Middle East Technical University Department of Computer Engineering

CENG 382

Analysis of Dynamic Systems

Spring 2024-2025

Take Home Exam 1

Due date: April 25, 2025, Friday, 23:55

1. (12 pts) Represent the following higher order equations as a system of first-order equations.
   (a) (6 pts) $y''(t) + 2y'(t) - 8y(t) = 0$
   (b) (6 pts) $y(k + 3) = 2y(k + 2) - y(k)$
2. (9 pts) Find an exact formula for $x(k)$, where $x(k + 1) = ax(k) + b$, $x(0) = x_0$, and a, b, and x_0 have the following values:
   (a) (3 pts) $a = 1, b = 2, x_0 = 5$.
   (b) (3 pts) $a = 0.8, b = 4, x_0 = 1$.
   (c) (3 pts) $a = -1, b = -1, x_0 = 2$.
3. (6 pts) For each of the discrete-time systems in Q2, determine whether or not $|x(k)| \to \infty$. Determine if the system has a fixed point and whether or not the system is approaching that fixed point.
4. (9 pts) Find the exact value of $x(t)$, where $x' = ax + b$, $x(0) = x_0$, and a, b, and x_0 have the following values:
   (a) (3 pts) $a = 2, b = 0, x_0 = 4$.
   (b) (3 pts) $a = 0, b = -3, x_0 = 1$.
   (c) (3 pts) $a = -2, b = 6, x_0 = 0$.
5. (6 pts) For each of the continuous-time systems in Q4, determine whether or not $|x(t)| \to \infty$. Determine if the system has a fixed point and whether or not the system is approaching that fixed point.
6. (9 pts) Find state transition matrix $\Phi(k, l)$ for system $x(k+1) = \begin{bmatrix} \frac{k+1}{k+2} & 0 \\ 0 & 3 \end{bmatrix} x(k)$. Comment on the behavior of the system as $k \to \infty$.

7. (9 pts) Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ and $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Consider the discrete-time dynamical system $x(k+1) = Ax(k)$.

   (a) (6 pts) Find an exact formula for $x(k)$.

   (b) (3 pts) Comment on the behavior of $x(k)$ as $k \to \infty$.

8. (9 pts) Let $A = \begin{bmatrix} 8 & 4 \\ 5 & 7 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Consider the continuous-time dynamical system $x'(t) = Ax(t) + b$.

   (a) (6 pts) Find an exact formula for $x(t)$.

   (b) (3 pts) Comment on the behavior of $x(t)$ as $t \to \infty$.

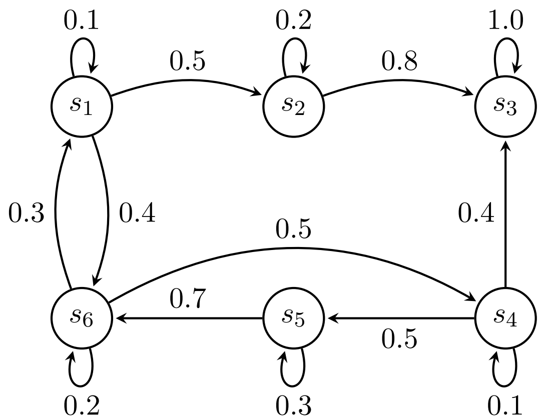
9. (12 pts) Consider the system shown below:

$$x(k+1) = Ax(k), \quad A = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}.$$

   (a) (6 pts) Show that the matrix A is diagonalizable. Justify your answer.

   (b) (6 pts) If we try to compute A^k in order to solve the system, what happens to the entries of A^k as $k \to \infty$? Justify your answer.

10. (9 pts) Consider the state diagram representing a Markov Chain shown below:



(a) (3 pts) Find the matrix representing the transition probabilities P of the state diagram above, where P_{ij} is the probability of transitioning from state i to state j.

(b) (6 pts) How does this Markov Chain behave in the long term? Identify any absorbing states and determine the long-term probability distribution or fate of the system. Justify your answer.

11. (15 pts) Consider the system given below:

$$x(k+1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(k)$$

(a) (5 pts) Show that this system is controllable. Justify your answer.

(b) (10 pts) Find the finite sequence of inputs $u(0), u(1), u(2)$ which drives the system from initial state $x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ to $x(3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

12. (10 pts) Determine whether the system below is observable or not. Justify your answer.

$$x(k+1) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x(k), \quad y(k) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x(k)$$

Regulations

1. You have to write your answers to the provided sections of the template answer file given in LaTeX. **Handwritten solutions will not be accepted.**
2. Do not write any extra stuff like question definitions to the answer file. Just give your solution to the question. Otherwise you will get 0 from that question.
3. Please justify your answers.
4. You can get 115 points in total. Your grade will be scaled down to 100.
5. **Late Submission: Not allowed!**
6. **Cheating: We have zero tolerance policy for cheating.** People involved in cheating will be punished according to the university regulations.
7. Submit a single PDF file named eXXXXXXX.pdf (7-digit student number). Submission that are not in the specified format will recieve a penalty of 10 points.
8. You may ask your questions in the course forum or by sending a mail to oguzhan@ceng.metu.edu.tr