APMA 1070 Spring 2019
FINAL PROJECT
PART 1

This take-home project is PART 1 of your Final Exam. As opposed to the homework assignments, where collaboration among students was allowed, you are to work on this take-home project entirely by yourself. You should **not** seek help from other people, in particular from your classmates.

Your full report must end with a signed statement saying "This is my own unaided work." Projects that do not end with this statement are subject to a failing grade.

Ching-Peng and I will be glad to address general questions about the material studied in class. Questions on the final project itself should be limited to requests for clarification, as is usual in exams (feel free to visit us during our office hours, or send us an e-mail).

PART 1 of the Final Exam consists of analyzing the mathematical model presented in the paper:
Crossing the Hopf Bifurcation in a Live Predator-Prey System
Gregor F. Fussmann, Stephen P. Ellner, Kyle W. Shertzer, Nelson G. Hairston Jr.
Science 2000, Vol 290, 1358–1360.

In what follows, this paper will be referred to as FESH2000.

1.1 Explain each term in each of the ODE's, equations (1) to (4); pay attention to the definitions of F_C and F_B. Is this system non-dimensionalized? Why does the term $F_C(N)C$, which appears first in eq. (3), appear again without being rescaled in eq. (4)? Why is parameter ε, in eq. (2), dimensionless?

1.2 Define: $S(t) = B(t) - R(t)$. S is the concentration of the non-reproductive, or "senescent," Brachionus. Show that: $dS/dt = -(\delta + m)S + \lambda R$. What happens if $\lambda = 0$?

As reported in FESH2000, for some parameter settings, the state converges to a non-trivial stable point attractor, i.e., a point attractor where all variables are positive. Prove that, for any such steady state: C satisfies $f_B(C) = \delta + m + \lambda$; the fraction of Brachionus that are reproductive is $R/B = (\delta + m)/(\delta + m + \lambda)$; the fraction of Brachionus that are senescent is $S/B = \lambda/(\delta + m)$.

1.3 Download the code rotifers_and_algae.m. Fill in the parameters, the conversion factors, the function definitions, and the derivative definitions. Attach all filled-in lines to your project. Pick a few parameter settings for δ and N_i in each of the four regions a, b, c and d of Fig. 1A, and confirm that the code behaves as described in the paper. Confirm that the Hopf bifurcations observed numerically are indeed supercritical, and explain how you do this.

Explain the curves shown in Fig. 1B of FESH2000. For instance, what is the meaning of the pair of values on the black curves and the pair of values on the grey curves at $\delta = .50$ per day? Confirm that these pairs of values are indeed produced by your simulations: attach to your report the (C, B) phase-plane plots and the time series for $\delta = .50$ per day and $N_i = 80$ μmol/liter. Observe that trajectories in the (C, B) phase plane sometimes cross each other. Why does this happen? Shouldn't it be the case that trajectories in a phase plane never cross?

1.4 Give an estimate of the two Hopf-bifurcation values of δ when N_i is fixed at 90. Give an estimate of the Hopf-bifurcation value of N_i when δ is fixed at 1.30. Based on these estimates—and on additional exploration if needed—how accurate do you think is the boundary between regions b and c in Fig. 1A of FESH2000?

What happens in region d? Do you see periodic behavior or extinction there? How is the "Minimum Value of B after Transient is Over," reported by the code, relevant to the biological interpretation of the model? Based on your answer, can you give an estimate of the boundary between regions c and d for δ = 1.30? Does this estimate agree with Fig. 1A of FESH2000?

1.5 When the system exhibits periodic behavior, what are, roughly, the phase relationships between N, C, R and B? Try to explain your observations.

When the system exhibits periodic behavior, does the period depend on the parameters? Try to explain your observations.

1.6 Set $\lambda = 0$. In view of 1.2, explain why the model is now essentially a 3-variable system rather than a 4-variable system.

Formulate a "reduced" (N, C, B) system that is equivalent to the original full (N, C, B, R) system of FESH2000 with $\lambda = 0$. Code this reduced (N, C, B) system, and run it with a few settings of your choice for δ and N_i, and all other parameters as in FESH2000.

How does the behavior of the reduced model differ from the behavior of the FESH2000 model with $\lambda = .4$? Are the behaviors qualitatively or only quantitatively different? Compare the reduced system with the 2-variable predator-prey system that we studied in HW 6, question 6.1.

1.7 Consider the following predator-prey model, where x is the prey and y is the predator:

$$\begin{cases} \dot{x} = x\left(1 - \dfrac{x}{30}\right) - y\dfrac{x}{x + 10} \\ \dot{y} = y\dfrac{x}{x + 10} - \beta y \end{cases}$$

The model is given here in dimensionless form. In addition, some of the parameters that were left after nondimensionalization have been given numerical values, so the only parameter left is β. We assume that $\beta \in (0,1)$.

Give a biological interpretation for each term in this system, and compare both with FESH2000 and with HW 6, question 6.1.

Do an exhaustive study of the system, using any and all appropriate methods, analytical and numerical, as we did for HW 6, question 6.1 and other examples studied in class and HWs. In particular, find precisely and characterize any possible bifurcation.

1.8 What were the goals of FESH2000 in studying their fairly complex 4-dimensional model, rather than a simpler 2-D or 3-D model as in 1.6 or 1.7 above?

Do a quick survey of the literature published after FESH2000 on the same predator-prey chemostat preparation. In view of the paper itself and of subsequent work, including by the same group, how would you characterize the success of FESH2000 in achieving the goals that were set by the authors?