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Answer 1

Let $G(x)$ be the generating function for the sequence $\{a_n\}$.

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

For this recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$ multiply each term by x^n and sum each term over all positive $n \geq 2$

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} 3a_{n-1} x^n + \sum_{n=2}^{\infty} 4a_{n-2} x^n$$

$$\sum_{n=2}^{\infty} a_n x^n + a_0 + a_1 x - (a_0 + a_1 x) = 3x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 4x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$

$$\sum_{n=0}^{\infty} a_n x^n - (a_0 + a_1 x) = 3x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 4x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$

$$\sum_{n=0}^{\infty} a_n x^n - (a_0 + a_1 x) = 3x(\sum_{n=2}^{\infty} a_{n-1} x^{n-1} + a_0 - a_0) + 4x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$

$$\sum_{n=0}^{\infty} a_n x^n - (a_0 + a_1 x) = 3x(\sum_{n=1}^{\infty} a_{n-1} x^{n-1} - a_0) + 4x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$

$$\sum_{n=0}^{\infty} a_n x^n - (a_0 + a_1 x) = 3x(\sum_{n=0}^{\infty} a_n x^n - a_0) + 4x^2 \sum_{n=0}^{\infty} a_n x^n$$

Right now we have

$$G(x) - a_0 - a_1 x = 3x(G(x) - a_0) + 4x^2 G(x)$$

We have given $a_0 = a_1 = 1$

$$G(x) - 1 - x = 3x(G(x) - 1) + 4x^2 G(x)$$

$$G(x)(1 - 3x - 4x^2) = 1 - 2x$$

$$G(x)((1 + x) \cdot (1 - 4x)) = 1 - 2x$$

Rewrite the equation to solve $G(x)$, and we have

$$G(x) = \frac{1 - 2x}{(1 - 4x) \cdot (1 + x)}$$

Expanding the right hand side of this equation into partial fractions

$$\frac{(1 - 2x)}{(1 - 4x) \cdot (1 + x)} = \frac{A}{(1 - 4x)} + \frac{B}{(1 + x)}$$

$$1 - 2x = A(1 + x) + B(1 - 4x)$$

$$A + B = 1$$

$$A - 4B = -2$$

We have found that

$$A = \frac{2}{5}, B = \frac{3}{5}$$

Substitute A and B

$$G(x) = \frac{2}{5} \cdot \frac{1}{1 - 4x} + \frac{3}{5} \cdot \frac{1}{1 + x}$$

Using the identity $\dfrac{1}{1 - ax} = \sum_{k=0}^{\infty} a^k x^k$

$$G(x) = \frac{2}{5} \sum_{n=0}^{\infty} 4^n x^n + \frac{3}{5} \sum_{n=0}^{\infty} (-1)^n x^n$$

$$G(x) = \sum_{n=0}^{\infty} (\frac{2}{5} 4^n + \frac{3}{5} (-1)^n) x^n$$

$$a_n = \frac{2}{5} 4^n + \frac{3}{5} (-1)^n$$

Answer 2

a)

First obtain

$$f(x) = 2 + 5x + 11x^2 + 29x^3 + 83x^4 + 245x^5 + \dots$$

Adding the following two series, we can get $f(x)$.

$$2 + 2x + 2x^2 + 2x^3 + 2x^4 + \dots \tag{1}$$

$$0 + 3x + 9x^2 + 27x^3 + 81x^4 \dots \tag{2}$$

We can write (1) and (2) as followings

$$2(1 + x + x^2 + x^3 + x^4 + \dots) = 2\sum_{k=0}^{\infty} x^k \tag{1}$$

$$3x(1 + 3x + 9x^2 + 27x^3 + \dots) = 3x\sum_{k=0}^{\infty} 3^k x^k \tag{2}$$

$$f(x) = 2\sum_{k=0}^{\infty} x^k + 3x\sum_{k=0}^{\infty} 3^k x^k$$

Using the identities $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ and $\dfrac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$

$$f(x) = 2 \cdot \frac{1}{1-x} + 3x \cdot \frac{1}{1-3x}$$

$$f(x) = \frac{-3x^2 - 3x + 2}{3x^2 - 4x + 1}$$

b)

$$G(x) = \frac{7 - 9x}{(1 - 2x) \cdot (1 - x)}$$

Expanding the right hand side of this equation into partial fractions

$$\frac{7 - 9x}{(1 - 2x) \cdot (1 - x)} = \frac{A}{1 - 2x} + \frac{B}{1 - x}$$

$$7 - 9x = A(1 - x) + B(1 - 2x)$$
$$7 - 9x = A + B + x(-A - 2B)$$
$$A + B = 7$$
$$-A - 2B = -9$$
$$A = 5, B = 2$$

Substitute A and B

$$G(x) = \frac{5}{1 - 2x} + \frac{2}{1 - x}$$

Using the identity $\dfrac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$

$$G(x) = 5\sum_{n=0}^{\infty} 2^n x^n + 2\sum_{n=0}^{\infty} x^n$$

$$G(x) = \sum_{n=0}^{\infty} (5 \cdot 2^n + 2)x^n$$

$$a_n = 5 \cdot 2^n + 2$$

$$< 7, 12, 22, 42, 82, 162, 322 \cdots >$$

Answer 3

a)

No, R is not an equivalence relation since transitivity property is not satisfied. A relation R on a set is called transitive if whenever aRb and bRc then aRc

To see this, consider the following counter example.

- let $a = 6, b = 8$ then there exists a right triangle with sides $a = 6, b = 8$ and hypotenuse $p = 10$ by Pythagorean theorem

- let $b = 8, c = 12$ then there exists a right triangle with sides $b = 8, c = 12$ and hypotenuse $q = 15$ by Pythagorean theorem

- since $a = 6, c = 12$ and hypotenuse $r = \sqrt{6^2 + 12^2} = 6\sqrt{5}$ and $r. \notin Z$. Transitivity property is not satisfied

b)

Yes, it is an equivalence relation since it is reflexive, symmetric, and transitive.

1. Reflexive:

   - Assume $x_1, y_1 \in R$. Then $(x_1, y_1)R(x_1, y_1)$ is true since $2x_1 + y_1 = 2x_1 + y_1$

2. Symmetric:

   - Assume $x_1, y_1 \in R$ and $x_2, y_2 \in R$
   - Suppose $(x_1, y_1)R(x_2, y_2)$. Then $2x_1 + y_1 = 2x_2 + y_2$
   - Then, $(x_2, y_2)R(x_1, y_1)$ is true since $2x_2 + y_2 = 2x_1 + y_1$
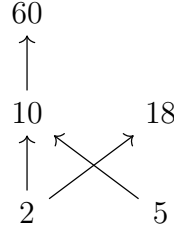
3. Transitive:

   - Let $x_1, x_2, x_3 \in R$ and $y_1, y_2, y_3 \in R$
   - Suppose $(x_1, y_1)R(x_2, y_2)$ and $(x_2, y_2)R(x_3, y_3)$
   - Then we have $2x_1 + y_1 = 2x_2 + y_2$ and $2x_2 + y_2 = 2x_3 + y_3$
   - We can write $(x_1, y_1)R(x_3, y_3)$ since $2x_1 + y_1 = 2x_2 + y_2 = 2x_3 + y_3$

Equivalence class of $(1, -2)$ is $[(1, -2)]_R = \{x, y | 2x - y = 0\}$
It represents a line passing through the origin and having a slope of -2.

Answer 4

a) Hasse diagram of R

$$
\begin{array}{ccc}
 & 60 & \\
 & \uparrow & \\
10 & & 18 \\
\uparrow & \times & \\
2 & & 5
\end{array}
$$

b) Matrix representation for R

$$
M_R = \begin{bmatrix}
1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

c)
- $R_s = R \cup R^{-1}$
- $R = \{(a,b)|a \; divides \; b\}$ and $R^{-1} = \{(a,b)|b \; divides \; a \}$
- $R_s = \{(a,b)|a \; divides \; b \; or \; b \; divides \; a\}$

Matrix representation for R_s

$$
M_{R_s} = \begin{bmatrix}
1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1
\end{bmatrix}
$$

All elements (x,y) where $(x,y) \in R_s$ and $(x,y) \notin R$

$$\{(10,2),(10,5),(18,2),(60,2),(60,5),(60,10)\}$$

d) To create a total ordering that includes all elements of A we need reflexivity, antisymmetry, transitivity and totality properties be satisfied. Since (2,5), (5,18), (18,60) are not related we need to remove from those elements to satisfy total ordering conditions. But it is not possible to remove an element and add an element to provide these situations. Since if we remove one of the elements of the listed pairs above, still there exists a not related pair. On the other hand, if we remove two elements and add one we can satisfy these conditions. For example if we remove 5 and 18. Add 1 we can satisfy all total ordering properties.