Intergenerational mobility measures in a bivariate normal model

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We model the joint log-income distribution of parents and children and derive analytic expressions for canonical relative and absolute intergenerational mobility measures. We find that both types of mobility measures can be expressed as a function of the other.

For the past several decades, many scholars have been studying economic intergenerational mobility [1, 5, 7–10]. The motivation for studying mobility stems from its relationship to concepts like equality of opportunity [4, 13], the so-called "American Dream" [3, 6] and income inequality [2, 7]. Typically measures of income intergenerational mobility are divided into two categories: relative – quantifying the propensity of individuals to change their position in the income distribution, and absolute – quantifying their propensity to change their income in money terms. The aim this note is to introduce a simple model for the joint income distribution of parents and children and use it for explicitly deriving canonical measures of relative and absolute mobility measures.

Our starting point is a population of N parent-child pairs. We denote by Y_p^i and Y_c^i the incomes of the parent and the child (at the same age), respectively, for family $i = 1 \ldots N$. We assume the incomes are all positive and move to define the log-incomes $X_p^i = \log Y_p^i$ and $X_c^i = \log Y_c^i$.

The canonical measure of relative mobility is the elasticity of child income with respect to parent income, known as the intergenerational earnings elasticity (IGE) [4, 9, 11] and defined as the slope (β) of the linear regression

$$X_c = \alpha + \beta X_p + \epsilon \,, \qquad (.1)$$

where α is the regression intercept and ϵ is the error term.

We note that IGE is a measure of immobility rather than of mobility and the larger it is, the stronger the relationship between the parent and child income. Therefore, $1 - \beta$ can be used as a measure of mobility.

A standard approach to measure absolute intergenerational mobility, recently used in [3] for studying the trends in absolute mobility in the United States is to measure the fraction of children earning more than their parents, denoted by A:

$$A = \frac{\sum_{j=1}^{N} 1_{\left\{i:Y_c^i > Y_p^i\right\}}\left(Y_c^j\right)}{N} \,, \qquad (.2)$$

where $1_S\left(x\right)$ is the indicator function for a set S and argument x and $\left\{i : Y_c^i > Y_p^i\right\}$ is the set of children earning more than their parents.

Since the logarithmic function preserves order we also get,

$$A = \frac{\sum_{j=1}^{N} 1_{\left\{i:X_c^i > X_p^i\right\}}\left(X_c^j\right)}{N} \,. \qquad (.3)$$

One hypothetical sample of such distribution is presented in Fig. 1. It also depicts graphically how A and β are defined. The blue line is $y = x$, hence the rate of absolute mobility is defined as the fraction of parent-child pairs which are above it. The red line is the linear regression $y = \alpha + \beta x$, for which β is the IGE.

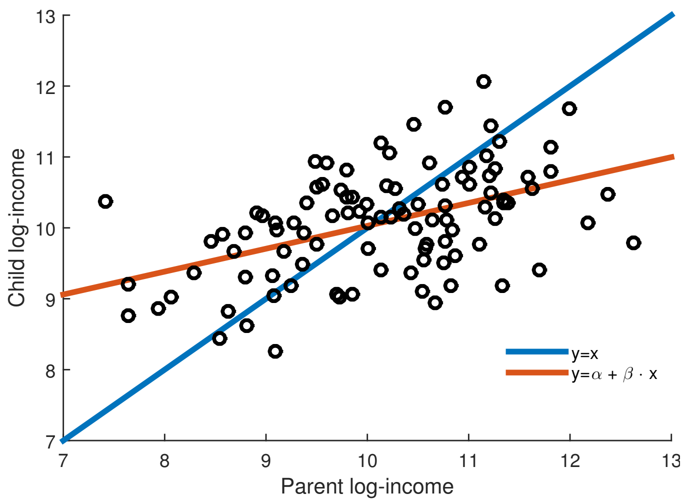

FIG. 1: An illustration of the absolute and relative mobility measures. The black circles are a randomly chosen sample of 100 parent-child log-income pairs. The sample was created assuming a bivariate normal distribution and the parameters used were $\mu_p = 10.1$, $\sigma_p = 0.78$ (for the parents marginal distribution) and $\mu_c = 10.25$, $\sigma_c = 1.15$ (for the children marginal distribution) with correlation of $\rho = 0.57$. The resulting α and β were 1.8 and 0.84, respectively.

Since income distributions are known to be well approximated by the log-normal distribution [12], a simple plausible model for the joint distribution of parent and child log-incomes is the bivariate normal distribution. Under this assumption, the marginal income distributions of both parents and children are log-normal and the correlation between their log-incomes is defined by a single parameter ρ. The marginal log-income distribution of the parents (children) follows $\mathcal{N}\left(\mu_p, \sigma_p^2\right)$ ($\mathcal{N}\left(\mu_c, \sigma_c^2\right)$), hence the joint distribution is fully characterized by 5 parameters: μ_p, σ_p, μ_c, σ_c and ρ.

Assuming the bivariate normal approximation for the joint distribution enables theoretically studying its properties. In particular, both measures of mobility, A and $1-\beta$, can be derived directly from the model and, notably, can both be expressed analytically as functions of the other. We first derive the IGE in terms of the distribution parameters:

Proposition 1 *For a bivariate normal distribution with parameters μ_p, σ_p (for the parents marginal distribution) and μ_c, σ_c (for the children marginal distribution) assuming correlation ρ, the IGE is*

$$1-\beta = 1-\frac{\sigma_c}{\sigma_p}\rho\,. \tag{.4}$$

Proof. First, by definition, the correlation ρ, between X_p and X_c equals to their covariance, divided by $\sigma_p\sigma_c$

$$\rho = \frac{\mathrm{Cov}\left[X_p, X_c\right]}{\sigma_p\sigma_c}\,. \tag{.5}$$

β can be directly calculated as follows, by the linear regression slope definition:

$$\beta = \frac{\sum_{i=1}^{N}\left(X_p^i-\bar{X}_p\right)\left(X_c^i-\bar{X}_c\right)}{\sum_{i=1}^{N}\left(X_p^i-\bar{X}_p\right)}\,, \tag{.6}$$

where $\bar{X}_p$ and $\bar{X}_c$ are the average parents and children log-incomes, respectively.

It follows that

$$\beta = \frac{\mathrm{Cov}\left[X_p, X_c\right]}{\sigma_p^2}\,. \tag{.7}$$

We immediately obtain

$$\beta = \frac{\sigma_c}{\sigma_p}\rho \tag{.8}$$

and therefore

$$1-\beta = 1-\frac{\sigma_c}{\sigma_p}\rho \tag{.9}$$

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Following Prop. 1 it is also possible to derive the rate of absolute mobility as a function of the distribution parameters and the IGE:

Proposition 2 *For a bivariate normal distribution with parameters μ_p, σ_p (for the parents marginal distribution), μ_c, σ_c (for the children marginal distribution) and $\rho = \sigma_p\beta/\sigma_c$ (where β is the IGE), the rate of absolute mobility is*

$$A = \Phi\left(\frac{\mu_c-\mu_p}{\sqrt{\sigma_p^2\left(1-2\beta\right)+\sigma_c^2}}\right)\,, \tag{.10}$$

where Φ is the cumulative distribution function of the standard normal distribution.

Proof. We start by defining a new random variable $Z = X_c - X_p$. It follows that calculating A is equivalent to calculating the probability $P\left(Z>0\right)$.

Subtracting two dependent normal distributions yields that $Z \sim \mathcal{N}\left(\mu_c-\mu_p, \sigma_p^2+\sigma_c^2-2\mathrm{Cov}\left[X_p, X_c\right]\right)$, so according to Prop. 1

$$Z \sim \mathcal{N}\left(\mu_c-\mu_p, \sigma_p^2\left(1-2\beta\right)+\sigma_c^2\right)\,. \tag{.11}$$

If follows that

$$\frac{Z-\left(\mu_c-\mu_p\right)}{\sqrt{\sigma_p^2\left(1-2\beta\right)+\sigma_c^2}} \sim \mathcal{N}\left(0,1\right)\,, \tag{.12}$$

so we can now write

$$\begin{aligned}
&P\left(Z>0\right) = \\
&P\left(\frac{Z-\left(\mu_c-\mu_p\right)}{\sqrt{\sigma_p^2\left(1-2\beta\right)+\sigma_c^2}} > -\frac{\mu_c-\mu_p}{\sqrt{\sigma_p^2\left(1-2\beta\right)+\sigma_c^2}}\right) = \\
&\Phi\left(\frac{\mu_c-\mu_p}{\sqrt{\sigma_p^2\left(1-2\beta\right)+\sigma_c^2}}\right)\,,
\end{aligned} \tag{.13}$$

where Φ is the cumulative distribution function of the standard normal distribution.

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Proposition 2 shows that the rate of absolute mobility can be explicitly described as a function of the relative mobility.

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