1 Jan 18

Linear programming duality

The problem is

$$\begin{aligned} V_P = \max_{x \geq 0} & \qquad x^\top c \\ s.t. & \qquad Ax \leq d \end{aligned}$$

We would like to write this as

$$\max_{x \geq 0} x^\top c + F(d - Ax)$$

where $F(u) = 0$ if $u \geq 0$, $F(u) = -\infty$ otherwise. The simplest choice is

$$F(u) = \min_{y \geq 0} \left\{ y^\top u \right\} = \min_{y_j \geq 0} \left\{ \sum_j y_j u_j \right\}.$$

Thus rewrite the problem as

$$\begin{aligned} V_P &= \max_{x \geq 0} x^\top c + \min_{y \geq 0} \left\{ y^\top (d - Ax) \right\} \\ &= \max_{x \geq 0} \min_{y \geq 0} \left\{ x^\top c + y^\top d - y^\top Ax \right\} \end{aligned}$$

By the minimax theorem, if there are feasible solutions, then

$$\begin{aligned} V_P &= \min_{y \geq 0} \max_{x \geq 0} \left\{ x^\top c + y^\top d - y^\top Ax \right\} \\ &= \min_{y \geq 0} y^\top d + \max_{x \geq 0} \left\{ x^\top c - y^\top Ax \right\} \\ &= \min_{y \geq 0} y^\top d + \max_{x \geq 0} \left\{ x^\top c - x^\top A^\top y \right\} \\ &= \min_{y \geq 0} y^\top d + \max_{x \geq 0} \left\{ x^\top \left(c - A^\top y \right) \right\} \end{aligned}$$

Now, we remark that

$$\begin{aligned} \max_{x \geq 0} \left\{ x^\top \left(c - A^\top y \right) \right\} &= +\infty \text{ if } c_i > \left(A^\top y \right)_i \text{ for some } i \\ &= 0 \text{ if } c \leq A^\top y \end{aligned}$$

we have derived the dual problem

$$\begin{aligned} V_P = V_D := \min_{y \geq 0} & \qquad y^\top d \\ s.t.\ c & \leq A^\top y \end{aligned}$$

Further, if x^* an optimal solution to primal and y^* an optimal solution to the dual problem, we have

$$(x^*)^\top \left(c - A^\top y^* \right) = 0$$

but $c \le A^\top y^*$ and $x^* \ge 0$ therefore we have for each i

$$x_i^* \left(c_i - \left(A^\top y^*\right)_i\right) = 0$$

therefore

$$x_i^* > 0 \implies \left(A^\top y^*\right)_i = c_i$$

and similarly

$$y_j^* > 0 \implies (Ax^*)_j = d_j$$

This is **complementary slackness**.

1.1 Gradient matrix

∇_{ax} $a \in A, x \in Z$

$$(\nabla f)_{xy} = \sum_z \nabla_{(xy)z} f_z = f_y - f_x$$

Mass balance equation. For every z,

$$\begin{aligned} q_z &= \text{(total mass arriving to } z \text{ from other nodes)} \\ &\quad - \text{(total mass departing from } z \text{ to other nodes)} \\ &= \sum_x \mu_{xz} - \sum_y \mu_{zy} \end{aligned}$$

This can be expressed in a matrix way! Indeed,

$$q_z = \sum_{a \in A} \nabla_{az} \mu_a$$

hence mass balance rewrites in a matrix notation as

$$q = \nabla^\top \mu.$$

By the way, if q satisfies mass balance, then

$$\begin{aligned} q_z &= \sum_x \mu_{xz} - \sum_y \mu_{zy} \\ \sum_{z \in Z} q_z &= \sum_{z \in Z} \sum_{x \in Z : xz \in A} \mu_{xz} - \sum_{z \in Z} \sum_{y : zy \in A} \mu_{zy} \\ &= \sum_{a \in A} \mu_a - \sum_{a \in A} \mu_a \\ &= 0 \end{aligned}$$

2 An equilibrium problem

Introduce p_z=price of the commodity at z.

Consider a trader operating on the arc xy.

trader's profit $= p_y - p_x - c_{xy}$

Assume that there is free entry of traders on any arc. Absence of rent implies that for any arc $xy \in A$ we have

$$p_y - p_x \leq c_{xy}$$

which can be written

$$\nabla p \leq c$$

No arbitrage condition.

Now assume at equilibrium, a quantity $\mu_{xy} > 0$ is shipped from x to y. This implies that the traders at arc xy break even, hence

$$p_y - p_x - c_{xy} = 0$$

To summarize, an equilbrium on the network is given by $(p_z)_{z \in Z}$ and $\left(\mu_{xy}\right)_{xy \in A}$ such that:

(i) balance of mass holds: $\nabla^\top \mu = q$

(ii) no arbitrage holds: $\nabla p \leq c$.

(iii) $\mu_{xy} > 0 \implies (\nabla p)_{xy} = p_y - p_x = c_{xy}$.

2.1 An optimal shippment problem

Consider the Soviet problem. They decide on μ_{xy} subject to mass balance $\nabla^\top \mu = q$.

In order to minimize costs, they try to achieve the minimum of $\sum_{xy \in A} \mu_{xy} c_{xy}$.

Thus, they do:

$$\begin{array}{rrl} \min\limits_{\mu \geq 0} & & \mu^\top c \\ s.t. & \nabla^\top \mu = & q \ \ [p] \end{array}$$

this is a linear programming problem (primal). Let's compute its dual.

$$\begin{array}{rrl} \max\limits_{p} & & p^\top q \\ s.t. & \nabla p \leq & c \ \ [\mu \geq 0] \end{array}$$

Theorem: if μ and p are respectively solutions to the primal and the dual problems, then they also solve the equilbrium problem above. Indeed, $\nabla^\top \mu = q$ and $\nabla p \leq c$ are immediately satisfies, and by complementary slackness, $\mu_{xy} > 0$ implies

$$p_y - p_x = c_{xy}.$$