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Answer 1

Suppose that $f : A \to B$

- To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.
- To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.
- To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.
- To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

a) $f_1 : \mathbb{R} \to \mathbb{R}, f(x) = x^2$

   a) Consider $-5 \in \mathbb{R}$(codomain) there is no $x \in \mathbb{R}$(domain) such that $x^2 = -5$ ($x^2 \geq 0$ for every $x \in \mathbb{R}$.) so f_1 is not surjective.

   b) Note that $-1, 1 \in \mathbb{R}$(domain) are distinct and $f(1) = 1 = f(-1)$. Hence f_1 is not injective.

b) $f_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}, f(x) = x^2$

   a) Consider $-5 \in \mathbb{R}$(codomain) there is no $x \in \mathbb{R}_{\geq 0}$(domain) such that $x^2 = -5$ ($x^2 \geq 0$ for every $x \in \mathbb{R}_{\geq 0}$.) so f_2 is not surjective.

   b) to show f_2 is injective we need to show $\forall a, b \in \mathbb{R}_{\geq 0}, f(a) = f(b) \implies a = b$

      - suppose $f(a) = f(b)$ for some arbitrary $a, b \in \mathbb{R}_{\geq 0}$(domain)
      - $f(a) = a^2$ and $f(b) = b^2$, $a^2 = b^2$
      - $|a| = |b|$ (take square root of both sides)
      - $a = b$ (note $a, b \geq 0$ because $a, b \in \mathbb{R}_{\geq 0}$)
      - since a, b were arbitrary this holds $\forall a, b \in \mathbb{R}_{\geq 0}$ thus f_2 is injective

c) $f_3 : \mathbb{R} \to \mathbb{R}_{\geq 0}, f(x) = x^2$

   a) to show f_3 is surjective we need to show $\forall y \in \mathbb{R}_{\geq 0} \exists x \in \mathbb{R}_{\geq 0} (y = f(x))$

      - take any $y \in \mathbb{R}_{\geq 0}$(codomain)
      - choose $x = \sqrt{y}$ and $x \in \mathbb{R}(domain)$

- $f(x) = f(\sqrt{y}) = (\sqrt{y})^2 = y$ so f_3 is surjective

b) Note that $-1, 1 \in \mathbb{R}$(domain) are distinct and $f(1) = 1 = f(-1)$. Hence f_3 is not injective.

d) $f_4 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, f(x) = x^2$

a) to show f_4 is surjective we need to show $\forall y \in \mathbb{R}_{\geq 0} \exists x \in \mathbb{R}_{\geq 0}(y = f(x))$

- take any $y \in \mathbb{R}_{\geq 0}$(codomain)
- choose $x = \sqrt{y}$ and $x \in \mathbb{R}_{\geq 0}$(domain)
- $f(x) = f(\sqrt{y}) = (\sqrt{y})^2 = y$ so f_4 is surjective

b) to show f_4 is injective we need to show $\forall a, b \in \mathbb{R}_{\geq 0}, f(a) = f(b) \implies a = b$

- suppose $f(a) = f(b)$ for some arbitrary $a, b \in \mathbb{R}_{\geq 0}$
- $f(a) = a^2$ and $f(b) = b^2$, $a^2 = b^2$
- $|a| = |b|$ (take square root of both sides)
- $a = b$ (note $a, b \geq 0$ because $a, b \in \mathbb{R}_{\geq 0}$)
- since a, b were arbitrary this holds $\forall a, b \in \mathbb{R}_{\geq 0}$ thus f_4 is injective

Answer 2

a) show that every function $f : A \subset \mathbb{Z} \to \mathbb{R}$ is continuous

- $\forall \epsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \forall x \in A(|x - x_0| < \delta \to |f(x) - f(x_0)| < \epsilon)$
- let $\epsilon > 0$ and choose arbitrary $x_0 \in A$
- if we take $\delta = 1$ the only point which fulfills $|x - x_0| < \delta$ must be $x = x_0$ itself
- also $|f(x) - f(x_0)| = 0 < \epsilon$ since $x = x_0$
- this equation holds since $x_0 \in A$ was arbitrary choice
- so we have shown $f : A \subset \mathbb{Z} \to \mathbb{R}$ is continuous

b) show that a necessary and sufficient condition for a function $f : \mathbb{R} \to \mathbb{Z}$ to be continuous is that f is a constant function

- suppose f is not constant
- $\exists x_1, x_2 \in \mathbb{R}$ such that $x_1 \neq x_2$ and $f(x_1) \neq f(x_2)$
- assume $f(x_2) > f(x_1))$ then $f(x_2) - f(x_1) > 1$ and there should be $f(x_2) > f(x_1) + 0.2 > f(x_1))$
- $f(x_1) + 0.2 \notin \mathbb{Z}$, so $\nexists x \in \mathbb{R}$ such that $f(x) = f(x_1) + 0.2$
- by intermediate value theorem f is not continuous this is a contradiction, therefore f is necessary to be constant function

Answer 3

a) $X_n = A_1 \times A_2 \times \ldots \ \times A_n$ for all $n \geq 2$ is countable

- we have given that in question $|\mathbb{Z} \times \mathbb{Z}| = |Z| = \aleph_0$ so cartesian product of two countable set is also countable
- so, for $n = 2$, $X_2 = A_1 \times A_2$ is countable
- assume that $2 < k < n$, and $B = X_k = A_1 \times A_2 \times \cdots \times A_k$ is countable
- $B \times A_{k+1} = A_1 \times A_2 \times \cdots \times A_k \times A_{k+1}$ is countable (since B and A_{k+1} is countable)
- so, by induction X_n is also countable

b) infinite countable product of the set $X = \{0, 1\}$ with itself is uncountable

- suppose that the infinite countable product of the set $X = \{0, 1\}$ with itself countable
- under this assumption we can list all all possible products $r_1, r_2, r_3 \ldots$
  - $r_1 =$ 00000...
  - $r_2 =$ 11111...
  - $r_3 =$ 01010...
  - $r_4 =$ 10101...
  - $r_5 =$ 11001...
  - $\ldots$
- we can construct a new product with choosing the complementary of the r_x's (x,x)th entry such that $r =$ 10110...
- contradiction since newly constructed r not in the list
- so by using Carter's diagonalization the infinite countable product of the set $X = \{0, 1\}$ with itself is uncountable

Answer 4

$$(\log n)^2, \sqrt{n} \log n, n^{50}, n^{51} + n^{49}, 2^n, 5^n, (n!)^2$$

a) $(\log n)^2$ is $O(\sqrt{n} \log n)$

1) $\log n \leq \sqrt{n}$ for $n > 1$
2) $(\log n)^2 \leq \sqrt{n} \log n$ (multiply each side with $\log n$)
3) we have choosen $k = 1$ and $C = 1$ to show $(\log n)^2$ is $O(\sqrt{n} \log n)$

b) $(\sqrt{n} \log n)$ is $O(n^{50})$

1) $\sqrt{n} \leq n$ and $\log n \leq n$ for $n > 1$
2) $\sqrt{n} \log n \leq n^2 \leq n^{50}$ (multiply previous equations)

3) we have choosen $k=1$ and $C=1$ to show $(\sqrt{n}\log n)$ is $O(n^{50})$

c) (n^{50}) is $O(n^{51}+n^{49})$

1) $n \leq (n^2+1)$ for $n>1$
2) $n^{50} \leq n^{49}(n^2+1)$ (multiply both sides with n^{49})
3) $n^{50} \leq (n^{51}+n^{49})$
4) we have choosen $k=1$ and $C=1$ to show (n^{50}) is $O(n^{51}+n^{49})$

d) $(n^{51}+n^{49})$ is $O(2^n)$

1) $n^{51}+n^{49} \leq 2n^{51}$ for $n>1$
2) $\lim\limits_{n\to\infty}\dfrac{2n^{51}}{2^n} = 2\lim\limits_{n\to\infty}\dfrac{n^{51}}{2^n}$
3) $2\cdot\lim\limits_{n\to\infty}\dfrac{51!}{(\ln 2)^{51}2^n}$(apply L Hospital's Rule until to get n^0)
4) $\dfrac{2\cdot 51!}{(\ln 2)^{51}}\cdot\lim\limits_{n\to\infty}\dfrac{1}{2^n}=0$ from ratio test we observed that 2^n grows faster than $n^{51}+n^{49}$
5) so we have shown that $(n^{51}+n^{49})$ is $O(2^n)$

e) (2^n) is $O(5^n)$

1) $2 \leq 5$
2) $2^n \leq 5^n$(take nth exponent)
3) so we have shown that (2^n) is $O(5^n)$

f) (5^n) is $O((n!)^2)$

1) let $f(n)=\dfrac{(n!)^2}{5^n}$
2) $L=\lim\limits_{n\to\infty}\dfrac{f(n+1)}{f(n)}=\lim\limits_{n\to\infty}\dfrac{((n+1)!)^2}{5^{(n+1)}}\cdot\dfrac{5^n}{(n!)^2}=\lim\limits_{n\to\infty}\dfrac{(n+1)^2}{5}=\infty$
3) for $n\in\mathbb{Z}^+, L\geq 1$
4) as a result from ratio test $(n!)^2$ is grows faster than 5^n
5) so we have shown that (5^n) is $O((n!)^2)$

Answer 5

a) gcd(94,134) $=2$

1) $134 = 94\times 1+40$
2) $94 = 40\times 2+14$

3) $40 = 14 \times 2 + 12$

4) $14 = 12 \times 1 + 2$

5) $12 = 2 \times 6 + 0$

b) Goldbach's conjecture states that every even integer greater than 2 is the sum of two primes. We will show that this statement is equvalent to that every integer greater than 5 is the sum of three primes

1) let n be an integer greater than 5

2) if n is odd

2.1) we can write $n = 3 + (n-3)$ and $(n-3)$ must be even.

2.2) $n-3 = p+q$ where p and q prime numbers (since $(n-3)$ is an even integer greater than 2 we can use Goldbach's conjecture)

2.3) therefore we have written $n = 3 + p + q$ as the sum of three primes

3) if n is even

3.1) we can write $n = 2 + (n-2)$ and $(n-2)$ must be even.

3.2) $n-2 = p+q$ where p and q prime numbers (since $(n-2)$ is an even integer greater than 2 we can use Goldbach's conjecture)

3.3) therefore we have written $n = 2 + p + q$ as the sum of three primes

4) for the converse assume that every integer greater than 5 is the sum of three primes

4.1) let n be an even integer greater than 2

4.2) by assumption we can write $(n+2)$ as the sum of three primes $(n+2) = p+q+r$ p, q, r prime numbers

4.3) since $(n+2)$ is even one of the prime numbers should be 2 (say $r = 2$)

4.4) so we have $n + 2 = 2 + p + q$

4.5) we have shown that $n = p + q$