1 Day 1

Linear programming duality

The problem is

$$\begin{aligned} V_P = \max_{x\geq 0} & \qquad x^\top c \\ s.t. & \qquad Ax \leq d \end{aligned}$$

We would like to write this as

$$\max_{x\geq 0} x^\top c + F\left(d - Ax\right)$$

where $F\left(u\right) = 0$ if $u \geq 0$, $F\left(u\right) = -\infty$ otherwise. The simplest choice is

$$F\left(u\right) = \min_{y\geq 0}\left\{y^\top u\right\} = \min_{y_j\geq 0}\left\{\sum_j y_j u_j\right\}.$$

Thus rewrite the problem as

$$\begin{aligned} V_P &= \max_{x\geq 0} x^\top c + \min_{y\geq 0}\left\{y^\top\left(d - Ax\right)\right\} \\ &= \max_{x\geq 0}\min_{y\geq 0}\left\{x^\top c + y^\top d - y^\top Ax\right\} \end{aligned}$$

By the minimax theorem, if there are feasible solutions, then

$$\begin{aligned} V_P &= \min_{y\geq 0}\max_{x\geq 0}\left\{x^\top c + y^\top d - y^\top Ax\right\} \\ &= \min_{y\geq 0} y^\top d + \max_{x\geq 0}\left\{x^\top c - y^\top Ax\right\} \\ &= \min_{y\geq 0} y^\top d + \max_{x\geq 0}\left\{x^\top c - x^\top A^\top y\right\} \\ &= \min_{y\geq 0} y^\top d + \max_{x\geq 0}\left\{x^\top\left(c - A^\top y\right)\right\} \end{aligned}$$

Now, we remark that

$$\begin{aligned} \max_{x\geq 0}\left\{x^\top\left(c - A^\top y\right)\right\} &= +\infty \text{ if } c_i > \left(A^\top y\right)_i \text{ for some } i \\ &= 0 \text{ if } c \leq A^\top y \end{aligned}$$

we have derived the dual problem

$$\begin{aligned} V_P = V_D := \min_{y\geq 0} & \qquad y^\top d \\ s.t.\ c & \leq A^\top y \end{aligned}$$

Further, if x^* an optimal solution to primal and y^* an optimal solution to the dual problem, we have

$$\left(x^*\right)^\top\left(c - A^\top y^*\right) = 0$$

but $c \leq A^\top y^*$ and $x^* \geq 0$ therefore we have for each i

$$x_i^* \left(c_i - \left(A^\top y^*\right)_i\right) = 0$$

therefore

$$x_i^* > 0 \implies \left(A^\top y^*\right)_i = c_i$$

and similarly

$$y_j^* > 0 \implies (Ax^*)_j = d_j$$

This is **complementary slackness**.

1.1 Gradient matrix

∇_{ax} $a \in A, x \in Z$

$$(\nabla f)_{xy} = \sum_z \nabla_{(xy)z} f_z = f_y - f_x$$

Mass balance equation. For every z,

$$\begin{aligned} q_z &= \text{(total mass arriving to } z \text{ from other nodes)} \\ &\quad - \text{(total mass departing from } z \text{ to other nodes)} \\ &= \sum_x \mu_{xz} - \sum_y \mu_{zy} \end{aligned}$$

This can be expressed in a matrix way! Indeed,

$$q_z = \sum_{a \in A} \nabla_{az} \mu_a$$

hence mass balance rewrites in a matrix notation as

$$q = \nabla^\top \mu.$$

By the way, if q satisfies mass balance, then

$$\begin{aligned} q_z &= \sum_x \mu_{xz} - \sum_y \mu_{zy} \\ \sum_{z \in Z} q_z &= \sum_{z \in Z} \sum_{x \in Z: xz \in A} \mu_{xz} - \sum_{z \in Z} \sum_{y: zy \in A} \mu_{zy} \\ &= \sum_{a \in A} \mu_a - \sum_{a \in A} \mu_a \\ &= 0 \end{aligned}$$

2 An equilibrium problem

Introduce p_z=price of the commodity at z.

Consider a trader operating on the arc xy.

trader's profit $= p_y - p_x - c_{xy}$

Assume that there is free entry of traders on any arc. Absence of rent implies that for any arc $xy \in A$ we have

$$p_y - p_x \leq c_{xy}$$

which can be written

$$\nabla p \leq c$$

No arbitrage condition.

Now assume at equilibrium, a quantity $\mu_{xy} > 0$ is shipped from x to y. This implies that the traders at arc xy break even, hence

$$p_y - p_x - c_{xy} = 0$$

To summarize, an equilbrium on the network is given by $(p_z)_{z \in Z}$ and $\left(\mu_{xy}\right)_{xy \in A}$ such that:

(i) balance of mass holds: $\nabla^\top \mu = q$

(ii) no arbitrage holds: $\nabla p \leq c$.

(iii) $\mu_{xy} > 0 \implies (\nabla p)_{xy} = p_y - p_x = c_{xy}$.

2.1 An optimal shippment problem

Consider the Soviet problem. They decide on μ_{xy} subject to mass balance $\nabla^\top \mu = q$.

In order to minimize costs, they try to achieve the minimum of $\sum_{xy \in A} \mu_{xy} c_{xy}$.

Thus, they do:

$$\begin{aligned} \min_{\mu \geq 0} & \qquad \mu^\top c \\ s.t. \quad \nabla^\top \mu = & \quad q \;\; [p] \end{aligned}$$

this is a linear programming problem (primal). Let's compute its dual.

$$\begin{aligned} \max_{p} & \qquad p^\top q \\ s.t. \quad \nabla p \leq & \quad c \;\; [\mu \geq 0] \end{aligned}$$

Theorem: if μ and p are respectively solutions to the primal and the dual problems, then they also solve the equilbrium problem above. Indeed, $\nabla^\top \mu = q$ and $\nabla p \leq c$ are immediately satisfies, and by complementary slackness, $\mu_{xy} > 0$ implies

$$p_y - p_x = c_{xy}.$$

3 Day 2

Solve the central planner's problem

$$\begin{aligned} \max_{\pi_{xy}\geq 0} \quad & \sum_{xy} \pi_{xy}\Phi_{xy} \\ s.t. \quad & \sum_{y} \pi_{xy} = p_x \;\; [u_x] \\ & \sum_{x} \pi_{xy} = q_y \;\; [v_y] \end{aligned}$$

by the maxmin formulation

$$\begin{aligned} & \max_{\pi_{xy}\geq 0} \min_{u_x,v_y} \sum_{xy} \pi_{xy}\Phi_{xy} + \sum_x p_x u_x + \sum_y q_y v_y - \sum_{xy} \pi_{xy}\left(u_x + v_y\right) \\ = \;& \min_{u_x,v_y} \sum_x p_x u_x + \sum_y q_y v_y + \max_{\pi_{xy}\geq 0} \sum_{xy} \pi_{xy}\left(\Phi_{xy} - u_x - v_y\right) \end{aligned}$$

this is

$$\begin{aligned} \min_{u_x,v_y} \quad & \sum_x p_x u_x + \sum_y q_y v_y \\ s.t. \quad & u_x + v_y \geq \Phi_{xy} \;\; [\pi_{xy} \geq 0] \end{aligned}$$

Remark: by complementary slackness, $\pi_{xy} > 0 \implies u_x + v_y = \Phi_{xy}$.

Remark 2: if (u, v) then $(u + c, v - c)$ is also a solution for any constant c.

Take a solution (u, v) of the dual problem. Then for any feasible solution we have

$$v_y \geq \max_x \left\{\Phi_{xy} - u_x\right\}$$

but for any optimal solution (u, v) we have

$$v_y = \max_x \left\{\Phi_{xy} - u_x\right\}.$$

But the same logic implies

$$u_x = \max_y \left\{\Phi_{xy} - v_y\right\}.$$

3.1 Interpretation 1: stable matching

(π_{xy}, u_x, v_y) is a stable matching if:

- $\pi \geq 0$ and $\sum_y \pi_{xy} = p_x$ and $\sum_x \pi_{xy} = q_y$
- $\forall x, y, u_x + v_y \geq \Phi_{xy}$
- If $u_x + v_y > \Phi_{xy} \implies \pi_{xy} = 0$; or in other words $\pi_{xy} > 0 \implies u_x + v_y = \Phi_{xy}$

If $u_x + v_y < \Phi_{xy}$ then xy would be a blocking pair, this should be ruled out

3.2 Interpretation 2: Wage equilibrium

Assume x is a worker and y is a firm, and interpret u_x as the wage of worker x, and v_y as the profit of firm y.

Then firm's problem is

$$\begin{aligned} v_y &= \max_x \{\Phi_{xy} - u_x\} \\ \pi_{xy} &> 0 \implies v_y = \Phi_{xy} - u_x \\ \pi &\geq 0, \sum_y \pi_{xy} = p_x, \sum_x \pi_{xy} = q_y \end{aligned}$$

Rewrite this as

$$\begin{aligned} v_y &\geq \Phi_{xy} - u_x \forall x, y \\ \pi_{xy} &> 0 \implies v_y = \Phi_{xy} - u_x \\ \pi &\geq 0, \sum_y \pi_{xy} = p_x, \sum_x \pi_{xy} = q_y \end{aligned}$$

3.3 Case with unassigned agents

Assume that people don't have to match, and if they remain unmatched they get utility 0

π_{x0} =mass of men of type x remaining unassigned

π_{0y} =mass of women of type y remaining unassigned

$\sum_y \pi_{xy} + \pi_{x0} = p_x$
$\sum_x \pi_{xy} + \pi_{0y} = q_y$

Optimal assignement problem

$$\begin{aligned} & \max_{\pi \geq 0} \sum_{xy} \pi_{xy} \Phi_{xy} \\ s.t. \quad & \sum_y \pi_{xy} + \pi_{x0} = p_x \\ & \sum_x \pi_{xy} + \pi_{0y} = q_y \end{aligned}$$

rewrite this as

$$\begin{aligned} & \max_{\pi \geq 0} \sum_{xy} \pi_{xy} \Phi_{xy} \\ s.t. \quad & \sum_y \pi_{xy} \leq p_x \;\; [u_x \geq 0] \\ & \sum_x \pi_{xy} \leq q_y \;\; [v_y \geq 0] \end{aligned}$$

whose dual is

$$\begin{aligned} \min_{u_x \geq 0, v_y \geq 0} \quad & \sum_x p_x u_x + \sum_y q_y v_y \\ s.t. \quad & u_x + v_y \geq \Phi_{xy} \;\; [\pi_{xy} \geq 0] \end{aligned}$$