

Coexistence of Centralized and Decentralized Markets

Berk Idem*

Penn State University

November 1, 2021

[Please click here for the latest version.](#)

Abstract

In this paper, I introduce a profit-maximizing centralized marketplace into a decentralized market with frictions. I analyze the equilibria of a market choice game where agents choose between the centralized marketplace and decentralized bilateral trade. I use a mechanism design approach to characterize the optimal rules of the profit-maximizing marketplace in this game. In the unique equilibrium, the centralized marketplace and decentralized trade coexist. The thickness of the centralized marketplace is not affected by the decentralized trade. The equilibrium profit of the marketplace is at least half of the baseline profit, independent of the distribution of valuations. The decrease in the profit as a result of decentralized trade only depends on the efficiency of the decentralized market and not on the distribution. Under some conditions, this equilibrium results in higher welfare than either institution on its own.

*Email address: berkidem@psu.edu

1 Introduction

Centralized marketplaces that bring buyers and sellers together have seen massive growth in the last decade. For instance, a recent report predicts that Amazon generates half of all e-commerce sales in the US. Another analysis estimates that usage of ride-share apps beat taxis in NYC as early as 2017. Airbnb and Vrbo’s market share in vacation rentals reached half of the market.¹ All of these suggest that the centralized marketplaces have significant positions in their respective markets. Indeed, a congressional panel investigating competition in the digital markets has asserted that Amazon has monopoly power as an intermediary in the US e-commerce market (Congress Majority Staff, 2020).

The success of these marketplaces points to considerable inefficiencies in the decentralized markets. There, an agent may not meet with a trading partner, and even when she meets with a partner, it may not be the right one. These search and information frictions in the decentralized trade allow profit-maximizing marketplaces to fill the vacuum. Centralized marketplaces solve these search and information frictions by collecting information from agents and making sure the realized matches are efficient enough to allow the marketplace to make some profit. Near certainty of trade on these platforms attract many agents and give the platforms significant market shares.

These centralized marketplaces are profit-maximizers who compete with the decentralized markets that have no agendas themselves. Literature on competing platforms sheds some light on the nature of the competition when there are multiple profit-maximizing platforms. However, we know less about the impact of a centralized marketplace that competes with numerous other trading venues. The decentralized markets do not have organizers; so they do not react to the actions of the centralized marketplaces directly. It is the traders who decide where to trade, given what the centralized marketplaces offer them. Would the centralized marketplace monopolize all trade or is there room for some forms of decentralized trade to coexist? Moreover, if coexistence is possible, what are the welfare and profit consequences of multiple trading modes?

To illustrate the point, consider the example of someone who wants to buy or sell a used car. She can check the prices offered by Carvana, a platform specialized in the used car market. If the price offered by Carvana is acceptable, she can simply take this deal. However, if she thinks she can get a better deal in the local market, she can choose to participate in the local market. Even though Carvana may have a large market share in this sector, there are still endless possibilities for trading outside the centralized market. When Carvana chooses

¹Detailed statistics about these market shares can be found in Congress Majority Staff (2020); Schneider (2021); Hinoje (2021), respectively.

what price to offer for each car, ignoring these possibilities would harm its profit. Moreover, it is hard to guess the impact of Carvana’s response to these other options.

Financial markets provide another example that fits within this framework. Many assets can be traded at the stock exchanges as well as over-the-counter. In the stock exchanges, there is essentially no uncertainty; agents can buy or sell at the posted prices. However, in the over-the-counter markets, trade is not as transparent. Dealers often don’t post prices in a public manner. Instead, they provide quotes when someone is interested in trading with them. The agents who trade with these dealers only observe the prices offered by a limited number of dealers before they trade. For this reason, parts of the over-the-counter trade are referred to as the *dark market*. This paper provides a framework to think about the problems faced by a stock exchange that competes with a dark market.

In this paper, I develop a model of a centralized marketplace that competes for agents who also have the option to trade in a decentralized manner. I consider a setup with a single, indivisible good where agents have some endowments and valuation for at most two units of the good. The endowments are common knowledge whereas the valuations are the private information of the agents. A designer chooses an individually rational mechanism to maximize revenue of the marketplace—say, the commissions charged for intermediation. By the revelation principle, without loss, I focus on direct mechanisms that are incentive compatible.

I first restrict attention to the case where the centralized marketplace is the only trading venue. I allow each agent to be either a buyer or a seller depending on their valuation, and the market conditions. I show that the revenue-maximizing allocation rule ranks the agents according to their virtual values and virtual costs. The optimal allocation allows a transaction from an agent to another if and only if one has a higher virtual value than the other’s virtual cost. The transfers are pinned down by an interplay of incentive compatibility, individual rationality, and profit maximization.

After studying the marketplace on its own, I introduce the possibility that agents can trade bilaterally among themselves in a decentralized manner—modeled as in Diamond-Mortensen-Pissarides (Diamond, 1971; Mortensen, 1970; Pissarides, 1979); agents are randomly matched among those who choose not to participate in the centralized market and then they use Nash Bargaining in each realized match. (Later I establish robustness by showing that the results extend to trading with a double auction, instead of Nash Bargaining under some distributions.) I consider a market choice game where (i) the marketplace designer announces a mechanism, (ii) agents choose whether to join the mechanism or to search for a trading partner, (iii) outcomes are realized in both markets. I first establish that in the unique equilibrium, centralized marketplace and decentralized trade coexist. Thus,

when the agents can choose between these two modes of trade, it is never an equilibrium for all agents to join the same market. In the equilibrium, the agents with low and high values join the marketplace while the agents with intermediate values choose to search.

One might expect that competition from the decentralized market will significantly decrease the profits of the marketplace. This is not the case. I show that the profit of the marketplace in the coexistence equilibrium is at least half of the profit that the marketplace would make if it had been the only market. Moreover, I show that the ratio of the reduction in profit of the marketplace as result of competition from the decentralized market is independent of the distribution of agents valuations. This is an interesting finding since every aspect of the model heavily depends on the distribution. This ratio is only a function of the efficiency parameter of the decentralized market. As the decentralized market gets more efficient, the profit of the marketplace decreases. However, even in the case of most efficient decentralized market, the profit of the marketplace is half of its profit when it operates on its own.

Next, I compare the welfare -measured as gains from trade- created in this equilibrium to welfare from either modes of trade operating on their own. I provide conditions under which the coexistence generates higher total welfare than either the marketplace or the search market generates alone. Essentially, the decentralized market extends the extensive margin of trade (more agents trades in the coexistence equilibrium than in the baseline, single marketplace) while the marketplace extends the intensive margin (some agents trade with a higher probability in the coexistence equilibrium than when the decentralized market operates alone). Thus, the combination of these leads to increased efficiency.

Finally, I discuss an alternative setup where there are multiple profit-maximizing marketplace designers competing with each other. First, I study the case where they are restricted to choose direct mechanisms. This environment is akin to Bertrand Competition and the insight from the Bertrand Equilibrium carries over: In the equilibrium, the designers make zero profit and the outcome is equivalent to a Walrasian Equilibrium. Next, I allow the designers to choose more complex mechanisms and construct one particular equilibrium using strategies that include ‘price-matching guarantees’. In standard Bertrand Competition, price-matching guarantees are known to allow monopoly pricing to be a Nash equilibrium. Here, each marketplace posts prices that are equivalent to the baseline marketplace where there is no decentralized market and the agents uniformly randomize over the marketplaces. In this case, the marketplaces share the baseline profit. This discontinuity between zero profit with direct mechanism and the collective baseline profit (similar to a cartel’s monopoly profit) reflects the inadequacy of direct mechanisms when there are multiple designers. If agents could also join the decentralized market, then the marketplaces would each post prices

equivalent to the coexistence equilibrium mechanism of the main model with price-matching guarantees, and the agents again randomize.

After the literature review below, the rest of the paper is organized as follows. In Section 2, I study the baseline model with no decentralized market. In Section 3, I introduce the decentralized market and study the equilibria of the market choice game. In Section 4, I consider the competition among multiple marketplace designers.

1.1 Literature Review

We have seen rapid developments in the literature on two-sided markets and platforms (Rochet and Tirole, 2003; Armstrong, 2006). In this literature, the questions mainly focus on the competition among platforms under numerous configurations of fee and price structures that could be employed by the platforms. The important distinction between this literature and this study is that in the context of competing platforms, the competition is between two profit-maximizing entities; they each react (or best respond) to the other's actions. However, here, the centralized marketplace is in competition with a fixed set of rules that cannot react to the marketplace's actions. This paper complements these studies by providing a insights into the nature of the competition in a decentralized market with a platform. More recently Hartline and Roughgarden (2014) also bridges the gap between mechanism design and two-sided markets with a model where sellers can choose to join a platform that sets a menu of selling procedures or to develop their own selling venue.

Peivandi and Vohra (2021) study a model where agents are allowed to deviate from a market mechanism to trade among themselves according to a different mechanism. Their main result states that almost every market mechanism is inherently unstable in the sense that there is always a positive measure of agents who would like to deviate from it. Our findings provide a partial counterpart to their result in that by restricting the possible deviations from the market mechanism, I am able to find a stable market structure.

Following the financial crisis of 2007-2008, a series of papers initiated by Philippon and Skreta (2012); Tirole (2012) focused on the ability of the public interventions to increase the efficiency of the investments in the financial markets. Specifically, they focus on markets with adverse selection (and moral hazard in the case of Tirole (2012)) in terms of the quality of the investments. Thus, without the public intervention, the level of investments is below the socially optimal level. To reduce the adverse selection and increase the level of investment, the government introduces a program by overpaying for some assets and removing the weakest assets from the market. Fuchs and Skrzypacz (2015) studies a similar problem but considers the effects of the dynamic nature of the markets. These studies consider a benevo-

lent market designer that improves the social welfare at a cost to itself. By contrast, I focus on the profit-maximization problem of a centralized marketplace. Moreover, the nature of the frictions in the decentralized markets are different from the ones considered here.

Literature on the efficient dissolution of partnerships has important parallels with this study. Starting with Cramton et al. (1987) and with contributions by many others (Mylovanov and Tröger, 2014; Kittsteiner, 2003; Fieseler et al., 2003; Loertscher and Wasser, 2019; Figueroa and Skreta, 2012), the setup in this literature includes a divisible good that is owned by many agents and a designer who wants to allocate the whole supply to one agent (or in some cases, to at least reduce the number of owners), thus dissolving the partnership. All of these papers have endogenous roles as buyers and sellers; each agent has some endowment which is less than the total endowment in the economy. Thus, like here, they obtain intermediate types who that are excluded from the trade and U-shaped utilities as a functions of agents types. Moreover, in extending the Nash bargaining, I used the efficient double auction which was introduced by Cramton et al. (1987) and was shown to have a unique equilibrium by Kittsteiner (2003).

Myerson and Satterthwaite (1983) studied the problem of choosing a trade mechanism to maximize the total welfare in the economy. Their main result shows that it is generically impossible to have an efficient trade mechanism -that allocates the good always to the agent who values it the most- without outside resources to finance it. In our model without a decentralized market, unsurprisingly, the welfare achieved is even less than what Myerson and Satterthwaite (1983) provides. However, introducing the option to search improves the efficiency of the market as a whole.

Miao (2006) also studies the coexistence of centralized and decentralized market in a different setup. Instead of designing mechanisms, the centralized marketplaces there are limited to choosing prices in a dynamic search environment. The paper shows that when the search and transactions costs vanish, the coexistence equilibrium converges to an equilibrium where everyone joins the decentralized market and obtain the Walrasian equilibrium outcomes. However, in this paper, I obtain coexistence even when the search cost vanishes.

2 Single Market

In this section, I study the centralized marketplace in isolation and restrict all trade to the centralized marketplace; hence the agents either trade on this marketplace or do not trade at all.

2.1 Setup

I consider a market with a single, indivisible good. There is a continuum of agents on $[0, 1]$. Each agent has 1 unit of endowment of the good and has a demand for 2 units of it. As the good is indivisible, each agent can sell 1 unit, buy 1 unit, or neither buy nor sell any. Each agent has some valuation $\theta \in [0, 1]$ for a unit of the good. The valuations are drawn from some continuous distribution F with support $[0, 1]$, and they are agents' private information. A mechanism designer knows the distribution of valuations, F , and wants to design a Bayesian Individually Rational mechanism to maximize its profit.

By Revelation Principle, I focus on direct, Bayesian Incentive Compatible, Bayesian Individually Rational mechanisms. Since there is a continuum of agents, there is no aggregate uncertainty. Thus, it is without loss to focus on direct, Ex-Post Incentive Compatible, Dominant-Strategy Individually Rational mechanisms. Moreover, as agents are symmetric other than their valuations, I restrict attention to anonymous mechanisms; that is, the designer does not condition the mechanism on agents' 'names'.

The designer will choose a mechanism described by the quantities and the transfers as functions of agents' reported valuations. Specifically, the allocations and transfers are given by functions $q : [0, 1] \rightarrow \mathbb{R}$ and $t : [0, 1] \rightarrow \mathbb{R}$, respectively. Thus, an agent who reports his valuation is θ will get $q(\theta)$ units of the good and will pay $t(\theta)$. Note that both the allocation and the transfer can be either positive or negative, depending on whether the agent is buying or selling. Hence, the net utility of the agent with the valuation θ from this mechanism is

$$u(\theta) = \theta \min\{1, q(\theta)\} - t(\theta).$$

As agents have demands for two units, having more than 2 units of the good is the same as having 2 unit for the agent. Therefore, the utility from the traded quantity is capped at 1 unit.

2.2 Analysis

The profit of the marketplace is the expected net payments. Thus, the designer seeks to maximize total payments, given incentive compatibility, individual rationality, and feasibility constraints.

$$\begin{array}{ll}
\max_{(q,t)} & \mathbb{E}_\theta [t(\theta)] \\
\text{s. t.} & \\
(\text{IC}) & \theta \min\{1, q(\theta)\} - t(\theta) \geq \theta \min\{1, q(\theta', \theta)\} - t(\theta') \\
(\text{IR}) & \theta \min\{1, q(\theta)\} - t(\theta) \geq 0 \\
(\text{Individual Feasibility}) & q(\theta) \geq -1 \\
(\text{Aggregate Feasability}) & \mathbb{E}_\theta [q(\theta)] \leq 0
\end{array}$$

In an Online Appendix, I develop a series of lemmata to simplify this problem.² They allow the problem to be restated in terms of the virtual values and virtual costs, defined as follows.

Definition 2.1. An agent with reported valuation θ has virtual value, $\mathcal{V}(\theta)$, and virtual cost, $\mathcal{C}(\theta)$, given by:

$$\mathcal{V}(\theta) = \theta - \frac{(1 - F(\theta))}{f(\theta)} \text{ and } \mathcal{C}(\theta) = \theta + \frac{F(\theta)}{f(\theta)}.$$

Virtual value and virtual cost can be thought of as the marginal revenue and marginal cost. When an agent is a seller, that is, when an agent has a negative allocation, $q(\theta) < 0$, his deduction from the profit of the marketplace is the virtual cost. Similarly, when an agent is a buyer, $q(\theta) > 0$, his contribution to the profit is the virtual value. Then, the problem can be restated in these terms as follows:

$$\begin{array}{ll}
\max_{q(\cdot)} & \mathbb{E} [q(\theta) (\mathbb{1}\{q(\theta) < 0\} \mathcal{C}(\theta) + \mathbb{1}\{q(\theta) > 0\} \mathcal{V}(\theta))] \\
\text{s. t.} & \\
& q(\theta) \text{ is increasing} \\
& q(\theta) \geq -1 \\
& \mathbb{E}_\theta [q(\theta)] = 0
\end{array}$$

Definition 2.2. The distribution of agents' valuations, F is **regular** if both \mathcal{V} and \mathcal{C} are increasing.

The regularity condition guarantees that the marketplace has a decreasing marginal revenue from having additional buyers and increasing marginal cost from having additional sellers. With this definition, we are ready to state the main result of this section, which characterizes the baseline optimal marketplace.

²Online Appendix is available at this link.

Theorem 2.1. *Suppose the distribution F is regular. Then, the optimal mechanism has the allocation rule*

$$q(\theta) = \begin{cases} -1 & \text{if } \theta \leq \underline{\theta} \\ 0 & \text{if } \underline{\theta} < \theta < \bar{\theta} \\ 1 & \text{if } \theta \geq \bar{\theta} \end{cases}$$

and the transfer rule

$$t(\theta) = \begin{cases} -\underline{\theta} & \text{if } \theta \leq \underline{\theta} \\ 0 & \text{if } \underline{\theta} < \theta < \bar{\theta} \\ \bar{\theta} & \text{if } \theta \geq \bar{\theta} \end{cases}$$

where $\underline{\theta}$ and $\bar{\theta}$ satisfies $C(\underline{\theta}) = \mathcal{V}(\bar{\theta})$ and solves the problem

$$\begin{aligned} \max_{\underline{\theta}, \bar{\theta}} \quad & \left[-\underline{\theta}F(\underline{\theta}) + \bar{\theta}(1 - F(\bar{\theta})) \right] \\ \text{s. t.} \quad & F(\underline{\theta}) = 1 - F(\bar{\theta}) \\ & 0 \leq \underline{\theta} \leq \bar{\theta} \leq 1. \end{aligned}$$

The proof can be found in Appendix B.

Notice that each agent below $\underline{\theta}$ sells 1 unit and gets paid $\underline{\theta}$, and each agent above $\bar{\theta}$ buys 1 unit and pays $\bar{\theta}$. If the designer posts $\underline{\theta}$ as the price for selling and $\bar{\theta}$ as the price for buying, and let agents choose what to do, the allocation above represents exactly what the agents would do. Thus, the designer can implement this mechanism in a very straightforward way by posting bid-ask prices.

2.3 The Simple Economics of Optimal Marketplaces

Here, I provide an analysis of the optimal profit-maximizing marketplaces in the same spirit as Bulow and Roberts (1989). They have shown that the optimal auction design problem can be understood as a monopoly pricing problem. In my setting, we will see that the optimal marketplace design problem can be understood as solving two simultaneous monopoly and monopsony problems.

The objective function of the designer is the expected payments of the agents. In the

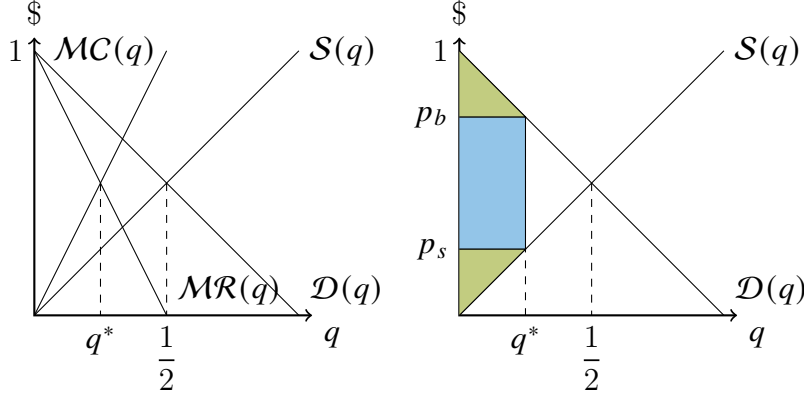


Figure 1: **Profit of the Marketplace**, **Information Rents**.

restatement of the problem, we have seen that an equivalent way of thinking about the expected payments is the expected difference between the virtual values and costs, or expected *virtual surplus*. Thus, I will now use virtual values and costs to find the optimal level of trade graphically.

For each quantity level, q , if the marketplace wants measure q of sellers, the price for selling should be $F^{-1}(q)$. Similarly, to have measure q of buyers, the price for buying should be $F^{-1}(1-q)$. Then, the inverse supply and demand in the economy are given by $S = F^{-1}(q)$ and $D = F^{-1}(1-q)$

Also, for each quantity level, q , let $MC(q) = C(F^{-1}(q))$ and $MR(q) = \mathcal{V}(F^{-1}(1-F(\theta)))$. Here, the virtual cost is the marginal cost of the marketplace so it represents the ‘effective supply’ in the marketplace. Similarly, the virtual value gives us the marginal revenue curve so it represents the ‘effective demand’ for the marketplace. Since the virtual values and costs are the marginal revenue and cost curves for the profit-maximization problem, instead of agents’ willingness to pay (or willingness to get paid), I use their virtual values and costs to find the optimal quantity of trade. This is very similar to using the marginal revenue instead of the demand in the monopoly problem. If the designer knew a buyer’s valuation, she would charge the buyer exactly his valuation. However, since the designer is uninformed about the valuations, she has to charge a buyer something less than his valuation. Similar reasoning holds for the sellers as well. So, each agent who trades, gets some information rent for his private information. This can be seen in the Figure 2.3.

Figure 2.3 shows D , S , MR , and MC together. As the marketplace is maximizing the profit, the optimal level of trade is given by θ^* such that MR and MC are equal to each other. Moreover, the area of the rectangle given by q^* and the difference $p_b - p_s$ is equal to the profit of the marketplace. The triangles above p_b and below p_s are the buyers’ and sellers’ information rents. Finally, the triangle between p_b and p_s to the right of the profit

is the deadweight loss created by (i) incomplete information and (ii) profit-maximization.

3 Decentralized Market

Now, I introduce a decentralized market and give the agents a choice between joining the mechanism or the decentralized market. The decentralized market is modeled as a search market with frictions as in the Diamond-Mortensen-Pissarides Model (Diamond, 1971; Mortensen, 1970; Pissarides, 1979).

Here is how the decentralized market works. Agents who join there are randomly matched to each other. This process is governed by a matching function, adapted from the search theory. Once matched, each pair of agents engage in Nash Bargaining; that is, the agents observe each other's valuations and then split the surplus equally.

Timeline is as follows. A mechanism designer announces a mechanism through which agents can trade the good and invite some of the agents to join it. Upon observing the mechanism, an agent joins either the mechanism or the search market, or does not join either market.

Agents who join the mechanism get what the announced mechanism promises. The mechanism can depend on the set of agents who join it. The simplest such mechanism would be one that operates if agents who join it are exactly those who were invited and shuts down otherwise. Results I obtain here would be valid for an equilibrium under this extreme mechanism. However, in that case it would also be an equilibrium for all agents to join the decentralized market. With proper handling of the off-path payoffs, I obtain an obedience principle: In the unique equilibrium, exactly the agents whom the designer wants to serve join the marketplace. I expand on this in the Section 3.3.

In this market choice game, I look at the subgame perfect equilibrium (SPE) with no bilateral deviations, meaning no pair of agents would prefer to change markets together. As before, I focus on deterministic mechanisms. Moreover, for technical reasons, I assume that the designer invites a closed set of agents to join the marketplace.

I first study these markets under the following assumption that restricts the set of mechanisms the designer can choose. Then, I consider the general case without this assumption. There, I show that this assumption is without loss. Starting the analysis under the assumption makes the environment more accessible; which is why I present the results in this order.

Assumption 3.1. Suppose a convex set of types, $(\underline{\theta}, \bar{\theta})$ joins the search market in the equilibrium.

So, the designer chooses $\underline{\theta}$ and $\bar{\theta}$ optimally, anticipating the agents' best responses to the announced mechanism. I call the equilibria that satisfy this assumption *simple equilibria* and the mechanisms that induce these equilibria *simple mechanisms*.

First, I show that the simple equilibria will be equivalent to posting bid and ask prices, i.e. prices for buying and selling. This structure will make sure that the mechanism has the agents with the high virtual values and the low virtual costs. Then, I prove that this assumption is without loss of profit: This is the structure we would have in the equilibrium without assuming it.

Although most of the paper focuses on a decentralized market with Nash bargaining, I later extend the analysis to a double auction. In this double auction, each agent makes a bid and the agent with the higher bid buys the other agent's endowment and pays the mid-point of their bids. I show that under some restrictions, with uniform distribution, all results obtained for the Nash Bargaining also applies to the environment with the double auction.

3.1 Matching Functions

In this section, I am going to introduce the matching technology in the search market. As in the search theory literature, a matching function determines the efficiency of the search process. Before defining this formally, I introduce some notation.

Suppose Θ^d is the set of agents who join the search market in a strategy profile and μ is the measure with respect to the distribution F . Then, the measure of meetings in the search market will be given by a matching function $M(\mu(\Theta^d))$ as a function of the measure of agents in the search market, $\mu(\Theta^d)$.

In search theory, matching functions are commonly assumed to have constant returns to scale (CRS). This means that doubling the size of the market also doubles the number of meetings. Since we focus on a market where every agent has the same endowment and demand, this is a one-sided market. In this setup, CRS matching functions are simply linear in the size of the market: $M(\mu(\Theta^d)) = m \times \mu(\Theta^d)$ where $m \in [0, 0.5]$ is the efficiency parameter of the matchings. Then, probability that an agent finds a match in the search market, p is equal to $2m$ since the total measure of meetings is $M(\mu(\Theta^d))$ and there will be two agents in each meeting. Notice that this is independent on the set of agents who join the search market as well as the measure of the set. Since there is a one-to-one relationship between p and m , from now, an agents probability of finding a match is simply denoted by p . For the same reason, in this setup, p itself can be thought of as the primitive of the search market and the efficiency parameter of the matching process.

3.2 Payoffs from Search

Let's fix values of $\underline{\theta}$ and $\bar{\theta}$ for now and consider agents' outside options given these values.

If an agent with valuation θ joins the search market, her expected payoff is

$$u^d(\theta) = p\mathbb{P}[x > \theta | x \in [\underline{\theta}, \bar{\theta}]]\mathbb{E}\left[\frac{x - \theta}{2} | x \in [\underline{\theta}, \bar{\theta}], x > \theta\right] + p\mathbb{P}[x < \theta | x \in [\underline{\theta}, \bar{\theta}]]\mathbb{E}\left[\frac{\theta - x}{2} | x \in [\underline{\theta}, \bar{\theta}], x < \theta\right]$$

where p is the probability of matching someone, $\mathbb{P}[x > \theta | x \in [\underline{\theta}, \bar{\theta}]]$ and $\mathbb{P}[x < \theta | x \in [\underline{\theta}, \bar{\theta}]]$ are the probabilities that the match has a higher or lower values, respectively, and $\mathbb{E}\left[\frac{x - \theta}{2} | x \in [\underline{\theta}, \bar{\theta}], x > \theta\right]$ and $\mathbb{E}\left[\frac{\theta - x}{2} | x \in [\underline{\theta}, \bar{\theta}], x < \theta\right]$ are the expected shares of the surplus given the match has a higher or a lower value, respectively. The denominator 2 in the expression for the surplus comes from the fact that each agent gets half of the surplus from the trade.

Suppose $\theta \in [\underline{\theta}, \bar{\theta}]$. Then,

$$\begin{aligned} u^d(\theta) &= p \frac{F(\bar{\theta}) - F(\theta)}{2(F(\bar{\theta}) - F(\underline{\theta}))} \left[\int_{\theta}^{\bar{\theta}} \frac{xf(x)dx}{F(\bar{\theta}) - F(\theta)} - \theta \right] + p \frac{F(\theta) - F(\underline{\theta})}{2(F(\bar{\theta}) - F(\underline{\theta}))} \left[\theta - \int_{\underline{\theta}}^{\theta} \frac{xf(x)dx}{F(\theta) - F(\underline{\theta})} \right] \\ &= \frac{p}{2(F(\bar{\theta}) - F(\underline{\theta}))} \left[\int_{\theta}^{\bar{\theta}} xf(x)dx - \theta[F(\bar{\theta}) - F(\theta)] + \theta[F(\theta) - F(\underline{\theta})] - \int_{\underline{\theta}}^{\theta} xf(x)dx \right] \\ &= \frac{p}{2(F(\bar{\theta}) - F(\underline{\theta}))} \left[\int_{\theta}^{\bar{\theta}} xf(x)dx - \int_{\underline{\theta}}^{\theta} xf(x)dx + \theta[2F(\theta) - F(\bar{\theta}) - F(\underline{\theta})] \right] \end{aligned}$$

Suppose $\theta \leq \underline{\theta}$. Then,

$$\begin{aligned} u^d(\theta) &= \frac{p}{2} \left[\int_{\underline{\theta}}^{\bar{\theta}} \frac{xf(x)dx}{F(\bar{\theta}) - F(\underline{\theta})} - \theta \right] \\ &= \frac{p}{2(F(\bar{\theta}) - F(\underline{\theta}))} \left[\int_{\underline{\theta}}^{\bar{\theta}} xf(x)dx - \theta[F(\bar{\theta}) - F(\underline{\theta})] \right] \end{aligned}$$

Similarly, if $\theta \geq \bar{\theta}$ then,

$$\begin{aligned} u^d(\theta) &= \frac{p}{2} \left[\theta - \int_{\underline{\theta}}^{\bar{\theta}} \frac{xf(x)dx}{F(\bar{\theta}) - F(\underline{\theta})} \right] \\ &= \frac{p}{2(F(\bar{\theta}) - F(\underline{\theta}))} \left[\theta[F(\bar{\theta}) - F(\underline{\theta})] - \int_{\underline{\theta}}^{\bar{\theta}} xf(x)dx \right] \end{aligned}$$

Notice that we write the net utilities from trade, not the total utilities. The total utility would also include the utility they get from consuming their endowment. To be consistent, we will also compute the net utilities mechanism promises to the agents. Hence, we ignore the utility they get from consuming their own endowment and simply focus on net utilities from each market.

We differentiate the utilities to understand their shape:

$$\frac{\partial u^d(\theta)}{\partial \theta} = \begin{cases} -\frac{p}{2} & \text{if } \theta \leq \underline{\theta}, \\ \frac{p(2F(\theta) - F(\underline{\theta}) - F(\bar{\theta}))}{2(F(\bar{\theta}) - F(\underline{\theta}))} & \text{if } \theta \in [\underline{\theta}, \bar{\theta}], \\ \frac{p}{2} & \text{if } \theta \geq \bar{\theta} \end{cases}$$

For $\theta \in [\underline{\theta}, \bar{\theta}]$, we have:

$$\begin{aligned} 2F(\bar{\theta}) &\geq 2F(\theta) \geq 2F(\underline{\theta}) \\ \iff F(\bar{\theta}) - F(\underline{\theta}) &\geq 2F(\theta) - F(\underline{\theta}) - F(\bar{\theta}) \geq F(\underline{\theta}) - F(\bar{\theta}) \\ \iff \frac{1}{2} &\geq \frac{2F(\theta) - F(\underline{\theta}) - F(\bar{\theta})}{2(F(\bar{\theta}) - F(\underline{\theta}))} \geq -\frac{1}{2} \end{aligned}$$

Thus, the slope of the expected utility from the search is always in the interval $[-\frac{p}{2}, \frac{p}{2}]$. This will be helpful when we want to show where the utilities the mechanism offers and the agents expect from search can cross each other.

The Figure 2 shows the utilities from the search market for all agents given that agents in $[\underline{\theta}, \bar{\theta}]$ join the search market.

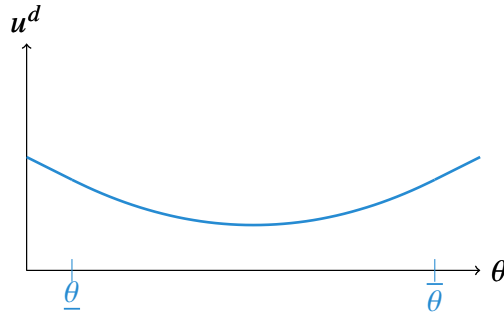


Figure 2: Agents' utilities from their outside options, given $(\underline{\theta}, \bar{\theta})$ join the decentralized market.

3.3 An Obedience Principle

Here I show that in a Simple Equilibrium, the agents who join the centralized marketplace are those who had been invited.

I assume that the designer announces the cutoffs $\underline{\theta}, \bar{\theta}$ as well as the mechanism and commits to operating the announced mechanism as long as (i) the reported types in the mechanism is a subset of the invited agents, (ii) the reported types in the mechanism is union of two intervals $[0, a] \cap [b, 1]$ where $a < \underline{\theta}$ and $\bar{\theta} < b$ and finally (iii) $F(a) = 1 - F(b)$. Otherwise, the designer shuts down the marketplace.

Proposition 3.1. *In a simple equilibrium, set of agents who join the marketplace are exactly those who were invited.*

This result will be useful in showing the uniqueness of the equilibrium. It is proved in Appendix C. However, the intuition is as follows: By committing to shut down the marketplace if there are unwanted type reports and designing an incentive compatible mechanism, the marketplace makes sure that there wouldn't be any unwanted agents there. In other words, the unwanted agents are offered utilities that are less than their expected payoff from the decentralized market. Since the marketplace is incentive compatible, joining there and misreporting cannot be better than joining the decentralized market by transitivity.

The more interesting part of the proof deals with showing that a strict subset of wanted agents joining the marketplace cannot be an equilibrium either. This is accomplished by showing that if there were such an equilibrium, then the realized cutoff types would have to be indifferent between two markets due to profit-maximization. But since the designer originally posts a mechanism that makes the cutoff types of the wanted agents indifferent between these two markets, it is not possible that the cutoff types of this subset are also indifferent.

3.4 Designing the Mechanism

Under Assumption 1, the agents who join the mechanism will have values in $[0, \underline{\theta}]$ and $[\bar{\theta}, 1]$. We can write their utilities as follows, using payoff equivalence from the Online Appendix.

$$u^m(\theta) = \begin{cases} u^m(\bar{\theta}) + \int_{\bar{\theta}}^{\theta} q(x)dx & \text{if } \theta \geq \bar{\theta} \\ u^m(0) + \int_0^{\theta} q(x)dx & \text{if } \theta \leq \underline{\theta} \end{cases}$$

Similarly, we can write the transfers:

$$t(\theta) = \begin{cases} \theta q(\theta) - u^m(\bar{\theta}) - \int_{\bar{\theta}}^{\theta} q(x) dx & \text{if } \theta \geq \bar{\theta} \\ \theta q(\theta) - u^m(0) - \int_0^{\theta} q(x) dx. & \text{if } \theta \leq \underline{\theta} \end{cases}$$

3.4.1 The Profit Function

Now, let us compute the profit from the optimal allocation given the cutoffs. The step-by-step derivation can be followed in the Appendix D but here is the end-result:

$$\begin{aligned} \Pi_{\underline{\theta}, \bar{\theta}} &= \mathbb{P}[\theta \in [0, \underline{\theta}]] \mathbb{E}[t(\theta) | \theta \in [0, \underline{\theta}]] + \mathbb{P}[\theta \in [\bar{\theta}, 1]] \mathbb{E}[t(\theta) | \theta \in [\bar{\theta}, 1]] \\ &= \int_0^{\underline{\theta}} t(\theta) f(\theta) d\theta + \int_{\bar{\theta}}^1 t(\theta) f(\theta) d\theta \\ &= -F(\underline{\theta}) u^d(\underline{\theta}) - (1 - F(\bar{\theta})) u^d(\bar{\theta}) + \int_0^{\underline{\theta}} \left[\left(x + \frac{F(x)}{f(x)} \right) q(x) \right] f(x) dx + \int_{\bar{\theta}}^1 \left[\left(x - \frac{1 - F(x)}{f(x)} \right) q(x) \right] f(x) dx \end{aligned}$$

Earlier, I defined the virtual cost, \mathcal{C} and the virtual value \mathcal{V} as:

$$\mathcal{C}(x) = \left(x + \frac{F(x)}{f(x)} \right) \text{ and } \mathcal{V}(x) = \left(x - \frac{1 - F(x)}{f(x)} \right).$$

I continue to assume both of them are increasing and call such distributions regular.

3.4.2 The Constraints

First of all, we are assuming that $0 \leq \underline{\theta} \leq \bar{\theta} \leq 1$, since the support of the distribution of values is $[0, 1]$.

Second, the feasibility requires that

$$\int_0^{\underline{\theta}} q(x) f(x) dx + \int_{\bar{\theta}}^1 q(x) f(x) dx \leq 0.$$

Third, we need $-1 \leq q(\theta) \leq 1$ for all agents respectively as they have a unit of endowment and up to two units of unit demand, in total, by assumption. In fact, since the good is indivisible, this means each agent can have either -1 , 0 or 1 as their allocation.

Fourth, we know that for incentive compatibility in the mechanism, we need the allocation to be increasing.

Finally, we need to consider the implications of the individual rationality.

Individual Rationality Notice that in any profit maximizing mechanism, individual rationality (IR) constraint for at least one type of agent in each segment who joins the mechanism ($[0, \underline{\theta}]$ and $[\bar{\theta}, 1]$) must bind. If not, then uniformly increasing the payment of all agents in the particular segment without a binding IR until there is a binding constraint increases the profit. Moreover, the binding IR constraints in the optimal mechanism must be $\underline{\theta}$ and $\bar{\theta}$. To see this, notice that under the Assumption 1, we want to construct an equilibrium such that agents in $[0, \underline{\theta}]$ and $[\bar{\theta}, 1]$ join the mechanism while the rest of the agents join the search market. Since agents will choose the market that offers them a higher utility, this means, $\underline{\theta}$ and $\bar{\theta}$ must be the points where u^d and u^m cross each other such that on $[0, \underline{\theta}]$ and $[\bar{\theta}, 1]$, u^m should be higher while on $(\underline{\theta}, \bar{\theta})$, u^d must be higher. Then, we must have: $u^d(\underline{\theta}) = u^m(\underline{\theta}) = u^m(0) + \int_0^{\underline{\theta}} q(x)dx$ so that $u^m(0) = u^d(\underline{\theta}) - \int_0^{\underline{\theta}} q(x)dx$ and $u^m(\bar{\theta}) = u^d(\bar{\theta})$.

Moreover, binding IR constraints for $\underline{\theta}$, $\bar{\theta}$, together with the monotonicity of the allocations significantly simplifies the problem since it can be used to obtain the optimal allocation in a simple equilibrium. Notice that around $\underline{\theta}$, the utility from the mechanism should have a left derivative below $-0.5p$:

$$\begin{aligned}
u^m(0) + \int_0^{\underline{\theta}} q(x)dx &\geq u^d(\underline{\theta}) \iff \\
u^d(\underline{\theta}) - \int_0^{\underline{\theta}} q(x)dx + \int_0^{\underline{\theta}} q(x)dx &\geq u^d(\underline{\theta}) \iff \\
u^d(\underline{\theta}) - \int_{\underline{\theta}}^{\bar{\theta}} q(x)dx &\geq u^d(\underline{\theta}) \iff \\
p \left[\frac{\int_{\underline{\theta}}^{\bar{\theta}} x f(x)dx}{2(F(\bar{\theta}) - F(\underline{\theta}))} - \frac{\underline{\theta}}{2} \right] - \int_{\underline{\theta}}^{\bar{\theta}} q(x)dx &\geq p \left[\frac{\int_{\underline{\theta}}^{\bar{\theta}} x f(x)dx}{2(F(\bar{\theta}) - F(\underline{\theta}))} - \frac{\underline{\theta}}{2} \right] \iff \\
p \frac{\bar{\theta} - \underline{\theta}}{2} &\geq \int_{\underline{\theta}}^{\bar{\theta}} q(x)dx \iff \\
p \frac{\bar{\theta} - \underline{\theta}}{2} &\geq u^m(\underline{\theta}) - u^m(\bar{\theta}).
\end{aligned}$$

Since only possible allocations are -1 , 0 , and 1 , this means we must have $q(\underline{\theta}) = -1$. Moreover, due to monotonicity of the allocation for incentive compatibility, this would only be true if for each $\theta \in [0, \underline{\theta}]$, $q(\theta) = -1$. Following the same steps around $\bar{\theta}$ also shows that $q(\theta) = 1$ for each $\theta \in [\bar{\theta}, 1]$. We note the observations we have made here in the following lemma.

Lemma 3.1. *In a simple equilibrium with cutoffs $\underline{\theta}, \bar{\theta}$, we have:*

$$q(\theta) = \begin{cases} -1 & \text{if } \theta \leq \underline{\theta} \\ 1 & \text{if } \theta \geq \bar{\theta} \end{cases}$$

3.5 Simple Equilibrium

Now we are going to show that a simple equilibrium exists.

Using the observations from the previous subsection, the profit from any such pairs of thresholds can be written as follows.³

$$\Pi = \frac{1}{2} \left([2 - p] \left[-\underline{\theta}F(\underline{\theta}) + \bar{\theta}(1 - F(\bar{\theta})) \right] - p \left[F(\underline{\theta}) - (1 - F(\bar{\theta})) \right] E[\theta | \underline{\theta} \leq \theta \leq \bar{\theta}] \right)$$

Our constraints are $0 \leq \underline{\theta} \leq \bar{\theta} \leq 1$ and $F(\underline{\theta}) \geq 1 - F(\bar{\theta})$. First, we can't have $\bar{\theta} < m(F)$ where $m(F)$ is the median of F since that would require $0.5 > F(\bar{\theta}) \geq F(\underline{\theta}) \geq 1 - F(\bar{\theta}) \geq 0.5$. Second, given that Π is strictly decreasing in $\underline{\theta}$, $F(\underline{\theta}) \geq 1 - F(\bar{\theta})$ should bind⁴. Third, $\underline{\theta} \leq m(F) \leq \bar{\theta}$ as a result of previous two observations. Thus, for a strictly increasing F , this is essentially a single parameter problem with a continuous objective and a compact domain. Hence, it has a solution by Weierstrass Theorem. Moreover, the solution is interior in the sense that the constraints $\underline{\theta} \leq m(F) \leq \bar{\theta}$ don't bind. If they did, then the profit would be 0 while it is possible to achieve a positive profit when they don't bind.⁵ The Appendix E shows this in more details but it is easy to see that when the feasibility condition binds, the expectation term disappears from the profit. Then, the profit in this equilibrium is equal to the profit when there was no search market, times a constant, $\frac{2-p}{2}$. Thus, the solution must still have $C(\underline{\theta}) = \mathcal{V}(\bar{\theta})$, as shown in the Theorem 2.1.

Theorem 3.1. *Suppose F is regular. Then, there exists $\underline{\theta}, \bar{\theta}$ such that in the simple equilibrium,*

- *agents in $[0, \underline{\theta}]$ and $[\bar{\theta}, 1]$ join the mechanism,*
- *agents in $(\underline{\theta}, \bar{\theta})$ join the search market,*
- *$C(\underline{\theta}) = \mathcal{V}(\bar{\theta})$ and $F(\underline{\theta}) = 1 - F(\bar{\theta})$.*

³The details can be followed in Appendix E.

⁴We show this in the Appendix E.

⁵Again, Appendix E shows this in more detail.

To illustrate the utilities agents are offered in the mechanism and expect from the search market, suppose $p = 1$ and $F(\theta) = \theta$ so that everyone gets a meeting and agents are uniformly distributed over the unit interval. Then, the optimal cutoffs are $\underline{\theta} = 0.25$ and $\bar{\theta} = 0.75$. Then, Figure 3.5 show the utilities agents can expect from either market.

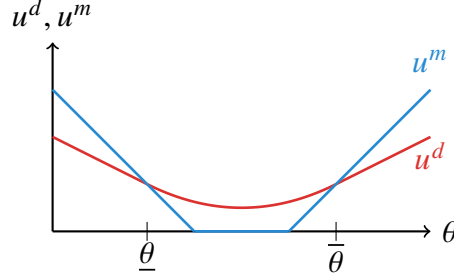


Figure 3: The utilities from the search market and the optimal mechanism under the simple equilibrium.

The preceding theorem describes the structure of the simple equilibrium. However, it doesn't completely describe the mechanism that induces this equilibrium. Although the allocations and transfers for the agents who join the mechanism are exactly pinned down by the discussions from the previous sections, especially in Lemma 3.1, we haven't described what is offered to agents in $[\underline{\theta}, \bar{\theta}]$ except for saying that they are offered lower utilities than u^d , since it is the only thing that matters for the equilibrium behavior. There are in fact several mechanisms that would induce the same equilibrium that only differ in the off-path payoffs: As long as what the mechanism promises to agents in $[\underline{\theta}, \bar{\theta}]$ is less than u^d , the outcomes would remain the same. The next proposition describes one such mechanism. As described before, it essentially offers ask and bid prices, i.e., prices for selling and buying as well as the option to not trade.

Proposition 3.2. *In the simple equilibrium, the mechanism the designer offers has the following allocation and transfer rules:*

$$\begin{aligned}
q(\theta) &= \begin{cases} -1 & \text{if } \theta \leq \frac{p\mathbb{E}[x|x \in [\underline{\theta}, \bar{\theta}]] + (2-p)\underline{\theta}}{2} \\ 0 & \text{if } \frac{p\mathbb{E}[x|x \in [\underline{\theta}, \bar{\theta}]] + (2-p)\underline{\theta}}{2} \leq \theta \leq \frac{p\mathbb{E}[x|x \in [\underline{\theta}, \bar{\theta}]] + (2-p)\bar{\theta}}{2} \\ 1 & \text{if } \theta \geq \frac{p\mathbb{E}[x|x \in [\underline{\theta}, \bar{\theta}]] + (2-p)\bar{\theta}}{2} \end{cases} \\
t(\theta) &= \begin{cases} -\frac{p\mathbb{E}[x|x \in [\underline{\theta}, \bar{\theta}]] + (2-p)\underline{\theta}}{2} & \text{if } \theta \leq \frac{p\mathbb{E}[x|x \in [\underline{\theta}, \bar{\theta}]] + (2-p)\underline{\theta}}{2} \\ 0 & \text{if } \frac{p\mathbb{E}[x|x \in [\underline{\theta}, \bar{\theta}]] + (2-p)\underline{\theta}}{2} \leq \theta \leq \frac{p\mathbb{E}[x|x \in [\underline{\theta}, \bar{\theta}]] + (2-p)\bar{\theta}}{2} \\ \frac{p\mathbb{E}[x|x \in [\underline{\theta}, \bar{\theta}]] + (2-p)\bar{\theta}}{2} & \text{if } \theta \geq \frac{p\mathbb{E}[x|x \in [\underline{\theta}, \bar{\theta}]] + (2-p)\bar{\theta}}{2} \end{cases}
\end{aligned}$$

Proof is discussed in the Appendix E.1. Essentially, it computes the transfers for the agents who join the mechanism and then extends the allocation and the transfer rules to the rest of the agents in a way that the allocations are increasing and we have $u^m(\theta) < u^d(\theta)$ for each $\theta \in (\underline{\theta}, \bar{\theta})$.

3.6 Profit of the Marketplace

Next, we are going to obtain the relationship between the profit in this equilibrium and the profit of the marketplace when there is no decentralized market.

Proposition 3.3. *Suppose F is regular. Then, the profit of the marketplace in coexistence (described in Theorem 3.1) is $1 - \frac{p}{2} = 1 - m$ times the profit the marketplace would make if there were no search market.*

Mathematically, this proposition is a direct consequence of the expression for the profit we obtain in the Appendix E.

It is surprising that the ratio of the profits is completely independent of the distribution, F . The ratio only depends on the probability that an agents finds a match in the search market, p . Clearly, the profit is decreasing in p : As the matching process becomes more efficient, the decentralized market becomes more attractive and the profit of the marketplace decreases. However, even when $p = 1$, which means when every agent finds a match in the decentralized market, the profit of the marketplace is half of its profit when it operate on its own. This is because even when each agent in the decentralized market gets a match with certainty, the matches themselves may not be very efficient. An agent might meet someone whose valuation is very close to her own. However, marketplace solves this problem and makes profit by creating efficient matches. At the other extreme, $p = 0$, when there is no

match in the decentralized market, the marketplace makes the profit it makes on its own, since agents cannot trade in the decentralized market anymore.

3.7 Efficiency of the Coexistence

Now, we focus on a concrete distribution for valuations to obtain some comparisons of the profit. In particular, we assume that $F(\theta) = \theta$ so that agents' valuations are drawn from the uniform distribution.

Proposition 3.4. *Suppose $F = U[0, 1]$. Then, the total welfare under the simple equilibrium is greater than when either market operates on its own.*

It is clear that the coexistence improves the gains from trade over the centralized marketplace alone: The same agents will trade in the marketplace, with or without the decentralized market. However, in the coexistence equilibrium, agents who do not get to trade in the marketplace join the search market and create some surplus. Thus, coexistence equilibrium improves the efficiency over the marketplace alone. What remains to be shown is that it also improves over the pure search market. We show this in the Appendix F.1.

Intuition behind this result is that the search market extends the extensive margin of trade while the centralized marketplace extends the intensive margin. Thus, when they operate together, everyone gets to trade with some probability (which causes this to create more surplus than the centralized marketplace alone, since it does not allow some agents to trade) and some agents trade for certain (which is why the coexistence equilibrium provides a higher total welfare than the pure search market).

Although the proposition only mentions the uniform distribution, the result is true for many other distribution. Next, we provide a condition on the distribution of the valuations which guarantees that the coexistence is more efficient than either the pure search market or the centralized marketplace alone. This condition is satisfied by most commonly used distributions.

Assumption 3.2. Under F , the following inequality holds:

$$2\mathbb{E}[\theta F(\theta)|\theta \in [\underline{\theta}, \bar{\theta}]] + \mathbb{E}[\theta|\theta > \bar{\theta}] \geq \mathbb{E}[\theta|\theta \in [\underline{\theta}, \bar{\theta}]] + \mathbb{E}[\theta F(\theta)|\theta \leq \underline{\theta}] + \mathbb{E}[\theta F(\theta)|\theta \geq \bar{\theta}],$$

where $\underline{\theta}$ and $\bar{\theta}$ are the optimal cutoffs for a simple equilibrium, given the distribution F .

As this condition depends on the optimal cutoffs, it is difficult to show that it is satisfied by any regular distribution. However, it is in fact hard to find a regular distribution that

doesn't satisfy it. Standard Normal Distribution, Logistic Distribution, Exponential Distribution, Standard Beta Distribution are among the distributions that has this property⁶.

Proposition 3.5. *If F satisfies Assumption 3.2, then the total welfare under the simple equilibrium is greater than when either market operates on its own.*

I prove this in the Appendix F.2.

3.8 Unrestricted Equilibrium

In this section, we remove the restriction to the ‘simple’ market structure that required an interval of agents to join the search market. Here, we allow the designer to invite any closed set of agents. As the following proposition shows, the restriction of the simple equilibrium is without loss of generality. Thus, it shows that the Assumption 3.1 was without loss of generality. Therefore, everything we have studied so far holds for all equilibria where both markets are active.

Theorem 3.2. *If both markets are active in an equilibrium, then the sets of agents in the search market must be an interval.*

This result is extremely helpful in reducing ‘the dimensionality’ of the problem. Without the structure provided by this theorem, searching for an equilibrium would require considering the consequences of every possible segmentation of a continuum of agents across markets. However, when we know that the agents in the search market will be an interval, the search is essentially reduced to finding a pair of cutoffs.

The proof is covered in the Appendix H. The intuition is as follows. The utilities from the decentralized market would be U -shaped, under any segmentation. The utilities from any mechanism would be ‘bucket-shaped’; meaning consisting of a decreasing line segment, a constant line segment, and an increasing line segment, in this order. Since under any segmentation, each type would choose the market that offers higher utilities, there are two possible segmentations that can happen with these utility shapes: Either one interval joins the decentralized market or the union of two intervals that satisfy some conditions join there. However, excluding the union of two intervals is not profit-maximizing: This would require having some agents with intermediate types in the marketplace who have small gains from trade. Thus, the unique equilibrium structure is excluding an interval of intermediate types.

⁶Mathematica codes for computations that show these distributions satisfy this assumption are available upon request.

3.9 Uniqueness of the Equilibrium

So far, we have seen that the simple equilibrium is the unique coexistence equilibrium. This leaves the question of whether there can be equilibria where only one of the markets is active. First, it is relatively easy to argue that there cannot be any equilibrium where all agents join the centralized marketplace. This is because the marketplace will exclude some agents from trade even if they are in the marketplace. But then, these agents would have a profitable bilateral deviation to the decentralized market where they would get a positive payoff, rather than getting 0 in the marketplace. Thus, the only other possibility is an equilibrium where all agents join the decentralized market. Next, we show that, everyone joining the decentralized market is not an equilibrium either.

Proposition 3.6.

1. *There is no equilibrium where all agents join the centralized marketplace.*
2. *There is no equilibrium where all agents join the decentralized market.*
3. *Thus, the simple equilibrium is the unique equilibrium.*

The uniqueness of the equilibrium increases the robustness of the predictions about the markets I study.

The proposition is proved in Appendix I. Here is the intuition. The equilibrium concept I focused on here does not allow there to be a pair of agents who would rather deviate to another market bilaterally. I show that even if everyone joins the decentralized market, the payoffs the centralized market offers to the lowest and the highest types are greater than what they expect in the decentralized market. Thus, they would deviate to the centralized marketplace. Therefore, everyone joining the decentralized market cannot be an equilibrium. On the other hand, if everyone joined the centralized marketplace, it must be the case that the marketplace still excludes the intermediate types from the trade. (Otherwise, the profit is at most 0). But given that the intermediate types have 0 utility in the centralized market, any pair of them would strictly prefer deviating to the decentralized market. Thus, there is no equilibrium without coexistence. Finally, since the simple equilibrium exists and it is the unique equilibrium with coexistence, it is indeed the unique equilibrium.

3.10 Simple Equilibrium with Double Auction

Suppose the agents who join the decentralized market are randomly matched to each other and then they participate in a double auction. The random matching process operates in the same manner as before. The double auction works as follows. Each agent makes some

bids, b_i . Agent with the higher bid buys the other's endowment and pays $\frac{b_i+b_j}{2}$, when bids are given by b_i and b_j ; so they are trading a unit of the good at the midpoint of the bids. This is a special case of the 'simple trading rule' studied by Cramton et al. (1987). They show that this game has a symmetric equilibrium where the bids are increasing in agents' valuations. Thus, the equilibrium is ex-post efficient in the sense that, the agent who values the good more ends up with the whole quantity. Moreover, Kittsteiner (2003) has shown that the symmetric equilibrium characterized by Cramton et al. (1987) is indeed the unique equilibrium. Now, I will describe the bidding functions in this equilibrium as well as agents' expected payoffs from the game.

Suppose the distribution of agents who participates in this double auction is given by some CDF, G . Suppose G is strictly increasing on its support, $[\underline{\theta}, \bar{\theta}]$ and is differentiable. Then, agents' bids in the unique equilibrium are given by:

$$b(\theta) = \theta - \frac{\int_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2}$$

It follows from the Proposition 5 of Cramton et al. (1987) that following this bidding strategy constitutes an equilibrium. Theorem 1 in Kittsteiner (2003) further shows that there is no other equilibrium.

As before, we will focus on simple mechanisms that exclude an interval of agents: We suppose agents in $[\underline{\theta}, \bar{\theta}]$ join the decentralized market and the rest of the agents join the centralized marketplace. Then, $G(\theta) = \frac{F(x) - F(\underline{\theta})}{F(\bar{\theta}) - F(\underline{\theta})}$ on its support $[\underline{\theta}, \bar{\theta}]$.

To compute all agents' expected payoffs from the decentralized market, we need to know the optimal bids of agents whose valuations lie outside $[\underline{\theta}, \bar{\theta}]$. For agents with values below $\theta \leq \underline{\theta}$, if they join the decentralized market and get matched to someone, the best response is to bid $b(\underline{\theta})$ and for agents with values above $\bar{\theta}$, the best response is to bid $b(\bar{\theta})$. I show this in two steps. First, I show that for any agent, the optimal bid must lie in $[b(\underline{\theta}), b(\bar{\theta})]$. Then, I show that for agents with values less than $\underline{\theta}$ the optimal bid is $b(\underline{\theta})$ while for agents with values above $\bar{\theta}$, the optimal bid is $b(\bar{\theta})$. This is stated in the next lemma and proved in the Appendix J.

Lemma 3.2. *For each $\theta \in [0, \underline{\theta}]$, $b(\theta) = b(\underline{\theta})$ and for each $\theta \in [\bar{\theta}, 1]$, $b(\theta) = b(\bar{\theta})$.*

Agents' utilities from the decentralized market:

Suppose $\theta \in [\underline{\theta}, \bar{\theta}]$. If the agent has the lower value, she has the lower bid, since the bidding function is monotone. Then, the agent gives up her endowment but gets paid. If the

agent has the higher value, then, she has the higher bid so she gets the other's endowment and pays for it. Thus, expected payoff is

$$\begin{aligned}
u^{da}(\theta) &= pG(\theta) \left[\theta - \int_{\underline{\theta}}^{\theta} \frac{\frac{1}{2}[b(x) + b(\theta)]g(x)dx}{G(\theta)} \right] + p(1 - G(\theta)) \left[\int_{\theta}^{\bar{\theta}} \frac{\frac{1}{2}[b(x) + b(\theta)]g(x)dx}{1 - G(\theta)} - \theta \right] \\
&= p\theta[2G(\theta) - 1] + \frac{p}{2} \int_{\underline{\theta}}^{\bar{\theta}} [b(x) + b(\theta)]g(x)dx - \frac{p}{2} \int_{\underline{\theta}}^{\theta} [b(x) + b(\theta)]g(x)dx \\
&= p\theta[2G(\theta) - 1] + \frac{p}{2}(1 - 2G(\theta))b(\theta) + \frac{p}{2} \int_{\theta}^{\bar{\theta}} b(x)g(x)dx - \frac{p}{2} \int_{\underline{\theta}}^{\theta} b(x)g(x)dx
\end{aligned}$$

If $\theta \leq \underline{\theta}$, then, as above argument shows, the optimal bid is $b(\underline{\theta})$ and the expected payoff from the decentralized market is:

$$u^{da}(\theta) = p \left[\int_{\underline{\theta}}^{\bar{\theta}} \frac{1}{2} [b(\underline{\theta}) + b(x)]g(x)dx - \theta \right]$$

Similarly, agents with $\theta \geq \bar{\theta}$ bid $b(\bar{\theta})$ and get the expected payoff:

$$u^{da}(\theta) = p \left[\theta - \int_{\underline{\theta}}^{\bar{\theta}} \frac{1}{2} [b(\bar{\theta}) + b(x)]g(x)dx \right]$$

Next, we compute $u^{da}(\underline{\theta})$ and $u^{da}(\bar{\theta})$ by using the formulas for $b(\cdot)$, as the IR constraints of $\underline{\theta}$ and $\bar{\theta}$ will again play an important role in the equilibrium. The details can be found in the Appendix K but here is the end result:

$$\begin{aligned}
u^{da}(\underline{\theta}) &= \frac{p}{2} \left[-\underline{\theta} + 4 \int_{\underline{\theta}}^{G^{-1}(\frac{1}{2})} \left[G(x) - \frac{1}{2} \right]^2 dx + \int_{\underline{\theta}}^{\bar{\theta}} b(x)g(x)dx \right] \\
u^{da}(\bar{\theta}) &= \frac{p}{2} \left[\bar{\theta} + 4 \int_{G^{-1}(\frac{1}{2})}^{\bar{\theta}} \left[G(x) - \frac{1}{2} \right]^2 dx - \int_{\underline{\theta}}^{\bar{\theta}} b(x)g(x)dx \right]
\end{aligned}$$

In the simple equilibrium, we are going to make the IR constraints of $\underline{\theta}$ and $\bar{\theta}$ bind, as

otherwise decreasing the payment until they bind increases the profit. For these cutoffs to work, we need the slope of the utility from the decentralized market, $\frac{\partial u^{da}(\theta)}{\partial \theta}$ for types below $\underline{\theta}$ to be greater than -1 , since -1 is the allocation they will be offered in the mechanism and we want u^{da} to be less than u^m on this region. Moreover, we need the slope of u^{da} to be greater than -1 around $\underline{\theta}$ and the slope should be increasing (thus the utility function should be convex). Finally, we need the slope of the utility from the decentralized market to be less than 1 for agents with values above $\bar{\theta}$.

Lemma 3.3.

$$\frac{\partial u^{da}(\theta)}{\partial \theta} = \begin{cases} -p & \text{if } \theta \leq \underline{\theta} \\ p(G(\theta) - 1) & \text{if } \underline{\theta} \leq \theta \leq \bar{\theta} \\ p & \text{if } \theta \geq \bar{\theta} \end{cases}$$

Proof can be found in the Appendix L.

Since this is between -1 and 1 , for agents with values in $[\underline{\theta}, \bar{\theta}]$, the designer can indeed offer lower utilities to these agents. One way of doing this would be offering the allocation -1 for agents between $\underline{\theta}$ and $G^{-1}(\frac{1}{2})$ and allocation 1 for agents between $G^{-1}(\frac{1}{2})$ and $\bar{\theta}$. This would make sure these agents are offered utilities lower than their expected payoff from the decentralized market since $u^m(\underline{\theta}) = u^{da}(\underline{\theta})$ and $u^m(\bar{\theta}) = u^{da}(\bar{\theta})$. (This may offer utilities below zero for some agents. The designer may not be concerned about this, since these agents are not wanted anyway. However, if the designer wishes the mechanism to offer nonnegative utilities, this can be achieved by flattening the utility when it reaches zero, as the Proposition 3.2 does.)

Now, we look at the profit function. As before, it is equal to

$$\begin{aligned} \Pi_{\underline{\theta}, \bar{\theta}} &= \mathbb{P}[\theta \in [0, \underline{\theta}]] \mathbb{E}[t(\theta) | \theta \in [0, \underline{\theta}]] + \mathbb{P}[\theta \in [\bar{\theta}, 1]] \mathbb{E}[t(\theta) | \theta \in [\bar{\theta}, 1]] \\ &= \int_0^{\underline{\theta}} t(\theta) f(\theta) d\theta + \int_{\bar{\theta}}^1 t(\theta) f(\theta) d\theta \\ &= -F(\underline{\theta}) u^d(\underline{\theta}) - (1 - F(\bar{\theta})) u^d(\bar{\theta}) + \int_0^{\underline{\theta}} C(x) q(x) f(x) dx + \int_{\bar{\theta}}^1 \mathcal{V}(x) q(x) f(x) dx \\ &= -F(\underline{\theta}) u^d(\underline{\theta}) - (1 - F(\bar{\theta})) u^d(\bar{\theta}) - \int_0^{\underline{\theta}} C(x) f(x) dx + \int_{\bar{\theta}}^1 \mathcal{V}(x) f(x) dx \end{aligned}$$

First two terms are the compensations for agents to join the centralized marketplace,

while the last two terms are the total virtual surplus. We know expression for the total surplus from before because that part is unchanged:

$$-\int_0^{\underline{\theta}} C(x)f(x)dx + \int_{\bar{\theta}}^1 \mathcal{V}(x)f(x)dx = -\underline{\theta}F(\underline{\theta}) + \bar{\theta}(1 - F(\bar{\theta})).$$

Next, we study the compensations, since now they will be different from what we had for the Nash bargaining.

$$\begin{aligned} & F(\underline{\theta})u^d(\underline{\theta}) + (1 - F(\bar{\theta}))u^d(\bar{\theta}) \\ &= F(\underline{\theta})p \left[\frac{1}{2} \left[b(\underline{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} b(x)g(x)dx \right] - \underline{\theta} \right] + (1 - F(\bar{\theta}))p \left[\bar{\theta} - \frac{1}{2} \left[b(\bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} b(x)g(x)dx \right] \right] \\ &= \frac{p}{2} \left[F(\underline{\theta}) \left[b(\underline{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} b(x)g(x)dx - 2\underline{\theta} \right] + (1 - F(\bar{\theta})) \left[2\bar{\theta} - b(\bar{\theta}) - \int_{\underline{\theta}}^{\bar{\theta}} b(x)g(x)dx \right] \right] \\ &= p \left[-\underline{\theta}F(\underline{\theta}) + \bar{\theta}(1 - F(\bar{\theta})) \right] \\ &+ \frac{p}{2} \left[\left[F(\underline{\theta}) + F(\bar{\theta}) - 1 \right] \int_{\underline{\theta}}^{\bar{\theta}} b(x)g(x)dx + F(\underline{\theta})b(\underline{\theta}) - (1 - F(\bar{\theta}))b(\bar{\theta}) \right] \end{aligned}$$

Remember that the optimal bidding strategy is given by

$$b(\theta) = \theta - \frac{\int_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2}$$

with $G(\theta) = \frac{F(x) - F(\underline{\theta})}{F(\bar{\theta}) - F(\underline{\theta})}$ on $[\underline{\theta}, \bar{\theta}]$ and $G^{-1}(\frac{1}{2}) = F^{-1}\left(\frac{F(\underline{\theta}) + F(\bar{\theta})}{2}\right)$. Although it is relatively easy to show that a simple equilibrium exists with arbitrary distributions, the general solution to the profit maximization problem is too complicated to provide some useful comparative statics. Hence, I focus on the uniform distribution, $U[0, 1]$ from here on to be able to find a closed form solution to the problem above.

Using the uniform distribution, with some algebra (see Appendix M), we can show that

$$b(\theta) = \frac{\bar{\theta} + \underline{\theta} + 4\theta}{6},$$

$$\int_{\underline{\theta}}^{\bar{\theta}} b(x)g(x)dx = \frac{\bar{\theta} + \underline{\theta}}{2}.$$

Now, we can plug these back into the expression for the profit to see that it is decreasing in $\underline{\theta}$. Thus, the feasibility must bind, which means $\underline{\theta} = 1 - \bar{\theta}$. Using this, we simplify the profit further and obtain the following simple expression for the profit. (The derivations can be followed in Appendix N.)

$$\Pi_{\underline{\theta}, \bar{\theta}} = \frac{6 - 5p}{6} [\underline{\theta}(\bar{\theta} - \underline{\theta})] = \frac{6 - 5p}{6} \Pi^M$$

Clearly, this problem has an interior solution, which is the same as the solution of the problem of the marketplace when it operated on its own: The profits in two cases are equal up to a constant multiplier. Thus, almost everything we have seen under the Nash bargaining hold here with the uniform distribution. I state them for this environment as well for completeness.

Theorem 3.3. *There exists $\underline{\theta}, \bar{\theta}$ such that in the simple equilibrium,*

- *agents in $[0, \underline{\theta}]$ and $[\bar{\theta}, 1]$ join the mechanism,*
- *agents in $(\underline{\theta}, \bar{\theta})$ join the decentralized market with double auction,*
- *$C(\underline{\theta}) = \mathcal{V}(\bar{\theta})$ and $F(\underline{\theta}) = 1 - F(\bar{\theta})$.*

Proposition 3.7. *The profit of the marketplace in coexistence is $1 - \frac{5p}{6} = 1 - \frac{5m}{3}$ times the profit the marketplace would make if there were no decentralized market with double auction.*

Proposition 3.8. *The total welfare under the simple equilibrium with the double auction is greater than when either market operates on its own.*

Proposition 3.9. *Same agents trade in the marketplace, with or without the decentralized market.*

4 Multiple Designers

A natural question to ask at this point is how the equilibrium would change if instead of one centralized marketplace competing with a decentralized market, there were two centralized marketplaces competing with each other. This is akin to the models extensively studied by the competing platforms literature. However, it is worthwhile to understand what the structure of the competition would look like within the framework of this paper.

Mechanism design problems with multiple designers are notoriously difficult to solve. The difficulty lies in the fact that when there are multiple designers, each one wants to condition her mechanism to the others' and neither of them wants to post a mechanism that only depends on the agents' types. As a result, revelation principle does not hold in environments with multiple designers. However, despite recent developments (Feng and Hartline, 2018), mechanism design without revelation principle is itself a notorious problem, even with a single designer.

One way the literature has dealt with this problem of infinite regress between competing mechanism is to explicitly assume that the designers each choose a direct mechanism. Although this is not without loss anymore, it at least provides a tractable way to study the problem. I will first follow this literature and then illustrate what else can be achieved with a non-direct mechanism.

Suppose there are n marketplace designers competing with each other for agents' participation. Each of them is a profit-maximizer. Suppose they are restricted to choose direct mechanisms. The timeline is as follows: First, designers announce their mechanisms simultaneously. Then, agents choose which marketplace to join simultaneously. Finally, trades in each marketplace happens simultaneously.

From the analysis of the Online Appendix⁷, we know that this is equivalent to saying that each designer i chooses a pair of prices (bid-ask prices) p_b^i and p_s^i for buying and selling in the marketplace i , respectively.

Since agents only care about their valuations and the prices they pay or get paid, this environment is essentially a model of Bertrand competition where roles as buyers and sellers is endogenously determined. The reasoning from the Bertrand Equilibrium still holds: If some designer i makes a positive profit, then another designer j can offer slightly more favorable prices to agents than i does and increase her own profit discontinuously by recruiting the agents who were trading on the marketplace i . Thus there cannot be an equilibrium where a designer makes a positive profit. Conversely, there cannot be an equilibrium where a designer

⁷Here is a link to the Online Appendix: https://berkidem.com/documents/coexistence_online_appendix.pdf.

makes a negative profit: It is always feasible to make 0 profit but setting a prohibitively high price to everyone.

Next, I argue that there are equilibria where all designers make 0 profit: Consider the strategy profiles where (i) at least one of the marketplaces set $p_b = p_s = m(F)$ where $m(F)$ is the median of the distribution of the valuations, (ii) the rest of the marketplaces either shut down or set prohibitively high prices, and (iii) all agents uniformly randomize over marketplaces where $p_b = p_s = m(F)$. Clearly, none of the designers or agents can improve their payoffs strictly and all such strategy profiles constitute equilibria. Finally, there is no equilibrium where all marketplaces are inactive: If this happened, then each marketplace would simply deviate to announcing the baseline mechanism and make a positive profit.⁸

Notice that $m(F)$ is the price that would clear the market if we had a competitive market: Setting the price at $m(F)$ makes sure that the supply is equal to demand. Thus, the equilibria of this game replicate the Walrasian equilibrium thanks to the nature of the competition considered here.

We have seen that introducing another profit-maximizing marketplace reduced the profits to zero. In the previous section, we have observed that when we instead considered a competing decentralized market, the profit only decreased by a constant ratio, always less than the half. This stark difference is a result of the restriction to direct mechanisms when they are not without loss of generality. Next, I will briefly discuss the simplest way in which mechanisms can depend on each other and how it allows the designers to recoup the profits.

Suppose now instead of simply announcing a pair of prices, the designers announce a pair of prices, and price-matching guarantees. In particular, the designer i announces prices p_b^i and p_s^i but promises to honor the prices $\min_j \{p_b^j : j = 1, \dots, n\}$ and $\max_j \{p_s^j : j = 1, \dots, n\}$ as well; so each agent can get the lowest price for buying and the highest price for selling from each marketplace. This is in line with introducing price-matching guarantees in a Bertrand Competition setting and obtaining an equilibrium with monopoly pricing. Similarly, in the framework of this paper, it is seen that there will be an equilibrium in which all designers announce the prices that are equivalent to the baseline mechanism and all agents uniformly randomize over all the marketplaces. Notice that, in this case, the designers are splitting the baseline profit equally. Thus, they essentially operate as a cartel. It is worth noting that this is not the unique equilibrium in this environment. Specifically, it is still an equilibrium for each designer to announce the median of the valuations as the posted price for both the buyers and sellers. However, requiring coalition-proofness for the designers as well would

⁸Notice that there are many other payoff-equivalent equilibria. For instance, for $n = 2$, suppose we have $p_b = p_s = m(F)$ on both marketplaces. Then, the agents are indifferent between the marketplaces. Any segmentation such that the market is cleared in each marketplace is also an equilibrium.

make sure that they are collectively achieving the baseline profit in the equilibrium.

Finally, in addition to choosing among these marketplaces, suppose agents also have the option to join the decentralized market as in the main model, after observing the posted mechanisms. In this case, the argument from the previous paragraph works with a minor modification: Now, the designers would each post the coexistence equilibrium mechanism with price-matching guarantees and the agents would still randomize over the marketplaces. Since no marketplace has any incentives to deviate, they collectively act as the centralized market from the main model. Then, given that the centralized market is the same (on the equilibrium-path), no agent has any incentives to deviate either. Thus, the same segmentation would constitute an equilibrium with the addition that the agents who joined the centralized market in the main model, now randomizes over all of the marketplaces, uniformly.

References

- Armstrong, M. (2006). Competition in two-sided markets. *The RAND Journal of Economics*, 37(3):668–691. 5
- Bulow, J. and Roberts, J. (1989). The simple economics of optimal auctions. *Journal of political economy*, 97(5):1060–1090. 9
- Congress Majority Staff (2020). Investigation of competition in digital markets. 2
- Cramton, P., Gibbons, R., and Klemperer, P. (1987). Dissolving a partnership efficiently. *Econometrica: Journal of the Econometric Society*, pages 615–632. 6, 24
- Diamond, P. A. (1971). A model of price adjustment. *Journal of economic theory*, 3(2):156–168. 3, 11
- Feng, Y. and Hartline, J. D. (2018). An end-to-end argument in mechanism design (prior-independent auctions for budgeted agents). In *2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 404–415. IEEE. 29
- Fieseler, K., Kittsteiner, T., and Moldovanu, B. (2003). Partnerships, lemons, and efficient trade. *Journal of Economic Theory*, 113(2):223–234. 6
- Figuerola, N. and Skreta, V. (2012). Asymmetric partnerships. *Economics Letters*, 115(2):268–271. 6
- Fuchs, W. and Skrzypacz, A. (2015). Government interventions in a dynamic market with adverse selection. *Journal of Economic Theory*, 158:371–406. 5
- Hartline, J. D. and Roughgarden, T. (2014). Optimal platform design. *arXiv preprint arXiv:1412.8518*. 5
- Hinote, A. (2021). Taxi and ridehailing usage in new york city. 2
- Kittsteiner, T. (2003). Partnerships and double auctions with interdependent valuations. *Games and Economic Behavior*, 44(1):54–76. 6, 24
- Loertscher, S. and Wasser, C. (2019). Optimal structure and dissolution of partnerships. *Theoretical Economics*, 14(3):1063–1114. 6
- Miao, J. (2006). A search model of centralized and decentralized trade. *Review of Economic dynamics*, 9(1):68–92. 6

- Mortensen, D. T. (1970). Job search, the duration of unemployment, and the phillips curve. *The American Economic Review*, 60(5):847–862. 3, 11
- Myerson, R. B. and Satterthwaite, M. A. (1983). Efficient mechanisms for bilateral trading. *Journal of economic theory*, 29(2):265–281. 6
- Mylovanov, T. and Tröger, T. (2014). Mechanism design by an informed principal: Private values with transferable utility. *The Review of Economic Studies*, 81(4):1668–1707. 6
- Peivandi, A. and Vohra, R. V. (2021). Instability of centralized markets. *Econometrica*, 89(1):163–179. 5
- Philippon, T. and Skreta, V. (2012). Optimal interventions in markets with adverse selection. *American Economic Review*, 102(1):1–28. 5
- Pissarides, C. A. (1979). Job matchings with state employment agencies and random search. *The Economic Journal*, 89(356):818–833. 3, 11
- Rochet, J.-C. and Tirole, J. (2003). Platform competition in two-sided markets. *Journal of the european economic association*, 1(4):990–1029. 5
- Schneider, T. W. (2021). Taxi and ridehailing usage in new york city. 2
- Tirole, J. (2012). Overcoming adverse selection: How public intervention can restore market functioning. *American economic review*, 102(1):29–59. 5

A Integrating the Virtual Surplus

$$\begin{aligned}
& - \int_0^{\underline{\theta}} C(x) f(x) dx + \int_{\bar{\theta}}^1 V(y) f(y) dy \\
& = - \int_0^{\underline{\theta}} C(x) f(x) dx + \int_{\bar{\theta}}^1 V(y) f(y) dy \\
& = - \int_0^{\underline{\theta}} \left[x + \frac{F(x)}{f(x)} \right] f(x) dx + \int_{\bar{\theta}}^1 \left[y - \frac{1 - F(y)}{f(y)} \right] f(y) dy \\
& = - \int_0^{\underline{\theta}} [x] f(x) dx - \int_0^{\underline{\theta}} [F(x)] dx + \int_{\bar{\theta}}^1 [y] f(y) dy - \int_{\bar{\theta}}^1 [1 - F(y)] dy \\
& = - \left([xF(x)]_0^{\underline{\theta}} - \int_0^{\underline{\theta}} [F(x)] dx \right) - \int_0^{\underline{\theta}} [F(x)] dx \\
& \quad + \left([yF(y)]_{\bar{\theta}}^1 - \int_{\bar{\theta}}^1 [F(y)] dy \right) - \int_{\bar{\theta}}^1 [1 - F(y)] dy \\
& = -\underline{\theta}F(\underline{\theta}) + 1 - \bar{\theta}F(\bar{\theta}) - \int_{\bar{\theta}}^1 [F(y)] dy - (1 - \bar{\theta}) + \int_{\bar{\theta}}^1 [F(y)] dy \\
& = -\underline{\theta}F(\underline{\theta}) + 1 - \bar{\theta}F(\bar{\theta}) - 1 + \bar{\theta} \\
& = -\underline{\theta}F(\underline{\theta}) - \bar{\theta}F(\bar{\theta}) + \bar{\theta}
\end{aligned}$$

B Optimal Baseline Marketplace

Proof of Theorem 2.1. Since the allocation needs to be increasing, if $q(\theta) < 0$ for some θ , we would have $q(\theta') < 0$ for each $\theta' \leq \theta$. Similarly, if $q(\theta) > 0$ for some θ , we would have $q(\theta') > 0$ for each $\theta' \geq \theta$. So, let $0 \leq \underline{\theta} \leq \bar{\theta} \leq 1$ such that $\underline{\theta}$ is the supremum of values with negative allocation and $\bar{\theta}$ is the infimum of the values with positive allocation. Then, we can write the objective function as follows:

$$\begin{aligned}
\Pi^M &= \mathbb{P}[\theta \in [0, \underline{\theta}]]\mathbb{E}[C(\theta)q(\theta)|\theta \in [0, \underline{\theta}]] + \mathbb{P}[\theta \in [\bar{\theta}, 1]]\mathbb{E}[\mathcal{V}(\theta)q(\theta)|\theta \in [\bar{\theta}, 1]] \\
&= \int_0^{\underline{\theta}} C(x)q(x)f(x)dx + \int_{\bar{\theta}}^1 \mathcal{V}(x)q(x)f(x)dx
\end{aligned}$$

Note that for $\theta \leq \underline{\theta}$, we must have $q(\theta) = -1$ as -1 is the only possible negative allocation with the indivisible good and similarly, $q(\theta) = 1$ for $\theta \geq \bar{\theta}$. Thus, the optimal allocation will have the following form and the next step is to choose the cutoffs, $\underline{\theta}$ and $\bar{\theta}$ optimally.

$$q(\theta) = \begin{cases} -1 & \text{if } \theta \leq \underline{\theta} \\ 0 & \text{if } \underline{\theta} \leq \theta \leq \bar{\theta} \\ 1 & \text{if } \theta \geq \bar{\theta} \end{cases}$$

Then, the problem can be restated as follows:

$$\begin{aligned}
&\max_{\underline{\theta}, \bar{\theta}} \left[-\int_0^{\underline{\theta}} C(x)f(x)dx + \int_{\bar{\theta}}^1 \mathcal{V}(x)f(x)dx \right] \\
&\text{s. t.} \\
&\quad F(\underline{\theta}) = 1 - F(\bar{\theta}) \\
&\quad 0 \leq \underline{\theta} \leq \bar{\theta} \leq 1
\end{aligned}$$

Integrating out the total virtual value and the virtual cost using integration by parts⁹ shows that the objective function is equal to:

$$-\underline{\theta}F(\underline{\theta}) + \bar{\theta}(1 - F(\bar{\theta})) = F(\underline{\theta})(\bar{\theta} - \underline{\theta})$$

where the equality is obtained by using the feasibility condition $F(\underline{\theta}) = 1 - F(\bar{\theta})$. Then, there exists a solution to this problem. Moreover, the solution is interior: If $\underline{\theta} = 0$ or $\underline{\theta} = \bar{\theta}$, the profit is 0. However, positive profit is feasible by any feasible interior solution as is clear from the objective function.

Moreover, by using the formula for the transfer rule and the optimal allocation from

⁹See Appendix A for details.

above, we can compute the transfers in the mechanism to be

$$t(\theta) = \begin{cases} -\underline{\theta} & \text{if } \theta \leq \underline{\theta} \\ 0 & \text{if } \underline{\theta} \leq \theta \leq \bar{\theta} \\ \bar{\theta} & \text{if } \theta \geq \bar{\theta} \end{cases}$$

Notice that each ‘seller’ gets the same payment while each ‘buyer’ pays the same amount. Thus, this mechanism is equivalent to offering bid-ask prices that the transfer rule above suggest, that is a price for buying and a price for selling, and letting agents choose whether they want to buy or sell or not trade.

Finally, notice that the solution must have $C(\underline{\theta}) = \mathcal{V}(\bar{\theta})$: If we had $C(\underline{\theta}) > \mathcal{V}(\bar{\theta})$, decreasing $\underline{\theta}$ and adjusting $\bar{\theta}$ accordingly for feasibility would increase the profit since buying from $\underline{\theta}$ is costlier than what selling to $\bar{\theta}$ pays off. Similarly, if we had $C(\underline{\theta}) < \mathcal{V}(\bar{\theta})$, then increasing $\underline{\theta}$ and adjusting $\bar{\theta}$ so that the feasibility binds would again increase the profit since there are more agents whose trade is profitable. \square

B.1 Illustrative Example with Uniform Distribution

Suppose θ is distributed uniformly over $[0, 1]$ with c.d.f. $F(\theta) = \theta$. Then, the virtual values and costs are given by

$$\mathcal{V}(\theta) = 2\theta - 1 \text{ and } C(\theta) = 2\theta.$$

It is easy to show that the objective function becomes $\underline{\theta}(1 - 2\underline{\theta})$ after substituting for $\bar{\theta} = 1 - \underline{\theta}$ (feasibility). So, the optimal cutoffs are $\underline{\theta} = \frac{1}{4}$ and $\bar{\theta} = \frac{3}{4}$.

Let us consider two agents with valuations θ_1 and θ_2 . Then, 1 will buy from 2 in the optimal marketplace when it operates on its own if and only if $\theta_1 \geq 0.75$ and $\theta_2 \leq 0.75$, and 2 will buy from 1 if and only if $\theta_2 \geq 0.75$ and $\theta_1 \leq 0.75$. The Figure 4 depicts the space of (θ_1, θ_2) where the shaded areas represent these trading regions.

Using the payment formula from the theorem above, we can compute the payments as below:

$$t(\theta) = \begin{cases} -\frac{1}{4}, & \text{if } \theta \leq \frac{1}{4}, \\ 0, & \text{if } \frac{1}{4} \leq \theta \leq \frac{3}{4}, \\ \frac{1}{4}, & \text{if } \theta \geq \frac{3}{4}. \end{cases}$$

The large area of the type space where there is no trade motivates our consideration for the decentralized market. The marketplace excludes these type profiles from trade because

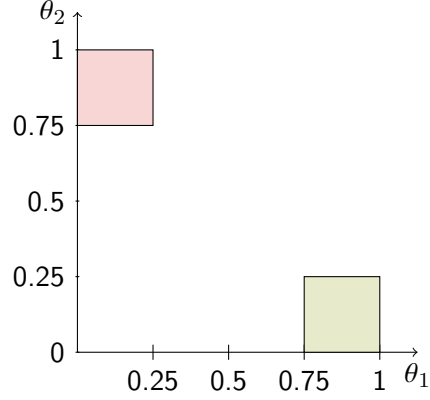


Figure 4: x -axis represents $\theta_1 \in [0, 1]$ and y -axis represents $\theta_2 \in [0, 1]$. Green and red areas show the type profiles at which agent 1 and 2 is the buyer, respectively.

their is not profitable; this is akin to a monopolist or a monopsonist excluding some agents from trade. Indeed, the marketplace acts as both a monopolist and a monopsonist. However, unlike the buyers a monopolist excludes, the agents the marketplace excludes actually have a surplus that they can create, if they were allowed to trade. Thus, assumption that they will remain in the marketplace even though they are not trading is not very realistic. Next, I am going to allow them to choose between the marketplace and a decentralized market.

C Proof of the Obedience Principle

Proof of Proposition 3.1. First, note that if some agents join the marketplace but the reported types in the marketplace does not satisfy the requirements for the operation of the announced mechanism, then, the marketplace shuts down and agents there gets 0 utility. Then, they would strictly prefer to be in the decentralized market since the expected payoff there is always positive. Thus, a different set of agents than the invited ones can only be in the marketplace if the marketplace is active. Moreover, notice that when the marketplace is active, the incentive compatibility of the mechanism ensures that if an agent joins there, the truthful revelation is better than misreporting one's type.

Suppose the designer announced $\underline{\theta}, \bar{\theta}$ (meaning she invited $[0, \underline{\theta}] \cap [\bar{\theta}, 1]$) and a strict subset of invited types, $[0, a] \cap [b, 1]$ that satisfy the above requirements joined the mechanism (and reported truthfully, by incentive compatibility). For this to be an equilibrium, agents with valuations a and b must be indifferent between the two market. To see this, suppose the utilities a or b get from the mechanism is strictly higher than their expected payoffs from search. Then, the designer can increase the transfers to strictly increase the profit. Now suppose the expected payoff from search is strictly lower than the utility from

the mechanism for a (b). Note that both the expected payoff from search and the utilities the mechanism offer are continuous in agents' own valuations. Then, a positive measure of agents with valuations just above a (below b) must also have been offered a higher utility in the mechanism than what they expect from search, which contradicts their joining the search market in an equilibrium. Thus, we conclude that in the equilibrium, the realized cutoffs in this potential equilibrium, a and b must be indifferent between two markets. We now write the utilities these agents expect from both markets.

$$\begin{aligned} u^d(a) &= \frac{p}{2} [\mathbb{E}[\theta | \theta \in [a, b]] - a] &= \frac{p}{2} [\mathbb{E}[\theta | \theta \in [\underline{\theta}, \bar{\theta}]] - \underline{\theta}] + \underline{\theta} - a = u^m(a) \\ u^d(b) &= \frac{p}{2} [b - \mathbb{E}[\theta | \theta \in [a, b]]] &= \frac{p}{2} [\bar{\theta} - \mathbb{E}[\theta | \theta \in [\underline{\theta}, \bar{\theta}]]] + b - \bar{\theta} = u^m(b) \end{aligned}$$

where we calculate the expected payoff from the mechanism using the payoff equivalence: The mechanism would make $\underline{\theta}$ and $\bar{\theta}$ indifferent between two markets and then have a slope of -0.5 for sellers and of 0.5 for buyers.

By adding these equations up, we end up with

$$\begin{aligned} \frac{p}{2} [b - a] &= \frac{p}{2} [\bar{\theta} - \underline{\theta}] + b + \underline{\theta} - a - \bar{\theta} \\ \frac{p}{2} [b - a - \bar{\theta} + \underline{\theta}] &= b + \underline{\theta} - a - \bar{\theta} \\ p &= 2, \end{aligned}$$

which is of course impossible since p is a probability. Thus, we conclude that there is no equilibrium where the set of agents in the mechanism is nonempty and different from the set of invited agents. \square

D Simplifying the Profit Function

Here we simplify the profit function.

$$\Pi_{\underline{\theta}, \bar{\theta}} = \int_0^{\underline{\theta}} t(\theta) f(\theta) d\theta + \int_{\bar{\theta}}^1 t(\theta) f(\theta) d\theta.$$

We will study each integral separately. We start with the first one.

$$\int_0^{\underline{\theta}} t(\theta) f(\theta) d\theta \quad (1)$$

$$= \int_0^{\underline{\theta}} \left[\theta q(\theta) - u^m(0) - \int_0^{\underline{\theta}} q(x) dx \right] f(\theta) d\theta \quad (2)$$

$$= \int_0^{\underline{\theta}} [\theta q(\theta) - u^m(0)] f(\theta) d\theta - \int_0^{\underline{\theta}} \int_0^{\underline{\theta}} q(x) f(\theta) dx d\theta \quad (3)$$

$$= \int_0^{\underline{\theta}} [\theta q(\theta) - u^m(0)] f(\theta) d\theta - \int_0^{\underline{\theta}} \int_x^{\underline{\theta}} q(x) f(\theta) d\theta dx \quad (4)$$

$$= \int_0^{\underline{\theta}} [\theta q(\theta) - u^m(0)] f(\theta) d\theta - \int_0^{\underline{\theta}} q(x) \int_x^{\underline{\theta}} f(\theta) d\theta dx \quad (5)$$

$$= \int_0^{\underline{\theta}} [\theta q(\theta) - u^m(0)] f(\theta) d\theta - \int_0^{\underline{\theta}} q(x) (F(\underline{\theta}) - F(x)) dx \quad (6)$$

$$= \int_0^{\underline{\theta}} \left[-u^m(0) + \left(x - \frac{F(\underline{\theta}) - F(x)}{f(x)} \right) q(x) \right] f(x) dx \quad (7)$$

$$= \int_0^{\underline{\theta}} \left[-u^d(\underline{\theta}) + \int_0^{\underline{\theta}} q(y) dy + \left(x - \frac{F(\underline{\theta}) - F(x)}{f(x)} \right) q(x) \right] f(x) dx \quad (8)$$

$$= \int_0^{\underline{\theta}} F(\underline{\theta}) q(y) \frac{f(y)}{f(y)} dy + \int_0^{\underline{\theta}} \left[-u^d(\underline{\theta}) + \left(x - \frac{F(\underline{\theta}) - F(x)}{f(x)} \right) q(x) \right] f(x) dx \quad (9)$$

$$= \int_0^{\underline{\theta}} \left[-u^d(\underline{\theta}) + \left(x + \frac{F(x)}{f(x)} \right) q(x) \right] f(x) dx \quad (10)$$

$$= -F(\underline{\theta}) u^d(\underline{\theta}) + \int_0^{\underline{\theta}} \left[\left(x + \frac{F(x)}{f(x)} \right) q(x) \right] f(x) dx \quad (11)$$

In line 4, we change the order of integration; in line 5, we isolate the inner integral by extracting the allocations out; in line 6, we replace the value of the inner integral; in line 7, we merge the sum back; in line 8, we replace the value of the utility of the lowest type; in

line 9, we integrate out the information rent for these types; in line 10, we cancel the new double integral with the $-\underline{\theta}q(x)$ as that integral turns out to be just the integral of $\underline{\theta}q(x)$ by changing the order of integration as above. We follow the similar steps for the transfers from $[\bar{\theta}, 1]$.

$$\begin{aligned}
& \int_{\bar{\theta}}^1 t(\theta) f(\theta) d\theta \\
&= \int_{\bar{\theta}}^1 \left[\theta q(\theta) - u^m(\bar{\theta}) - \int_{\bar{\theta}}^{\theta} q(x) dx \right] f(\theta) d\theta \\
&= \int_{\bar{\theta}}^1 \left[\theta q(\theta) - u^m(\bar{\theta}) \right] f(\theta) d\theta - \int_{\bar{\theta}}^1 \int_{\bar{\theta}}^{\theta} q(x) f(\theta) dx d\theta \\
&= \int_{\bar{\theta}}^1 \left[\theta q(\theta) - u^m(\bar{\theta}) \right] f(\theta) d\theta - \int_{\bar{\theta}}^1 \int_x^1 q(x) f(\theta) d\theta dx \\
&= \int_{\bar{\theta}}^1 \left[\theta q(\theta) - u^m(\bar{\theta}) \right] f(\theta) d\theta - \int_{\bar{\theta}}^1 q(x) \int_x^1 f(\theta) d\theta dx \\
&= \int_{\bar{\theta}}^1 \left[\theta q(\theta) - u^m(\bar{\theta}) \right] f(\theta) d\theta - \int_{\bar{\theta}}^1 q(x) (1 - F(x)) dx \\
&= \int_{\bar{\theta}}^1 \left[-u^m(\bar{\theta}) + \left(x - \frac{1 - F(x)}{f(x)} \right) q(x) \right] f(x) dx \\
&= -(1 - F(\bar{\theta})) u^d(\bar{\theta}) + \int_{\bar{\theta}}^1 \left[\left(x - \frac{1 - F(x)}{f(x)} \right) q(x) \right] f(x) dx
\end{aligned}$$

E Profit from the simple equilibrium

From the Appendix A, we have

$$-\int_0^{\underline{\theta}} C(x)f(x)dx + \int_{\bar{\theta}}^1 V(y)f(y)dy = -\underline{\theta}F(\underline{\theta}) - \bar{\theta}F(\bar{\theta}) + \bar{\theta}$$

Moreover, under a CRS matching function M for the search market, p is independent of the segmentation of the market. Then, the compensations paid to the agents will be given by:

$$\begin{aligned} & F(\underline{\theta})u^d(\underline{\theta}) + (1 - F(\bar{\theta}))u^d(\bar{\theta}) \\ &= p \frac{F(\underline{\theta})}{2} \left[\frac{\frac{\bar{\theta}}{2} \int_{\underline{\theta}}^{\bar{\theta}} \theta f(\theta) d\theta}{F(\bar{\theta}) - F(\underline{\theta})} - \underline{\theta} \right] + p \frac{1 - F(\bar{\theta})}{2} \left[\bar{\theta} - \frac{\frac{\bar{\theta}}{2} \int_{\underline{\theta}}^{\bar{\theta}} \theta f(\theta) d\theta}{F(\bar{\theta}) - F(\underline{\theta})} \right] \\ &= p \frac{1}{2} \left(F(\underline{\theta}) \left[\frac{\frac{\bar{\theta}}{2} \int_{\underline{\theta}}^{\bar{\theta}} \theta f(\theta) d\theta}{F(\bar{\theta}) - F(\underline{\theta})} - \underline{\theta} \right] + (1 - F(\bar{\theta})) \left[\bar{\theta} - \frac{\frac{\bar{\theta}}{2} \int_{\underline{\theta}}^{\bar{\theta}} \theta f(\theta) d\theta}{F(\bar{\theta}) - F(\underline{\theta})} \right] \right) \\ &= \frac{p}{2} \left(F(\underline{\theta}) E[\theta | \underline{\theta} \leq \theta \leq \bar{\theta}] - \underline{\theta} F(\underline{\theta}) + \bar{\theta} (1 - F(\bar{\theta})) - (1 - F(\bar{\theta})) E[\theta | \underline{\theta} \leq \theta \leq \bar{\theta}] \right) \\ &= \frac{p}{2} \left(\left(F(\underline{\theta}) - (1 - F(\bar{\theta})) \right) E[\theta | \underline{\theta} \leq \theta \leq \bar{\theta}] - \underline{\theta} F(\underline{\theta}) + \bar{\theta} (1 - F(\bar{\theta})) \right) \end{aligned}$$

Thus,

Π

$$\begin{aligned} &= -\frac{p}{2} \left(\left(F(\underline{\theta}) - (1 - F(\bar{\theta})) \right) E[\theta | \underline{\theta} \leq \theta \leq \bar{\theta}] - \underline{\theta} F(\underline{\theta}) + \bar{\theta} (1 - F(\bar{\theta})) \right) + \left[-\underline{\theta} F(\underline{\theta}) - \bar{\theta} F(\bar{\theta}) + \bar{\theta} \right] \\ &= \frac{1}{2} \left((2 - p) \left[-\underline{\theta} F(\underline{\theta}) + \bar{\theta} (1 - F(\bar{\theta})) \right] - p \left[F(\underline{\theta}) - (1 - F(\bar{\theta})) \right] E[\theta | \underline{\theta} \leq \theta \leq \bar{\theta}] \right) \end{aligned}$$

Notice that

$$\begin{aligned}
\mathbb{E}[\theta|\theta \in [\underline{\theta}, \bar{\theta}]] &= \frac{\int_{\underline{\theta}}^{\bar{\theta}} x f(x) dx}{F(\bar{\theta}) - F(\underline{\theta})} \\
\frac{\partial \mathbb{E}[\theta|\theta \in [\underline{\theta}, \bar{\theta}]]}{\partial \underline{\theta}} &= \frac{-\underline{\theta} f(\underline{\theta}) [F(\bar{\theta}) - F(\underline{\theta})] + f(\underline{\theta}) \left[\int_{\underline{\theta}}^{\bar{\theta}} x f(x) dx \right]}{[F(\bar{\theta}) - F(\underline{\theta})]^2} \\
&= f(\underline{\theta}) \frac{\int_{\underline{\theta}}^{\bar{\theta}} x f(x) dx - \underline{\theta} [F(\bar{\theta}) - F(\underline{\theta})]}{[F(\bar{\theta}) - F(\underline{\theta})]^2} \\
&= f(\underline{\theta}) \frac{\mathbb{E}[\theta|\theta \in [\underline{\theta}, \bar{\theta}]] - \underline{\theta}}{[F(\bar{\theta}) - F(\underline{\theta})]} > 0.
\end{aligned}$$

Then, using this,

$$\begin{aligned}
&2 \frac{\partial \Pi}{\partial \underline{\theta}} \\
&= -(2-p)F(\underline{\theta}) - (2-p)\underline{\theta}f(\underline{\theta}) - p[f(\underline{\theta})E[\theta|\underline{\theta} \leq \theta \leq \bar{\theta}] - p \left[F(\underline{\theta}) - (1 - F(\bar{\theta})) \right] \frac{\partial \mathbb{E}[\theta|\theta \in [\underline{\theta}, \bar{\theta}]]}{\partial \underline{\theta}}
\end{aligned}$$

For a feasible mechanism, we need $F(\underline{\theta}) - (1 - F(\bar{\theta})) \geq 0$. Thus, each term above is negative, and Π is decreasing in $\underline{\theta}$. Given that Π is decreasing in $\underline{\theta}$, $F(\underline{\theta}) - (1 - F(\bar{\theta})) \geq 0$ will bind in any equilibrium. Therefore,

$$\begin{aligned}
\Pi &= \frac{1}{2}(2-p) \left[-\underline{\theta}F(\underline{\theta}) + \bar{\theta}(1 - F(\bar{\theta})) \right] \\
&= \frac{2-p}{2} \left[-\underline{\theta}F(\underline{\theta}) + \bar{\theta}(F(\underline{\theta})) \right] \\
&= \frac{2-p}{2} \left[F(\underline{\theta})[\bar{\theta} - \underline{\theta}] \right]
\end{aligned}$$

Notice that for interior values with $\bar{\theta} > \underline{\theta}$, $\Pi > 0$. Thus, positive profit is feasible and will be achieved in the equilibrium.

E.1 The Mechanism That Induces The Simple Equilibrium

Proof of Proposition 3.2. First, we compute the transfers for the agents who join the mechanism using the binding IR constraints for $\underline{\theta}$ and $\bar{\theta}$.

In the simple equilibrium we construct, for an agent with $\theta \in [0, \underline{\theta}]$,

$$\begin{aligned}
t(\theta) &= \theta q(\theta) - u^m(0) - \int_0^\theta q(x) dx \\
&= \theta(-1) - u^m(0) - \theta(-1) \\
&= -u^m(0) = -u^d(\underline{\theta}) + \int_0^{\underline{\theta}} q(x) dx = -u^d(\underline{\theta}) + \int_0^{\underline{\theta}} (-1) dx \\
&= -\frac{p}{2} \left[E[x|x \in [\underline{\theta}, \bar{\theta}]] - \underline{\theta} \right] - \underline{\theta} \\
&= -\frac{p}{2} E[x|x \in [\underline{\theta}, \bar{\theta}]] - \underline{\theta} \frac{2-p}{2}
\end{aligned}$$

Similarly, we can compute the transfer of agents with $\theta \in [\bar{\theta}, 1]$,

$$\begin{aligned}
t(\theta) &= \theta q(\theta) - u^m(\bar{\theta}) - \int_{\bar{\theta}}^\theta q(x) dx \\
&= \theta(1) - u^m(\bar{\theta}) - 1(\theta - \bar{\theta}) \\
&= \bar{\theta} - u^m(\bar{\theta}) = \bar{\theta} - \frac{p}{2} \left[\bar{\theta} - E[x|x \in [\underline{\theta}, \bar{\theta}]] \right] \\
&= \frac{p}{2} E[x|x \in [\underline{\theta}, \bar{\theta}]] + \bar{\theta} \frac{2-p}{2}
\end{aligned}$$

Knowing these, the designer can offer -1 allocation to all agents whose valuations are below $\frac{p\mathbb{E}[x|x \in [\underline{\theta}, \bar{\theta}]] + (2-p)\underline{\theta}}{2}$. Similarly, for agents with valuations above $\frac{p\mathbb{E}[x|x \in [\underline{\theta}, \bar{\theta}]] + (2-p)\bar{\theta}}{2}$, 1 unit of allocation can be offered. In between, they aren't offered any trade. This allocation is clearly increasing. Moreover, accompanied by the transfers $-\frac{p\mathbb{E}[x|x \in [\underline{\theta}, \bar{\theta}]] + (2-p)\underline{\theta}}{2}$ for agents with negative allocations and $\frac{p\mathbb{E}[x|x \in [\underline{\theta}, \bar{\theta}]] + (2-p)\bar{\theta}}{2}$ for agents with positive allocations, agents with values in $(\underline{\theta}, \bar{\theta})$ would strictly prefer the search market. To see this, note that these agents have utilities with slopes -1 until u^m hits 0, then it is constant at 0 and then it has the slope 1, after $\frac{p\mathbb{E}[x|x \in [\underline{\theta}, \bar{\theta}]] + (2-p)\bar{\theta}}{2}$. Moreover, u^m and u^d are equal at $\underline{\theta}$ and $\bar{\theta}$. Since the slope of u^d is bounded between $-\frac{p}{2}$ and $\frac{p}{2}$, and u^d is positive, u^m and u^d cannot cross each other at any point other than $\underline{\theta}$ and $\bar{\theta}$. Thus, u^d remains below u^m for values in $(\underline{\theta}, \bar{\theta})$. \square

F Efficiency of Coexistence

F.1 Under Uniform Distribution

Proof of Proposition 3.4. Suppose everyone is in the search market. Then, the total welfare can be computed as follows:

$$\begin{aligned}\mathbb{E}[u^d(\theta)] &= p \int_0^1 [\theta\theta - (1 - \theta)\theta] d\theta \\ &= p \int_0^1 [2\theta^2 - \theta] d\theta \\ &= p \left[\frac{2\theta^3}{3} - \frac{\theta^2}{2} \right]_0^1 = \frac{p}{6}.\end{aligned}$$

Next, we compute the welfare created by the marketplace alone in the coexistence equilibrium. The welfare marketplace generates will be more than enough to exceed the total welfare of the pure search market, so we don't need to compute the welfare created by the search market in the coexistence.

The profit function from the simple equilibrium under the uniform distribution is a constant times $\underline{\theta}(\bar{\theta} - \underline{\theta}) = \underline{\theta}(1 - 2\underline{\theta})$ using the fact that the feasibility binds so that $\underline{\theta} = 1 - \bar{\theta}$. This is maximized at $\underline{\theta} = \frac{1}{4}$ and $\bar{\theta} = \frac{3}{4}$. Then, the welfare generated by the marketplace is

$$\int_{0.75}^1 \theta_b d\theta_b - \int_0^{0.25} \theta_s d\theta_s = \left[\frac{\theta^2}{2} \right]_{0.75}^1 - \left[\frac{\theta^2}{2} \right]_0^{0.25} = \frac{3}{16}.$$

The total welfare from the search market is $\frac{p}{6} \leq \frac{1}{6}$ for any matching function since the probability a meeting will be less than or equal to 1. Moreover, $\frac{3}{16} > \frac{1}{6}$. Thus, for any matching function, the coexistence equilibrium creates a welfare higher than the pure search market. \square

F.2 Under General Distribution

Proof of Proposition 3.5. For the pure search market, the total welfare created is given by

$$\begin{aligned}
& \int_0^1 [pF(\theta)\theta - p(1 - F(\theta))\theta] f(\theta)d\theta \\
&= p \int_0^1 \theta [2F(\theta) - 1] f(\theta)d\theta
\end{aligned}$$

In the first line above, $pF(\theta)$ is the probability that the agent with the value θ meets with an agent with a value less than θ , so she gets θ in the trade and $p(1 - F(\theta))$ is the probability that she meets with an agent whose value is higher so she loses θ . As with the uniform case, we ignore the transfers in the utilities as the transfers will cancel in the search market.

In the coexistence equilibrium, the total welfare created in the search market is

$$\begin{aligned}
& \int_0^1 \left[p \left[\frac{F(\theta) - F(\underline{\theta})}{F(\bar{\theta}) - F(\underline{\theta})} \right] \theta - p \left[\frac{F(\bar{\theta}) - F(\theta)}{F(\bar{\theta}) - F(\underline{\theta})} \right] \theta \right] f(\theta)d\theta \\
&= p \int_{\underline{\theta}}^{\bar{\theta}} \theta \left[\frac{2F(\theta) - F(\underline{\theta}) - F(\bar{\theta})}{F(\bar{\theta}) - F(\underline{\theta})} \right] f(\theta)d\theta \\
&= p \int_{\underline{\theta}}^{\bar{\theta}} \theta \left[\frac{2F(\theta) - 1}{F(\bar{\theta}) - F(\underline{\theta})} \right] f(\theta)d\theta
\end{aligned}$$

The welfare generated by the marketplace in the coexistence is

$$\int_{\bar{\theta}}^1 xf(x)dx - \int_0^{\underline{\theta}} xf(x)dx$$

$$\begin{aligned}
& p \int_{\underline{\theta}}^{\bar{\theta}} \theta \left[\frac{2F(\theta) - 1}{F(\bar{\theta}) - F(\underline{\theta})} \right] f(\theta) d\theta + \int_{\bar{\theta}}^1 x f(x) dx - \int_0^{\underline{\theta}} x f(x) dx \geq p \int_0^1 \theta [2F(\theta) - 1] f(\theta) d\theta \\
\iff & p \int_{\underline{\theta}}^{\bar{\theta}} \theta \left[\frac{2F(\theta)}{F(\bar{\theta}) - F(\underline{\theta})} \right] f(\theta) d\theta + \int_{\bar{\theta}}^1 x f(x) dx + p \int_0^1 \theta f(\theta) d\theta \\
& \geq p \int_{\underline{\theta}}^{\bar{\theta}} \left[\frac{\theta f(\theta) d\theta}{F(\bar{\theta}) - F(\underline{\theta})} \right] + p \int_0^1 \theta [2F(\theta)] f(\theta) d\theta + \int_0^{\underline{\theta}} x f(x) dx \\
\iff & p \int_{\underline{\theta}}^{\bar{\theta}} \theta \left[\frac{2F(\theta)}{F(\bar{\theta}) - F(\underline{\theta})} \right] f(\theta) d\theta + (1+p) \int_{\bar{\theta}}^1 \theta f(\theta) d\theta \\
& \geq p \left[\frac{2F(\underline{\theta})}{1 - 2F(\underline{\theta})} \right] \int_{\underline{\theta}}^{\bar{\theta}} \theta f(\theta) d\theta + 2p \int_0^1 \theta [F(\theta)] f(\theta) d\theta + (1-p) \int_0^{\underline{\theta}} x f(x) dx
\end{aligned}$$

In the first line above, either we have

$$\int_{\underline{\theta}}^{\bar{\theta}} \theta \left[\frac{2F(\theta) - 1}{F(\bar{\theta}) - F(\underline{\theta})} \right] f(\theta) d\theta \geq \int_0^1 \theta [2F(\theta) - 1] f(\theta) d\theta$$

in which case the inequality is satisfied for any p , since $\int_{\bar{\theta}}^1 x f(x) dx - \int_0^{\underline{\theta}} x f(x) dx \geq 0$ when $F(\underline{\theta}) = 1 - F(\bar{\theta})$ or

$$\int_{\underline{\theta}}^{\bar{\theta}} \theta \left[\frac{2F(\theta) - 1}{F(\bar{\theta}) - F(\underline{\theta})} \right] f(\theta) d\theta < \int_0^1 \theta [2F(\theta) - 1] f(\theta) d\theta.$$

Thus, to show that for any p , the coexistence is more efficient, it is enough to show it with $p = 1$.

□

G Slope of Utilities from Search Market

Lemma G.1. *Under any equilibrium, $-\frac{p}{2} \leq \frac{\partial u^d(\theta)}{\partial \theta} \leq \frac{p}{2}$ for each agent.*

Proof. Suppose Θ^d is the set of agents who join the decentralized market, $\mu(\Theta^d) = \mathbb{P}[x \in \Theta^d]$ their measure, and let $\theta \in \text{Cov}(\Theta^d)$. Then,

$$\begin{aligned}
u^d(\theta) &= \frac{p}{2} \mathbb{P}[x > \theta | x \in \Theta^d] [\mathbb{E}[x | x > \theta, x \in \Theta^d] - \theta] \\
&\quad + \frac{p}{2} \mathbb{P}[x < \theta | x \in \Theta^d] [\theta - \mathbb{E}[x | x < \theta, x \in \Theta^d]] \\
&= \frac{p}{2} \frac{\mathbb{P}[x > \theta, x \in \Theta^d]}{\mu(\Theta^d)} [\mathbb{E}[x | x > \theta, x \in \Theta^d] - \theta] \\
&\quad + \frac{p}{2} \frac{\mathbb{P}[x < \theta, x \in \Theta^d]}{\mu(\Theta^d)} [\theta - \mathbb{E}[x | x < \theta, x \in \Theta^d]] \\
&= \frac{p}{2\mu(\Theta^d)} \left[\int_{\{x \in \Theta^d : x > \theta\}} x f(x) dx - \theta \mathbb{P}[x > \theta, x \in \Theta^d] \right] \\
&\quad + \frac{p}{2\mu(\Theta^d)} \left[\theta \mathbb{P}[x < \theta, x \in \Theta^d] - \int_{\{x \in \Theta^d : x < \theta\}} x f(x) dx \right] \\
\frac{\partial u^d(\theta)}{\partial \theta} &= \frac{p}{2\mu(\Theta^d)} [\mathbb{P}[x < \theta, x \in \Theta^d] - \mathbb{P}[x > \theta, x \in \Theta^d]].
\end{aligned}$$

If $\theta \leq \theta'$ for each $\theta' \in \Theta^d$, then

$$\begin{aligned}
u^d(\theta) &= \frac{p}{2} \mathbb{P}[x > \theta | x \in \Theta^d] [\mathbb{E}[x | x \in \Theta^d] - \theta] \\
&= \frac{p}{2} \left[\int_{\{x \in \Theta^d\}} \frac{x f(x) dx}{\mu(\Theta^d)} - \theta \right] \\
\frac{\partial u^d(\theta)}{\partial \theta} &= -\frac{p}{2}.
\end{aligned}$$

Finally, if $\theta \geq \theta'$ for each $\theta' \in \Theta^d$, then

$$\begin{aligned}
u^d(\theta) &= \frac{p}{2} \mathbb{P}[x < \theta | x \in \Theta^d] [\theta - \mathbb{E}[x | x \in \Theta^d]] \\
&= \frac{p}{2} \left[\theta - \int_{\{x \in \Theta^d\}} \frac{x f(x) dx}{\mu(\Theta^d)} \right] \\
\frac{\partial u^d(\theta)}{\partial \theta} &= \frac{p}{2}.
\end{aligned}$$

□

H Unrestricted Mechanisms

Proof of Theorem 3.2. We start with a simple observation: For the mechanism to make a positive profit, there has to be both agents who buy and sell at the marketplace. This means for a positive measure of agents, $u^m(\theta) \geq u^d(\theta)$ on both regions with $\frac{\partial u^m(\theta)}{\partial \theta} = 1$ and $\frac{\partial u^m(\theta)}{\partial \theta} = -1$.

We have shown in Appendix G that for an arbitrary segmentation of agents, the expected utility from search has a slope between -0.5 and 0.5 .

Notice that if for an agent $\frac{\partial u^m(\theta)}{\partial \theta} = 1$, then for each $\theta' > \theta$, $q(\theta') = 1$ by the envelope condition and monotonicity of the allocation for an IC mechanism. Similarly, if $\frac{\partial u^m(\theta)}{\partial \theta} = -1$, then for each $\theta' < \theta$, $q(\theta') = -1$.

Given this, if for θ , $u^m(\theta) \geq u^d(\theta)$ and $\frac{\partial u^m(\theta)}{\partial \theta} = 1$, then for each $\theta' > \theta$, $u^m(\theta') \geq u^d(\theta')$ and similarly for the sellers. Thus, let $\underline{\theta}$ be the highest value such that $u^m(\theta) \geq u^d(\theta)$ and $q(\theta) = -1$ in the equilibrium. Similarly, let $\bar{\theta}$ be the lowest value such that $u^m(\theta) \geq u^d(\theta)$ and $q(\theta) = 1$.

There must be at least one type such that $u^m(\theta) = u^d(\theta)$. If not, either the utilities from search are above the utilities from the mechanism everywhere so that no one comes to the marketplace and the profit of the marketplace is zero or the utilities from the mechanism is strictly higher everywhere so everyone is in the mechanism and the mechanism can reduce the utilities until some IR constraint binds to strictly increase the profit.

Next, we argue that it cannot be the case that u^m and u^d are only tangent at $\underline{\theta}$ and $\bar{\theta}$, the utilities have to cross each other at these cutoffs: If they were only tangent but didn't cross each other, then $\underline{\theta}$ or $\bar{\theta}$ would have to be the point of a kink on u^m . Then, there are two cases: Either (i) every agent joins the mechanism or (ii) only one of the cutoffs is at a kink, and an interval of agents near the other cutoff join the search market. (i) In the former case, the outside option is zero for every agent so the mechanism doesn't compensate them. But then, they bilaterally deviate for a positive payoff. (ii) In the latter case, suppose there is a kink at $\bar{\theta}$. Then, if the search market is active, u^d has to be increasing at $\bar{\theta}$ since everyone above it is in the mechanism so that an agent with the value $\bar{\theta}$ can only be a buyer in the search market. For there to be agents in the search market, u^d should cross u^m at a point θ such that u^d is decreasing since it cannot cross u^m on the part it is constant or has a slope of 1 below $\bar{\theta}$. But if there is such a point, then u^d would be increasing at θ , since all agents in the search market will be below it as well, which shows this case is impossible as well. Thus,

u^m and u^d cannot be tangent at $\underline{\theta}$ and $\bar{\theta}$, they have to cross each other at these points.

Then, due to the shape of the feasible utility functions (u^m can have slopes -1 , 0 , and 1 in this order and u^d is first decreasing and then increasing -with a potentially constant 0 slope in the middle- with a slope that remains between $-\frac{1}{2}$ and $\frac{1}{2}$), either all agents with values in $[\underline{\theta}, \bar{\theta}]$ join the search market or the flat part of the u^m crosses u^d twice again, in which case agents with values in $[\underline{\theta}, a]$ and $[b, \bar{\theta}]$ join the search market for some $\underline{\theta} < a < b < \bar{\theta}$ and agents with values in a, b join the mechanism as well. Moreover, in the latter case, we need $F(a) - F(\underline{\theta}) = F(\bar{\theta}) - F(b)$; otherwise the u^d would be either strictly decreasing or strictly increasing for agents with values in $[a, b]$, in which case u^d and u^m wouldn't cross at both a and b , as this case requires. When $F(a) - F(\underline{\theta}) = F(\bar{\theta}) - F(b)$, u^d would be flat, as it can be seen from the slope we computed above. So, we can write the profit as follows where the case with $a = b$ corresponds to the situation where the flat part of u^m doesn't cross u^d .

$$\Pi = -F(\underline{\theta})u^d(\underline{\theta}) - (1 - F(\bar{\theta}))u^d(\bar{\theta}) - (F(b) - F(a))u^d(a) - \int_0^{\underline{\theta}} C(x)f(x)dx + \int_{\bar{\theta}}^1 V(x)f(x)dx$$

Moreover, the constraints are $0 \leq \underline{\theta} \leq a \leq b \leq \bar{\theta} \leq 1$, $F(\underline{\theta}) \geq 1 - F(\bar{\theta})$ and $F(a) - F(\underline{\theta}) = F(\bar{\theta}) - F(b)$.

Let $\Theta^d = [\underline{\theta}, a] \cup [b, \bar{\theta}]$. Then,

$$\begin{aligned}
\Pi &= -F(\underline{\theta})u^d(\underline{\theta}) - (1 - F(\bar{\theta}))u^d(\bar{\theta}) - (F(b) - F(a))u^d(a) - \int_0^{\underline{\theta}} C(x)f(x)dx + \int_{\bar{\theta}}^1 V(x)f(x)dx \\
&= -\frac{pF(\underline{\theta})}{2} \left[\int_{\{x \in \Theta^d\}} \frac{xf(x)dx}{[F(\bar{\theta}) - F(b) + F(a) - F(\underline{\theta})]} - \underline{\theta} \right] \\
&\quad - \frac{p(1 - F(\bar{\theta}))}{2} \left[\bar{\theta} - \int_{\{x \in \Theta^d\}} \frac{xf(x)dx}{[F(\bar{\theta}) - F(b) + F(a) - F(\underline{\theta})]} \right] \\
&\quad - \frac{p(F(b) - F(a))}{2[F(\bar{\theta}) - F(b) + F(a) - F(\underline{\theta})]} \left[\int_b^{\bar{\theta}} xf(x)dx - \int_{\underline{\theta}}^a xf(x)dx \right] - [-\underline{\theta}F(\underline{\theta}) - \bar{\theta}F(\bar{\theta}) + \bar{\theta}] \\
&= \frac{p[(1 - F(\bar{\theta}) - F(\underline{\theta}))]}{2} \left[\int_{\{x \in \Theta^d\}} \frac{xf(x)dx}{[F(\bar{\theta}) - F(b) + F(a) - F(\underline{\theta})]} \right] + \frac{p(F(\underline{\theta})\underline{\theta} - (1 - F(\bar{\theta}))\bar{\theta})}{4} \\
&\quad - \frac{p(F(b) - F(a))}{2[F(\bar{\theta}) - F(b) + F(a) - F(\underline{\theta})]} \left[\int_b^{\bar{\theta}} xf(x)dx - \int_{\underline{\theta}}^a xf(x)dx \right] + \frac{1}{2} [-\underline{\theta}F(\underline{\theta}) - \bar{\theta}F(\bar{\theta}) + \bar{\theta}] \\
\frac{\partial \Pi}{\partial \underline{\theta}} &= \frac{p[(1 - F(\bar{\theta}) - F(\underline{\theta}))]}{2} \left[\frac{-\underline{\theta}f(\underline{\theta})[F(\bar{\theta}) - F(b) + F(a) - F(\underline{\theta})] + f(\underline{\theta}) \int_{\{x \in \Theta^d\}} xf(x)dx}{[F(\bar{\theta}) - F(b) + F(a) - F(\underline{\theta})]^2} \right] \\
&\quad - \frac{pf(\underline{\theta})}{2} \left[\int_{\{x \in \Theta^d\}} \frac{xf(x)dx}{[F(\bar{\theta}) - F(b) + F(a) - F(\underline{\theta})]} \right] + \frac{p(\underline{\theta}f(\underline{\theta}) + F(\underline{\theta}))}{2} \\
&\quad - \frac{p(F(b) - F(a))}{2[F(\bar{\theta}) - F(b) + F(a) - F(\underline{\theta})]} [\underline{\theta}f(\underline{\theta})] \\
&\quad - \frac{pf(\underline{\theta})(F(b) - F(a))}{2[F(\bar{\theta}) - F(b) + F(a) - F(\underline{\theta})]^2} \left[\int_b^{\bar{\theta}} xf(x)dx - \int_{\underline{\theta}}^a xf(x)dx \right] - (\underline{\theta}f(\underline{\theta}) + F(\underline{\theta})) \\
&= \frac{pf(\underline{\theta})[(1 - F(\bar{\theta}) - F(\underline{\theta}))]}{2} \left[\frac{\mathbb{E}[x|x \in \Theta^d] - \underline{\theta}}{[F(\bar{\theta}) - F(b) + F(a) - F(\underline{\theta})]} \right] \\
&\quad - \frac{pf(\underline{\theta})}{2} \left[\int_{\{x \in \Theta^d\}} \frac{xf(x)dx}{[F(\bar{\theta}) - F(b) + F(a) - F(\underline{\theta})]} \right] - \frac{p(F(b) - F(a))}{2[F(\bar{\theta}) - F(b) + F(a) - F(\underline{\theta})]} [\underline{\theta}f(\underline{\theta})] \\
&\quad - \frac{pf(\underline{\theta})(F(b) - F(a))}{2[F(\bar{\theta}) - F(b) + F(a) - F(\underline{\theta})]^2} \left[\int_b^{\bar{\theta}} xf(x)dx - \int_{\underline{\theta}}^a xf(x)dx \right] - [\underline{\theta}f(\underline{\theta}) + F(\underline{\theta})] \left[1 - \frac{p}{2} \right]
\end{aligned}$$

By noting that $(1 - F(\bar{\theta}) - F(\underline{\theta})) \leq 0$ by feasibility, each term in the above sum is negative and hence Π is decreasing in $\underline{\theta}$. We will use this to show that the feasibility binds.

Next, we consider the Lagrangian problem to study the KKT conditions. Here, we will initially relax the problem by relaxing the equality constraint $F(a) - F(\underline{\theta}) = F(\bar{\theta}) - F(b)$ to $F(a) + F(b) \geq F(\underline{\theta}) + F(\bar{\theta})$ but focus on solutions where it binds. From here, we are going to learn that the feasibility constraint must bind. We will use this to observe that $a = b$ should hold in the equilibrium, which will reduce the unrestricted equilibrium to a simple equilibrium.

$$\begin{aligned}
\mathcal{L}(\underline{\theta}, a, b, \bar{\theta}, \lambda) &= \Pi + \lambda_1(F(\underline{\theta}) + F(\bar{\theta}) - 1) + \lambda_2(1 - \bar{\theta}) + \lambda_3(\bar{\theta} - b) + \lambda_4(b - a) + \lambda_5(a - \underline{\theta}) + \lambda_6\underline{\theta} \\
&\quad + \lambda_7(F(a) + F(b) - F(\underline{\theta}) - F(\bar{\theta})) \\
\frac{\partial \mathcal{L}}{\partial \underline{\theta}} &= \frac{\partial \Pi}{\partial \underline{\theta}} + \lambda_1 f(\underline{\theta}) - \lambda_5 + \lambda_6 - \lambda_7 f(\underline{\theta}) = 0 \\
\frac{\partial \mathcal{L}}{\partial a} &= \frac{\partial \Pi}{\partial a} - \lambda_4 + \lambda_5 + \lambda_7 f(a) = 0 \\
\frac{\partial \mathcal{L}}{\partial b} &= \frac{\partial \Pi}{\partial b} - \lambda_3 + \lambda_4 + \lambda_7 f(b) = 0 \\
\frac{\partial \mathcal{L}}{\partial \bar{\theta}} &= \frac{\partial \Pi}{\partial \bar{\theta}} + \lambda_1 f(\bar{\theta}) - \lambda_3 + \lambda_4 - \lambda_7 f(\bar{\theta}) = 0 \\
\lambda_i &\geq 0 \\
\lambda_1(F(\underline{\theta}) + F(\bar{\theta}) - 1) &= 0 \\
\lambda_2(1 - \bar{\theta}) &= 0 \\
\lambda_3(\bar{\theta} - b) &= 0 \\
\lambda_4(b - a) &= 0 \\
\lambda_5(a - \underline{\theta}) &= 0 \\
\lambda_6\underline{\theta} &= 0 \\
\lambda_7(F(a) + F(b) - F(\underline{\theta}) - F(\bar{\theta})) &= 0.
\end{aligned}$$

First, we note that for $\Pi > 0$, we need $1 > \bar{\theta}$. Moreover, for $1 > \bar{\theta}$, we need $\underline{\theta} > 0$ by feasibility. Then, we have $\lambda_2 = \lambda_6 = 0$ by complementary slackness conditions.

Remember that for $\underline{\theta} > 0$, we have $\frac{\partial \Pi}{\partial \underline{\theta}} < 0$. Then, since $\lambda_6 = 0$ and $\lambda_5, \lambda_7 \geq 0$, for $\frac{\partial \mathcal{L}}{\partial \underline{\theta}} = 0$, we need $\lambda_1 > 0$. By complementary slackness, this implies the feasibility constraint must bind.

Then, the profit function becomes:

$$\Pi = -\frac{p(F(b) - F(a))}{2[F(\bar{\theta}) - F(b) + F(a) - F(\underline{\theta})]} \left[\int_b^{\bar{\theta}} xf(x)dx - \int_{\underline{\theta}}^a xf(x)dx \right] + \frac{2-p}{2} [-\underline{\theta}F(\underline{\theta}) - \bar{\theta}F(\bar{\theta}) + \bar{\theta}].$$

Next we are going to argue that in any solution to the above problem with a positive profit, we must have $a = b$. Suppose $(\underline{\theta}, a, b, \bar{\theta})$ maximizes Π and $b > a$. Remember that we reject any solution that doesn't satisfy $F(a) - F(\underline{\theta}) = F(\bar{\theta}) - F(b)$, since this is an equilibrium requirement. Then, the differences of integrals in the above equation is nonnegative. Moreover, it is strictly positive if $\Pi > 0$:

Notice that for $\Pi > 0$, we need $\bar{\theta} > \underline{\theta}$. If $\bar{\theta} = \underline{\theta}$, then it must be the case that $\bar{\theta} = a = b = \underline{\theta} = F^{-1}(0.5)$ and then we can verify that $\Pi = 0$. $\bar{\theta} > \underline{\theta}$ implies $\bar{\theta} > b$ and $a > \underline{\theta}$ because (i) we need $F(a) - F(\underline{\theta}) = F(\bar{\theta}) - F(b)$ in the equilibrium and (ii) $\bar{\theta} = b > a = \underline{\theta}$ cannot happen in the equilibrium as shown before stating the Lagrangian problem. But when we have $\bar{\theta} > b \geq a > \underline{\theta}$ and $F(a) - F(\underline{\theta}) = F(\bar{\theta}) - F(b)$, we have:

$$\begin{aligned} \left[\int_b^{\bar{\theta}} xf(x)dx - \int_{\underline{\theta}}^a xf(x)dx \right] &= \left[(F(\bar{\theta}) - F(b)) \int_b^{\bar{\theta}} \frac{xf(x)dx}{(F(\bar{\theta}) - F(b))} - (F(a) - F(\underline{\theta})) \int_{\underline{\theta}}^a \frac{xf(x)dx}{(F(a) - F(\underline{\theta}))} \right] \\ &= (F(\bar{\theta}) - F(b))\mathbb{E}[x|x \in [b, \bar{\theta}]] - (F(a) - F(\underline{\theta}))\mathbb{E}[x|x \in [\underline{\theta}, a]] \\ &= (F(\bar{\theta}) - F(b)) \left[\mathbb{E}[x|x \in [b, \bar{\theta}]] - \mathbb{E}[x|x \in [\underline{\theta}, a]] \right] > 0. \end{aligned}$$

Then, we must have $a = b$, since this doesn't effect the virtual surplus but minimizes the cost. Moreover, it must be the case that $F(a) = F(b) = F^{-1}(\frac{1}{2})$ since we need $F(a) - F(\underline{\theta}) = F(\bar{\theta}) - F(b)$ and the feasibility binds. Thus, we have reduced the unrestricted equilibrium to a simple equilibrium. \square

I Uniqueness of the Equilibrium

Proof of Proposition 3.6. 1. Suppose all agents join the centralized market. Then, everyone's outside option is 0, since no agent trades in the decentralized market. Thus, the marketplace must be operating the baseline mechanism (when it operated without the competition from the decentralized market) to maximize the profit. Then, agents who don't get to trade get 0 utility. However, any two agent can profitably deviate to the decentralized market: They almost surely have different valuations and as long as they have different valuations, they have a positive surplus to share. Thus, it isn't a best

response for them to stay in the marketplace.

2. Suppose all agents joined the decentralized market. We are going to show that there exist $a \leq \underline{\theta}$ and $b \geq \bar{\theta}$ such that agents in $[0, a) \cup (b, 1]$ strictly prefer joining the centralized marketplace instead.

When all agents join the decentralized market,

$$\begin{aligned} u^d(0) &= \frac{p}{2} \mathbb{E} [\theta | \theta \in [0, 1]] \\ u^d(1) &= \frac{p}{2} [1 - \mathbb{E} [\theta | \theta \in [0, 1]]] \end{aligned}$$

Next, we compute the utilities the centralized marketplace promises them.

$$\begin{aligned} u^m(0) &= \frac{p}{2} \left[\mathbb{E} [\theta | \theta \in [\underline{\theta}, \bar{\theta}]] - \underline{\theta} \right] + \underline{\theta} \\ u^m(1) &= \frac{p}{2} \left[\bar{\theta} - \mathbb{E} [\theta | \theta \in [\underline{\theta}, \bar{\theta}]] \right] + 1 - \bar{\theta} \end{aligned}$$

Now, we check whether these two types can profitably deviate from the decentralized market to the marketplace:

$$\begin{aligned} u^m(0) + u^m(1) &> u^d(0) + u^d(1) \\ \iff \frac{p}{2} [-\underline{\theta}] + \underline{\theta} + \frac{p}{2} [\bar{\theta}] + 1 - \bar{\theta} &> \frac{p}{2} [1] \\ \iff \underline{\theta} + 1 - \bar{\theta} &> \frac{p}{2} [1 - \bar{\theta} + \underline{\theta}] \\ \iff 1 &> \frac{p}{2} \end{aligned}$$

In fact, since both u^m and u^d are continuous functions, the above inequality show that there are indeed a, b such that pairs of agents with values in $[0, a)$ and in $(b, 1]$ strictly prefer the mechanism to the decentralized market. Thus, all agents joining the decentralized market cannot be an equilibrium.

3. Earlier, I have shown that the only possible coexistence equilibrium is the simple equilibrium, and that a simple equilibrium exists. Now, I have further shown that

there is no equilibrium without coexistence. Thus, the unique equilibrium is the simple equilibrium.

□

J Optimal Bids for Agents in the Marketplace

Proof of Lemma 3.2. Step 1: Bidding below $b(\underline{\theta})$ can never be optimal: For any bid $b \leq b(\underline{\theta})$, the agent sells her endowment with certainty but get paid less than she would get if she bid $b(\underline{\theta})$. Mathematically, the expected utility of an agent who bids $b \leq b(\underline{\theta})$ is given by:

$$\frac{1}{2} \int_{\underline{\theta}}^{\bar{\theta}} [b + b(x)]g(x)dx - \theta$$

Thus, bids strictly below $b(\underline{\theta})$ cannot be optimal.

Similarly, bids strictly above $b(\bar{\theta})$ cannot be optimal either. In that case, the agent's expected payoff would be

$$\theta - \frac{1}{2} \int_{\underline{\theta}}^{\bar{\theta}} [b + b(x)]g(x)dx$$

Hence, for each $\theta \in [0, \underline{\theta}] \cup [\bar{\theta}, 1]$, the optimal bid must be the bid of some type joins the decentralized market, that is: $b(\theta) \in [b(\underline{\theta}), b(\bar{\theta})]$.

Step 2: Let us first define the following notation: If an agent bids $b(\theta')$, then the expected price for selling is $p_s(\theta') = \frac{1}{2} \int_{\underline{\theta}}^{\theta'} \frac{[b(\theta') + b(x)]g(x)dx}{1 - G(\theta')}$ and the expected price for buying is $p_b(\theta') =$

$$\frac{1}{2} \int_{\theta'}^{\bar{\theta}} \frac{[b(\theta') + b(x)]g(x)dx}{G(\theta')}.$$

Since the best response of agent with value $\underline{\theta}$ is $b(\underline{\theta})$, her expected payoff from this bid should be higher than any other $b(\theta')$ by revealed preference. Then,

$$\begin{aligned} p_s(\underline{\theta}) - \underline{\theta} &\geq (1 - G(\theta')) [p_s(\theta') - \underline{\theta}] + G(\theta') [\underline{\theta} - p_b(\theta')] \\ &= \underline{\theta}[2G(\theta') - 1] + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \\ \iff p_s(\underline{\theta}) - (1 - G(\theta'))p_s(\theta') + G(\theta')p_b(\theta') &\geq 2\underline{\theta}G(\theta') \end{aligned}$$

Suppose $\theta \leq \underline{\theta}$. We want to show that bidding $b(\underline{\theta})$ gives a higher payoff than any other type's bid $b(\theta')$:

$$\begin{aligned} p_s(\underline{\theta}) - \theta &\geq (1 - G(\theta')) [p_s(\theta') - \theta] + G(\theta') [\underline{\theta} - p_b(\theta')] \\ &= \theta[2G(\theta') - 1] + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \\ p_s(\underline{\theta}) - (1 - G(\theta'))p_s(\theta') + G(\theta')p_b(\theta') &\geq 2\theta G(\theta') \end{aligned}$$

But this is true since $p_s(\underline{\theta}) - (1 - G(\theta'))p_s(\theta') + G(\theta')p_b(\theta') \geq 2\underline{\theta}G(\theta') \geq 2\theta G(\theta')$ where the first inequality follows from the revealed preference argument above and the second one follows from $\underline{\theta} \geq \theta$ and $G(\theta') \geq 0$.

Similarly, the best response of an agent with value $\underline{\theta}$ is $b(\bar{\theta})$. Thus,

$$\begin{aligned} \bar{\theta} - p_b(\bar{\theta}) &\geq (1 - G(\theta')) [p_s(\theta') - \bar{\theta}] + G(\theta') [\bar{\theta} - p_b(\theta')] \\ &= \bar{\theta}[2G(\theta') - 1] + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \\ \iff 2\bar{\theta}(1 - G(\theta')) &\geq p_b(\bar{\theta}) + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \end{aligned}$$

Suppose $\theta \geq \bar{\theta}$. In this case, we want to show that bidding $b(\bar{\theta})$ gives a higher payoff than any other type's bid $b(\theta')$:

$$\begin{aligned} \theta - p_b(\bar{\theta}) &\geq (1 - G(\theta')) [p_s(\theta') - \theta] + G(\theta') [\theta - p_b(\theta')] \\ &= \theta[2G(\theta') - 1] + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \\ \iff 2\theta(1 - G(\theta')) &\geq p_b(\bar{\theta}) + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta') \end{aligned}$$

Again, this is true since $2\theta(1 - G(\theta')) \geq 2\bar{\theta}(1 - G(\theta')) \geq p_b(\bar{\theta}) + (1 - G(\theta'))p_s(\theta') - G(\theta')p_b(\theta')$ where the first inequality again follows from the revealed preference argument above and the second one follows from $\theta \geq \bar{\theta}$ and $1 - G(\theta') \geq 0$. \square

K Binding IR constraints

$$\begin{aligned}
u^{da}(\underline{\theta}) &= p \left[\int_{\underline{\theta}}^{\bar{\theta}} \frac{1}{2} [b(\underline{\theta}) + b(x)] g(x) dx - \underline{\theta} \right] \\
&= p \left[\frac{1}{2} \int_{\underline{\theta}}^{\bar{\theta}} [b(\underline{\theta}) + b(x)] g(x) dx - \underline{\theta} \right] \\
&= p \left[\frac{1}{2} \left[b(\underline{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} b(x) g(x) dx \right] - \underline{\theta} \right] \\
&= p \left[\frac{1}{2} \left[\underline{\theta} - \frac{\int_{\underline{\theta}}^{\bar{\theta}} [G(x) - \frac{1}{2}]^2 dx}{[G(\underline{\theta}) - \frac{1}{2}]^2} + \int_{\underline{\theta}}^{\bar{\theta}} b(x) g(x) dx \right] - \underline{\theta} \right] \\
&= \frac{p}{2} \left[-\underline{\theta} + 4 \int_{\underline{\theta}}^{G^{-1}(\frac{1}{2})} \left[G(x) - \frac{1}{2} \right]^2 dx + \int_{\underline{\theta}}^{\bar{\theta}} b(x) g(x) dx \right]
\end{aligned}$$

Similarly, we obtain,

$$\begin{aligned}
u^{da}(\bar{\theta}) &= p \left[\bar{\theta} - \int_{\underline{\theta}}^{\bar{\theta}} \frac{1}{2} [b(\bar{\theta}) + b(x)] g(x) dx \right] \\
&= p \left[\bar{\theta} - \frac{1}{2} \left[b(\bar{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} b(x) g(x) dx \right] \right] \\
&= p \left[\bar{\theta} - \frac{1}{2} \left[\bar{\theta} - \frac{\int_{G^{-1}(\frac{1}{2})}^{\bar{\theta}} \left[G(x) - \frac{1}{2} \right]^2 dx}{\left[G(\bar{\theta}) - \frac{1}{2} \right]^2} + \int_{\underline{\theta}}^{\bar{\theta}} b(x) g(x) dx \right] \right] \\
&= \frac{p}{2} \left[\bar{\theta} + 4 \int_{G^{-1}(\frac{1}{2})}^{\bar{\theta}} \left[G(x) - \frac{1}{2} \right]^2 dx - \int_{\underline{\theta}}^{\bar{\theta}} b(x) g(x) dx \right]
\end{aligned}$$

L Slope of Utilities from the Double Auction

Proof of Lemma 3.3. It is easy to verify that for agents with values less than $\underline{\theta}$, $\frac{\partial u^{da}(\theta)}{\partial \theta} = -p \geq -1$ and for agents with values above $\bar{\theta}$, $\frac{\partial u^{da}(\theta)}{\partial \theta} = p \leq 1$. Next we show that $\frac{\partial u^{da}(\theta)}{\partial \theta}$ is greater than -1 for $\underline{\theta}$ and less than 1 for $\bar{\theta}$.

First, we need the derivative of the bidding function:

$$\begin{aligned}
b(\theta) &= \theta - \frac{\int_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2} \\
b'(\theta) &= 1 - \frac{[G(\theta) - \frac{1}{2}]^4 - 2g(\theta)(G(\theta) - \frac{1}{2}) \int_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^4} \\
&= \frac{2g(\theta)(G(\theta) - \frac{1}{2}) \int_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^4} \\
&= 2g(\theta) \frac{\int_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^3}
\end{aligned}$$

$$u^{da}(\theta) = p\theta[2G(\theta) - 1] + \frac{p}{2}(1 - 2G(\theta))b(\theta) + \frac{p}{2} \int_{\theta}^{\bar{\theta}} b(x)g(x)dx - \frac{p}{2} \int_{\underline{\theta}}^{\theta} b(x)g(x)dx$$

$$\begin{aligned}
\frac{\partial u^{da}(\theta)}{\partial \theta} &= p [(2G(\theta) - 1) + 2\theta g(\theta)] + \frac{p}{2} [-2g(\theta)b(\theta) + (1 - 2G(\theta))b'(\theta)] - \frac{p}{2} 2b(\theta)g(\theta) \\
&= p(2G(\theta) - 1) + 2pg(\theta)(\theta - b(\theta)) + \frac{p}{2} [(1 - 2G(\theta))b'(\theta)] \\
&= p \left[G(\theta) - 1 + 2g(\theta)(\theta - b(\theta)) + \frac{1}{2} [(1 - 2G(\theta))b'(\theta)] \right] \\
&= p \left[G(\theta) - 1 + 2g(\theta) \left[\theta - \theta + \frac{\int_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2} \right] + \frac{1}{2} (1 - 2G(\theta))b'(\theta) \right] \\
&= p \left[G(\theta) - 1 + 2g(\theta) \frac{\int_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2} + \frac{1}{2} (1 - 2G(\theta)) \left[2g(\theta) \frac{\int_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^3} \right] \right] \\
&= p \left[G(\theta) - 1 + 2g(\theta) \frac{\int_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2} - 2g(\theta) \frac{\int_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2} \right] \\
&= p [G(\theta) - 1]
\end{aligned}$$

□

M Bids with Uniform Distribution

$$\begin{aligned}
b(\theta) &= \theta - \frac{\int_{G^{-1}(\frac{1}{2})}^{\theta} [G(x) - \frac{1}{2}]^2 dx}{[G(\theta) - \frac{1}{2}]^2} &= \theta - \frac{\int_{\frac{\underline{\theta} + \bar{\theta}}{2}}^{\theta} \left[\frac{x - \underline{\theta}}{\bar{\theta} - \underline{\theta}} - \frac{1}{2} \right]^2 dx}{\left[\frac{\theta - \underline{\theta}}{\bar{\theta} - \underline{\theta}} - \frac{1}{2} \right]^2} \\
&= \theta - \frac{\int_{\frac{\underline{\theta} + \bar{\theta}}{2}}^{\theta} \left[\frac{2x - \underline{\theta} - \bar{\theta}}{2(\bar{\theta} - \underline{\theta})} \right]^2 dx}{\left[\frac{2\theta - \underline{\theta} - \bar{\theta}}{2(\bar{\theta} - \underline{\theta})} \right]^2} &= \theta - \frac{\int_{\frac{\underline{\theta} + \bar{\theta}}{2}}^{\theta} [2x - \underline{\theta} - \bar{\theta}]^2 dx}{[2\theta - \underline{\theta} - \bar{\theta}]^2} \\
&= \theta - \frac{\left[\frac{1}{2 \times 3} [2x - \underline{\theta} - \bar{\theta}]^3 \right]_{\frac{\underline{\theta} + \bar{\theta}}{2}}^{\theta}}{[2\theta - \underline{\theta} - \bar{\theta}]^2} &= \theta + \frac{1}{6} \frac{[2\theta - \underline{\theta} - \bar{\theta}]^3}{[2\theta - \underline{\theta} - \bar{\theta}]^2} \\
&= \theta - \frac{2\theta - \underline{\theta} - \bar{\theta}}{6} &= \frac{4\theta + \underline{\theta} + \bar{\theta}}{6}
\end{aligned}$$

Next, we compute another expression from the profit function:

$$\begin{aligned}
\int_{\underline{\theta}}^{\bar{\theta}} b(x)g(x)dx &= \int_{\underline{\theta}}^{\bar{\theta}} \frac{4\theta + \underline{\theta} + \bar{\theta}}{6} \frac{1}{\bar{\theta} - \underline{\theta}} dx = \frac{1}{6(\bar{\theta} - \underline{\theta})} \int_{\underline{\theta}}^{\bar{\theta}} (4\theta + \underline{\theta} + \bar{\theta}) dx \\
&= \frac{1}{6(\bar{\theta} - \underline{\theta})} [2\bar{\theta}^2 - 2\underline{\theta}^2 + (\bar{\theta} + \underline{\theta})(\bar{\theta} - \underline{\theta})] = \frac{1}{6(\bar{\theta} - \underline{\theta})} [3(\bar{\theta} + \underline{\theta})(\bar{\theta} - \underline{\theta})] = \frac{\bar{\theta} + \underline{\theta}}{2}
\end{aligned}$$

N Profit from Simple Equilibrium under Double Auction

$$\begin{aligned}
\Pi_{\underline{\theta}, \bar{\theta}} &= - \int_0^{\underline{\theta}} C(x) f(x) dx + \int_{\bar{\theta}}^1 \mathcal{V}(x) f(x) dx - F(\underline{\theta}) u^d(\underline{\theta}) - (1 - F(\bar{\theta})) u^d(\bar{\theta}) \\
&= \left[-\underline{\theta} F(\underline{\theta}) + \bar{\theta} (1 - F(\bar{\theta})) \right] - p \left[-\underline{\theta} F(\underline{\theta}) + \bar{\theta} (1 - F(\bar{\theta})) \right] \\
&\quad - \frac{p}{2} \left[\left[F(\underline{\theta}) + F(\bar{\theta}) - 1 \right] \int_{\underline{\theta}}^{\bar{\theta}} b(x) g(x) dx + F(\underline{\theta}) b(\underline{\theta}) - (1 - F(\bar{\theta})) b(\bar{\theta}) \right] \\
&= (1 - p) \left[-\underline{\theta} F(\underline{\theta}) + \bar{\theta} (1 - F(\bar{\theta})) \right] \\
&\quad - \frac{p}{2} \left[\left[F(\underline{\theta}) + F(\bar{\theta}) - 1 \right] \int_{\underline{\theta}}^{\bar{\theta}} b(x) g(x) dx + F(\underline{\theta}) b(\underline{\theta}) - (1 - F(\bar{\theta})) b(\bar{\theta}) \right] \\
&= (1 - p) \left[-\underline{\theta}^2 + \bar{\theta} (1 - \bar{\theta}) \right] \\
&\quad - \frac{p}{2} \left[\left[\underline{\theta} + \bar{\theta} - 1 \right] \frac{\bar{\theta} + \underline{\theta}}{2} + \underline{\theta} \frac{5\underline{\theta} + \bar{\theta}}{6} - (1 - \bar{\theta}) \frac{5\bar{\theta} + \underline{\theta}}{6} \right]
\end{aligned}$$

The profit is decreasing in $\underline{\theta}$:

$$\begin{aligned}
\frac{\partial \Pi_{\underline{\theta}, \bar{\theta}}}{\partial \underline{\theta}} &= (1 - p) \left[-2\underline{\theta} \right] - \frac{p}{2} \left[\frac{\bar{\theta} + \underline{\theta}}{2} + \frac{1}{2} \left[\underline{\theta} + \bar{\theta} - 1 \right] + \frac{5\underline{\theta} + \bar{\theta}}{6} + \underline{\theta} \frac{5}{6} - (1 - \bar{\theta}) \frac{1}{6} \right] \\
&= (1 - p) \left[-2\underline{\theta} \right] - \frac{p}{2} \left[\frac{2}{3} (4\underline{\theta} + 2\bar{\theta} - 1) \right] \leq 0
\end{aligned}$$

Notice that the first summand is negative and inside the brackets of the second summand is positive since $\underline{\theta} \geq 1 - \bar{\theta}$ by feasibility. Thus, the profit is decreasing in $\underline{\theta}$. Hence, the feasibility binds and we have $\underline{\theta} = 1 - \bar{\theta}$, otherwise decreasing $\underline{\theta}$ until the feasibility binds strictly increases the profit. Then, we have

$$\begin{aligned}
\Pi_{\underline{\theta}, \bar{\theta}} &= (1-p) \left[-\underline{\theta}^2 + \bar{\theta}(1-\bar{\theta}) \right] - \frac{p}{2} \left[\underline{\theta} \frac{5\underline{\theta} + \bar{\theta}}{6} - (1-\bar{\theta}) \frac{5\bar{\theta} + \underline{\theta}}{6} \right] \\
&= (1-p) \left[-\underline{\theta}^2 + \bar{\theta}(1-\bar{\theta}) \right] - \frac{p}{2} \left[\underline{\theta} \frac{5\underline{\theta} + \bar{\theta} - 5\bar{\theta} - \underline{\theta}}{6} \right] \\
&= (1-p) \left[-\underline{\theta}^2 + (1-\underline{\theta})\underline{\theta} \right] - \frac{p}{2} \left[\underline{\theta} \frac{5\underline{\theta} + (1-\underline{\theta}) - 5(1-\underline{\theta}) - \underline{\theta}}{6} \right] \\
&= (1-p) \left[\underline{\theta}(1-2\underline{\theta}) \right] - \frac{p}{2} \left[\underline{\theta} \frac{8\underline{\theta} - 4}{6} \right] = (1-p) \left[\underline{\theta}(1-2\underline{\theta}) \right] + \frac{p}{6} \left[\underline{\theta}(1-2\underline{\theta}) \right] \\
&= \frac{6-5p}{6} \left[\underline{\theta}(1-2\underline{\theta}) \right] = \frac{6-5p}{6} \Pi^M
\end{aligned}$$