Class notes for math+econ+code

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1 Day 1

Linear programming duality

The problem is

$$\begin{aligned} V_P = \max_{x \geq 0} & \quad x^\top c \\ s.t. & \quad Ax \leq d \end{aligned}$$

We would like to write this as

$$\max_{x \geq 0} x^\top c + F(d - Ax)$$

where $F(u) = 0$ if $u \geq 0$, $F(u) = -\infty$ otherwise. The simplest choice is

$$F(u) = \min_{y \geq 0} \left\{ y^\top u \right\} = \min_{y_j \geq 0} \left\{ \sum_j y_j u_j \right\}.$$

Thus rewrite the problem as

$$\begin{aligned} V_P &= \max_{x \geq 0} x^\top c + \min_{y \geq 0} \left\{ y^\top (d - Ax) \right\} \\ &= \max_{x \geq 0} \min_{y \geq 0} \left\{ x^\top c + y^\top d - y^\top Ax \right\} \end{aligned}$$

By the minimax theorem, if there are feasible solutions, then

$$\begin{aligned} V_P &= \min_{y \geq 0} \max_{x \geq 0} \left\{ x^\top c + y^\top d - y^\top Ax \right\} \\ &= \min_{y \geq 0} y^\top d + \max_{x \geq 0} \left\{ x^\top c - y^\top Ax \right\} \\ &= \min_{y \geq 0} y^\top d + \max_{x \geq 0} \left\{ x^\top c - x^\top A^\top y \right\} \\ &= \min_{y \geq 0} y^\top d + \max_{x \geq 0} \left\{ x^\top \left(c - A^\top y \right) \right\} \end{aligned}$$

Now, we remark that

$$\begin{aligned}\max_{x\geq 0}\left\{x^{\top}\left(c-A^{\top}y\right)\right\} &= +\infty \text{ if } c_i > \left(A^{\top}y\right)_i \text{ for some } i\\ &= 0 \text{ if } c \leq A^{\top}y\end{aligned}$$

we have derived the dual problem

$$\begin{aligned}V_P = V_D := \min_{y\geq 0} & \quad y^{\top}d\\ s.t.\ c & \leq A^{\top}y\end{aligned}$$

Further, if x^* an optimal solution to primal and y^* an optimal solution to the dual problem, we have

$$\left(x^*\right)^{\top}\left(c-A^{\top}y^*\right)=0$$

but $c \leq A^{\top}y^*$ and $x^* \geq 0$ therefore we have for each i

$$x_i^*\left(c_i-\left(A^{\top}y^*\right)_i\right)=0$$

therefore

$$x_i^* > 0 \implies \left(A^{\top}y^*\right)_i = c_i$$

and similarly

$$y_j^* > 0 \implies \left(Ax^*\right)_j = d_j$$

This is **complementary slackness**.

1.1 Gradient matrix

∇_{ax} $a \in A, x \in Z$

$$(\nabla f)_{xy} = \sum_z \nabla_{(xy)z} f_z = f_y - f_x$$

Mass balance equation. For every z,

$$\begin{aligned}q_z &= \text{(total mass arriving to } z \text{ from other nodes)}\\ &\quad -\text{(total mass departing from } z \text{ to other nodes)}\\ &= \sum_x \mu_{xz} - \sum_y \mu_{zy}\end{aligned}$$

This can be expressed in a matrix way! Indeed,

$$q_z = \sum_{a\in A}\nabla_{az}\mu_a$$

hence mass balance rewrites in a matrix notation as

$$q = \nabla^{\top}\mu.$$

By the way, if q satisfies mass balance, then

$$
\begin{aligned}
q_z &= \sum_x \mu_{xz} - \sum_y \mu_{zy} \\
\sum_{z\in Z} q_z &= \sum_{z\in Z} \sum_{x\in Z:xz\in A} \mu_{xz} - \sum_{z\in Z} \sum_{y:zy\in A} \mu_{zy} \\
&= \sum_{a\in A} \mu_a - \sum_{a\in A} \mu_a \\
&= 0
\end{aligned}
$$

1.2 An equilibrium problem

Introduce p_z=price of the commodity at z.

Consider a trader operating on the arc xy.

trader's profit $= p_y - p_x - c_{xy}$

Assume that there is free entry of traders on any arc. Absence of rent implies that for any arc $xy \in A$ we have

$$p_y - p_x \leq c_{xy}$$

which can be written

$$\nabla p \leq c$$

No arbitrage condition.

Now assume at equilibrium, a quantity $\mu_{xy} > 0$ is shipped from x to y. This implies that the traders at arc xy break even, hence

$$p_y - p_x - c_{xy} = 0$$

To summarize, an equilbrium on the network is given by $(p_z)_{z\in Z}$ and $\left(\mu_{xy}\right)_{xy\in A}$ such that:

(i) balance of mass holds: $\nabla^\top \mu = q$

(ii) no arbitrage holds: $\nabla p \leq c$.

(iii) $\mu_{xy} > 0 \implies (\nabla p)_{xy} = p_y - p_x = c_{xy}$.

1.3 An optimal shippment problem

Consider the Soviet problem. They decide on μ_{xy} subject to mass balance $\nabla^\top \mu = q$.

In order to minimize costs, they try to achieve the minimum of $\sum_{xy\in A} \mu_{xy} c_{xy}$.

Thus, they do:

$$
\begin{aligned}
\min_{\mu\geq 0} & \qquad & \mu^\top c \\
s.t. & \quad \nabla^\top \mu = & q \ \ [p]
\end{aligned}
$$

this is a linear programming problem (primal). Let's compute its dual.

$$\begin{aligned} \max_{p} \quad & & p^\top q \\ s.t. \quad & \nabla p \leq & c \ \ [\mu \geq 0] \end{aligned}$$

Theorem: if μ and p are respectively solutions to the primal and the dual problems, then they also solve the equilbrium problem above. Indeed, $\nabla^\top \mu = q$ and $\nabla p \leq c$ are immediately satisfies, and by complementary slackness, $\mu_{xy} > 0$ implies

$$p_y - p_x = c_{xy}.$$

2 Day 2

Solve the central planner's problem

$$\begin{aligned} \max_{\pi_{xy} \geq 0} \quad & \sum_{xy} \pi_{xy} \Phi_{xy} \\ s.t. \quad & \sum_{y} \pi_{xy} = p_x \ \ [u_x] \\ & \sum_{x} \pi_{xy} = q_y \ \ [v_y] \end{aligned}$$

by the maxmin formulation

$$\begin{aligned} & \max_{\pi_{xy} \geq 0} \min_{u_x, v_y} \sum_{xy} \pi_{xy} \Phi_{xy} + \sum_{x} p_x u_x + \sum_{y} q_y v_y - \sum_{xy} \pi_{xy} \left(u_x + v_y \right) \\ = \quad & \min_{u_x, v_y} \sum_{x} p_x u_x + \sum_{y} q_y v_y + \max_{\pi_{xy} \geq 0} \sum_{xy} \pi_{xy} \left(\Phi_{xy} - u_x - v_y \right) \end{aligned}$$

this is

$$\begin{aligned} \min_{u_x, v_y} \quad & \sum_{x} p_x u_x + \sum_{y} q_y v_y \\ s.t. \quad & u_x + v_y \geq \Phi_{xy} \ \ [\pi_{xy} \geq 0] \end{aligned}$$

Remark: by complementary slackness, $\pi_{xy} > 0 \implies u_x + v_y = \Phi_{xy}$.

Remark 2: if (u, v) then $(u + c, v - c)$ is also a solution for any constant c.

Take a solution (u, v) of the dual problem. Then for any feasible solution we have

$$v_y \geq \max_{x} \left\{ \Phi_{xy} - u_x \right\}$$

but for any optimal solution (u, v) we have

$$v_y = \max_{x} \left\{ \Phi_{xy} - u_x \right\}.$$

But the same logic implies

$$u_x = \max_y \{\Phi_{xy} - v_y\}.$$

2.1 Interpretation 1: stable matching

(π_{xy}, u_x, v_y) is a stable matching if:

- $\pi \geq 0$ and $\sum_y \pi_{xy} = p_x$ and $\sum_x \pi_{xy} = q_y$
- $\forall x, y, u_x + v_y \geq \Phi_{xy}$
- If $u_x + v_y > \Phi_{xy} \implies \pi_{xy} = 0$; or in other words $\pi_{xy} > 0 \implies u_x + v_y = \Phi_{xy}$

If $u_x + v_y < \Phi_{xy}$ then xy would be a blocking pair, this should be ruled out

2.2 Interpretation 2: Wage equilibrium

Assume x is a worker and y is a firm, and interpret u_x as the wage of worker x, and v_y as the profit of firm y.

Then firm's problem is

$$\begin{aligned}
v_y &= \max_x \{\Phi_{xy} - u_x\} \\
\pi_{xy} &> 0 \implies v_y = \Phi_{xy} - u_x \\
\pi &\geq 0, \sum_y \pi_{xy} = p_x, \sum_x \pi_{xy} = q_y
\end{aligned}$$

Rewrite this as

$$\begin{aligned}
v_y &\geq \Phi_{xy} - u_x \forall x, y \\
\pi_{xy} &> 0 \implies v_y = \Phi_{xy} - u_x \\
\pi &\geq 0, \sum_y \pi_{xy} = p_x, \sum_x \pi_{xy} = q_y
\end{aligned}$$

2.3 Case with unassigned agents

Assume that people don't have to match, and if they remain unmatched they get utility 0

π_{x0} =mass of men of type x remaining unassigned

π_{0y} =mass of women of type y remaining unassigned

$\sum_y \pi_{xy} + \pi_{x0} = p_x$

$\sum_x \pi_{xy} + \pi_{0y} = q_y$

Optimal assignement problem

$$\max_{\pi\geq 0}\sum_{xy}\pi_{xy}\Phi_{xy}$$
$$s.t. \qquad \sum_{y}\pi_{xy}+\pi_{x0}=p_x$$
$$\sum_{x}\pi_{xy}+\pi_{0y}=q_y$$

rewrite this as

$$\max_{\pi\geq 0}\sum_{xy}\pi_{xy}\Phi_{xy}$$
$$s.t. \qquad \sum_{y}\pi_{xy}\leq p_x \;\; [u_x\geq 0]$$
$$\sum_{x}\pi_{xy}\leq q_y \;\; [v_y\geq 0]$$

whose dual is

$$\min_{u_x\geq 0, v_y\geq 0} \qquad \sum_x p_x u_x+\sum_y q_y v_y$$
$$s.t. \qquad u_x+v_y\geq \Phi_{xy} \;\; [\pi_{xy}\geq 0]$$

3 Day 3

Compute primal problem

$$\max_{\pi\geq 0}\left\{\sum_{ij}\pi_{ij}\Phi_{ij}-\sigma\sum_{ij}\pi_{ij}\log\pi_{ij}\right\}$$
$$s.t. \qquad \sum_j \pi_{ij}=p_i$$
$$\sum_i \pi_{ij}=q_j$$

Write Lagrangian

$$\max_{\pi\geq 0}\left\{\sum_{ij}\pi_{ij}\Phi_{ij}-\sigma\sum_{ij}\pi_{ij}\log\pi_{ij}\right\}+\min_{u_i,v_j}\left\{\sum_i u_i\left(p_i-\sum_j\pi_{ij}\right)+\sum_j v_j\left(q_j-\sum_i\pi_{ij}\right)\right\}$$

$$\min_{u_i,v_j}\sum_i p_i u_i+\sum_j q_j v_j+\max_{\pi\geq 0}\left\{\sum_{ij}\pi_{ij}\left(\Phi_{ij}-u_i-v_j\right)-\sigma\sum_{ij}\pi_{ij}\log\pi_{ij}\right\}$$

FOC in the inner problem. We have

$$\Phi_{ij} - u_i - v_j = \sigma \left(1 + \log \pi_{ij}\right)$$

that is

$$\pi_{ij} = \Pi_{ij}\left(u_i, v_j\right) := \exp\left(\frac{\Phi_{ij} - u_i - v_j - \sigma}{\sigma}\right)$$

and we have

$$\pi_{ij}\left(\Phi_{ij} - u_i - v_j\right) = \sigma\pi_{ij} + \sigma\pi_{ij}\log\pi_{ij}$$

therefore

$$\pi_{ij}\left(\Phi_{ij} - u_i - v_j\right) - \sigma\pi_{ij}\log\pi_{ij} = \sigma\pi_{ij}$$

hence the previous problem becomes

$$\min_{u_i, v_j} \sum_i p_i u_i + \sum_j q_j v_j + \sum_{ij} \sigma \Pi_{ij}\left(u_i, v_j\right)$$

where $\Pi_{ij}\left(u_i, v_j\right) := \exp\left(\frac{\Phi_{ij} - u_i - v_j - \sigma}{\sigma}\right)$, that is

$$\min_{u_i, v_j} F\left(u, v\right) := \sum_i p_i u_i + \sum_j q_j v_j + \sum_{ij} \sigma \exp\left(\frac{\Phi_{ij} - u_i - v_j - \sigma}{\sigma}\right)$$

F is smooth and convex but ** not ** strictly convex because

$$F\left(u + c, v - c\right) = F\left(u, v\right)$$

Consider the problem of

$$\min_{\left(\theta_i\right)_{1 \leq i \leq n}} F\left(\theta\right)$$

1. Gradient descent / tatonnement

$$\theta^{t+1} = \theta^t - \epsilon \nabla F\left(\theta^t\right)$$

2. Newton descent

$$\theta^{t+1} = \theta^t - \epsilon\left(D^2 F\left(\theta^t\right)\right)^{-1} \nabla F\left(\theta^t\right)$$

3. Coordinate descent.
For each t
For each i
Fix θ_j^t for $j \neq i$ and consider the problem

$$\min_{\theta_i \in \mathbb{R}} F\left(\theta_i, \theta_{-i}^t\right)$$

and call it θ_i^t
4. [later on] proximal gradient descent – will talk about it later

3.1 Gradient descent for regularized OT

$F(u,v) = \sum_i p_i u_i + \sum_j q_j v_j + \sigma \sum_{ij} \exp\left(\frac{\Phi_{ij} - u_i - v_j - \sigma}{\sigma}\right)$

$$
\begin{aligned}
\frac{\partial F}{\partial u_i}(u,v) &= p_i - \sum_j \underbrace{\exp\left(\frac{\Phi_{ij} - u_i - v_j - \sigma}{\sigma}\right)}_{\pi_{ij}} \\
\frac{\partial F}{\partial v_j}(u,v) &= q_j - \sum_i \underbrace{\exp\left(\frac{\Phi_{ij} - u_i - v_j - \sigma}{\sigma}\right)}_{\pi_{ij}}
\end{aligned}
$$

therefore (u,v) is optimal iff

$$
\begin{aligned}
\sum_j \Pi_{ij}(u_i, v_j) &= p_i \\
\sum_i \Pi_{ij}(u_i, v_j) &= q_j
\end{aligned}
$$

Gradient descent:

$$
\begin{aligned}
u_i^{t+1} &= u_i^t + \epsilon \left(\sum_j \exp\left(\frac{\Phi_{ij} - u_i - v_j - \sigma}{\sigma}\right) - p_i \right) \\
v_j^{t+1} &= v_j^t + \epsilon \left(\sum_i \exp\left(\frac{\Phi_{ij} - u_i - v_j - \sigma}{\sigma}\right) - q_j \right)
\end{aligned}
$$

Coordinate descent.
We have u_i^{t+1} such that

$$
\frac{\partial F\left(\left(u_i^{t+1}; u_{-i}^t\right), v^t\right)}{\partial u_i^{t+1}} = 0
$$

that is

$$
p_i = \sum_j \exp\left(\frac{\Phi_{ij} - u_i^{t+1} - v_j^t - \sigma}{\sigma}\right)
$$

Introduce $A_i = \exp(-u_i/\sigma)$ and $B_j = \exp(-v_j/\sigma)$, and

$$
K_{ij} = \exp\left(\frac{\Phi_{ij} - \sigma}{\sigma}\right)
$$

we can rewrite the algorithm as

$$
p_i = \sum_j K_{ij} A_i^{t+1} B_j^t,
$$

thus

$$A_i^{t+1} = \frac{1}{\sum_j K_{ij} B_j^t}$$

Similarly, optimality wrt v_j^{t+1} / B_j^{t+1} yields

$$B_j^{t+1} = \frac{1}{\sum_i K_{ij} A_i^{t+1}}.$$

This is the IPFP algorithm / Sinkhorn's algorithm.
Solution when $\Phi = 0$:

$$\pi_{ij} = p_i q_j$$

3.2 The log-sum-exp trick

We have that

$$\sigma \log \left(\exp\left(\frac{a}{\sigma}\right) + \exp\left(\frac{b}{\sigma}\right) \right) \to_{\sigma \to 0^+} \max\{a, b\}$$

because of this $\sigma \log \left(\exp\left(\frac{a}{\sigma}\right) + \exp\left(\frac{b}{\sigma}\right) \right)$ is called smooth-max of a and b.

The idea is that for any $c \in \mathbb{R}$

$$\sigma \log \left(\exp\left(\frac{a+c}{\sigma}\right) + \exp\left(\frac{b+c}{\sigma}\right) \right) = c + \sigma \log \left(\exp\left(\frac{a}{\sigma}\right) + \exp\left(\frac{b}{\sigma}\right) \right)$$

Take $c = -\max\{a, b\}$ will get

$$\begin{aligned} & \sigma \log \left(\exp\left(\frac{a}{\sigma}\right) + \exp\left(\frac{b}{\sigma}\right) \right) - \max\{a, b\} \\ = \; & \sigma \log \left(\exp\left(\frac{\min\{0, a-b\}}{\sigma}\right) + \exp\left(\frac{\min\{0, b-a\}}{\sigma}\right) \right) \end{aligned}$$

But we have

$$0 \le \sigma \log \left(\exp\left(\frac{\min\{0, a-b\}}{\sigma}\right) + \exp\left(\frac{\min\{0, b-a\}}{\sigma}\right) \right) \le \sigma \log 2$$

For practical purposes, we will use

$$\sigma \log \left(\exp\left(\frac{a}{\sigma}\right) + \exp\left(\frac{b}{\sigma}\right) \right) = \sigma \log \left(\exp\left(\frac{a+c}{\sigma}\right) + \exp\left(\frac{b+c}{\sigma}\right) \right) - c$$

with $c = -\max\{a, b\}$, thus

$$\begin{aligned} & \sigma \log \left(\exp\left(\frac{a}{\sigma}\right) + \exp\left(\frac{b}{\sigma}\right) \right) \\ = \; & \max\{a, b\} + \sigma \log \left(\exp\left(\frac{\min\{0, a-b\}}{\sigma}\right) + \exp\left(\frac{\min\{0, b-a\}}{\sigma}\right) \right) \end{aligned}$$

Back to the IPFP algorithm. We had

$$
\begin{aligned}
u_i^{t+1} &= -\sigma \log \left(\frac{1}{p_i} \sum_j \exp \left(\frac{\Phi_{ij} - v_j^t - \sigma}{\sigma} \right) \right) \\
u_i^{t+1} &= \sigma \log p_i - \sigma \log \left(\sum_j \exp \left(\frac{\Phi_{ij} - v_j^t - \sigma}{\sigma} \right) \right)
\end{aligned}
$$

3.3 Discrete choice

Consider

$$\sigma_y (U) = \Pr (U_y + \varepsilon_y \geq U_z + \varepsilon_z \forall z \in \mathcal{Y}_0)$$

IF the distribution of (ε_y) has a density, then

$$\sum_{y \in \mathcal{Y}_0} \sigma_y (U) = 1$$

When the distribution of (ε_y) has a density, then $\Pr (U_y + \varepsilon_y = U_z + \varepsilon_z) = 0$ for $y \neq z$, therefore

$$\sigma_y (U) = \Pr (U_y + \varepsilon_y > U_z + \varepsilon_z \forall z \in \mathcal{Y}_0 \backslash \{y\})$$

and we have

$$\sum_{y \in \mathcal{Y}_0} \Pr (U_y + \varepsilon_y > U_z + \varepsilon_z \forall z \in \mathcal{Y}_0 \backslash \{y\}) \leq 1 \leq \sum_{y \in \mathcal{Y}_0} \sigma_y (U)$$

For instance when $\mathcal{Y}_0 = \{0, 1\}$

$$\Pr (U_1 + \varepsilon_1 > U_0 + \varepsilon_0) + \Pr (U_0 + \varepsilon_0 \geq U_1 + \varepsilon_1) = 1$$

but then

$$\Pr (U_1 + \varepsilon_1 \geq U_0 + \varepsilon_0) + \Pr (U_0 + \varepsilon_0 \geq U_1 + \varepsilon_1) = 1$$

3.4 Daly-Zachary-Williams

Compute the overall indirect utility of the consumers (social welfare). It is

$$G (U) = E \left[\max_{y \in \mathcal{Y}} \{U_y + \varepsilon_y, \varepsilon_0\} \right]$$

This is called the Emax operator. It is a convex function. Let's see how a change in U_y affects the social welfare. This is given by

$$\begin{aligned}\frac{\partial G}{\partial U_y}(U) &= E\left[\frac{\partial}{\partial U_y}\max_{z\in\mathcal{Y}}\{U_z+\varepsilon_z,\varepsilon_0\}\right] \\ &= E\left[1\left\{y\in\arg\max_{z\in\mathcal{Y}}\{U_z+\varepsilon_z,\varepsilon_0\}\right\}\right] \\ &= \sigma_y(U)\end{aligned}$$

3.5 The Logit model

If (ε_y) are iid Gumbel ie if their joint cdf

$$\begin{aligned}F_\varepsilon(a) &= \Pr(\varepsilon_y\le a_y\forall y) = \prod_{y\in\mathcal{Y}}\exp(-\exp(-a_y+\gamma)) \\ &= \exp\left(-e^\gamma\sum_{y\in\mathcal{Y}}e^{-a_y}\right)\end{aligned}$$

then

Proposition: One has

$$Z=\max_y\{U_y+\varepsilon_y\} =_D \log\sum_y\exp(U_y)+\varepsilon$$

therefore

$$\begin{aligned}\max_y\{U_y+\sigma\varepsilon_y\} &= \sigma\max_y\left\{\frac{U_y}{\sigma}+\varepsilon_y\right\} \\ &= {}_D\sigma\log\sum_y\exp\left(\frac{U_y}{\sigma}\right)+\sigma\varepsilon\end{aligned}$$

Proof of the proposition. Let's compute the c.d.f. of Z. We have

$$\begin{aligned}F_Z(z) &= \Pr(Z\le z)=\Pr\left(\max_y\{U_y+\varepsilon_y\}\le z\right)=\Pr(U_y+\varepsilon_y\le z,\forall y) \\ &= \Pr(\varepsilon_y\le z-U_y:\forall y)=\exp\left(-e^\gamma\sum_{y\in\mathcal{Y}}e^{U_y-z}\right)\end{aligned}$$

Now let's compute the cdf of $Z'=\log\sum_y\exp(U_y)+\varepsilon$, where ε is a Gumbel. We have

$$\begin{aligned}F_{Z'}(z) &= \Pr\left(\log\sum_y\exp(U_y)+\varepsilon\le z\right)=\Pr\left(\varepsilon\le z-\log\sum_y\exp(U_y)\right) \\ &= \exp\left(-\exp\left(\log\sum_y\exp(U_y)-z+\gamma\right)\right)=\exp\left(-e^{-\gamma}\sum_{y\in\mathcal{Y}}e^{U_y-z}\right)\end{aligned}$$

Thus these two cdfs are the same!

As a result, we have that in the logit model,

$$G(U) = \log \sum_{y \in \mathcal{Y}_0} \exp(U_y)$$

and if we assume $U_0 = 0$, we will get

$$G(U) = \log \left(1 + \sum_{y \in \mathcal{Y}} \exp(U_y) \right).$$

Let's deduce the market shares (choice probabilities) in the logit model. We have

$$\begin{aligned} \sigma_y(U) &= \frac{\partial}{\partial U_y} \log \left(\sum_{y \in \mathcal{Y}_0} \exp(U_y) \right) \\ &= \frac{\exp(U_y)}{\sum_{y \in \mathcal{Y}_0} \exp(U_y)} \end{aligned}$$

and if $U_0 = 0$,

$$\sigma_y(U) = \frac{\exp(U_y)}{1 + \sum_{y \in \mathcal{Y}} \exp(U_y)} \text{ and } \sigma_0(U) = \frac{1}{1 + \sum_{y \in \mathcal{Y}} \exp(U_y)}$$

which is Gibbs distribution.

3.6 Extending the logit model

Let's assume that η_i for $i = 1, ..., n$ are i.i.d. Gumbel random variables. We would like to combine the η_i in order to create a model for some ε_y. We will take

$$\varepsilon_y = \max_i \{a_{iy} + \eta_i\} - \log \sum_i \exp(a_{iy}) \tag{1}$$

We have that the each of the ε_y is a Gumbel random variable, BUT they are not independent. Let's compute their c.d.f.

We have

$$
\begin{aligned}
F_\varepsilon(c) &= \Pr\left(\varepsilon_y \le c_y \forall y\right) \\
&= \Pr\left(\max_i \left\{a_{iy} + \eta_i\right\} - \log \sum_i \exp\left(a_{iy}\right) \le c_y, \forall y\right) \\
&= \Pr\left(a_{iy} + \eta_i \le c_y + \log \sum_i \exp\left(a_{iy}\right), \forall y, \forall i\right) \\
&= \Pr\left(\eta_i \le c_y - a_{iy} + \log \sum_i \exp\left(a_{iy}\right), \forall y, \forall i\right) \\
&= \Pr\left(\eta_i \le c_y + \min_y \left\{-a_{iy} + \log \sum_i \exp\left(a_{iy}\right)\right\}, \forall i\right) \\
&= \exp\left(-e^{\gamma} \sum_i e^{-c_y} e^{-\min_y\left\{-a_{iy} + \log \sum_i \exp(a_{iy})\right\}}\right)
\end{aligned}
$$

that is

$$
\begin{aligned}
F_\varepsilon(c) &= \exp\left(-e^{\gamma} \sum_i e^{-c_y} e^{-\min_y\left\{-a_{iy} + \log \sum_i \exp(a_{iy})\right\}}\right) \\
&= \exp\left(-e^{\gamma} \sum_i e^{-c_y} e^{\max_y\left\{a_{iy} - \log \sum_i \exp(a_{iy})\right\}}\right) \\
&= \exp\left(-\sum_i e^{\gamma} \max_y \left\{e^{-c_y} \frac{e^{a_{iy}}}{\sum_i \exp\left(a_{iy}\right)}\right\}\right)
\end{aligned}
$$

therefore, we get that

$$
F_\varepsilon(c) = \exp\left(-g\left(e^{-a}\right)\right)
$$

where

$$
g(b) = \sum_i e^{\gamma} \max_y \left\{b_y \frac{e^{a_{iy}}}{\sum_i \exp\left(a_{iy}\right)}\right\}
$$

We have that $g(b)$ is positive homogenous of degree one and is such that $\exp\left(-g\left(e^{-a}\right)\right)$ is a c.d.f.

This is what is called the Generalized Extreme Value model of McFadden, also called the Multivariate Extreme Value model. The representation (1) is called Pickand's representation.

Definition. The distribution of ε belongs to the GEV distribution if there is a function $g(b)$ such that $g(b)$ is positive homogenous of degree one and is such that $\exp\left(-g\left(e^{-a}\right)\right)$ is a c.d.f.

Theorem (McFadden 1978). If ε has a GEV distribution associated with homogeneous function g, one has

$$Z = \max_{y} \{U_y + \varepsilon_y\} =_D \log g\left(e^U\right) + \varepsilon$$

Proof of the theorem. Let's compute the c.d.f. of Z. We have

$$\begin{aligned}
F_Z(z) &= \Pr(Z \leq z) = \Pr\left(\max_{y} \{U_y + \varepsilon_y\} \leq z\right) = \Pr(U_y + \varepsilon_y \leq z, \forall y) \\
&= \Pr(\varepsilon_y \leq z - U_y : \forall y) = \exp\left(-g\left(e^{U_y - z}\right)\right) \\
&= \exp\left(-g\left(e^{-z} e^{U_y}\right)\right) \\
&= \exp\left(-e^{-z} g\left(e^U\right)\right)
\end{aligned}$$

Now let's compute the cdf of $Z' = \log g\left(e^U\right) + \varepsilon$, where ε is a Gumbel. We have

$$\begin{aligned}
F_{Z'}(z) &= \Pr\left(\log g\left(e^U\right) + \varepsilon \leq z\right) = \Pr\left(\varepsilon \leq z - \log g\left(e^U\right)\right) \\
&= \exp\left(-e^{-z} g\left(e^U\right)\right)
\end{aligned}$$

Thus these two cdfs are the same!

Consequence: We have a closed-form expression for G and σ_y which is

$$\begin{aligned}
G(U) &= \log g\left(e^U\right) \\
\sigma_y(U) &= \frac{\partial_y g\left(e^U\right)}{g\left(e^U\right)} e^{U_y}
\end{aligned}$$

3.7 Tomorrow

Random coefficient logit model of Berry Levinsohn Pakes.

$U_y + \varepsilon_y$

$\varepsilon_y = \xi_y^\top \eta + \epsilon_y$

The inversion of this model ie the problem of going from

(s_y) to U_y

is an entropic regularized OT problem. More specifically,

$$\begin{aligned}
\max_{\pi \geq 0} \quad & \sum_{iy} \pi_{iy} \underbrace{\left(\xi_y^\top \eta_i\right)}_{\Phi_{iy}} - \sum_{iy} \pi_{iy} \log \pi_{iy} \\
s.t. \quad & \sum_{i} \pi_{iy} = s_y \;\; [U_y] \\
& \sum_{y} \pi_{iy} = \frac{1}{n}
\end{aligned}$$

4 Day 4

An announcement:

Looking for RAing opportunity at the frontier of economics, ML and computation?

Equiprice is hiring!

$\Phi(x,y) = x^\top A y$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}$

$y' = Ay$

4.1 Inverting demand maps

Recall from B06 that (DZW theorem)

$$\sigma(U) = \nabla G(U)$$

where $G(U) = E[\max\{U_y + \varepsilon_y\}]$.

Now we are looking for U such that

$$\sigma(U) = s,$$

that is

$$\nabla G(U) = s$$

This can be solved by a convex optimization problem.

If $s = 0$ this is $\nabla G(U) = 0$ hence that is the foc associated with $\min_U G(U)$.

In general, this is

$$G^*(s) = \max_U \left\{ s^\top U - G(U) \right\}$$

hence

$$\sigma^{-1}(s) = \arg\max_U \left\{ s^\top U - G(U) \right\}$$

and we have that

$$\nabla G^*(s) = \sigma^{-1}(s).$$

Example: logit model (with normalization $U_0 = 0$). $G(U) = \log\left(1 + \sum_{y\in\mathcal{Y}} e^{U_y}\right)$.

A straightfoward calculation shows that

$$G^*(s) = \sum_{y\in\mathcal{Y}} s_y \log s_y + s_0 \ln s_0$$

where $s_0 = 1 - \sum_{y\in\mathcal{Y}} s_y$. Cupid's paper Galichon-Salanie.

In this case,

$$\begin{aligned} U_y &= \sigma_y^{-1}(s) = \frac{\partial G^*}{\partial s_y}(s) = (1+\log s_y) - (1+\log s_0) \\ &= \log \frac{s_y}{s_0} \end{aligned}$$

4.2 More on G^*

In the general case. We have

$$\begin{aligned} G^*(s) &= \max_U \left\{ s^\top U - G(U) \right\} \\ &= \max_{U:U_0=0} \left\{ \sum_{y\in\mathcal{Y}} s_y U_y - E\left[\max_{y\in\mathcal{Y}_0} \{U_y + \varepsilon_y\} \right] \right\} \end{aligned}$$

$$-G^*(s) = \min_{U:U_0=0} \left\{ -\sum_{y\in\mathcal{Y}} s_y U_y + E\left[\max_{y\in\mathcal{Y}_0} \{U_y + \varepsilon_y\} \right] \right\}$$

Define $v_y = -U_y$, we have

$$-G^*(s) = \min_{(v_y):v_0=0} \left\{ \sum_{y\in\mathcal{Y}} s_y v_y + E\left[\max_{y\in\mathcal{Y}_0} \{-v_y + \varepsilon_y\} \right] \right\}$$

This can be recast as $-G^*(s) =$

$$\begin{array}{cc} \min\limits_{\substack{(v_y):v_0=0 \\ u(\varepsilon)}} & \left\{ \sum\limits_{y\in\mathcal{Y}} s_y v_y + E\left[u(\varepsilon)\right] \right\} \\ & u(\varepsilon) \geq -v_y + \varepsilon_y \end{array}$$

equivalently

$$\begin{array}{cc} \min\limits_{\substack{(v_y):v_0=0 \\ u(\varepsilon)}} & \left\{ \sum\limits_{y\in\mathcal{Y}} s_y v_y + E_P\left[u(\varepsilon)\right] \right\} \\ & u(\varepsilon) + v_y \geq \varepsilon_y \end{array}$$

In the discrete case, if we sample P, into $\varepsilon^1, ..., \varepsilon^N$, we have, setting $u\left(\varepsilon^i\right) = u_i$

$$\begin{array}{cc} \min\limits_{\substack{(v_y):v_0=0 \\ u_i}} & \left\{ \frac{1}{N} \sum\limits_{i=1}^N u_i + \sum\limits_{y\in\mathcal{Y}} s_y v_y \right\} \\ & u_i + v_y \geq \varepsilon_y^i \end{array}$$

Theorem (Galichon-Salanie 2011). The $U_y = -v_y$ are identified by an optimal transport problem of matching

$\varepsilon \sim P$ ("distribution of firms") and $Y \sim s$ ("distribution of workers") in order to optimize total surplus $\Phi_{\varepsilon y} = \varepsilon_y$.

Primal version

$$\begin{array}{rcl} & & \max E_\pi \left[\varepsilon_Y\right] \\ (\varepsilon, Y) & \sim & \pi : \varepsilon \sim P, Y \sim s \end{array}$$

and dual version

$$\begin{array}{cc} \min\limits_{\substack{(v_y):v_0=0 \\ u_i}} & \left\{ E_P\left[u\left(\varepsilon\right)\right] + \sum\limits_{y\in\mathcal{Y}} s_y v_y \right\}. \\ & u\left(\varepsilon\right) + v_y \geq \varepsilon_y \ \forall \varepsilon, y \end{array}$$

Sampled version:

Primal version

$$\begin{array}{ll} \max\limits_{\pi_{iy}} & \sum\limits_{iy} \pi_{iy} \varepsilon_y^i \\ s.t. & \sum\limits_{y\in\mathcal{Y}_0} \pi_{iy} = \dfrac{1}{N} \\ & \sum\limits_{i} \pi_{iy} = s_y \end{array}$$

and the dual version

$$\begin{array}{cc} \min\limits_{\substack{(v_y):v_0=0 \\ u_i}} & \left\{ \dfrac{1}{N} \sum\limits_{i=1}^N u_i + \sum\limits_{y\in\mathcal{Y}} s_y v_y \right\}. \\ & u_i + v_y \geq \varepsilon_y^i \end{array}$$

Probit example. $\varepsilon_{iy} = \left(Z\epsilon_i\right)_y$ where $\epsilon_i \sim N\left(0, I_K\right)$, Z is a $|\mathcal{Y}_0| \times K$ matrix

$cov\left(\varepsilon_{iy}, \varepsilon_{iy'}\right) = \left(Z^\top Z\right)_{y,y'}$ in particular if we want to impose the covariance matrix Σ between alternatives, one should take $Z = \Sigma^{1/2}$.

Then we can identify the U_y by

$$\begin{array}{cc} \arg \min\limits_{\substack{(U_y):v_0=0 \\ u_i}} & \left\{ \dfrac{1}{N} \sum\limits_{i=1}^N u_i - \sum\limits_{y\in\mathcal{Y}} s_y U_y \right\}. \\ & u_i - U_y \geq \sum\limits_{k} Z_{yk} \epsilon_{ki} \end{array}$$

4.3 Random coefficient logit model

RCL model = pure characteristics + logit

in the sense that

Random coefficient logit model of Berry Levinsohn Pakes.

$U_y + \varepsilon_y$

$\varepsilon_y = (Z\epsilon_i)_y + T\eta_y$ with $T > 0$

where $\epsilon_i \sim \nu$ a fixed distribution over R^K

and Z is a $|Y_0| \times K$ matrix.

and $(\eta_{iy})_y$ iid Gumbel

and ϵ_i and $(\eta_{iy})_y$ are independent.

We recall that

$$\sigma_y(U) = \frac{\partial G(U)}{\partial U_y}$$

where

$$G(U) = E\left[\max_{y\in\mathcal{Y}_0}\left\{U_y + (Z\epsilon_i)_y + T\eta_y\right\}\right]$$

by the law of iterated expectations, we have

$$G(U) = E\left[E\left[\max_{y\in\mathcal{Y}_0}\left\{U_y + (Z\epsilon_i)_y + T\eta_y\right\}|\epsilon\right]\right]$$

by independence,

$$E\left[\max_{y\in\mathcal{Y}_0}\left\{U_y + (Z\epsilon_i)_y + T\eta_y\right\}|\epsilon\right] = T\log\left(1 + \sum_{y\in\mathcal{Y}}\exp\left(\frac{U_y + (Z\epsilon_i)_y}{T}\right)\right)$$

hnce

$$G(U) = E_\epsilon\left[T\log\left(1 + \sum_{y\in\mathcal{Y}}\exp\left(\frac{U_y + (Z\epsilon_i)_y}{T}\right)\right)\right]$$

hence

$$\sigma_y(U) = E_\epsilon\left[\frac{\exp\left(\frac{U_y + (Z\epsilon_i)_y}{T}\right)}{1 + \sum_{y\in\mathcal{Y}}\exp\left(\frac{U_y + (Z\epsilon_i)_y}{T}\right)}\right]$$

Now lets compute the inversion of the model. We have

$$\sigma^{-1}(s) = \arg\min_U\left\{G(U) - s^\top U\right\}$$

thus we are looking for U that solves

$$\min_U\left\{E_\epsilon\left[T\log\left(1 + \sum_{y\in\mathcal{Y}}\exp\left(\frac{U_y + (Z\epsilon_i)_y}{T}\right)\right)\right] - s^\top U\right\}$$

that is $v_y = -U_y$ solves

$$\min_{v:v_0=0} \left\{ E_\epsilon \left[T \log \left(1 + \sum_{y \in \mathcal{Y}} \exp \left(\frac{(Z\epsilon_i)_y - v_y}{T} \right) \right) \right] + \sum_{y \in \mathcal{Y}\int} s_y v_y \right\} \tag{2}$$

this problem is equivalent to

$$\begin{array}{lcl} \max & & E_\pi \left[(Z\epsilon)_Y \right] - T E_\pi \left[\log \pi \left(\epsilon, Y \right) \right] \\ (\epsilon, Y) & \sim & \pi \in M \left(P_\epsilon, s \right) \end{array}$$

Indeed the FOC in the (2) problem yield

$$s_y = \int f(\epsilon) \frac{\exp \left(\frac{-v_y + (Z\epsilon_i)_y}{T} \right)}{1 + \sum_{y \in \mathcal{Y}} \exp \left(\frac{-v_y + (Z\epsilon_i)_y}{T} \right)} d\epsilon$$

introduce

$$a(\epsilon) = -T \log \left(\frac{1 + \sum_{y \in \mathcal{Y}} \exp \left(\frac{-v_y + (Z\epsilon_i)_y}{T} \right)}{f(\epsilon)} \right)$$

so that the problem becomes

$$s_y = \int \exp \left(\frac{(Z\epsilon_i)_y - v_y - a(\epsilon)}{T} \right) d\epsilon$$

and we note that

$$\sum_y \exp \left(\frac{(Z\epsilon_i)_y - v_y - a(\epsilon)}{T} \right) = \sum_y \frac{f(\epsilon) \exp \left(\frac{(Z\epsilon_i)_y - v_y}{T} \right)}{1 + \sum_{y \in \mathcal{Y}} \exp \left(\frac{-v_y + (Z\epsilon_i)_y}{T} \right)} = f(\epsilon)$$

4.4 Parametric choice with no individual-specific observable heterogenity

Assume

$$U_y = \sum_k \Phi_{yk} \beta_k$$

that is $U = \Phi\beta$.

Let us see how this works with maximum likelihood. The log-likelihood of the sample is

$$l(\beta) = N \sum_y \hat{s}_y \log \sigma_y (\Phi\beta)$$

and therefore the max-likelihood estimator is given by

$$\max_\beta l(\beta).$$

Let's see this in the logit model. In the logit model

$$\sigma_y(\Phi\beta) = \frac{\exp\left((\Phi\beta)_y\right)}{\sum_y \exp\left((\Phi\beta)_y\right)}$$

therefore we get

$$\max_\beta \sum_y \hat{s}_y (\Phi\beta)_y - \log \sum_y \exp\left((\Phi\beta)_y\right)$$

In the logit model, the max-likelihood problem is therefore a convex problem. In fact, this is

$$\max_\beta \left\{\hat{s}^\top \Phi\beta - G(\Phi\beta)\right\}.$$

However, in the general case, there is no guarantee that

$$\frac{l(\beta)}{N} = \sum_y \hat{s}_y \log \sigma_y(\Phi\beta)$$

should be concave. In fact, we are no longer going to have
$\sum_y \hat{s}_y \log \sigma_y(\Phi\beta) = \hat{s}^\top \Phi\beta - G(\Phi\beta)$

Instead of doing max-likelihood in general, let's estimate β using the moment-matching estimator

$$\max_\beta \left\{\hat{s}^\top \Phi\beta - G(\Phi\beta)\right\}$$

FOC of this problem wrt β_k:

$$\left(\hat{s}^\top \Phi\right)_k = \left(\sigma^\top(\Phi\beta)\Phi\right)_k$$

These are moment conditions; indeed

$$\sum_y \hat{s}_y \Phi_{yk} = \sum_y \sigma_y(\Phi\beta)\Phi_{yk}$$

Let's see how to compute this in practice in a probit model. In that case, recall that

$\varepsilon_{iy} = (Z\epsilon_i)_y$ where $\epsilon_i \sim N(0, I_K)$, Z is a $|\mathcal{Y}_0| \times K$ matrix, $i = 1, ..., N$.

In that case

$$G_N(U) = \frac{1}{N}\sum_{i=1}^N \max_{y\in Y_0}\left\{U_y + (Z\epsilon_i)_y\right\}$$

then the simulator for the moment-matching estimator

$$\max_{\beta\in R^K}\left\{\hat{s}^\top \Phi\beta - G_N(\Phi\beta)\right\}$$

ie

$$\max_{\beta\in R^K}\left\{\hat{s}^\top\Phi\beta-\frac{1}{N}\sum_{i=1}^N\max_{y\in Y_0}\left\{(\Phi\beta)_y+(Z\epsilon_i)_y\right\}\right\}$$

which reformulates as

$$\begin{aligned}\max_{(u_i)\in R^N,\beta\in R^K}\quad & \left\{\hat{s}^\top\Phi\beta-\frac{1}{N}\sum_{i=1}^N u_i\right\}\\ s.t.\quad & u_i-(\Phi\beta)_y\geq(Z\epsilon_i)_y\ \ \forall i\in\{1,...,N\},\forall y\in Y_0\end{aligned}$$

4.5 Parametric choice with individual-specific observable heterogenity

Assume

$$u_{iy}=\sum_k\beta_k\Phi_{iyk}+\varepsilon_{iy}$$

The analysis is left unchanged for the most part. In the logit case, the log-likelihood associated with observation i is

$$l_i(\beta)=\sum_{y\in\mathcal{Y}}\hat{\mu}_{iy}(\Phi\beta)_{iy}-\log\sum_{y\in\mathcal{Y}}\exp(\Phi\beta)_{iy}$$

and the max-likelihood rewrites as

$$\max_\beta\left\{\sum_{i\in\mathcal{I},y\in\mathcal{Y}}\hat{\mu}_{iy}(\Phi\beta)_{iy}-\sum_{i\in\mathcal{I}}\log\sum_{y\in\mathcal{Y}}\exp(\Phi\beta)_{iy}\right\}$$

With other random utility structures, this yields a moment matching procedure to estimate β, namely

$$\max_\beta\left\{\hat{\mu}^\top\Phi\beta-\sum_{i\in\mathcal{I}}G\left((\Phi\beta)_{i.}\right)\right\},$$

where G is the Emax operator associated with the distribution of the random utility.

5 Day 5

TU problem w heterogeneities (sample version)

$$\begin{aligned}&\min_{u_i,v_j}\sum_i u_i+\sum_j v_j\\ s.t.\quad & u_i+v_j\geq\tilde{\Phi}_{ij}=\Phi_{x_iy_j}+\varepsilon_{iy_j}+\eta_{x_ij}\\ & u_i\geq\varepsilon_{i0}\\ & v_j\geq\eta_{0j}\end{aligned}$$

Consider the constraint

$$u_i + v_j \geq \Phi_{x_i y_j} + \varepsilon_{i y_j} + \eta_{x_i j} \ \forall i, j$$

which becomes

$$u_i - \varepsilon_{iy} + v_j - \eta_{xj} \geq \Phi_{xy} \ \forall i : x_i = x, \forall j : y_j = y, \forall x, y$$

this holds if and only if

$$\min_{\substack{i:x_i=x \\ j:y_j=y}} \left\{ u_i - \varepsilon_{iy} + v_j - \eta_{xj} \right\} \geq \Phi_{xy} \ \forall x, y$$

because of separabilty, this becomes

$$\min_{i:x_i=x} \{u_i - \varepsilon_{iy}\} + \min_{j:y_j=y} \left\{ v_j - \eta_{xj} \right\} \geq \Phi_{xy} \ \forall x, y$$

hence call

$$\begin{aligned} U_{xy} &= \min_{i:x_i=x} \{u_i - \varepsilon_{iy}\} \\ V_{xy} &= \min_{j:y_j=y} \left\{ v_j - \eta_{xj} \right\} \end{aligned}$$

and the constraint rewrites

$$\begin{aligned} U_{xy} + V_{xy} &\geq \Phi_{xy} \\ u_i &= \max_y \{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\} \\ v_j &= \max_x \left\{ V_{xy} + \eta_{xj}, \eta_{0j} \right\} \end{aligned}$$

As a result, the dual of the individual matching problem can be rewritten as

$$\begin{aligned} & \min_{u_i, v_j, U_{xy}, V_{xy}} \sum_i u_i + \sum_j v_j \\ s.t. \quad & U_{xy} + V_{xy} \geq \Phi_{xy} \\ & u_i \geq U_{xy} + \varepsilon_{iy} \\ & v_j \geq V_{xy} + \eta_{xj} \\ & u_i \geq \varepsilon_{i0} \\ & v_j \geq \eta_{0j} \end{aligned}$$

we can further rewrite as

$$\begin{aligned} & \min_{U_{xy}, V_{xy}} \sum_i \max_y \{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\} + \sum_j \max_x \left\{ V_{xy} + \eta_{xj}, \eta_{0j} \right\} \\ s.t. \quad & U_{xy} + V_{xy} \geq \Phi_{xy} \end{aligned}$$

and the constraint can be taken binding wlog so

$$\begin{aligned}
&\min_{U_{xy},V_{xy}} \sum_i \max_y \{U_{xy}+\varepsilon_{iy},\varepsilon_{i0}\} + \sum_j \max_x \left\{V_{xy}+\eta_{xj},\eta_{0j}\right\} \\
s.t.\quad & U_{xy}+V_{xy}=\Phi_{xy}
\end{aligned}$$

which can be even further simplified into

$$\min_{U_{xy}} \sum_i \max_y \{U_{xy}+\varepsilon_{iy},\varepsilon_{i0}\} + \sum_j \max_x \left\{\Phi_{xy}-U_{xy}+\eta_{xj},\eta_{0j}\right\}$$

Large market assumption: assume that the number of individuals per type tends to $+\infty$. Recall

$$\begin{aligned}
&\min_{U_{xy},V_{xy}} \sum_x \frac{N_x}{S}\frac{1}{N_x}\sum_{i:x_i=x} \max_y \{U_{xy}+\varepsilon_{iy},\varepsilon_{i0}\} \\
&+\sum_y \frac{M_y}{S}\frac{1}{M_y}\sum_j \max_x \left\{V_{xy}+\eta_{xj},\eta_{0j}\right\} \\
s.t.\quad & U_{xy}+V_{xy}=\Phi_{xy}
\end{aligned}$$

where $N_x = |\{i : x_i = x\}|$ and $M_y = |\{j : y_j = y\}|$ and $S = \sum N_x + \sum M_y$. When $S \to +\infty$, one has

$$\begin{aligned}
\frac{1}{N_x}\sum_{i:x_i=x} \max_y \{U_{xy}+\varepsilon_{iy},\varepsilon_{i0}\} &\quad\to\quad E\left[\max_y \{U_{xy}+\varepsilon_y,\varepsilon_0\}\right] = G_x(U) \\
\frac{1}{M_y}\sum_j \max_x \left\{V_{xy}+\eta_{xj},\eta_{0j}\right\} &\quad\to\quad E\left[\max_x \{V_{xy}+\eta_x,\eta_0\}\right] = H_y(V)
\end{aligned}$$

and calling $n_x = \lim_{S\to\infty}\frac{N_x}{S}$ and $m_y = \lim_{S\to\infty}\frac{M_y}{S}$, we get that the limit of the previous problem is

$$\begin{aligned}
&\min_{U_{xy},V_{xy}} \sum_x n_x G_x(U) + \sum_y m_y H_y(V) \\
s.t.\quad & U_{xy}+V_{xy}=\Phi_{xy}
\end{aligned}$$

Let us denote $G(U) = \sum_x n_x G_x(U)$ and $H(V) = \sum_y m_y H_y(V)$ the total indirect utilities of the men and the women, respectively. The problem rewrites as

$$\begin{aligned}
&\min_{U_{xy},V_{xy}} G(U) + H(V) \\
s.t.\quad & U_{xy}+V_{xy}=\Phi_{xy} \quad [\mu_{xy}]
\end{aligned}$$

1. Derive the FOC and interpret.

This is $\min_U \{G(U) + H(\Phi - U)\}$, so by FOC

$$\frac{\partial G}{\partial U_{xy}}(U) - \frac{\partial H}{\partial V_{xy}}(\Phi - U) = 0$$

Next

$$\begin{aligned}
\frac{\partial G}{\partial U_{xy}}(U) &= \sum_{x'} n_{x'} \frac{\partial G_{x'}}{\partial U_{xy}}(U) = n_x \frac{\partial G_x}{\partial U_{xy}}(U) = n_x \sigma_{y|x}(U) \\
\frac{\partial H}{\partial V_{xy}}(V) &= m_y \sigma_{x|y}(V)
\end{aligned}$$

and therefore, the FOC become

$$n_x \sigma_{y|x}(U) = m_y \sigma_{x|y}(\Phi - U)$$

which is an equilibrium condition, and both terms are equal to

$$\mu_{xy}$$

which is the number of xy pairs formed at equilibrium.

2. Derive the primal problem associated with the above min. Write this as a minimax problem

$$\min_{U_{xy}, V_{xy}} \max_{\mu} G(U) + H(V) + \sum_{xy} \mu_{xy} (\Phi_{xy} - U_{xy} - V_{xy})$$

and

$$\max_{\mu} \sum_{xy} \mu_{xy} \Phi_{xy} + \min_{U_{xy}, V_{xy}} G(U) + H(V) - \sum_{xy} \mu_{xy} U_{xy} - \sum_{xy} \mu_{xy} V_{xy}$$

this is

$$\max_{\mu} \sum_{xy} \mu_{xy} \Phi_{xy} + \min_{U} \left\{ G(U) - \sum_{xy} \mu_{xy} U_{xy} \right\} + \min_{V} \left\{ H(V) - \sum_{xy} \mu_{xy} V_{xy} \right\}$$

and therefore

$$\max_{\mu} \sum_{xy} \mu_{xy} \Phi_{xy} - \max_{U} \left\{ \sum_{xy} \mu_{xy} U_{xy} - G(U) \right\} - \max_{V} \left\{ \sum_{xy} \mu_{xy} V_{xy} - H(V) \right\}$$

Now let's focus on

$$\begin{aligned}
\max_{U} \left\{ \sum_{xy} \mu_{xy} U_{xy} - G(U) \right\} &= \max_{(U_{xy})} \left\{ \sum_{xy} \mu_{xy} U_{xy} - \sum_{x} n_x G_x(U) \right\} \\
&= \sum_{x} \max_{(U_{xy})_y} \left\{ \mu_{xy} U_{xy} - n_x G_x(U) \right\} \\
&= \sum_{x} n_x \max_{(U_{xy})_y} \left\{ \frac{\mu_{xy}}{n_x} U_{xy} - G_x(U) \right\}
\end{aligned}$$

introducing $\mu_{y|x} = \mu_{xy}/n_x$ the conditional share of choosing y given x, this is

$$\begin{aligned} & \sum_x n_x \max_{(U_{xy})_y} \left\{ \mu_{y|x} U_{xy} - G_x(U) \right\} \\ = & \sum_x n_x G_x^* \left(\mu_{.|x} \right) \\ = & : G^*(\mu) \end{aligned}$$

Therefore the minimax problem becomes

$$\max_\mu \left\{ \sum_{xy} \mu_{xy} \Phi_{xy} - \left(G^*(\mu) + H^*(\mu) \right) \right\}$$

where we have defined H^* analogously, ie

$$H^*(\mu) = \sum_y m_y H_y^* \left(\mu_{.|y} \right).$$

Just to recap:
Primal problem

$$\max_\mu \left\{ \sum_{xy} \mu_{xy} \Phi_{xy} - \left(G^*(\mu) + H^*(\mu) \right) \right\}$$

FOC in the primal problem are ** identification equation **

$$\Phi_{xy} = \frac{\partial G^*}{\partial \mu_{xy}}(\mu) + \frac{\partial H^*}{\partial \mu_{xy}}(\mu).$$

Dual problem

$$\min_{U+V=\Phi} G(U) + H(V)$$

FOC in the dual are ** equilibrium conditions **

$$n_x \mu_{y|x} = m_y \mu_{x|y}$$

In Choo-Siow model, we have

$$\begin{aligned} G^*(\mu) &= \sum_{x \in X} n_x \sum_{y \in Y_0} \frac{\mu_{xy}}{n_x} \log\left(\frac{\mu_{xy}}{n_x} \right) \\ &= \sum_{x \in X} \sum_{y \in Y_0} \mu_{xy} \log \mu_{xy} - \sum n_x \log n_x \end{aligned}$$

and thus

$$U_{xy} = \frac{\partial G^*}{\partial \mu_{xy}}(\mu) = 1 + \log \mu_{xy} - (1 + \log \mu_{x0}) = \log \frac{\mu_{xy}}{\mu_{x0}}$$

and similarly,

$$V_{xy} = \frac{\partial H^*}{\partial \mu_{xy}}(\mu) = \log \frac{\mu_{xy}}{\mu_{0y}}$$

hence the identification formula

$$\Phi_{xy} = \frac{\partial G^*}{\partial \mu_{xy}}(\mu) + \frac{\partial H^*}{\partial \mu_{xy}}(\mu).$$

becomes

$$\Phi_{xy} = \log \frac{\mu_{xy}}{\mu_{x0}} + \log \frac{\mu_{xy}}{\mu_{0y}}$$

and therefore we get Choo-Siow's formula

$$\Phi_{xy} = \log \frac{\mu_{xy}^2}{\mu_{x0}\mu_{0y}}$$

equivalently

$$\mu_{xy} = \sqrt{\mu_{x0}\mu_{0y}} \exp\left(\frac{\Phi_{xy}}{2}\right)$$

5.1 Parametric estimation

Now assume a parametric form for Φ_{xy}

$$\Phi_{xy}^{\lambda} = \sum_k \phi_{xy}^k \lambda_k$$

Social surplus

$$W(\lambda) = \min_U \left\{G(U) + H\left(\Phi^{\lambda} - U\right)\right\} = \max_{\mu} \left\{\sum_{xy} \mu_{xy}\Phi_{xy}^{\lambda} - \{G^*(\mu) + H^*(\mu)\}\right\}$$

Let's compute

$$\frac{\partial W(\lambda)}{\partial \lambda_k} = \sum_{xy} \mu_{xy}^{\lambda} \phi_{xy}^k$$

Moment matching method: set λ such that

$$\sum_{xy} \mu_{xy}^{\lambda} \phi_{xy}^k = \sum_{xy} \hat{\mu}_{xy} \phi_{xy}^k$$

that is

$$\frac{\partial W(\lambda)}{\partial \lambda_k} = \sum_{xy} \hat{\mu}_{xy} \phi_{xy}^k$$

We can view this as the FOC of an optimization problem:

$$\min_{\lambda} \left\{ W(\lambda) - \sum_{kxy} \hat{\mu}_{xy} \phi_{xy}^k \lambda_k \right\}$$

that is

$$\min_{\lambda, U} \left\{ G(U) + H\left(\Phi^{\lambda} - U\right) - \sum_{kxy} \hat{\mu}_{xy} \phi_{xy}^k \lambda_k \right\}$$

Logit case. We can show that this problem [here, with singles] can be rephrased as

$$\min_{(u_x),(v_y),(\lambda_k)} \left\{ \begin{array}{c} \sum_x n_x u_x + \sum_y m_y v_y + 2\sum_{xy} \exp\left(\frac{\Phi_{xy}^{\lambda} - u_x - v_y}{2}\right) \\ + \sum_x \exp(-u_x) + \sum_y \exp(-v_y) - \sum_{kxy} \hat{\mu}_{xy} \phi_{xy}^k \lambda_k \end{array} \right\}$$

Indeed, foc with u_x, v_y, λ_k respectively yield

$$\begin{aligned} n_x &= \sum_y \mu_{xy}^{\lambda,u,v} + \mu_{x0}^{\lambda,u,v} \\ m_y &= \sum_x \mu_{xy}^{\lambda,u,v} + \mu_{0y}^{\lambda,u,v} \\ \sum_{xy} \mu_{xy}^{\lambda,u,v} \phi_{xy}^k &= \sum_{xy} \hat{\mu}_{xy} \phi_{xy}^k \end{aligned}$$

where

$$\mu_{xy}^{\lambda,u,v} = \exp\left(\frac{\Phi_{xy}^{\lambda} - u_x - v_y}{2}\right), \mu_{x0}^{\lambda,u,v} = \exp(-u_x), \text{ and } \mu_{0y}^{\lambda,u,v} = \exp(-v_y)$$

we recover Choo-Siow's optimality conditions

$$\mu_{xy}^{\lambda,u,v} = \sqrt{\mu_{x0}^{\lambda,u,v} \mu_{0y}^{\lambda,u,v}} \exp\left(\frac{\Phi_{xy}^{\lambda}}{2}\right)$$

Now version without singles

$$\min_{(u_x),(v_y),(\lambda_k)} \left\{ \sum_x n_x u_x + \sum_y m_y v_y + 2\sum_{xy} \exp\left(\frac{\Phi_{xy}^{\lambda} - u_x - v_y}{2}\right) - \sum_{kxy} \hat{\mu}_{xy} \phi_{xy}^k \lambda_k \right\}$$

5.2 Continuous version

Dupuy and G is a continuous version of Choo and Siow

Man i's characteristics: $x_i \in R^d$

Woman j's characteristics: $y_j \in R^d$

Assume $\Phi(x, y) = x^\top A y = \sum_{k,l} A_{kl} x^k y^l$.

Equivalently, $\Phi(x, y) = -\frac{1}{2} \sum_{k,l} A_{kl} (x^k - y^l)^2$.

Becker's ability model: assume that there are two single-dimensional "ability indices" for both men and women which are formed using all the characteristics, that is

$$\begin{aligned} \bar{x}_i &= \lambda^\top x_i = \sum_k \lambda_k x_i^k \\ \bar{y}_j &= \nu^\top y_j = \sum_l \nu_l y_i^l \end{aligned}$$

Then the surplus will then be

$$\begin{aligned} \Phi(x_i, y_j) &= \bar{x}_i \bar{y}_j \\ &= x_i^\top \lambda \nu^\top y_j \\ &= x_i^\top \left(\lambda \nu^\top \right) y_j \end{aligned}$$

that is, it implies that the affinity matrix $A = \lambda \nu^\top$. That is A is a rank-one matrix.

So one strategy to compute the indices is to do a single value decomposition of A:

$$A = \sum_{p=1}^{d} \sigma_p \lambda^p \left(\nu^p \right)^\top$$

where $|\lambda^p| = |\nu^p| = 1$ and $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_d \geq 0$ are the singular values of A, and the number of nonzero σ_p is the rank of A.

Now PCA consists of

$$\Sigma_{XY} = E_{\hat{\mu}} \left[XY^\top \right]$$

and take a singular value decomposition of Σ_{XY} – canonical correlation analysis.

However, one should do SVD of A, not of Σ, see https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2167564

5.3 Gravity equation from regularized optimal transport

$$\begin{aligned} \max \quad & \sum \pi_{ij} \Phi_{ij} - \sum_{ij} \pi_{ij} \log \pi_{ij} \\ & \sum \pi_{ij} = p_i \;\; [a_i] \\ & \sum \pi_{ij} = q_j \;\; [b_j] \end{aligned}$$

foc

$$\log \pi_{ij} = \Phi_{ij} - a_i - b_j$$

therefore

$$\begin{aligned}
\pi_{ij} &= \exp\left(\Phi_{ij} - a_i - b_j\right) \\
&= A_i B_j K_{ij}
\end{aligned}$$

where $A_i = \exp(-a_i)$ and $B_j = \exp(-b_j)$ and $K_{ij} = \exp(\Phi_{ij})$.

$$\begin{aligned}
\sum_j A_i B_j K_{ij} &= p_i \\
\sum_i A_i B_j K_{ij} &= q_j
\end{aligned}$$