, \theta_{-i}, e) \geq 0$, each type's IR condition must also hold.

□

Lemma 2.5. *If an IC and IR mechanism maximizes the expected revenue of the designer, then for each $e \in [0,1]^n$, for each agent $i \in N$,*

$$t_i(\theta_i^*, \theta_{-i}, e) = \theta_i^* q_i(\theta_i^*, \theta_{-i}, e)$$

where θ_i^* is defined as

1. *$\theta_i^* = 0$ if $q_i^*(\underline{\theta_i}, \theta_{-i}, e) \geq 0$,*
2. *$\theta_i^* = \overline{\theta_i}$ if $q_i^*(\overline{\theta_i}, \theta_{-i}, e) < 0$,*
3. *the solution to $q_i^*(\theta_i^*, \theta_{-i}, e) = 0$ if such a type exists.*

Proof. The previous proposition shows that IC and IR mechanisms must have $\theta_i^* q_i(\theta_i^*, \theta_{-i}, e)$ greater than $t_i(\theta_i^*, \theta_{-i}, e)$. However, if $\theta_i^* q_i(\theta_i^*, \theta_{-i}, e) > t_i(\theta_i^*, \theta_{-i}, e)$, then the seller can increase the expected revenue by increasing $t_i(\underline{\theta_i}, \theta_{-i}, e)$ and keeping the allocation rule the same. This would increase all types' payments and the revenue strictly, contradicting revenue maximization.

□

We can use the previous lemmata to see that

$$\begin{aligned}
\theta_i^* q_i^*(\theta_i^*, \theta_{-i}, e) \quad &= t_i(\theta_i^*, \theta_{-i}, e) \\
= \quad & -u_i(\underline{\theta_i}, \theta_{-i}, e) + \theta_i^* q_i^*(\theta_i^*, \theta_{-i}, e) - \int_{\underline{\theta_i}}^{\theta_i^*} q_i^*(x, \theta_{-i}, e) dx \\
\iff \quad & u_i(\underline{\theta_i}, \theta_{-i}, e) = -\int_{\underline{\theta_i}}^{\theta_i^*} q_i^*(x, \theta_{-i}, e) dx \\
\iff \quad & t_i(\theta_i, e) = \int_{\underline{\theta_i}}^{\theta_i^*} q_i^*(x, \theta_{-i}, e) dx + \theta_i q_i^*(\theta_i, \theta_{-i}, e) - \int_{\underline{\theta_i}}^{\theta_i} q_i^*(x, \theta_{-i}, , e) dx
\end{aligned}$$

Now we are ready to show that the allocation rule in a revenue-maximizing mechanism is not 'wasteful'.

Proposition 2.3. *Let $(q_i, t_i)_{i \in N}$ be a direct mechanism that maximizes the revenue of the designer. Then, for each $\theta \in \theta$, $e \in [0,1]^n$ for each $i \in N$, $q_i(\theta, e) \leq 1 - e_i$*

Proof. First, suppose that in the optimal mechanism, there exists θ, e and j such that $q_j(\theta, e) > 1 - e_j$. Notice that decreasing the allocation to $1 - e_j$ has no effect on the agent's payoff. Hence, it doesn't effect any IC or IR constraints.

Next, let us examine the transfer rule in a direct, IC mechanism:

$$t_i(\theta_i, \theta_{-i}, e) = \int_{\underline{\theta_i}}^{\theta_i^*} q_i^*(x, \theta_{-i}, e)dx + \theta_i q_i^*(\theta, e) - \int_{\underline{\theta_i}}^{\theta_i} q_i^*(x, e)dx.$$

If we have $q_j(\theta, e) > 1 - e_j$ for a positive measure of types, then we must have $q_i(\theta, e) < 0$ for a positive measure of types by the aggregate feasibility constraint. Hence, if we reduced $q_j(\theta, e) = 1 - e_j$ for a positive measure of types, this wouldn't decrease profit everywhere but instead increase profit as it allows us to increase $q_i(\theta, e) < 0$ for a positive measure of types, contradicting the optimality of the mechanism.

□

Now we can restate the problem as follows.

$$\begin{aligned}
\max_{(q_i,t_i)_{i\in N}} \quad & \sum_{i=1}^{n} \int_{\Theta} \Bigg[\int_{\{\theta_i | q_i(\theta_i, \theta_{-i}, e) \leq 0\}} q_i(x, \theta_{-i}, e)dx \\
& + \left(\theta_i q_i(\theta_i, \theta_{-i}, e) - \int_{\underline{\theta_i}}^{\theta_i} q_i(x, \theta_{-i}, e)dx \right) \Bigg] f(\theta)d\theta \\
\text{s. t.} \quad & \\
& q_i(\theta, e) \text{ is increasing in } \theta_i \\
& q_i(\theta, e) \geq -e_i \\
& 0 = \textstyle\sum_{i=1}^{n} q_i(\theta, e)
\end{aligned} \tag{1}$$

After some transformations[3], the problem above can be rewritten as follows:

$$\begin{aligned}
\max_{(q_i,t_i)_{i\in N}} \sum_{i=1}^{n} \quad & \left[\int_{\Theta} q_i(\theta_i, \theta_{-i}, e) \left[\frac{\mathbb{1}\{q_i(\theta_i, \theta_{-i}, e) \leq 0\}}{f_i(\theta_i)} + \left(\theta_i - \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right) \right] f(\theta)d\theta \right] \\
\text{s. t.} \quad & \\
& q_i(\theta, e) \text{ is increasing in } \theta_i \\
& q_i(\theta, e) \geq -e_i \\
& 0 = \textstyle\sum_{i=1}^{n} q_i(\theta, e_i)
\end{aligned}$$

Let $B_i(\theta_i) = \left(\theta_i - \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right)$, which is the virtual value and $S_i(\theta_i) = \left(\theta_i + \frac{(F_i(\theta_i))}{f_i(\theta_i)} \right)$, which is the virtual cost.

Definition 2.1. The distribution of an agent i's type, F_i is **regular** if both B_i and S_i are increasing.

[3]The details can be followed in Appendix D.

2.3 An Algorithm to Calculate the Optimal Allocation

We now describe an algorithm that would determine the allocation rule that maximizes the expected revenue.

Given some type profile, θ, suppose without loss of generality that $B_1(\theta_1) \geq \cdots \geq B_i(\theta_i) \geq \cdots \geq B_n(\theta_n)$ and that $S_{s(1)}(\theta_{s(1)}) \geq \cdots \geq S_{s(i)}(\theta_{s(i)}) \geq \cdots \geq S_{s(n)}(\theta_{s(n)})$ where $s(i)$ denotes the agent with i-th highest virtual cost.

Let us define auxiliary allocation functions among the agents: $q_i^k(\theta)$ denotes the allocation the agent i receives from agent k. We start the algorithm at $q_i^k(\theta) = 0$ for each $i, k \in N$.

We will define the allocation as follows.

Set $i = 1$ and $j = s(n)$.

1. If $B_i(\theta_i) \not> S_j(\theta_j)$, go to (3).

2. If $B_i(\theta_i) > S_j(\theta_j)$,

$$q_i^j(\theta) = \min\{1 - e_i - \sum_{\{k:s^{-1}(k) \leq s^{-1}(j)\}} q_i^k(\theta), e_j + \sum_{k=1}^{i-1} q_j^k\}.$$

   i. If

$$1 - e_i - \sum_{\{k:s^{-1}(k) \leq s^{-1}(j)\}} q_i^k(\theta) < e_j + \sum_{k=1}^{i-1} q_j^k$$

   $i = i + 1$ and go to (1).

   ii. If

$$1 - e_i - \sum_{\{k:s^{-1}(k) \leq s^{-1}(j)\}} q_i^k(\theta) > e_j + \sum_{k=1}^{i-1} q_j^k$$

   $j = s(s^{-1}(j) - 1)$ and go to (1).

   iii. If

$$1 - e_i - \sum_{\{k:s^{-1}(k) \leq s^{-1}(j)\}} q_i^k(\theta) = e_j + \sum_{k=1}^{i-1} q_j^k$$

   $i = i + 1$ and $j = s(s^{-1}(j) - 1)$ and go to (1).

3. For each $k \in N$, let $q_k(\theta) = \sum_{a=1}^{n} q_k^a(\theta)$ and exit.

We are going to see that this algorithm does not admit any cycles.

Proposition 2.4. *The allocations algorithm described above does not admit any cycles.*

Proof. Given some type profile, θ, suppose without loss of generality that $B_1(\theta_1) \geq \cdots \geq B_i(\theta_i) \geq \cdots \geq B_n(\theta_n)$ and that $S_{s(1)}(\theta_{s(1)}) \geq \cdots \geq S_{s(i)}(\theta_{s(i)}) \geq \cdots \geq S_{s(n)}(\theta_{s(n)})$.

Suppose there was a cycle: $\mathcal{B}_i(\theta_i) \geq \mathcal{S}_j(\theta_j); \mathcal{B}_j(\theta_j) \geq \mathcal{S}_k(\theta_k), \ldots, \mathcal{B}_y(\theta_y) \geq \mathcal{S}_z(\theta_z)$ but $\mathcal{B}_z(\theta_z) > \mathcal{S}_i(\theta_i)$.

Notice that for any agent α, we have

$$\mathcal{S}_\alpha(\theta_\alpha) = \mathcal{B}_\alpha(\theta_\alpha) + \frac{1}{f_\alpha(\theta_\alpha)} \geq \mathcal{B}_\alpha(\theta_\alpha).$$

Then, above cycle implies that

$$\mathcal{B}_i(\theta_i) \geq \mathcal{S}_j(\theta_j) \geq \mathcal{B}_j(\theta_j) \geq \mathcal{S}_k(\theta_k), \ldots, \mathcal{B}_y(\theta_y) \geq \mathcal{S}_z(\theta_z) \geq \mathcal{B}_z(\theta_z) > \mathcal{S}_i(\theta_i),$$

a contradiction as it comes to mean $\mathcal{B}_i(\theta_i) > \mathcal{S}_i(\theta_i)$.

□

Now, we can prove the optimality of this algorithm in maximizing the platform's profit.

Theorem 2.1. *Suppose each agent's type is drawn from a regular distribution. Then, the revenue-maximizing mechanism has the allocation rule described by the above algorithm and the following transfer rule:*

$$t_i^*(\theta, e) = \int_{\{\theta_i | q_i(\theta_i, \theta_{-i}, e) \leq 0\}} q_i(x, \theta_{-i}, e)dx + \theta_i q_i(\theta, e) - \int_{\underline{\theta_i}}^{\theta_i} q_i(x, \theta_{-i}, e)dx.$$

Proof. Consider the point-wise maximization problem given (θ, e) and ignore the constraint that q_i is increasing in θ_i for a moment.

Claim 2.1. *Let i and j be such that $B_i(\theta_i) > S_j(\theta_j)$. If $q_j(\theta_j) > -e_j$, then $q_i(\theta, e) = 1 - e_i$.*

Proof. Let i and j be such that $B_i(\theta_i) > S_j(\theta_j)$. If $q_i(\theta, e) < 1 - e_i$ and $q_j(\theta, e) > -e_j$, then increasing $q_i(\theta, e)$ and decreasing $q_j(\theta, e)$ by the same amount, to the extent that it is possible under the constraints, strictly increases the revenue. □

□

2.4 Illustrative Example

Suppose there are two agents, $N = \{1, 2\}$ with endowments $(e_1, e_2) = (0.6, 0.5)$. Each θ_i is distributed uniformly over $[0, 1]$ with c.d.f. $F_i(\theta_i) = \theta_i$. Then, the virtual values and costs are given by

$$\mathcal{B}_i(\theta_i) = 2\theta_i - 1 \text{ and } \mathcal{S}_i(\theta_i) = 2\theta_i.$$

The Figure 2 depicts the space of (θ_1, θ_2) where the shaded areas represent $\mathcal{B}_1(\theta_1) > \mathcal{S}_2(\theta_2)$ and $\mathcal{B}_2(\theta_2) > \mathcal{S}_1(\theta_1)$ respectively.

Straightforward calculations show that

$$\mathcal{B}_i(\theta_i) > \mathcal{S}_j(\theta_j) \iff \theta_i > \theta_j + 0.5.$$

By Theorem 2.1, we have

$$q(\theta, e) = \begin{cases} (0.4, -0.4), & \text{if } \theta_1 > \theta_2 + 0.5, \\ (-0.5, 0.5), & \text{if } \theta_2 - 0.5 > \theta_1, \\ (0, 0), & \text{otherwise.} \end{cases}$$

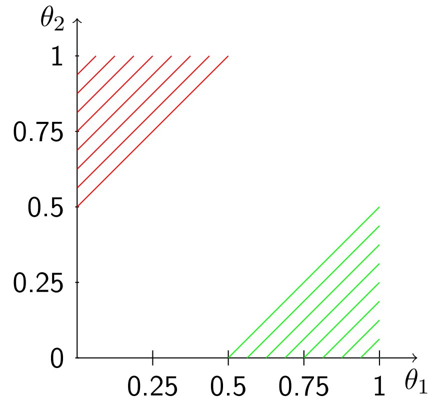

Figure 2: x-axis represents Θ_1 and y-axis represents Θ_2. Green and red areas show the type profiles at which agent 1 and 2 is the buyer, respectively.

Now, we can calculate the transfers.

$$t_1(\theta, e) = \begin{cases} -0.5\max\{0, \theta_2 - 0.5\} + \theta_1 0.4 - \int_0^{\theta_1} q_1(x, \theta_2, e)dx, & \text{if } \theta_1 > \theta_2 + 0.5, \\ -0.5\max\{0, \theta_2 - 0.5\} - \theta_1 0.5 - \int_0^{\theta_1} q_1(x, \theta_2, e)dx, & \text{if } \theta_2 - 0.5 > \theta_1, \\ -0.5\max\{0, \theta_2 - 0.5\} - \int_0^{\theta_1} q_1(x, \theta_2, e)dx, & \text{otherwise.} \end{cases}$$

and

$$t_2(\theta, e) = \begin{cases} -0.4\max\{0, \theta_1 - 0.5\} - \theta_2 0.4 - \int_0^{\theta_2} q_2(x, \theta_1, e)dx, & \text{if } \theta_1 > \theta_2 + 0.5, \\ -0.4\max\{0, \theta_1 - 0.5\} + \theta_2 0.5 - \int_0^{\theta_2} q_2(x, \theta_1, e)dx, & \text{if } \theta_2 - 0.5 > \theta_1, \\ -0.4\max\{0, \theta_1 - 0.5\} - \int_0^{\theta_2} q_2(x, \theta_1, e)dx, & \text{otherwise.} \end{cases}$$

We can actually calculate the integrals in the above expressions since we know q precisely. Then, the transfers are as follows:

$$t_1(\theta, e) = \begin{cases} 0.4(\theta_2 + 0.5), & \text{if } \theta_1 > \theta_2 + 0.5, \\ -0.5(\theta_2 - 0.5), & \text{if } \theta_2 - 0.5 > \theta_1, \\ 0, & \text{otherwise.} \end{cases}$$

$$t_2(\theta, e) = \begin{cases} -0.4(\theta_1 - 0.5), & \text{if } \theta_1 > \theta_2 + 0.5, \\ 0.5(\theta_2 + 0.5), & \text{if } \theta_2 - 0.5 > \theta_1, \\ 0, & \text{otherwise.} \end{cases}$$

3 Welfare Effect of Platforms

The platforms we consider maximize their own profit while providing a venue for agents to trade. However, their welfare effect compared to a decentralized market is not clear. To be able to make some assertions and comparisons about the welfare, we consider a decentralized market where agents are randomly matched in groups of two. Once they are matched, we assume that agents truthfully reveal their valuation of the good to each other. Following this revelation, agents make the efficient trade, i.e., the agent with the higher valuation buys as many units as possible in each pair. The payment is such that agents share the welfare created by this trade equally. We now specify a concrete example to work through these steps.

Before studying an example, it is important to note that examples where the platform decreases the welfare gains are very easy to construct. For instance, consider the previous example with two agents. If they were to be in the decentralized market we just described, they would always trade efficiently. However, the platform only allows them to trade when their valuation differ by more than 0.5. Hence, the platform definitely decreases the welfare compared to this specification of a decentralized market. Hence, the point is whether this effect is always in this direction or not.

Suppose we have four agents with identical endowments, $e_i = \frac{3}{4}$ and they all have valuations drawn from $U[0,1]$. To illustrate, consider two agents, 1 and 2 matched to each other. Suppose their valuations are 0.4 an 0.8, respectively. Then, agent 2 would buy 0.25 unit of good from agent 1, as that is the maximum level of trade possible under the constraints and demands. The payment agent 2 makes to agent 1 would be equal to $0.4 \times 0.25 + \frac{(0.8-0.4)\times 0.25}{2}$ where the first term compensates agent 1 for her utility loss and the second term gives her half of the welfare created by this trade.

Expected total welfare created in the decentralized markets is 2 times $\frac{1}{4}$ times $\frac{1}{3}$ since there will be 2 groups of 2 agents where the trade volume is 0.25 and the expected difference of valuations is $\frac{1}{3}$.

We calculate the expected welfare gains from the platform by partitioning the type space to regions where total trade is equal to 0.75, 0.5, 0.25 and 0, and calculate the gains from each one to be summed up later.

Welfare gains from total trade is equal to 0.75:

$$\int_0^{0.5} \int_{i+0.5}^{1} \int_j^1 \int_k^1 6(l+k+j-3i)dldkdjdi = 0.0328125$$

Welfare gains from total trade is equal to 0.5:

$$\int_0^{0.5} \int_i^{i+0.5} \int_{\max j, i+0.5}^{1} \int_k^1 6(l+k-2i)dldkdjdi = 0.0859375$$

Welfare gains from total trade is equal to 0.25:

$$\int_0^{0.5} \int_i^{i+0.5} \int_j^{i+0.5} \int_{\max k, i+0.5}^{1} 6(l-i)dldkdjdi = 0.0625$$

Therefore, total welfare gains from the platform is 0.18125. However, total welfare gains in the decentralized market was $\frac{1}{6}$, which is clearly less than the gains from the platform. So, the platform improves the welfare by centralizing the trade. As we have already argued that in a 2 agent environment, such as in the previous section, the platform would decrease the welfare, we conclude that the welfare effect is ambiguous with this specification of the decentralized market and with this definition of welfare.

4 Equilibria with Decentralized Market

In our platform, agents with 'intermediate' valuations do not trade frequently; profit maximization and incentive compatibility requires excluding them, similar to an auction with a reserve price in which a good is not always sold. The reason this is in principal's interest is that excluding the intermediate values from trade decreases the information rent of the agents who get to trade. However, even though the platform is IR in the sense that 'it does no harm', the initial model excludes the possibility that agents who don't get to trade in the platform may want to engage in bilateral trades. We allow this possibility in this section.

Considering agents who can choose between trading in the platform or doing bilateral trade also enables us to relate our results to [Peivandi and Vohra, 2017] where they argue that centralized markets are not stable in the sense that there is -almost- always a positive measure of agents who would like to deviate from any given centralized trading mechanism to another one. However, except for special cases they investigate, the nature of a stable market structure in this sense is unknown. Our results here give one particular way a stable structure can be achieved.

Suppose we have a continuum of agents where each agent $i \in [0,1]$ holds an endowment equal to 0.5 and has a valuation for the good that is drawn from the uniform distribution over the set of types $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$. Suppose agents and the principal faces the following game. The principal announces a direct mechanism. Agents, observing the announced mechanism, choose whether to join this mechanism or to search for a trade partner in the decentralized market. In the decentralized market, agents are randomly matched with each other in pairs. After they are matched, each agent announces a valuation for the good. If they announce the same valuation, nothing happens; if they announce different valuations, they do the efficient trade implied by the declared valuations and they split the surplus created by the trade equally. I.e., if the announced valuations are θ_i and θ_j with $\theta_i > \theta_j$, then agent i gets j's endowment and pays

$$0.5\theta_j + \frac{\theta_i - \theta_j}{4}$$

to agent j (where the first summand is the compensation of agent j and the second summand is the half of the surplus created.) We are going to construct an equilibrium where agents with types $\frac{1}{3}$ and $\frac{2}{3}$ opt for the decentralized market and the agents with types 0 and 1 join the centralized market. We start by calculating the payoffs of type $\frac{1}{3}$ in the decentralized market from claiming to have a valuation of $0, \frac{1}{3}, \frac{2}{3}, 1$ given that the rest of the players are playing truthfully:

$$\begin{aligned}
0 :& \frac{1}{2}\frac{1}{3}\frac{1}{2}\frac{1}{2} + 0 - \frac{1}{2}\frac{1}{3}\frac{1}{2} + \frac{1}{2}\frac{2}{3}\frac{1}{2}\frac{1}{2} + 0 - \frac{1}{2}\frac{1}{3}\frac{1}{2} && = -\frac{1}{24} \\
\frac{1}{3} :& \frac{1}{2}0 + \frac{1}{2}\frac{1}{3}\frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{3}\frac{1}{2} - \frac{1}{2}\frac{1}{3}\frac{1}{2} && = \frac{1}{24} \\
\frac{2}{3} :& -\frac{1}{2}\frac{1}{3}\frac{1}{2}\frac{1}{2} - \frac{1}{2}\frac{1}{3}\frac{1}{2} + \frac{1}{2}\frac{1}{3}\frac{1}{2} + \frac{1}{2}0 && = -\frac{1}{24} \\
1 :& -\frac{1}{2}\frac{2}{3}\frac{1}{2}\frac{1}{2} - \frac{1}{2}\frac{1}{3}\frac{1}{2} + \frac{1}{2}\frac{1}{3}\frac{1}{2} - \frac{1}{2}\frac{1}{3}\frac{1}{2}\frac{1}{2} - \frac{1}{2}\frac{2}{3}\frac{1}{2} + \frac{1}{2}\frac{1}{3}\frac{1}{2} && = -\frac{5}{24}
\end{aligned}$$

Clearly, truthful revelation benefits type $\frac{1}{3}$. As the calculations are tedious, we only summarize the results from here. When we consider this incentives for each type, we see that every type would reveal truthfully. When we do the corresponding calculations for types 0 and 1, these give us the outside options, which are important for the individually rationality constraints of the platform. It turns out that the outside options of types 0 and 1 are both equal to $\frac{1}{8}$. Given this and incentive compatibility constraints for both types, we maximize the profit. Notice that due to the continuum of agents, there is no aggregate uncertainty in the platform. Hence, choosing the interim and ex-post allocation and transfer rules are equivalent. Thus, we focus on the ex-post problem directly.

$$\begin{array}{cllll}
\max\limits_{\{q(0),t(0),q(1),t(1)\}} & \mathbb{E}\left[t(\theta_i)\right] & & \\
\text{s. t.} & & & \\
& -t(0) & \geq \dfrac{1}{8} & \mathrm{IR}_0 \\
& t(1) & \geq t(0) & \mathrm{IC}_0 \\
& q(1)-t(1) & \geq \dfrac{1}{8} & \mathrm{IR}_1 \\
& q(1)-t(1) & \geq q(0)-t(0) & \mathrm{IC}_1
\end{array}$$

This is a straightforward problem and it turns out that the only binding constraints are the IR constraints with the solution being $t(0) = -\dfrac{1}{8}$, $t(1) = \dfrac{3}{8}$, $q(0) = -0.5$ and $q(1) = 0.5$.

There are a couple of things worth mentioning here. First, introducing possibility of bilateral trades forces the platform to run the efficient trade, as it can be seen from the allocations above. Second, even though this is one equilibrium, there are several other equilibria. Cases where all agents join either the platform or the decentralized market are both equilibria as in both cases, deviations can only give 0 utility. Moreover, there are two more equilibria, (i) types $\frac{1}{3}$ and 1 joins the platform, the rest goes to the decentralized market, (ii) types 0 and $\frac{2}{3}$ joins the platform, the rest goes to the decentralized market. Notice that there is no equilibrium where types 0 and 1 joins the decentralized market and types $\frac{1}{3}$ and $\frac{2}{3}$ joins the platform.

5 Future Work

We plan to work on (i) Finding a more general framework where a platform and a decentralized market coexist, (ii) Finding conditions under which welfare effect is unambiguous, (iii) A model where several principals compete against each other as well as a decentralized market, and (iv) A dynamic model that incorporates stochastic arrivals.

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A Proof of Lemma 1

Proof. Let $\theta \in \Theta$ and $\theta_i' \in \Theta_i$. Then, by incentive compatibility

$$\theta_i \min\{1, q_i(\theta, e) + e_i\} - t_i(\theta_i, \theta_{-i}, e) \geq \theta_i \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} - t_i(\theta_i', \theta_{-i}, e)$$

and

$$\theta_i' \min\{1, q_i(\theta_i, \theta_{-i}, e) + e_i\} - t_i(\theta_i, \theta_{-i}, e) \leq \theta_i' \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} - t_i(\theta_i', \theta_{-i}, e).$$

Subtracting the second inequality from the first one leads to:

$$(\theta_i - \theta_i') \min\{1, q_i(\theta_i, \theta_{-i}, e) + e_i\} \geq (\theta_i - \theta_i') \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\}$$

Suppose $q(\theta, e) + e_i < 1$ and $\theta_i > \theta_i'$. Then, we have

$$
\begin{aligned}
\min\{1, q_i(\theta,e)+e_i\} &\geq \min\{1, q_i(\theta_i', \theta_{-i}, e)+e_i\} \iff \\
1 > q_i(\theta,e)+e_i &\geq \min\{1, q_i(\theta_i', \theta_{-i}, e)+e_i\} \iff \\
q_i(\theta,e) &\geq q_i(\theta_i', \theta_{-i}, e)
\end{aligned}
$$

Now suppose $q(\theta,e)+e_i \geq 1$ and $\theta_i' \geq \theta_i$. Then,

$$
\begin{aligned}
\min\{1, q_i(\theta_i', \theta_{-i}, e)+e_i\} &\geq \min\{1, q_i(\theta,e)+e_i\} \iff \\
\min\{1, q_i(\theta_i', \theta_{-i}, e)+e_i\} &\geq 1 \iff \\
q_i(\theta_i', \theta_{-i}, e)+e_i &\geq 1.
\end{aligned}
$$

□

B Proof of Lemma 2

Proof. First, suppose $q_i(\theta,e)+e_i \ < 1$. Then, IC implies that for a type θ_i agent:

$$
\begin{aligned}
u_i(\theta,e) &= \max_{\theta_i' \in \Theta_i} \min\{1, q_i(\theta_i', \theta_{-i}, e)+e_i\}\theta_i - t_i(\theta_i', \theta_{-i}, e) - e_i\theta_i \\
&= \max_{\theta_i' \in \Theta_i} q_i(\theta_i', \theta_{-i}, e)\theta_i - t_i(\theta_i', \theta_{-i}, e) - e_i\theta_i.
\end{aligned}
$$

Notice that the RHS is the maximum of affine functions of θ_i, so $u_i(\theta_i, e)$ is convex in θ_i in this region. Hence, $u_i(\theta_i, \theta_{-i}, e)$ is differentiable almost everywhere in θ_i on this region. For any θ_i at which it is differentiable, for $\delta > 0$, IC implies that

$$
\begin{aligned}
& \lim_{\delta\to 0} \frac{u_i(\theta_i+\delta, \theta_{-i}, e) - u_i(\theta,e)}{\delta} \\
\geq \ & \lim_{\delta\to 0} \frac{(q_i(\theta,e)(\theta_i+\delta) - t_i(\theta,e)) - (q_i(\theta,e)\theta_i - t_i(\theta,e))}{\delta} = q_i(\theta,e). \\
& \lim_{\delta\to 0} \frac{u_i(\theta,e) - u_i(\theta_i-\delta, \theta_{-i}, e)}{\delta} \\
\leq \ & \lim_{\delta\to 0} \frac{(q_i(\theta,e)\theta_i - t_i(\theta,e)) - (q_i(\theta,e)(\theta_i-\delta) - t_i(\theta,e))}{\delta} = q_i(\theta,e).
\end{aligned}
$$

Then, two inequalities together imply that

$$
\frac{\partial u_i(\theta,e)}{\partial \theta_i} = q_i(\theta,e).
$$

Now suppose $q_i(\theta,e)+e_i \geq 1$. Then,

$$
u_i(\theta,e) = \min\{1, q_i(\theta,e)+e_i\}\theta_i - t_i(\theta,e) - e_i\theta_i = \theta_i - t_i(\theta,e) - e_i\theta_i.
$$

Then, of course,

$$
\frac{\partial u_i(\theta,e)}{\partial \theta_i} = 1 - e_i.
$$

□

C Proof of Proposition 1

Proof. We want to show that for each $i \in N$, for each $\theta_i, \theta_i' \in \Theta_i$ and $\theta_{-i} \in \Theta_i$, we have

$$
\begin{aligned}
& u_i(\theta, e) \geq \theta_i \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} - t_i(\theta_i', \theta_{-i}, e) - \theta_i e_i \\
\iff \quad & u_i(\theta, e) \geq \theta_i \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} + \theta_i' \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} \\
& -\theta_i' \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} - t_i(\theta_i', \theta_{-i}, e) - \theta_i e_i + \theta_i' e_i - \theta_i' e_i \\
\iff \quad & u_i(\theta, e) \geq \theta_i \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} - \theta_i' \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} \\
& + u_i(\theta_i', \theta_{-i}, e) - e_i(\theta_i - \theta_i') \\
\iff \quad & u_i(\theta_i, \theta_{-i}, e) - u_i(\theta_i', \theta_{-i}, e) \geq (\theta_i - \theta_i') \min\{1, q_i(\theta_i', \theta_{-i}, e) + e_i\} \\
& -e_i(\theta_i - \theta_i') \\
\iff \quad & \int_{\theta_i'}^{\theta_i} q_i^*(x, \theta_{-i}, e) dx \geq \int_{\theta_i'}^{\theta_i} q_i^*(\theta_i', \theta_{-i}, e) dx
\end{aligned}
$$

Suppose $\theta_i > \theta_i'$. Since $q_i^*(\cdot, \theta_{-i}, e)$ is increasing, $q_i^*(x, \theta_{-i}, e) \geq q_i^*(\theta_i', \theta_{-i}, e)$ for each $x \in [\theta_i', \theta_i]$. Then, the last inequality above holds. Similar analysis holds for the case of $\theta_i < \theta_i'$.

□

D Transformations of the Principal's Problem

We start with the following problem in Equation 1 and make the following transformation:

$$
\begin{aligned}
& \int_\Theta \int_{\underline{\theta_i}}^{\theta_i} q_i(x, \theta_{-i}, e) dx f(\theta) d\theta \\
= \quad & \int_{\Theta_{-i}} \int_{\Theta_i} \int_x^{\overline{\theta_i}} q_i(x, \theta_{-i}, e) f(\theta) d\theta dx \\
= \quad & \int_{\Theta_{-i}} \int_{\Theta_i} q_i(x, \theta_{-i}, e) \int_x^{\overline{\theta_i}} f_i(\theta_i) d\theta_i f_{-i}(\theta_{-i}) d\theta_{-i} dx \\
= \quad & \int_\Theta q_i(\theta_i, , \theta_{-i}, e) \left(\frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right) f(\theta) d\theta
\end{aligned}
$$

So, the second part of the objective function becomes:

$$
\begin{aligned}
\sum_{i=1}^n \quad & \int_\Theta \theta_i q_i(\theta_i, \theta_{-i}, e) f(\theta) d\theta - q_i(\theta_i, \theta_{-i}, e) \left(\frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right) f(\theta) d\theta \\
= \quad & \sum_{i=1}^n \int_\Theta \left(\theta_i q_i(\theta_i, \theta_{-i}, e) - q_i(\theta_i, \theta_{-i}, e) \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right) f(\theta) d\theta \\
= \quad & \sum_{i=1}^n \int_\Theta q_i(\theta_i, \theta_{-i}, e) \left(\theta_i - \frac{(1 - F_i(\theta_i))}{f_i(\theta_i)} \right) f(\theta) d\theta
\end{aligned}
$$

Next we look at the first summand in the objective function above. Notice that inside is actually a constant given θ_{-i} so it can be expressed as below:

$$
\begin{aligned}
& \int_\Theta \left[\int_{\{\theta_i | q_i(\theta_i, \theta_{-i}, e) \leq 0\}} q_i(x, \theta_{-i}, e) dx \right] f(\theta) d\theta \\
= \quad & \int_{\Theta_{-i}} \left[\int_{\{\theta_i | q_i(\theta_i, \theta_{-i}, e) \leq 0\}} q_i(x, \theta_{-i}, e) dx \right] f_{-i}(\theta_{-i}) d\theta_{-i} \\
= \quad & \int_{\Theta_{-i}} \left[\int_{\Theta_i} q_i(x, \theta_{-i}, e) \mathbb{1}\{q_i(x, \theta_{-i}, e) \leq 0\} dx \right] f_{-i}(\theta_{-i}) d\theta_{-i} \\
= \quad & \int_\Theta q_i(\theta_i, \theta_{-i}, e) \frac{\mathbb{1}\{q_i(\theta_i, \theta_{-i}, e) \leq 0\}}{f_i(\theta_i)} f(\theta) d\theta
\end{aligned}
$$

Finally, the objective function can be written as:

$$\sum_{i=1}^n \left[\int\limits_\Theta q_i(\theta_i, \theta_{-i}, e)\frac{\mathbb{1}\{q_i(\theta_i, \theta_{-i}, e) \leq 0\}}{f_i(\theta_i)} f(\theta)d\theta\right]$$
$$+\sum_{i=1}^n \left[\int\limits_\Theta q_i(\theta_i, \theta_{-i}, e)\left(\theta_i - \frac{(1-F_i(\theta_i))}{f_i(\theta_i)}\right) f(\theta)d\theta\right]$$
$$= \sum_{i=1}^n \left[\int_\Theta q_i(\theta_i, \theta_{-i}, e)\left[\frac{\mathbb{1}\{q_i(\theta_i, \theta_{-i}, e) \leq 0\}}{f_i(\theta_i)} + \left(\theta_i - \frac{(1-F_i(\theta_i))}{f_i(\theta_i)}\right)\right] f(\theta)d\theta\right]$$

Hence, the revenue maximization problem can be expressed as

$$\max_{(q_i,t_i)_{i\in N}} \sum_{i=1}^n \left[\int\limits_\Theta q_i(\theta_i, \theta_{-i}, e)\left[\frac{\mathbb{1}\{q_i(\theta_i, \theta_{-i}, e) \leq 0\}}{f_i(\theta_i)} + \left(\theta_i - \frac{(1-F_i(\theta_i))}{f_i(\theta_i)}\right)\right] f(\theta)d\theta\right]$$

s. t.

$q_i(\theta, e)$ is increasing in θ_i

$q_i(\theta, e) \geq -e_i$

$0 = \sum_{i=1}^n q_i(\theta, e_i)$