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Answer 1

To construct the proof let $P(n)$ be the proposition "$6^{2n} - 1$ is divisible by both 5 and 7. For $n \in N^{+}$"

1) **Basis Step**

We must show the statement is true for $n = 1$. The statement $P(1)$ is true because $6^2 - 1 = 35$ and it is divisible by both 5 and 7.

2) **Inductive Step**

For the inductive hypothesis we assume that $P(k)$ is true; that is we assume that $6^{2k} - 1$ is divisible by both 5 and 7. For arbitrary $k \in N^{+}$
We must show $P(k+1) = 6^{2\cdot(k+1)} - 1$ is divisible by both 5 and 7.

$$\begin{aligned} P(k+1) &= 6^{2\cdot(k+1)} - 1 \\ &= 6^{2k+2} - 1 \\ &= 36 \cdot 6^{2k} - 1 \\ &= (35 \cdot 6^{2k}) + (6^{2k} - 1) \end{aligned}$$

clearly the first term $35 \cdot 6^{2k}$ is divisible by both 5 and 7. Using the inductive hypothesis, we conclude that the second term $6^{2k} - 1$ is divisible by both 5 and 7. This completes the inductive step.

Hence, $6^{2n} - 1$ is divisible by both 5 and 7 by induction.

Answer 2

1) **Basis Step**

We must show the statement is true for $n = 0$ which is the smallest value of n for this inequality.
$H_n \leq 9^n$ must be satisfied for $n = 0$
$H_0 \leq 9^0$ is true since H_0 is given as 1. Same as H_1 and H_2
$H_1 = 5 \leq 9^1$
$H_2 = 7 \leq 9^2$

2) **Inductive Step**

Assume that $H_j \leq 9^j$ for $0 \leq j \leq k$ where $k \geq 2$ We must show that $H_{k+1} \leq 9^{k+1}$ to prove it is true for all values of n.

$$H_{k+1} = 8H_k + 8H_{k-1} + 9H_{k-2}$$

since we assumed H_j is true where $0 \leq j \leq k$, $H_k \leq 9^k, H_{k-1} \leq 9^{k-1}, H_{k-2} \leq 9^{k-2}$ we can write this inequality

$$8H_k + 8H_{k-1} + 9H_{k-2} \leq 8 \cdot 9^k + 8 \cdot 9^{k-1} + 9 \cdot 9^{k-2}$$
$$8H_k + 8H_{k-1} + 9H_{k-2} \leq 8 \cdot 9^k + 8 \cdot 9^{k-1} + 9^{k-1}$$
$$8H_k + 8H_{k-1} + 9H_{k-2} \leq 8 \cdot 9^k + 9 \cdot 9^{k-1}$$
$$8H_k + 8H_{k-1} + 9H_{k-2} \leq 8 \cdot 9^k + 9^k$$
$$8H_k + 8H_{k-1} + 9H_{k-2} \leq 9 \cdot 9^k$$
$$8H_k + 8H_{k-1} + 9H_{k-2} \leq 9^{k+1}$$
$$H_{k+1} \leq 9^{k+1}$$

Hence, $H_n \leq 9^n$, for all $n \in N$ by induction.

Answer 3

To construct solution we can group possible arrangements according to $k =$ first zero position of consecutive four zero part.

- For $k = 1$ we have 2^4 strings (0000****)
- For $k = 2$ we have 2^3 strings (10000***)
- For $k = 3$ we have 2^3 strings (*10000**)
- For $k = 4$ we have 2^3 strings (**10000*)
- For $k = 5$ we have 2^3 strings (***10000)

This grouping gives us to the number of strings with at least 4 consecutive zeros is $2^4 + 4 \cdot 2^3 = 48$. By symmetry, the number of strings with at least 4 consecutive ones is the same. However, in this case we would have counted 11110000 and 00001111 twice.

Hence the total number is $48 + 48 - 2 = 94$

Answer 4

We need to choose 1 star from 10 distinct stars, 2 habitable planet from 20 distinct habitable planets and 8 non-habitable planet from 80 distinct habitable planets

- 1 star from 10 distinct stars $C(10,1) = \dfrac{10!}{1! \cdot 9!}$
- 2 habitable planet from 20 distinct habitable planets $C(20,2) = \dfrac{20!}{2! \cdot 18!}$
- 8 non-habitable planet from 80 distinct non-habitable planets $C(80,8) = \dfrac{80!}{8! \cdot 72!}$

We can choose $\dfrac{10!}{1! \cdot 9!} \cdot \dfrac{20!}{2! \cdot 18!} \cdot \dfrac{80!}{8! \cdot 72!}$ different star and planet sets by product rule.

Now we need to arrange planets around the star such that there are at least 6 non-habitable planets between 2 habitable planets. There can be 6, 7 or 8 planets between 2 habitable planets

Let H_i represents habitable planet and $*$ represents non-habitable planets.

- 6 non-habitable planets between 2 habitable planets. There can be 3 different arrangement.
  - $H_1, *, *, *, *, *, *, H_2, *, *$
  - $*, H_1, *, *, *, *, *, *, H_2, *$
  - $*, *, H_1, *, *, *, *, *, *, H_2$
- 7 non-habitable planets between 2 habitable planets.There can be 2 different arrangement.
  - $H_1, *, *, *, *, *, *, *, H_2, *$
  - $*, H_1, *, *, *, *, *, *, *, H_2$
- 8 non-habitable planets between 2 habitable planets. There can be 1 different arrangement.
  - $H_1, *, *, *, *, *, *, *, *, H_2$
- We can distribute 2 habitable planets 2! and 8 non-habitable planets 8! and there is 6 six different arrangements shown above. Total number of arrangement of planets around star is $6 \cdot 2! \cdot 8!$ for one set.

We showed above we can choose $\frac{10!}{1! \cdot 9!} \cdot \frac{20!}{2! \cdot 18!} \cdot \frac{80!}{8! \cdot 72!}$ different star and planet set and, for one set we have $6 \cdot 2! \cdot 8!$ different arrangements. So by product rule our general result is :

$$\frac{10!}{1! \cdot 9!} \cdot \frac{20!}{2! \cdot 18!} \cdot \frac{80!}{8! \cdot 72!} \cdot 6 \cdot 2! \cdot 8!$$

Answer 5

a) Let $S(n)$ is the number of ways to jump n cells. Robot can move 1st cell with 1 way. 2nd cell with 2 ways and 3rd cell with 4 ways.

1	2	3
1	1+1	1+1+1
	2	1+2
		2+1
		3

Robot can go nth cell by

- go (n-1)th cell and jump 1 cell
- go (n-2)th cell and jump 2 cell
- go (n-3)th cell and jump 3 cell

We can write this recurrence relation as follows for $n > 3$:

$$S(n) = S(n-1) + S(n-2) + S(n-3)$$

b) Initial conditions are

- $S(1) = 1$
- $S(2) = 2$
- $S(3) = 4$

c)

$$S(4) = S(3) + S(2) + S(1) = 7$$
$$S(5) = S(4) + S(3) + S(2) = 13$$
$$S(6) = S(5) + S(4) + S(3) = 24$$
$$S(7) = S(6) + S(5) + S(4) = 44$$
$$S(8) = S(7) + S(6) + S(5) = 81$$
$$S(9) = S(8) + S(7) + S(6) = 149$$