CENG 382 - Analysis of Dynamic Systems
Spring 2024
Take Home Exam 1
Student's Solution

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1. (a) Let $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$. Then $\dot{x}_1(t) = x_2(t)$ and, from the given $y''(t) + 2y'(t) - 8y(t) = 0$,

$$\dot{x}_2(t) = y''(t) = -2y'(t) + 8y(t) = -2x_2(t) + 8x_1(t).$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(b) Let $x_1(k) = y(k)$, $x_2(k) = y(k+1)$ and $x_3(k) = y(k+2)$. Using the difference equation $y(k+3) = 2y(k+2) - y(k)$ we obtain

$$\begin{aligned} x_1(k+1) &= y(k+1) = x_2(k), \\ x_2(k+1) &= y(k+2) = x_3(k), \\ x_3(k+1) &= y(k+3) = 2y(k+2) - y(k) = 2x_3(k) - x_1(k). \end{aligned}$$

Therefore, we can represent this system with

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

2. (a) The general first order difference equation

$$x(k+1) = ax(k) + b, \qquad x(0) = x_0$$

has the solution

$$x(k) = \begin{cases} a^k x_0 + b\frac{1-a^k}{1-a}, & a \neq 1, \\ x_0 + kb, & a = 1. \end{cases}$$

For $a = 1,\ b = 2,\ x_0 = 5$ we are in the case $a = 1$; hence

$$x(k) = x_0 + kb \ = \ 5 + 2k.$$

(b) With $a = 0.8,\ b = 4,\ x_0 = 1$ we use the $a \neq 1$ formula:

$$x(k) = 0.8^{\,k} x_0 + 4\frac{1 - 0.8^{\,k}}{1 - 0.8} = 0.8^{\,k} + 20(1 - 0.8^{\,k}) = 20 - 19 \cdot 0.8^{\,k}.$$

(c) For $a = -1,\ b = -1,\ x_0 = 2$ (again $a \neq 1$):

$$x(k) = (-1)^k x_0 + (-1)\frac{1 - (-1)^k}{1 - (-1)} = 2(-1)^k - \frac{1 - (-1)^k}{2}.$$

Since $(-1)^k = 1$ for even k and $(-1)^k = -1$ for odd k, this simplifies to following form

$$x(k) = \begin{cases} -3, & k \text{ odd}, \\ 2, & k \text{ even}. \end{cases}$$

3. We have to check following limit
$$\lim_{k\to\infty} |x(k)|$$
for each discrete time system in Q2 and decide (i) whether $|x(k)| \to \infty$, (ii) whether a fixed point $x^\star$ exists and (iii) whether the system approaches that fixed point.

   (a) $a = 1,\ b = 2,\ x_0 = 5$
   From Q2 we have
   $$x(k) = 5 + 2k.$$
   Hence
   $$\lim_{k\to\infty} |x(k)| = \lim_{k\to\infty} (5 + 2k) = \infty,$$
   so the sequence diverges.
   A fixed point satisfies $x^\star = 1\cdot x^\star + 2$, is impossible; therefore no fixed point exists Because the state grows without bound, the system clearly does not approach a fixed point.

   (b) $a = 0.8,\ b = 4,\ x_0 = 1$
   The solution is
   $$x(k) = 20 - 19(0.8)^k.$$
   Since $|0.8| < 1$,
   $$\lim_{k\to\infty} |x(k)| = \big|\, 20 - 19 \cdot 0 \big| = 20 < \infty$$
   thus it is bounded.
   The fixed point equation $x^\star = 0.8x^\star + 4$ gives
   $$x^\star = \frac{4}{1 - 0.8} = 20.$$
   the term $19(0.8)^k$ decays to zero and
   $$\lim_{k\to\infty} x(k) = 20 = x^\star.$$
   Hence the system has a fixed point at $x^\star = 20$ and the system approaches that fixed point.

   (c) $a = -1,\ b = -1,\ x_0 = 2$
   From Q2,
   $$x(k) = \begin{cases} 2, & k \text{ even}, \\ -3, & k \text{ odd}. \end{cases}$$
   As a result $|x(k)|$ alternates between 2 and 3; it is bounded so $|x(k)| \not\to \infty$ and $\lim_{k\to\infty} x(k)$ does not exist.
   A fixed point solves $x^\star = -x^\star - 1$, yielding
   $$x^\star = -\frac{1}{2}.$$
   the orbit keeps switching between 2 and -3, the sequence does not converge to $x^\star = -\frac{1}{2}$. Thus the system has a fixed point, but the it does not approach it.

4. The linear first order ODE
$$x'(t) = ax(t) + b, \qquad x(0) = x_0$$
has the solution
$$x(t) = \begin{cases} (x_0 + \dfrac{b}{a})e^{at} - \dfrac{b}{a}, & a \neq 0, \\ x_0 + bt, & a = 0. \end{cases}$$

   (a) $a = 2,\ b = 0,\ x_0 = 4$. Applying the $a \neq 0$ formula:
   $$x(t) = x_0 e^{at} = 4e^{2t}.$$

   (b) $a = 0,\ b = -3,\ x_0 = 1$. Applying the $a = 0$ formula:
   $$x(t) = x_0 + bt = 1 - 3t.$$

   (c) $a = -2,\ b = 6,\ x_0 = 0$. Applying the $a \neq 0$ formula:
   $$x(t) = \Big(x_0 + \frac{b}{a}\Big)e^{at} - \frac{b}{a} = \Big(0 + \frac{6}{-2}\Big)e^{-2t} - \frac{6}{-2} = -3e^{-2t} + 3 = 3\big(1 - e^{-2t}\big).$$

5. We have to check the value of the following limit
$$\lim_{t\to\infty} x(t)$$
for each continuous time system in Q4, decide whether $|x(t)| \to \infty$, and examine the existence and whether the system approaching that fixed point.

(a) For part (a), $x(t) = 4e^{2t}$, so:

$$\lim_{t\to\infty} 4e^{2t} = +\infty.$$

Therefore, the system goes to infinity.

To find a fixed point we solve the equilibrium condition $x'(t) = 0$. Since $x'(t) = 2x(t)$, we must have $2x = 0 \Rightarrow x^\star = 0$. The system has a fixed point at $x = 0$. The state diverges, so the system does not approach its fixed point.

(b) For part (b), $x(t) = 1 - 3t$, so:

$$\lim_{t\to\infty} (1 - 3t) = -\infty.$$

Hence the system again goes to infinity in magnitude.

Setting $x'(t) = 0$ gives $b = 0$, but here $b = -3$, so the differential equation $x' = b$ has no fixed point. Therefore, it also cannot approach a fixed point.

(c) For part (c), $x(t) = 3\big(1 - e^{-2t}\big) = 3 - 3e^{-2t}$, so:

$$\lim_{t\to\infty} \big(3 - 3e^{-2t}\big) = 3.$$

Therefore, the system does not go to infinity, it converges to a finite value.

To locate the fixed point we again set $x'(t) = 0$. Because $x'(t) = -2x + 6$, we obtain $-2x + 6 = 0 \Rightarrow x^\star = 3$. Since $x(t) \to 3$ as $t \to \infty$, the system approaches this fixed point.

6. The state transition matrix for $k \geq \ell$ is the ordered product

$$\Phi(k, \ell) = A(k-1)A(k-2)\cdots A(\ell).$$

Because $A(k)$ is diagonal, multiply the diagonal entries separately:

$$\prod_{i=\ell}^{k-1} \frac{i+1}{i+2} = \frac{\ell+1}{\ell+2}\cdot\frac{\ell+2}{\ell+3}\cdots\frac{k}{k+1} = \frac{\ell+1}{k+1}, \qquad \prod_{i=\ell}^{k-1} 3 = 3^{\,k-\ell}$$

Hence

$$\Phi(k, \ell) = \begin{bmatrix} \dfrac{\ell+1}{k+1} & 0 \\ 0 & 3^{\,k-\ell} \end{bmatrix}, \qquad k \geq \ell,$$

Assume that $\begin{bmatrix} a \\ b \end{bmatrix}$ is a fixed point Then it must satisfy

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \dfrac{k+1}{k+2} & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

The first row gives $a = \dfrac{k+1}{k+2}\, a$, which is possible for every k only when $a = 0$ because the factor $\frac{k+1}{k+2} \neq 1$. The second row yields $b = 3\,b$, so $b = 0$. Hence the $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the unique fixed point of the system.

If the state starts at this point it stays there forever. Otherwise, with the state transition matrix

$$\Phi(k, 0) = \begin{bmatrix} \dfrac{1}{k+1} & 0 \\ 0 & 3^{\,k} \end{bmatrix},$$

the system is

$$x(k) = \Phi(k, 0)\, x(0) = \begin{bmatrix} \dfrac{1}{k+1} & 0 \\ 0 & 3^{\,k} \end{bmatrix} x(0).$$

The term $3^{\,k} \to \infty$ as $k \to \infty$, so any initial state with a non zero second component causes the system to diverge. The first component, scaled by $\frac{1}{k+1}$, tends to 0, but there is unbounded growth in the second direction. As a result, apart from the equilibrium at the origin, the system is unstable.

7. (a)

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

Hence the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$.

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (\lambda_1 = 2), \qquad v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad (\lambda_2 = 3).$$

Assemble $S = \begin{bmatrix} v_1 \; v_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, with $S^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$ (because $\det S = -1$).
Since $A = S\Lambda S^{-1}$ with $\Lambda = \mathrm{diag}(2, 3)$, we have $A^k = S\Lambda^k S^{-1}$, where $\Lambda^k = \mathrm{diag}(2^k,\, 3^k)$.

$$S^{-1}x_0 = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Therefore

$$x(k) = A^k x_0 = S\Lambda^k S^{-1} x_0 = S \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = S \begin{bmatrix} -2^k \\ 3^k \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2^k \\ 3^k \end{bmatrix}.$$

Multiplying out,

$$x(k) = \begin{bmatrix} -2^{\,k} + 2 \cdot 3^{\,k} \\ -2^{\,k} + 3^{\,k} \end{bmatrix}$$

(b) Behavior as $k \to \infty$
Because 3^k grows faster than 2^k,

$$x(k) = 3^{\,k} \begin{bmatrix} 2 - \left(\frac{2}{3}\right)^k \\ 1 - \left(\frac{2}{3}\right)^k \end{bmatrix} \xrightarrow[k \to \infty]{} 3^{\,k} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus $x(k) \to \infty$ exponentially (rate $\lambda_2 = 3$), and the state vector eventually points in the direction of the dominant eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. It diverges to infinity along that eigen direction.

8. (a)

$$\det(A - \lambda I) = \begin{vmatrix} 8 - \lambda & 4 \\ 5 & 7 - \lambda \end{vmatrix} = (8 - \lambda)(7 - \lambda) - 20 = \lambda^2 - 15\lambda + 36 = (\lambda - 12)(\lambda - 3).$$

Hence the eigenvalues are $\lambda_1 = 12$ and $\lambda_2 = 3$.

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (\lambda_1 = 12), \qquad v_2 = \begin{bmatrix} -4 \\ 5 \end{bmatrix} \quad (\lambda_2 = 3).$$

$$A = S\Lambda S^{-1}, \qquad \Lambda = \begin{bmatrix} 12 & 0 \\ 0 & 3 \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & -4 \\ 1 & 5 \end{bmatrix}, \qquad S^{-1} = \frac{1}{9} \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix}.$$

Let $u(t) = S^{-1}x(t)$ so that $x(t) = S\,u(t)$. Differentiate:

$$\begin{aligned} \dot{x}(t) &= S\Lambda S^{-1}x(t) + b \\ S^{-1}\dot{x}(t) &= \Lambda S^{-1}x(t) + S^{-1}b \\ \dot{u}(t) &= \Lambda u(t) + c \end{aligned}$$

where $c \; = \; S^{-1}b = \frac{1}{9} \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{9} \\ \frac{1}{9} \end{bmatrix}$.
Thus each component satisfies the scalar non-homogeneous ODE $\dot{u}_i = \lambda_i u_i + c_i, \quad i = 1, 2$.
For $\lambda_i \neq 0$ the solution is

$$u_i(t) = e^{\lambda_i t}\Big(u_i(0) + \tfrac{c_i}{\lambda_i}\Big) - \frac{c_i}{\lambda_i}.$$

$$u(0) = S^{-1}x_0 = \tfrac{1}{9} \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$u_1(t) = e^{12t}\Big(1 + \frac{(4/9)}{12}\Big) - \frac{4/9}{12} \;=\; e^{12t}\Big(1 + \frac{1}{27}\Big) - \frac{1}{27} \;=\; e^{12t}\frac{28}{27} - \frac{1}{27},$$

$$u_2(t) = e^{3t}\Big(0 + \frac{(1/9)}{3}\Big) - \frac{1/9}{3} \;=\; e^{3t}\frac{1}{27} - \frac{1}{27}.$$

$$x(t) = S\,u(t) = \begin{bmatrix} 1 & -4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

$$x(t) = \begin{bmatrix} \dfrac{28e^{12t} - 4e^{3t} + 3}{27} \\[2ex] \dfrac{28e^{12t} + 5e^{3t} - 6}{27} \end{bmatrix}$$

(b) Behaviour as $t \to \infty$

The first eigenvalue $\lambda_1 = 12$ is positive and dominates $\lambda_2 = 3$. As a result e^{12t} outruns both e^{3t} and the constant terms:

$$x(t) = e^{12t} \frac{28}{27} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{v_1} \Big[1 + \dots\Big], \qquad t \to \infty.$$

Hence $\|x(t)\|$ grows exponentially (rate 12) and the state vector eventually points along the eigen direction $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The equilibrium is therefore unstable the trajectory diverges to infinity.

9. (a)

$$\lambda_1 = 2, \qquad \lambda_2 = 5 \quad (\lambda_1 \neq \lambda_2).$$

$$(A - 2I)v = 0 \implies \begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \implies v_2 = 0,\ v_1 \text{ free} \implies v^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$(A - 5I)v = 0 \implies \begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \implies v_1 = v_2 \implies v^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The two eigenvectors are linearly independent, so

$$P = \begin{bmatrix} v^{(1)} & v^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad D = \operatorname{diag}(2, 5), \qquad P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \qquad A = PDP^{-1}.$$

Because P is invertible, A is diagonalizable

(b)

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & 5^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

$$A^k = \begin{bmatrix} 2^k & 5^k - 2^k \\ 0 & 5^k \end{bmatrix},$$

As $k \to \infty$

Since both eigenvalues exceed 1, $2^k \to \infty$ and $5^k \to \infty$, with 5^k dominating because $5 > 2$. More precisely,

$$\frac{2^k}{5^k} = \left(\tfrac{2}{5}\right)^k \longrightarrow 0, \qquad 5^k - 2^k \sim 5^k.$$

Hence every non–zero entry of A^k grows without bound, and the matrix behaves like

$$A^k \approx 5^k \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad k \to \infty.$$

The growth rate of $\|A^k\|$ is therefore controled by the dominant eigenvalue $\lambda_2 = 5$ all entries blow up essentially at the rate 5^k.

10. (a) Transition probability matrix P:

$$P = \begin{bmatrix} 0.1 & 0.5 & 0 & 0 & 0 & 0.4 \\ 0 & 0.2 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.1 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0.7 \\ 0.3 & 0 & 0 & 0.5 & 0 & 0.2 \end{bmatrix}$$

(b) Find transpose of matrix P found in part (a):

$$P^T = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 & 0.3 \\ 0.5 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 1 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.3 & 0 \\ 0.4 & 0 & 0 & 0 & 0.7 & 0.2 \end{bmatrix}$$

Since we know $\lambda = 1$ is an eigenvalue for stochastic matrices,

$$P^T v = v$$

$$\begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 & 0.3 \\ 0.5 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 1 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.3 & 0 \\ 0.4 & 0 & 0 & 0 & 0.7 & 0.2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}$$

$$\begin{bmatrix} 0.1v_1 + 0.3v_6 \\ 0.5v_1 + 0.2v_2 \\ 0.8v_2 + v_3 + 0.4v_4 \\ 0.1v_4 + 0.5v_6 \\ 0.5v_4 + 0.3v_5 \\ 0.4v_1 + 0.7v_5 + 0.2v_6 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This means in the long run, no matter which state we start in, we always end up in state 3. Long term probability matrix P^∞ is:

$$P^\infty = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

11. (a) For a discrete-time system $x(k+1) = Ax(k) + Bu(k)$ the system is controllable iff the controllability matrix $M = [\, B \; AB \; A^2B \,]$ has rank n (the dimension of x).

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \qquad A^2B = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \qquad M = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}.$$

Row–reducing M:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The echelon form has three pivots, hence rank $M = 3 = n$. Therefore the system is controllable

(b) Using the state–transition expansion

$$x(3) = A^3x(0) + A^2Bu(0) + ABu(1) + Bu(2),$$

set $x(3) = 0$ and define the reachability matrix for three steps

$$M = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}.$$

$$A^3x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

so we must solve

$$M \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Row reduction:

$$\begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Hence

$$u(0) = 1, \qquad u(1) = -3, \qquad u(2) = 2$$

Applying this input sequence drives the state from $x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ to $x(3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, as required.

12. In order to decide whether the system is observable, we must check whether the observability matrix

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

has rank n, where n is the dimension of the state vector $x(k)$ ($n = 3$ here)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 2 & 2 & 1 \end{bmatrix}$$

Hence

$$O = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}.$$

Row-reducing O to echelon form:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The reduced matrix has three pivot columns, so $rank(O) = 3 = n$

Because the observability matrix has $rank(O) = 3 = n$, the system is observable