Middle East Technical University Department of Computer Engineering

CENG 382

Analysis of Dynamic Systems

Spring 2024-2025

Take Home Exam 2

Due date: May 31, 2025, Saturday, 23:59

Questions:

1. (10 pts) Consider the following continuous-time system:

$$\dot{x}_1 = -x_2$$

$$\dot{x}_2 = x_1 - x_2^3$$

   And consider the Lyapunov function candidate (Is it a Lyapunov function or not? Why? Prove it.):

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$$

   Show whether the fixed point:

$$\tilde{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   is stable.

2. (20 pts) Consider the continuous time system:

$$\dot{x}_1 = -4x_2$$

$$\dot{x}_2 = Ax_2 + 4x_1 - 3x_1^2 x_2$$

   Explain the behavior of this system for different values of A using Poincare-Bendixson theorem. You can use the Lyapunov function $V(x_1, x_2) = x_1^2 + x_2^2$ after analyzing it.

3. (10 pts) Consider the following discrete-time system:

$$x(k+1) = x(k)^2 - 1$$

   (a) Find the fixed points of the system and determine their stability.
   (b) Find the periodic points of prime period 2.
   (c) Determine the stability of the period-2 orbit.

4. (20 pts) **Strong Convexity Implies Unique Global Minimum**

   If $f : \mathbb{R}^d \to \mathbb{R}$ is strongly convex with parameter $\mu > 0$, prove that f is strictly convex and has a unique global minimum.

5. (20 pts) **Newton Step Lemma**

   Let f be convex and twice differentiable at $\mathbf{x}_t \in \text{dom}(f)$, with $\nabla^2 f(\mathbf{x}_t) \succ 0$ being invertible. Prove that the vector $\mathbf{x}_{t+1}$ resulting from the Newton step satisfies:

$$\mathbf{x}_{t+1} = \arg\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}_t) + \nabla f(\mathbf{x}_t)^\top(\mathbf{x}-\mathbf{x}_t) + \frac{1}{2}(\mathbf{x}-\mathbf{x}_t)^\top \nabla^2 f(\mathbf{x}_t)(\mathbf{x}-\mathbf{x}_t).$$

6. (20 pts) **Gradient Descent vs Stochastic Gradient Descent**
   Consider a simple linear regression model $h_\theta(x) = \theta x$ (where θ is a single parameter). The loss function for a dataset with N points (x_i, y_i) using Mean Squared Error is:

$$L(\theta) = \frac{1}{2N}\sum_{i=1}^{N}(h_\theta(x_i) - y_i)^2 = \frac{1}{2N}\sum_{i=1}^{N}(\theta x_i - y_i)^2$$

   For Stochastic Gradient Descent (SGD), the loss for a single data point (x_j, y_j) is:

$$L_j(\theta) = \frac{1}{2}(h_\theta(x_j) - y_j)^2 = \frac{1}{2}(\theta x_j - y_j)^2$$

   a) Derive the general update rule for θ using batch Gradient Descent (GD). Start by expressing the gradient $\nabla L(\theta) = \frac{\partial L(\theta)}{\partial \theta}$.

   b) Derive the general update rule for θ using Stochastic Gradient Descent (SGD) for a single data point (x_j, y_j). Start by expressing the gradient $\nabla L_j(\theta) = \frac{\partial L_j(\theta)}{\partial \theta}$.

   c) Now, consider a specific dataset consisting of two points: $(x_1, y_1) = (1, -2)$ and $(x_2, y_2) = (2, 3)$. Analytically find the value of θ (denoted θ^*) that minimizes the batch loss function $L(\theta)$ for this dataset.

   d) Using the specific dataset from part (c), an initial parameter $\theta_0 = 0$, and a learning rate $\alpha = 0.1$, calculate the value of θ_1 after one iteration of batch Gradient Descent using the update rule derived in part (a).

   e) Using the specific dataset from part (c), the initial parameter $\theta_0 = 0$, and learning rate $\alpha = 0.1$, calculate the value of θ_1 after one iteration of Stochastic Gradient Descent using the update rule derived in part (b). For this SGD iteration, use only the first data point $(x_1, y_1) = (1, -2)$.

   f) Compare the direction of the first step taken by batch GD (from θ_0 to θ_1^{GD} in part d) and SGD (from θ_0 to θ_1^{SGD} in part e) relative to the optimal θ^* found in part (c). Does SGD move towards or away from θ^* in this first step?

Regulations

1. You have to write your answers to the provided sections of the template answer file given in LaTeX. **Handwritten solutions will not be accepted.**

2. Do not write any extra stuff like question definitions to the answer file. Just give your solution to the question. Otherwise you will get 0 from that question.

3. Please justify your answers.

4. **Late Submission: Not allowed!**

5. **Cheating: We have zero tolerance policy for cheating.** People involved in cheating will be punished according to the university regulations.

6. Submit a single PDF file named eXXXXXXX.pdf (7-digit student number). Submission that are not in the specified format will recieve a penalty of 10 points.

7. You may ask your questions in the course forum or by sending a mail to oguzhan@ceng.metu.edu.tr