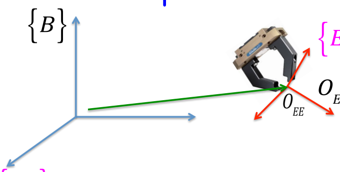
More on representation of orientation



$\{EE\}$;*End effector pose*

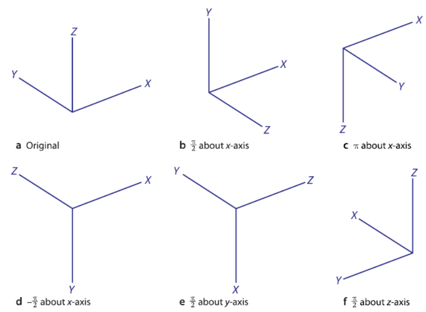
O_{EE} :*Operational* point

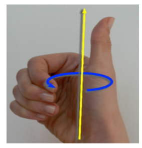
$\{EE\} \rightarrow Description\ {}^{B}_{EE}T$

Position $\rightarrow$ $vector(x,y,z)$

$$Orientation \rightarrow \begin{cases} Three\ \text{angles}\ (\alpha,\phi,\gamma) \\ Angle\ \text{vector}\ (\vec{K},\theta) \\ Euler\ paramenters \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \end{bmatrix} \end{cases}$$

Changing orientation





Direction of the increasing angle when rotating about a vector

Rotation representation

$$Rotation\ Matrix \rightarrow {}^{A}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}_{3x3} = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}$$

$$Direction\ Cosine \rightarrow x_r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}_{9x1}$$

Constrains

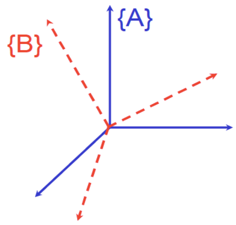
${}^{A}_{B}R_{3x3} \rightarrow 9\ parameters\ but\ they\ are\ not\ independent$

$$|r_1| = |r_2| = |r_3| = 1$$

$$r_1.r_2 = r_1.r_3 = r_3.r_2 = 0 \qquad r_1 \times r_2 = r_3; r_2 \times r_3 = r_1; r_3 \times r_1 = r_2$$

Three angles representation

$$\{A\} \rightarrow {}^{A}_{B}R_{(\alpha\beta\gamma)} \rightarrow \{B\}$$

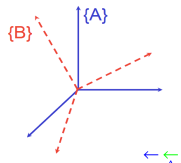


12+12 Possibles sequences

X Y Z	X Y X
X Z Y	X Z X
Y X Z	Y X Y
Y Z X	Y Z Y
Z X Y	Z X Z
Z Y X	Z Y Z

Note:
The order of multiplication is important. See appendix B. The 24 angle-set convention of John J. Craight book

Three angles representation

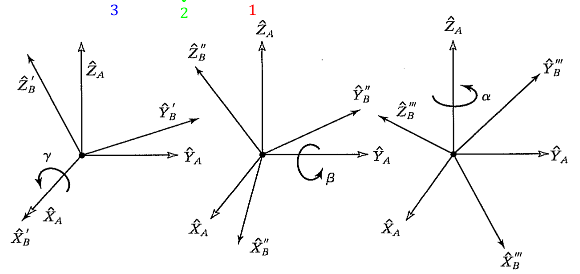


$${}^{A}_{B}R_{XYZ} = \overbrace{\underbrace{R_z(\alpha)}_{3}\underbrace{R_y(\beta)}_{2}\underbrace{R_x(\gamma)}_{1}}^{\leftarrow\leftarrow\leftarrow};\ (X-Y-Z)\ Fixed\ \text{angles}$$

$${}^{A}_{B}R_{Z'Y'X'} = \overbrace{\underbrace{R_z(\alpha)}_{1}\underbrace{R_y(\beta)}_{2}\underbrace{R_x(\gamma)}_{3}}^{\rightarrow\rightarrow\rightarrow};(Z-Y-X)\ \text{Moving frame}$$

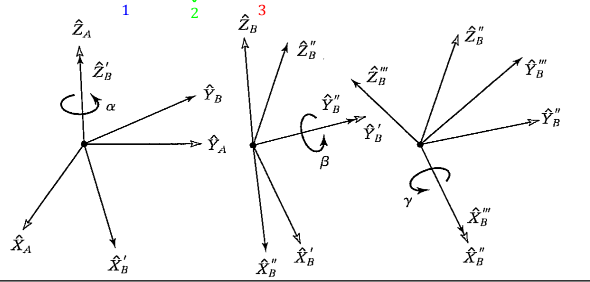
Three angles representation

$${}^{A}_{B}R_{XYZ} = \overbrace{\underbrace{R_z(\alpha)}_{3}\underbrace{R_y(\beta)}_{2}\underbrace{R_x(\gamma)}_{1}}^{\leftarrow\leftarrow\leftarrow};\ (X-Y-Z)\ Fixed\ \text{angles}$$



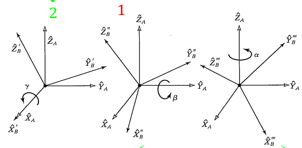
Three angles representation

$${}^{A}_{B}R_{Z'Y'X'} = \overbrace{\underbrace{R_z(\alpha)}_{1}\underbrace{R_y(\beta)}_{2}\underbrace{R_x(\gamma)}_{3}}^{\rightarrow\rightarrow\rightarrow};(Z-Y-X)\ \text{Moving frame}$$



Three angles representation

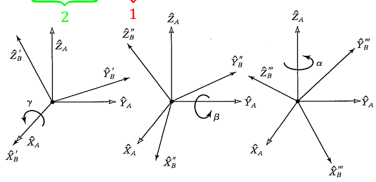
$${}^{A}_{B}R_{XYZ} = \overbrace{\underbrace{R_z(\alpha)}_{3}\underbrace{R_y(\beta)}_{2}\underbrace{R_x(\gamma)}_{1}}^{\leftarrow\leftarrow\leftarrow};\ (X-Y-Z)\ \textit{Fixed}\ \text{angles}$$



$${}^{A}_{B}R_{XYZ} = \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{pmatrix}$$

Three angles representation

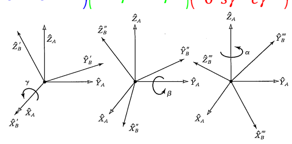
$${}^{A}_{B}R_{XYZ} = \overbrace{\underbrace{R_z(\alpha)}_{3}\underbrace{R_y(\beta)}_{2}\underbrace{R_x(\gamma)}_{1}}^{\leftarrow\leftarrow\leftarrow};\ (X-Y-Z)\ \textit{Fixed}\ \text{angles}$$



$${}^{A}_{B}R_{XYZ} = \begin{pmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma - c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{pmatrix}$$

Three angles representation

$${}^{A}_{B}R_{XYZ} = \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{pmatrix};\textit{Fixed frames}$$

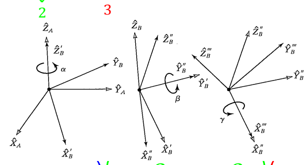


$$R_{XY} = \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{pmatrix} = \begin{pmatrix} c\beta & s\beta s\gamma & s\beta c\gamma \\ 00 & c\gamma & -s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{pmatrix}$$

$${}^{A}_{B}R_{XYZ} = \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c\beta & s\beta s\gamma & s\beta c\gamma \\ 00 & c\gamma & -s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{pmatrix} = \begin{pmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma - c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{pmatrix}$$

Three angles representation

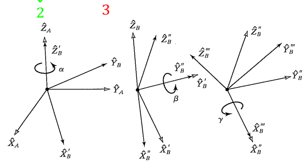
$${}^{A}_{B}R_{Z'Y'X'} = \overbrace{\underbrace{R_z(\alpha)}_{1}\underbrace{R_y(\beta)}_{2}\underbrace{R_x(\gamma)}_{3}}^{\rightarrow\rightarrow\rightarrow};(Z-Y-X)\ \text{Moving frame}$$



$${}^{A}_{B}R_{XYZ} = \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{pmatrix}$$

Three angles representation

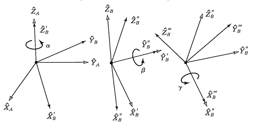
$${}^{A}_{B}R_{Z'Y'X'} = \overbrace{\underbrace{R_z(\alpha)}_{1}\underbrace{R_y(\beta)}_{2}\underbrace{R_x(\gamma)}_{3}}^{\rightarrow\rightarrow\rightarrow};(Z-Y-X)\ \text{Moving frame}$$



$${}^{A}_{B}R_{Z'Y'X'} = \begin{pmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma - c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{pmatrix}$$

Three angles representation

$$ {}^{A}_{B}R_{ZYX} = \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{pmatrix} \text{;Moving frames} $$



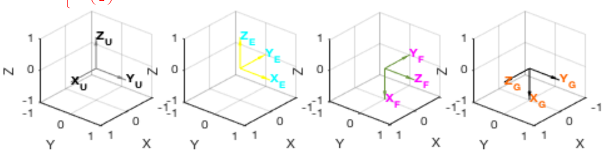
$$ R_{Z'Y'} = \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix} = \begin{pmatrix} c\alpha c\beta & -s\alpha & c\alpha s\beta \\ s\alpha c\beta & c\alpha & s\alpha s\beta \\ -s\beta & 00 & c\beta \end{pmatrix} $$

$$ {}^{A}_{B}R_{Z'Y'X'} = \begin{pmatrix} c\alpha c\beta & -s\alpha & c\alpha s\beta \\ s\alpha c\beta & c\alpha & s\alpha s\beta \\ -s\beta & 00 & c\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{pmatrix} = \begin{pmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma - c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{pmatrix} $$

Three angles representation: example

$$ {}^{A}_{B}R_{Z'Y'X'} = \overbrace{\underbrace{R_z(\alpha)}_{1}\underbrace{R_y(\beta)}_{2}\underbrace{R_x(\gamma)}_{3}}^{\rightarrow\rightarrow\rightarrow};(Z-Y-X) \text{ Moving frame} $$

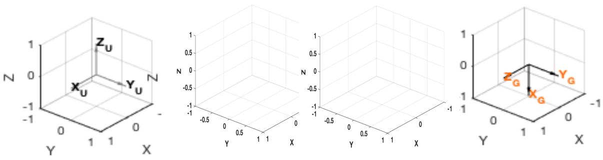
$$ \alpha=\beta=\gamma=\begin{cases} \cos\left(\frac{\pi}{2}\right)=0 \\ \sin\left(\frac{\pi}{2}\right)=1 \end{cases} $$



$$ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{Rotz(pi/2)} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{Roty(pi/2)} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}_{Rotx(pi/2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} $$

Three angles representation: example

$$ {}^{A}_{B}R_{XYZ} = \overbrace{\underbrace{R_z(\alpha)}_{3}\underbrace{R_y(\beta)}_{2}\underbrace{R_x(\gamma)}_{1}}^{\leftarrow\leftarrow\leftarrow};\ (X-Y-Z) \text{ } \textit{Fixed} \text{ angles} $$



Note: Do the same exercise and give a interpretation of the results

Three angles, inverse problem

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix};\ {}^{A}_{B}R_{XYZ} = \begin{pmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma - c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{pmatrix} $$

How to extrac X-Y-Z fixed angles

There are 9 equation (6 dependencies) and three unkonw

$$ \beta = \mathrm{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2});\ \left\{ \begin{array}{c} r_{31} = -sen\beta \\ \sqrt{r_{11}^2 + r_{21}^2} = \cos\beta \end{array} \right\} $$

$$ \alpha = \mathrm{Atan2}(\tfrac{r_{21}}{\cos\beta}, \tfrac{r_{11}}{\cos\beta});\ \left\{ \begin{array}{c} \frac{-r_{21}}{\cos\beta} = sen\alpha \\ \frac{r_{11}}{\cos\beta} = \cos\alpha \end{array} \right\} $$

$$ \gamma = \mathrm{Atan2}(\tfrac{r_{32}}{\cos\beta}, \tfrac{r_{33}}{\cos\beta});\ \left\{ \begin{array}{c} \frac{r_{32}}{\cos\beta} = sen\gamma \\ \frac{r_{33}}{\cos\beta} = \cos\gamma \end{array} \right\} $$

if $c\beta=0 \rightarrow (\beta=\pm\frac{\pi}{2}) \rightarrow$ *Saingularity of the representation*

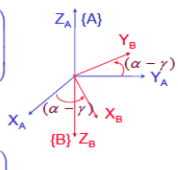
Only $(\alpha+\gamma)$ *or* $(\alpha-\gamma)$ *is defined*

Three angles, inverse problem

$$ {}^{A}_{B}R_{XYZ} = \begin{pmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma - c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{pmatrix} $$

$$ c\beta = 0,\ s\beta = +1 $$

$$ {}^{A}_{B}R = \begin{pmatrix} 0 & -s(\alpha-\gamma) & c(\alpha-\gamma) \\ 0 & c(\alpha-\gamma) & s(\alpha-\gamma) \\ -1 & 0 & 0 \end{pmatrix} $$

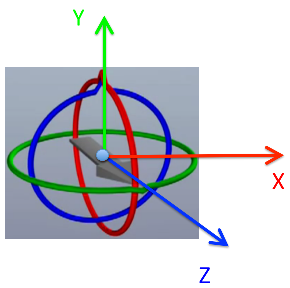


$$ c\beta = 0,\ s\beta = -1 $$

$$ {}^{A}_{B}R = \begin{pmatrix} 0 & -s(\alpha+\gamma) & -c(\alpha+\gamma) \\ 0 & c(\alpha+\gamma) & -s(\alpha+\gamma) \\ 1 & 0 & 0 \end{pmatrix} $$

Singularity

Singularity of the representation



X Y Z

X Z Y

Y X Z

Y Z X

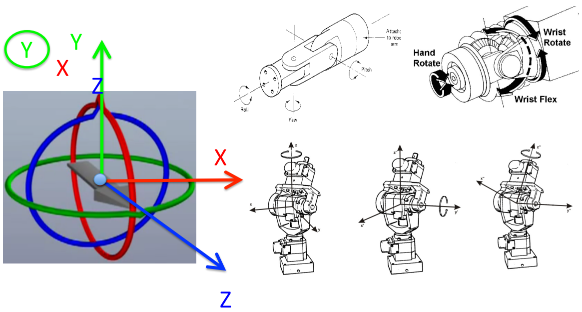
Z X Y

Z Y X

The broad arrow is attached to Z

Sngularity of the representation

Y X Z



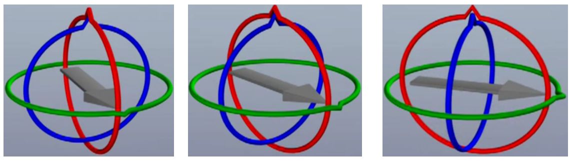
The broad arrow is attached to Z

Singularity of the representation

Y X Z

Notice the hierarchy and dependences

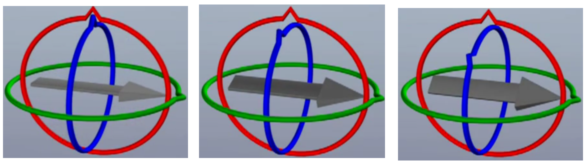
Rotation around Y: everything moves (X and Z) because it is on top of the hierarchy (outer ring)



Singularity of the representation

Y X Z

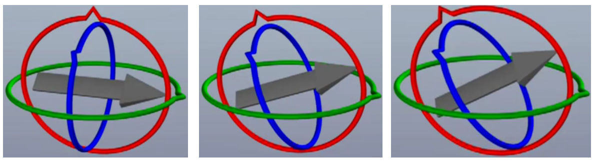
Rotation around Z(inner ring): it only rotates the the broad arrow about its axis.



Singularity of the representation

Y X Z

Rotation around X

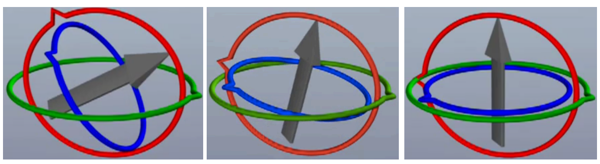


Singularity of the representation

Y X Z

Guimbal lock

Occurs when the first rotation and the last rotation axes (Y and Z in this case) are aligned.

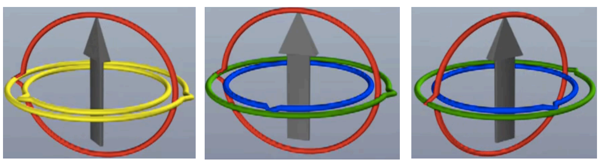


Singularity of the representation

Y X Z

Guimbal lock

The rotations around Y and Z are redundant.
One degree of freedom is lost.

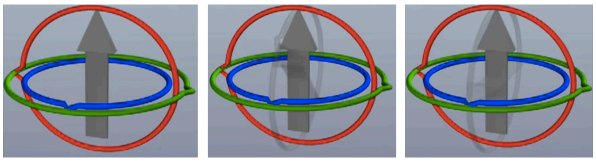


Singularity of the representation

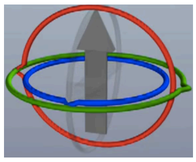
Y X Z

Guimbal lock: Y aligned whith Z

It is impossible to point the arrow down



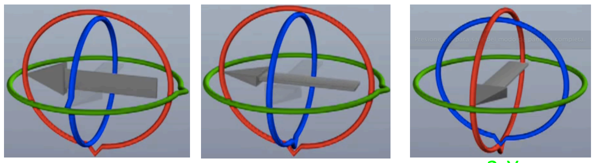
Singularity of the representation



Guimbal unlock Sequence to point down the arrow

Remeber the hierarchy

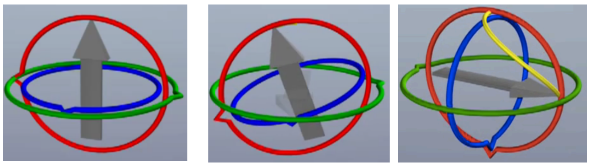
Y X Z



1-X 2-Z 3-Y

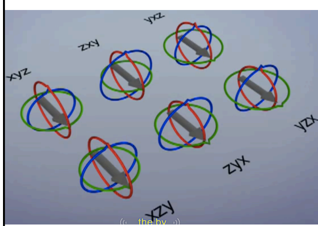
Singularity of the representation

Guimbal unlock: Y-X-Z simultaneous movements to point the arrow down

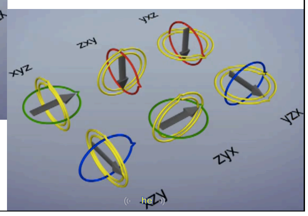


X Z Y

Singularity of the representation



Guimbal locks occur in all possible sequences of Euler Angles



In Mathematical term…

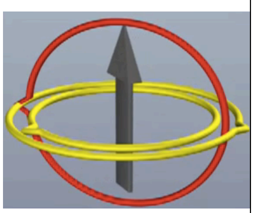
$$R_{zyx}=\begin{bmatrix}1&0&0\\0&c\alpha&-s\alpha\\0&s\alpha&c\alpha\end{bmatrix}\begin{bmatrix}c\beta&0&s\beta\\0&1&0\\-s\beta&0&c\beta\end{bmatrix}\begin{bmatrix}c\gamma&-s\gamma&0\\s\gamma&c\gamma&0\\0&0&1\end{bmatrix} if\ \beta=\pi/2$$

$$R_{zyx}=\begin{bmatrix}1&0&0\\0&c\alpha&-s\alpha\\0&s\alpha&c\alpha\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}\begin{bmatrix}c\gamma&-s\gamma&0\\s\gamma&c\gamma&0\\0&0&1\end{bmatrix}$$

$$R_{zyx}=\begin{bmatrix}0&0&1\\s\alpha&c\alpha&0\\-c\alpha&s\alpha&0\end{bmatrix}\begin{bmatrix}c\gamma&-s\gamma&0\\s\gamma&c\gamma&0\\0&0&1\end{bmatrix}$$

$$R_{zyx}=\begin{bmatrix}0&0&1\\s\alpha c\gamma+c\alpha s\gamma&-s\alpha s\gamma+c\alpha c\gamma&0\\-c\alpha c\gamma+s\alpha s\gamma&c\alpha s\gamma+s\alpha c\gamma&0\end{bmatrix}$$

$$R_{zyx}=\begin{bmatrix}0&0&1\\s(\alpha+\gamma)&c(\alpha+\gamma)&0\\-c(\alpha+\gamma)&-s(\alpha+\gamma)&0\end{bmatrix}$$



Changing the values of α and γ has the same effects.
The rotation axis remains in Z direction.
First row and last column will not change.
We are loosing one degree of freedom.

Rotating around and arbitrary axis

$Av=v; A\ is\ a\ Rotation\ Matrix\ throught\ some\ angle\ about\ some\ vector\ v.$

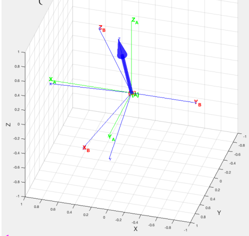
$$Av=v=A^Tv\rightarrow\underbrace{\left(A-A^T\right)}_{Q}v=0\rightarrow Qv=0$$

$$(A-\lambda I)v=0\rightarrow(A-\lambda I)=0;\ \lambda=1\rightarrow(A-I)v=0\rightarrow A=\frac{1}{2}(S+Q)\begin{cases}Q=\left(A-A^T\right)\\S=\left(A+A^T\right)\end{cases}$$

$$Q=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&\end{pmatrix}-\begin{pmatrix}a_{11}&a_{21}&a_{31}\\a_{12}&a_{22}&a_{32}\\a_{13}&a_{23}&a_{33}\end{pmatrix}$$

$$Qv=\begin{pmatrix}0&q_3&-q_2\\-q_3&0&q_1\\q_2&-q_1&0\end{pmatrix}\begin{pmatrix}v_x\\v_y\\v_z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix};\left\{\begin{matrix}q_1=a_{23}-a_{32}\\q_2=a_{31}-a_{13}\\q_3=a_{12}-a_{21}\end{matrix}\right\}$$

$$\rightarrow\begin{cases}let\ v_z=1\quad v_x=\dfrac{q_1}{q_3}\quad v_y=\dfrac{q_2}{q_3}\\ if\ \ A\ \mathrm{Rotz}\rightarrow\underbrace{a_{11}}_{c\theta}+\underbrace{a_{22}}_{c\theta}+\underbrace{a_{33}}_{1}=2c\theta+1\rightarrow c\theta=\dfrac{1}{2}Trace(A)-1\end{cases}$$

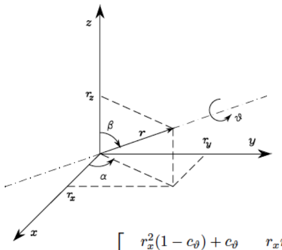


Rotating around and arbitrary axis

$$R(\vartheta, r) = R_z(\alpha) R_y(\beta) R_z(\vartheta) R_y(-\beta) R_z(-\alpha)$$

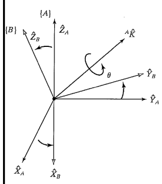
$$\sin\alpha = \frac{r_y}{\sqrt{r_x^2 + r_y^2}} \qquad \cos\alpha = \frac{r_x}{\sqrt{r_x^2 + r_y^2}}$$

$$\sin\beta = \sqrt{r_x^2 + r_y^2} \qquad \cos\beta = r_z.$$



$$R(\vartheta, r) = \begin{bmatrix} r_x^2(1-c_\vartheta) + c_\vartheta & r_x r_y(1-c_\vartheta) - r_z s_\vartheta & r_x r_z(1-c_\vartheta) + r_y s_\vartheta \\ r_x r_y(1-c_\vartheta) + r_z s_\vartheta & r_y^2(1-c_\vartheta) + c_\vartheta & r_y r_z(1-c_\vartheta) - r_x s_\vartheta \\ r_x r_z(1-c_\vartheta) - r_y s_\vartheta & r_y r_z(1-c_\vartheta) + r_x s_\vartheta & r_z^2(1-c_\vartheta) + c_\vartheta \end{bmatrix}.$$

Rotating around and arbitrary axis



$$C = \begin{pmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

C is a transformation matrix such:

$\hat{k}$ axis is the z vector of C

Thats means: C brings $\hat{Z}_A$ aligned to $\hat{k}$

Rotating around $\hat{k} \rightarrow Rot(\hat{k},\theta) = C Rot(\hat{z},\theta) C^{-1}$;

(once rotated we need to reverse the operation)

$$Rot(\hat{k},\theta) = \begin{pmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_x & n_y & n_z & 0 \\ o_x & o_y & o_z & 0 \\ a_x & a_y & a_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

Consider:

1.- dot produc of any column or row with any row o column is 'zero'

2.- dot produc of any column or row with itseft is 'one'

3.- cross product: $a = n \otimes o$;

$a_x = n_y o_z - n_z o_y$

$a_y = n_z o_y - n_y o_z$

$a_z = n_x o_y - n_y o_x$

4.- The versine $\rightarrow vers\theta = (1 - \cos\theta)$

5.- The $\hat{k}$ vector

$k_x = a_x$

$k_y = a_x$

$k_y = a_z$

Rotating around and arbitrary axis

$$Rot(\hat{k},\theta) = \begin{pmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta & 0 \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta & 0 \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Check that rotating around: $X \rightarrow \begin{cases} k_x = 1 \\ k_y = 0 \\ k_z = 0 \end{cases}; Y \rightarrow \begin{cases} k_x = 0 \\ k_y = 1 \\ k_z = 0 \end{cases}; Z \rightarrow \begin{cases} k_x = 0 \\ k_y = 0 \\ k_z = 1 \end{cases}$

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}; R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}; R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotating around and arbitrary axis

What are the vector and rotation angle given: $C = \begin{pmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta & 0 \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta & 0 \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1.– Equating the diagonal

$n_x + o_y + a_z + 1 = k_x^2 v\theta + c\theta + k_y^2 v\theta + c\theta + k_z^2 v\theta + c\theta + 1$

$n_x + o_y + a_z = \left(k_x^2 + k_y^2 + k_z^2\right)(1 - c\theta) + 3c\theta = 1 + 2c\theta$

$c\theta = \frac{n_x + o_y + a_z - 1}{2};$

2.– Diference of off diagonal terms

$o_z - a_y = 2k_x s\theta \rightarrow k_x = \frac{o_z - a_y}{2s\theta}$

$a_x - n_z = 2k_y s\theta \rightarrow k_y = \frac{a_x - n_z}{2s\theta}$

$n_y - o_x = 2k_x s\theta \rightarrow k_z = \frac{n_y - o_x}{2s\theta}$

2*bis*.– From diference of off diagonal terms

$\left(o_z - a_y\right)^2 + \left(a_y - n_z\right)^2 + \left(n_y - o\right)^2 = 4s^2\theta$

$s\theta = \pm\frac{1}{2}\sqrt{\left(o_z - a_y\right)^2 + \left(a_y - n_z\right)^2 + \left(n_y - o\right)^2}$

Singularity for $\sin\theta = 0$

Euler parameters

$$\left.\begin{aligned} k_x &= \frac{o_z - a_y}{2s\theta} \\ k_y &= \frac{a_x - n_z}{2s\theta} \\ k_z &= \frac{n_y - o_x}{2s\theta} \end{aligned}\right\} {}^A K = \frac{1}{2s\theta}\begin{bmatrix} o_z - a_y \\ a_x - n_z \\ n_y - o_x \end{bmatrix}; \theta = \cos^{-1}\left(\frac{n_x + o_y + a_z - 1}{2}\right)$$

Angle/vector

$$\left.\begin{aligned} \varepsilon_1 &= k_x \sin\frac{\theta}{2} \\ \varepsilon_2 &= k_y \sin\frac{\theta}{2} \\ \varepsilon_3 &= k_z \sin\frac{\theta}{2} \\ \varepsilon_4 &= \cos\frac{\theta}{2} \end{aligned}\right\} \rightarrow Q = \left(\vec{\varepsilon} \;,\; \varepsilon_4\right) \rightarrow |Q| = \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1$$

Euler parameters

Euler parameters: Inverse Problem

Given a ${}^A_B R$ *find the Euler parameters*

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \equiv \begin{bmatrix} 1 - 2\varepsilon_2^2 - 2\varepsilon_3^2 & 2(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) & 1 - 2\varepsilon_1^2 - 2\varepsilon_3^2 & 2(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4) & 1 - 2\varepsilon_1^2 - 2\varepsilon_2^2 \end{bmatrix}$$

$$r_{11} + r_{22} + r_{33} = 3 - 4(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)$$

$$(1 - \varepsilon_4^2)$$

$$\varepsilon_4 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$

$$\varepsilon_1 = \frac{r_{32} - r_{23}}{4\varepsilon_4}, \quad \varepsilon_2 = \frac{r_{13} - r_{31}}{4\varepsilon_4}, \quad \varepsilon_3 = \frac{r_{21} - r_{12}}{4\varepsilon_4}$$

$\varepsilon_4 = 0?$

Robotics Tool Box RTB Overvie: