"Optimal Control"

Sulav Duwal
PhD Candidate
System pharmacology and disease control group
Freie Universität Berlin

Bioinformatics Social Meeting, HU Berlin

Outline

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Optimization and Optimal Control

- Optimization
- Optimal Control in Nutshell

Optimal Control for Markov Process

- Disease and drug model - Markov Process
- Optimal control of Markov Process
- Brute Force Algorithm
- Branch and bound Algorithm

Optimization

- What is optimization ?

Optimization

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- 3 Ingredients

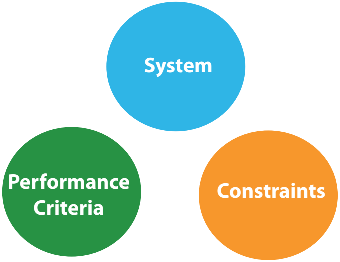

Figure: Ingredients of optimization

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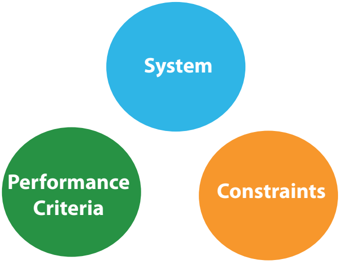

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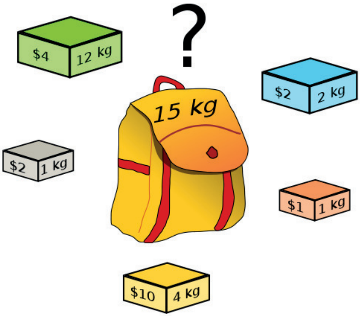

Figure: source:wikimedia commons

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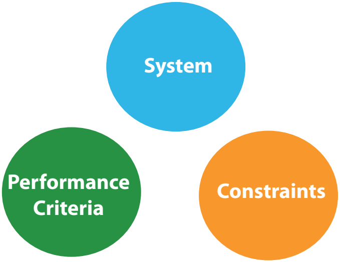

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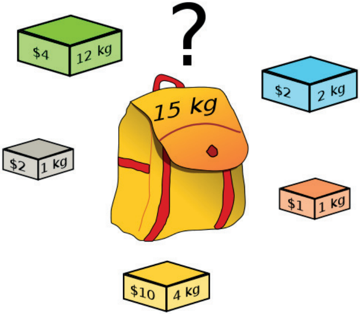
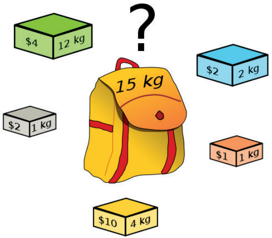


Figure: source:wikimedia commons

- Description of system : (weight , value)
- Performance criteria : Maximize total value
- Constraints : Total weights $\leq$ 15 Kg

Optimization and Optimal Control

- Static optimization
  - The system is independent of time.
  - Knapsack problem



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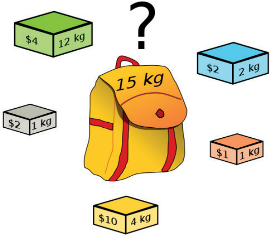

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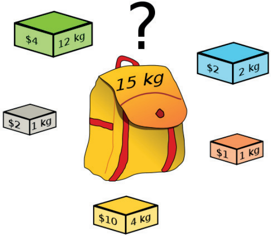

  - Linear programming, Integer programming

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  - **A.k.a. Optimal Control**

Optimal Control in Nutshell

- System

$$\frac{dx(t)}{dt} = f(x(t), u(t)) \quad \text{and} \quad x(t_0) = x_0 \tag{1}$$

  $x(t)$ is called state variable and $u(t)$ is called control variable.

- Performance criteria

$$J(x_0, u) = \int_{t_0}^{t_f} \mathcal{L}(x(t), u(t), t)dt + \phi(x(t_f), t_f) \tag{2}$$

  $\mathcal{L}(x(t), u(t), t)$ is the running cost and $\phi(x(t_f), t_f)$ is the terminal cost.

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- Optimal control
$$\begin{aligned} \min_u \quad & \int_{t_0}^{t_f} \mathcal{L}(x(t), u(t), t)dt + \phi(x(t_f), t_f) \\ \text{subject to} \quad & \frac{dx}{dt} = f(x(t), u(t)) \\ & x(t_0) = x0 \end{aligned} \tag{3}$$

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- *Optimal control problem* is a *two-point boundary value problem*.
- Sufficient condition of optimality
  1. For maximization problem
  $$\frac{\partial^2 H}{\partial u^2} < 0 \quad \text{at} \quad u^*$$
  2. For minimization problem
  $$\frac{\partial^2 H}{\partial u^2} > 0 \quad \text{at} \quad u^*$$

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- A broad field with history of 300 years. The theory was developed in parallel in USA and former USSR during cold war.
- Application in biology : control of epidemics, treatment design etc. largely underexplored !!

Outline

PLOS | COMPUTATIONAL BIOLOGY

RESEARCH ARTICLE

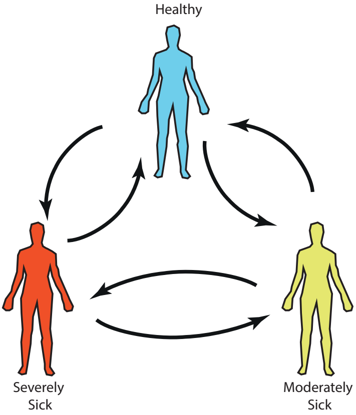
Optimal Treatment Strategies in the Context of ‘Treatment for Prevention’ against HIV-1 in Resource-Poor Settings

Sulav Duwal[1,2], Stefanie Winkelmann[1], Christof Schütte[1,3], Max von Kleist[1,2]*

1 Department of Mathematics and Computer Science, Freie Universität Berlin, Germany, **2** Junior Research Group “Systems Pharmacology & Disease Control”, **3** Zuse Institute Berlin, Germany

Disease and drug model - Markov Process

- Stochastic process with memorylessness
- State space $\mathcal{S}$: The set of all possible states
- Action space $\mathcal{A}$: The set of all possible actions (treatment option).



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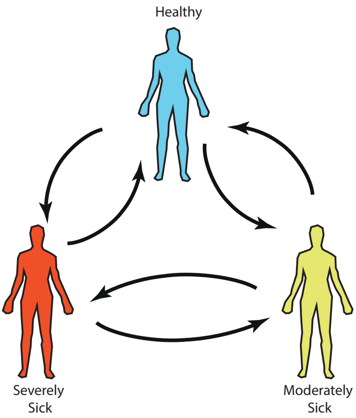

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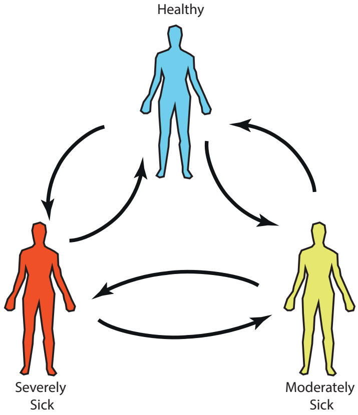

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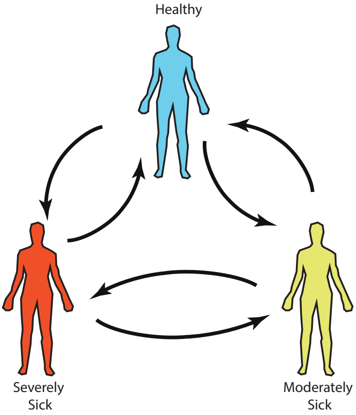

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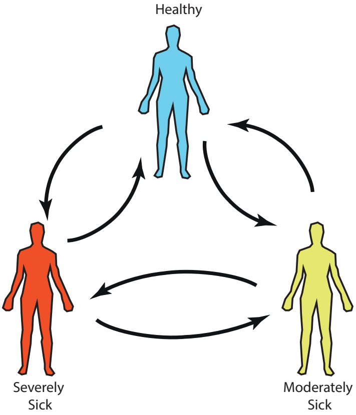
where T_a is the transition matrix specific to action a.



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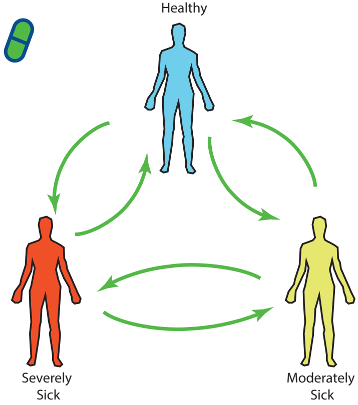

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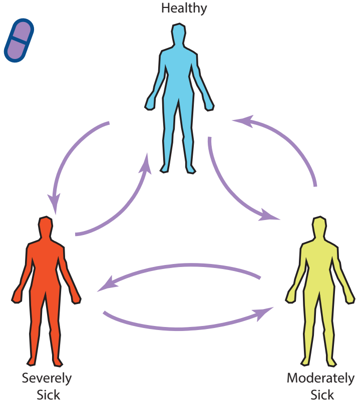

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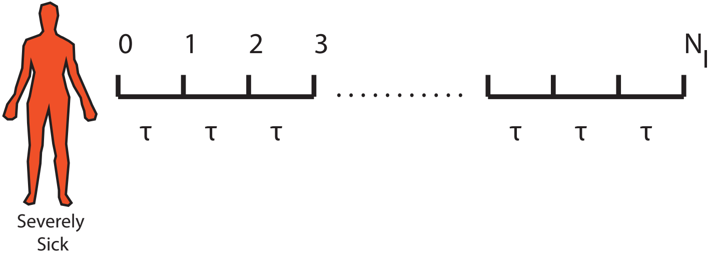
Optimal control of Markov Process



- System : Discrete Dynamics

$$p_{j+1} = T_{u_j} \cdot p_j \text{ and } p(0) = p_0$$

where $u_j \in \mathcal{A}$

- Performance Criteria

$N_{\mathcal{I}} :=$ is a number of intervals and $q_{u_j,j}, q_{N_{\mathcal{I}}} \in \mathbb{R}^{|\mathcal{S}|}$ are cost vectors.

$$J(p_0, u) = \sum_{j=0}^{N_{\mathcal{I}}-1} q'_{u_j,j} \cdot p_j + q'_{N_{\mathcal{I}}} \cdot p_{N_{\mathcal{I}}}$$

Optimal control of Markov Process

- Optimal Control Problem

$$\begin{aligned} &\operatorname*{argmin}_{u\in\mathcal{U}} && \sum_{j=0}^{N_{\mathcal{I}}-1} q'_{u_j,j}\cdot p_j + q'_{N_{\mathcal{I}}}\cdot p_{N_{\mathcal{I}}} \\ &\text{w.r.t} && p_{j+1} = T_{u_j}\cdot p_j \\ & && p(0) = p_0 \end{aligned}$$

Optimal control of Markov Process

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\text{w.r.t} \quad & p_{j+1} = T_{u_j} \cdot p_j \\
& p(0) = p_0
\end{aligned}
$$

- Hamiltonian

$$
H_j = \xi'_{j+1} \cdot T_{u_j} \cdot p_j + q'_{u_j,j} \cdot p_j
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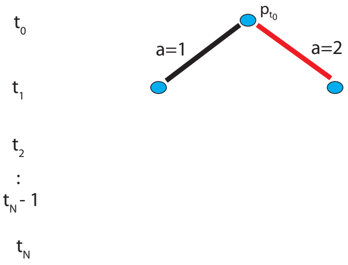
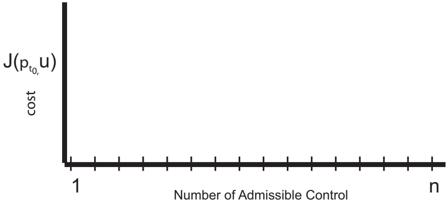
- Optimal control

$$u_j^* = \underset{a \in \mathcal{A}}{\operatorname{argmin}} \{ \xi^{*\prime}_{j+1} \cdot T_a \cdot p_j^* + q'_{a,j} \cdot p_j^* \}$$

Brute Force Algorithm





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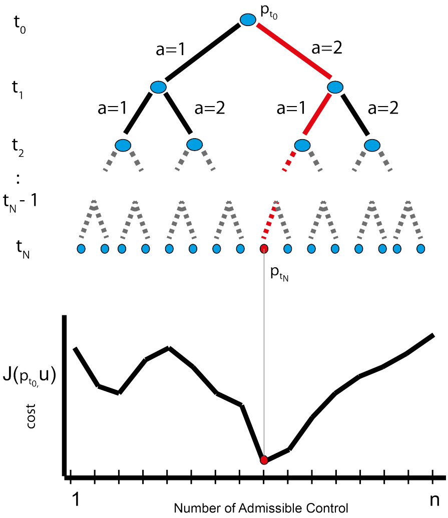
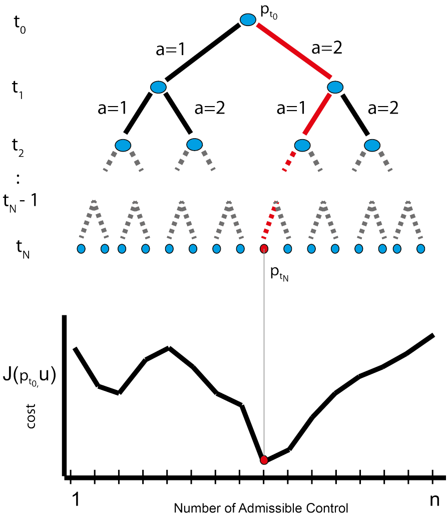


Brute Force Algorithm

Problem :
- Curse of dimensionality
- Given $|\mathcal{A}|$ number of action options and $N_{\mathcal{I}}$ number of intervals
  $|\mathcal{A}|^{N_{\mathcal{I}}}$ number of controls available

Idea : ??

Branch and bound algorithm (Redundancy)



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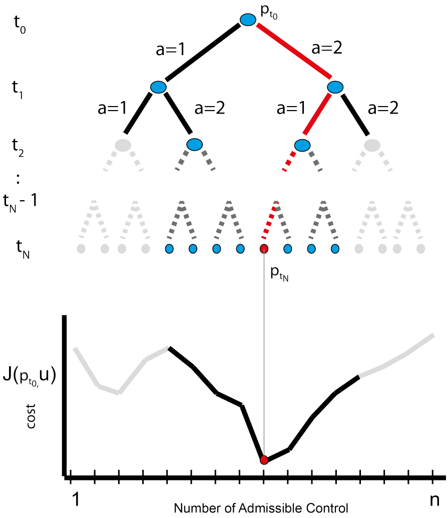

Santa’s problem

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- In 7 days, he has to grow exactly 999 strands. Otherwise ....
- He has 2 options
  - $a_1 : +3$ Naturally, he has +3 strands more in a day.
  - $a_2 : \times 3$ If he takes a magic drug, he multiplies his beard 3 times in a day.

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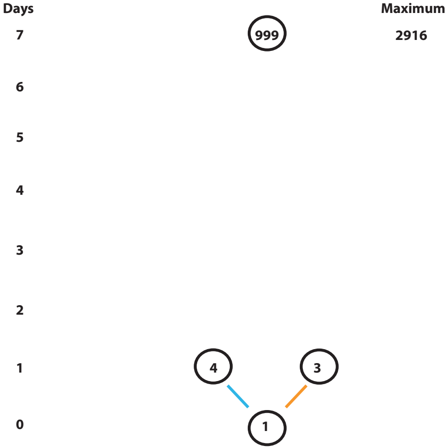
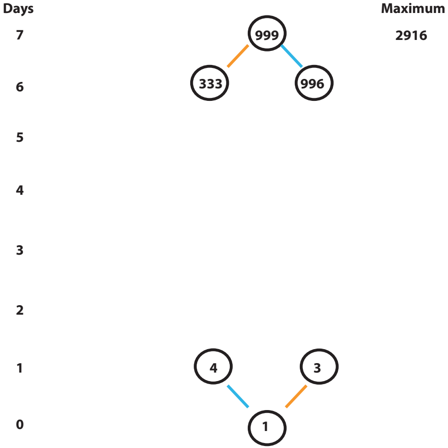
- In 7 days, he has to grow exactly 999 strands. Otherwise ....
- He has 2 options
  - $a_1 : +3$ Naturally, he has +3 strands more in a day.
  - $a_2 : \times 3$ If he takes a magic drug, he multiplies his beard 3 times in a day.
- How can he get 999 beard in 7 days starting from a single beard ?

Santa's problem

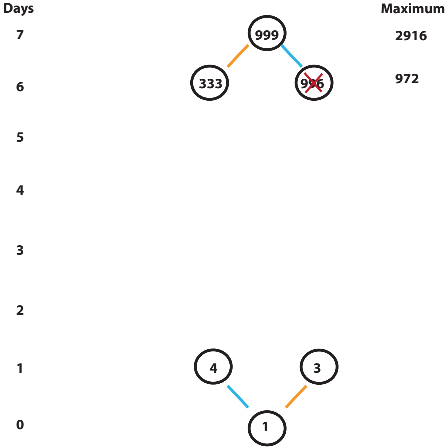
Santa’s problem



Santa’s problem



Santa's problem



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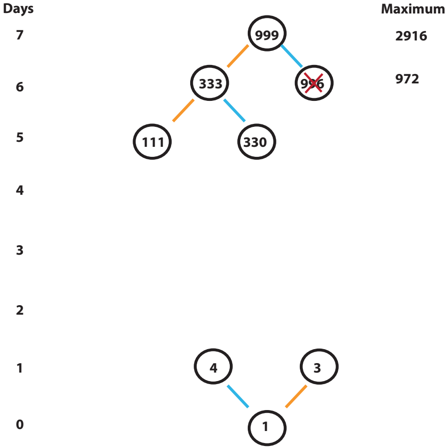

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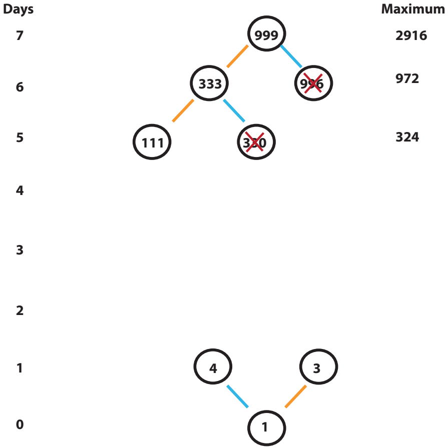
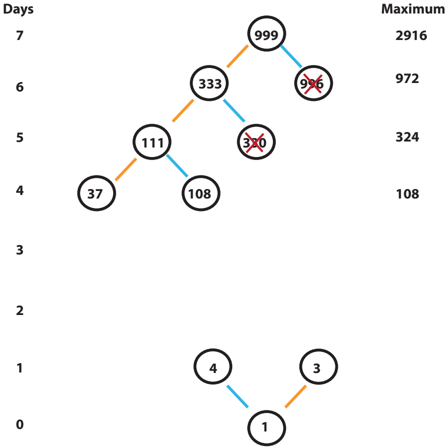
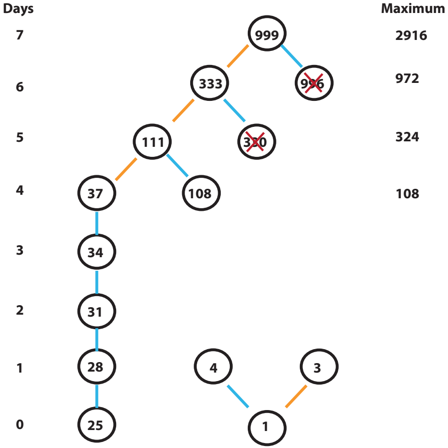


Santa’s problem



Santa’s problem



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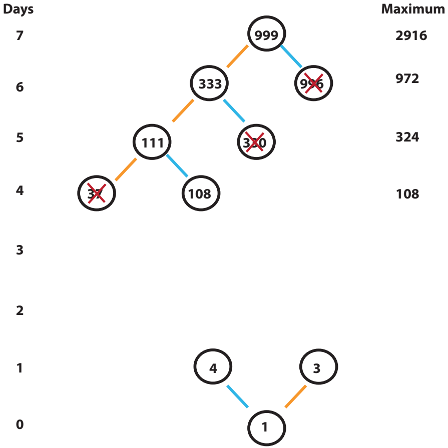
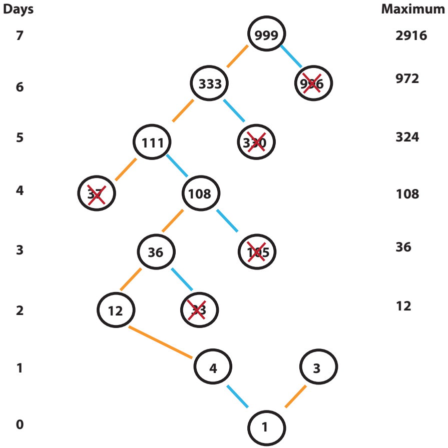


Santa's problem



Santa's problem

Dear Santa,
your drug schedule

- Day 1 no drug
- Day 2 magic drug
- Day 3 magic drug
- Day 4 magic drug
- Day 5 no drug
- Day 6 magic drug
- Day 7 magic drug

and you will have exactly 999 strands of beard.
Merry Christmas

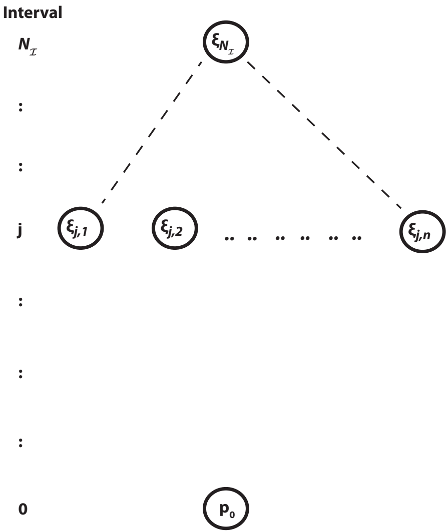
Optimal Control for Markov Process

- Discrete Dynamics

$$\begin{aligned} p_{j+1} &= T_{u_j} \cdot p_j \\ p(0) &= p_0 \end{aligned}$$

- Adjoint equation

$$\begin{aligned} \xi_j' &= \xi_{j+1}' \cdot T_{u_j} + q_{u_j,j}' \\ \xi_{N_\mathcal{I}}' &= q_{N_\mathcal{I}}' \end{aligned}$$



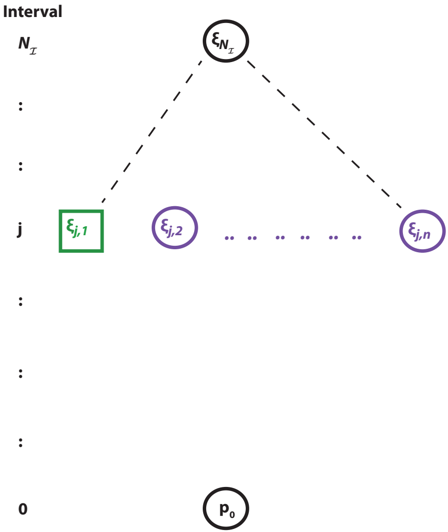
Optimal Control for Markov Process

- Discrete Dynamics

$$
\begin{aligned}
p_{j+1} &= T_{u_j} \cdot p_j \\
p(0) &= p_0
\end{aligned}
$$

- Adjoint equation

$$
\begin{aligned}
\xi'_j &= \xi'_{j+1} \cdot T_{u_j} + q'_{u_j,j} \\
\xi'_{N_\mathcal{I}} &= q'_{N_\mathcal{I}}
\end{aligned}
$$



Optimal Control for Markov Process

- Redundancy Test - Linear programing

$$\mu = \min_{\rho,\pi,\zeta} \begin{bmatrix} \xi'_{j,1} & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} \rho \\ \pi \\ \zeta \end{bmatrix}$$

$$\text{w.r.t} \begin{bmatrix} \xi'_{j,2} & 1 & -1 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \xi'_{j,n} & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} \rho \\ \pi \\ \zeta \end{bmatrix} \geq \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \tag{5}$$

$$\sum_i \rho[i] = 1$$

$$p_{min} \leq \rho \leq p_{max}$$

$$0 \leq \pi \leq J_{max}$$

$$J_{min} \leq \zeta \leq J_{max},$$

where p_{min},p_{max} and J_{min},J_{max} are lower and upper bounds for p^*_m and $J^*(p_0, u^*(p_0))$ respectively.

- if $\mu > 0$, the adjoint vector $\xi_{j,1}$ is redundant.

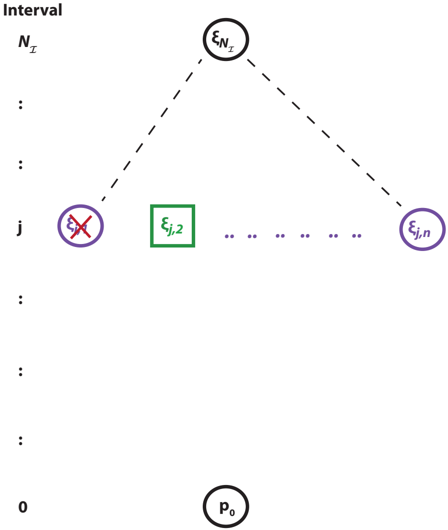
Optimal Control for Markov Process

- Discrete Dynamics

$$\begin{aligned} p_{j+1} &= T_{u_j} \cdot p_j \\ p(0) &= p_0 \end{aligned}$$

- Adjoint equation

$$\begin{aligned} \xi'_j &= \xi'_{j+1} \cdot T_{u_j} + q'_{u_j,j} \\ \xi'_{N_\mathcal{I}} &= q'_{N_\mathcal{I}} \end{aligned}$$



Optimal Control for Markov Process

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RESEARCH ARTICLE

Optimal Treatment Strategies in the Context of ‘Treatment for Prevention’ against HIV-1 in Resource-Poor Settings

Sulav Duwal[1,2], Stefanie Winkelmann[1], Christof Schütte[1,3], Max von Kleist[1,2]*

1 Department of Mathematics and Computer Science, Freie Universität Berlin, Germany, **2** Junior Research Group “Systems Pharmacology & Disease Control”, **3** Zuse Institute Berlin, Germany

Codes can be downloaded from https://github.com/SulavDuwal/OptimalTreatmentStrategies

@SulavDuwal

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