Jacobian

- Differential Motion
- Lineal & Angular Motion
- Velocity Propagation
- Kinematic Singularity
- Statics Forces

Differential Motion

Forward Kinematics

$\theta's_{Given} \rightarrow x_{Pose}$

Intantaneous Kinematics

$\theta + \delta\theta \rightarrow x + \delta x$

What are the relationship: $\boxed{\delta\theta \leftrightarrow \delta x}$

$$\boxed{\dot{\theta} \leftrightarrow \dot{x}} \left\{ \begin{array}{l} \textit{Lineal Velocity} \\ \textit{Angular velocity} \end{array} \right.$$

Differential Motion

Small angles

$$Rot(X,\delta_x) \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta_x & 0 \\ 0 & \delta_x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; Rot(Y,\delta_y) \approx \begin{bmatrix} 1 & 0 & \delta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\delta_y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; Rot(Z,\delta_z) \approx \begin{bmatrix} 1 & -\delta_z & 0 & 0 \\ \delta_z & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; Trans(dx,dy,dz) = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Motion due to a robot's joints moving during Δt

$$T + \Delta T = R_{\vec{K}}(\theta) transl\left(d_x, d_y, d_z\right) T; T \text{ original frame}$$

$$T + \Delta T \approx \begin{bmatrix} 1 & -\delta_z & \delta_y & d_x \\ \delta_z & 1 & -\delta_x & d_y \\ -\delta_y & \delta_x & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} T$$

Angular differential Motion

$$\left. \begin{array}{l} \dot{R} = \lim_{\Delta t \to 0} \dfrac{R(t+\Delta t) - R(t)}{\Delta t} \\ R(t+\Delta t) = R_{\hat{K}}(\Delta\theta) R(t) \end{array} \right\} \textit{Differentiation of a rotation matrix}$$

$$\textit{Considering } Rot(\hat{k},\theta)_{\Delta\theta \approx Small} = \begin{pmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta & 0 \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta & 0 \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\Delta\theta \approx Small}$$

$$S = \dot{R}R^{-1} = \begin{pmatrix} 0 & -k_z\dot{\theta} & k_y\dot{\theta} \\ k_z\dot{\theta} & 0 & -k_x\dot{\theta} \\ -k_y\dot{\theta} & k_x\dot{\theta} & 0 \end{pmatrix} \rightarrow \textit{Skew symetric Matrix}; \Omega = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \begin{bmatrix} k_x\dot{\theta} \\ k_y\dot{\theta} \\ k_z\dot{\theta} \end{bmatrix} = \dot{\theta}\hat{K}$$

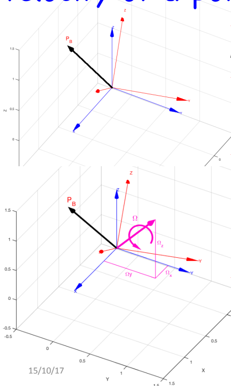
The change in orientation of a rotating frame occurs about some axis $\hat{K}$ *at rate* $\dot{\theta}$

Deduction of S

For any orthogonal Matrix : $RR^T = I$

Differentiating : $\dot{R}R^T + R\dot{R}^T = 0 \rightarrow S + S^T = 0 \rightarrow$ *S must be a Skew Symetric Matrix*

Velocity of a point if the RF is rotating



$${}^AP = {}^A_BR\,{}^BP \rightarrow {}^A\dot{P} = {}^A_B\dot{R}\,{}^BP$$

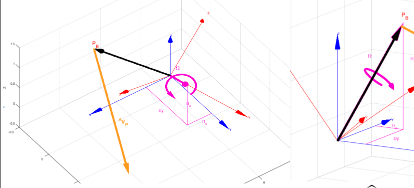
So

$${}^AV_P = {}^A_B\dot{R}\,{}^BP = {}^A_B\dot{R}\underbrace{{}^A_BR^T\,{}^AP}_{{}^BP} = \underbrace{{}^A_BS\,{}^AP = \Omega \times {}^AP = \widehat{\Omega}\,{}^AP}_{\textit{Cross Product}}$$

$${}^AV_P = \underbrace{\begin{pmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{pmatrix}}_{\substack{\Omega\times \equiv \widehat{\Omega} \\ \textit{Matrix form}}} \begin{bmatrix} {}^AP_x \\ {}^AP_y \\ {}^AP_z \end{bmatrix}$$

$${}^AV_P = \underbrace{\begin{bmatrix} \Omega_y\,{}^AP_z - \Omega_z\,{}^AP_y \\ \Omega_z\,{}^AP_x - \Omega_x\,{}^AP_z \\ \Omega_x\,{}^AP_y - \Omega_y\,{}^AP_x \end{bmatrix}}_{\textit{Cross product}} = \underbrace{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \Omega_x & \Omega_y & \Omega_z \\ {}^AP_x & {}^AP_y & {}^AP_z \end{vmatrix}}_{\textit{Cross product}}$$

Velocity of a point if the RF is rotating



$${}^AV_P \perp {}^BP \;\&\; {}^AV_P \perp \Omega$$

$${}^AV_P = {}^A_B\dot{R}\,{}^BP = {}^A_B\dot{R}\underbrace{{}^A_BR^T\,{}^AP}_{{}^BP} = \underbrace{{}^A_BS\,{}^AP = \Omega \times {}^AP = \widehat{\Omega}\,{}^AP}_{\textit{Cross Product}}$$

$${}^AV_P = \underbrace{\begin{pmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{pmatrix}}_{\substack{\Omega\times \equiv \widehat{\Omega} \\ \textit{Matrix form}}} \begin{bmatrix} {}^AP_x \\ {}^AP_y \\ {}^AP_z \end{bmatrix} = \underbrace{\begin{bmatrix} \Omega_y\,{}^AP_z - \Omega_z\,{}^AP_y \\ \Omega_z\,{}^AP_x - \Omega_x\,{}^AP_z \\ \Omega_x\,{}^AP_y - \Omega_y\,{}^AP_x \end{bmatrix}}_{\textit{Cross product}}$$

Differential Motion

Joint Coordinates

$$coordinate - i : \begin{cases} \theta_i & \text{revolute} \\ d_i & \text{prismatic} \end{cases}$$

Joint coordinate-i: $\boxed{q_i = \bar{\varepsilon}_i \theta_i + \varepsilon_i d_i}$

with $\varepsilon_i = \begin{cases} 0 & \text{revolute} \\ 1 & \text{prismatic} \end{cases}$

and $\bar{\varepsilon}_i = 1 - \varepsilon_i$

Joint Coordinate Vector: $\boxed{q = (q_1 q_2 \ldots q_n)^T}$

Jacobian

Jacobians: Direct Differentiation

$$x = f(q); \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} f_1(q) \\ f_2(q) \\ \vdots \\ f_m(q) \end{pmatrix}$$

$$\delta x_1 = \frac{\partial f_1}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f_1}{\partial q_n}\delta q_n$$

$$\vdots$$

$$\delta x_m = \frac{\partial f_m}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f_m}{\partial q_n}\delta q_n$$

$$\delta x = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial q_1} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix} . \delta q$$

$$\boxed{\delta x_{(m\times 1)} = J_{(m\times n)}(q)\, \delta q_{(n\times 1)}}$$

Jacobian

$$\delta x_{(m\times 1)} = J_{(m\times n)}(q)\, \delta q_{(n\times 1)}$$

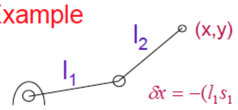
$$\dot{x}_{(m\times 1)} = J_{(m\times n)}(q)\, \dot{q}_{(n\times 1)}$$

where

$$J_{ij}(q) = \frac{\partial}{\partial q_j} f_i(q)$$

Jacobian

Example



$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

$$\delta x = -(l_1 s_1 + l_2 s_{12})\delta\theta_1 - l_2 s_{12}\delta\theta_2$$

$$\delta y = (l_1 c_1 + l_2 c_{12})\delta\theta_1 + l_2 c_{12}\delta\theta_2$$

$$\delta X = \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} -y & -l_2 s_{12} \\ x & l_2 c_{12} \end{bmatrix} \begin{pmatrix} \delta\theta_1 \\ \delta\theta_2 \end{pmatrix}$$

$$\boxed{\delta x = J(\theta)\delta\theta}$$

$$\dot{x} = J(\theta)\dot{\theta}$$

$$J \equiv \begin{pmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} \end{pmatrix} = \begin{bmatrix} -y & -l_2 s_{12} \\ x & l_2 c_{12} \end{bmatrix}$$

Jacobian repreasentations

$$X = \begin{bmatrix} x_P \\ x_R \end{bmatrix}$$

- Cartesian
- Spherical
- Cylindrical
- ….

- Euler Angles
- Direction Cosines
- Euler Parameters

Jacobian

Jacobian for X

$$\dot{x}_P = J_{X_P}(q)\dot{q}$$

$$\dot{x}_R = J_{X_R}(q)\dot{q}$$

$$\begin{pmatrix} \dot{x}_P \\ \dot{x}_R \end{pmatrix} = \begin{pmatrix} J_{X_P}(q) \\ J_{X_R}(q) \end{pmatrix} \dot{q}$$

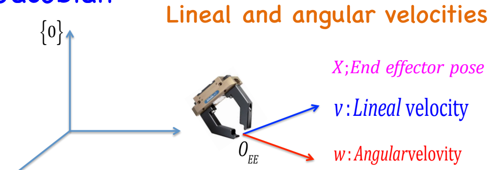
Cartesian & Direction Cosines

$$\dot{X}_{(12x1)} = J_X(q)_{(12x6)}\, \dot{q}_{(6x1)}$$

The Jacobian is dependent on the representation

Jacobian

Lineal and angular velocities



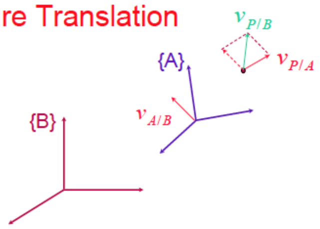
O_{EE} :Operational point

$$\dot{X} = \begin{pmatrix} v \\ w \end{pmatrix}_{6x1} = J(q)_{6xn}\,\dot{q}_{nx1}$$

$$v = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}; w = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}; \dot{q}_{n\times1} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times1}$$

Lineal and angular velocities

Pure Translation

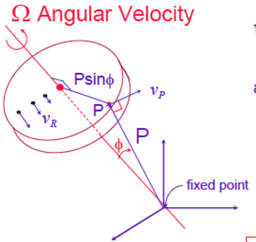


$$v_{P/B} = v_{A/B} + v_{P/A}$$

Lineal and angular velocities

Rotational Motion

Ω Angular Velocity



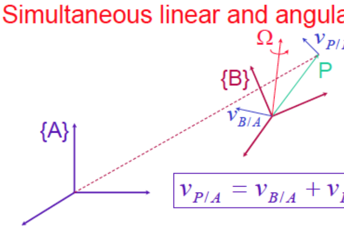
v_P is proportional to:
- ||Ω||
- ||Psinϕ||

and
- $v_P \perp \Omega$
- $v_P \perp P$

$$\boxed{v_P = \Omega \times P}$$

Lineal and angular velocities

Simultaneous linear and angular motion



$$\boxed{v_{P/A} = v_{B/A} + v_{P/B} + \Omega \times P_B}$$

$${}^{A}v_{P/A} = {}^{A}v_{B/A} + {}^{A}_{B}R.{}^{B}v_{P/B} + {}^{A}\Omega_B \times {}^{A}_{B}R.{}^{B}P_B$$

Velocity propagation

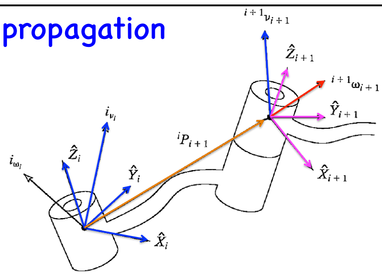
Spatial Mechanisms

Propagation of velocities

$\dot{x}$ < v : linear velocity; ω : angular velocity

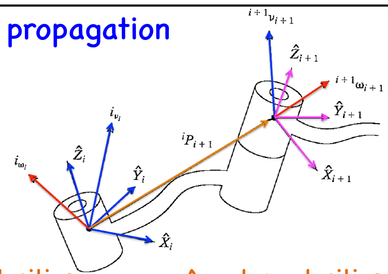
$$\dot{x} = J(\theta).\dot{\theta}$$

Velocity propagation



We can compute the velocity of each link in order, starting from the base.
The velocity of link i + 1 will be that of link i, plus whatever new velocity components were added by joint i + 1

Velocity propagation



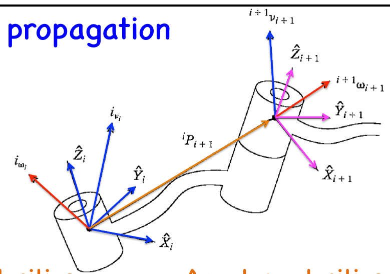
Lineal velocities

$$^{i}v_{i+1} = {}^{i}v_i + {}^{i}w_i \times {}^{i}P_{i+1}$$
$$^{i+1}_{i}R\,{}^{i}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_i + {}^{i}w_i \times {}^{i}P_{i+1}\right)$$
$$^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_i + {}^{i}w_i \times {}^{i}P_{i+1}\right)$$

Angular velocities

$$^{i}w_{i+1} = {}^{i}w_i + {}_{1+1}^{\;\;i}R\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{1+1}; \quad \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{1+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}$$
$$^{i+1}_{i}R\,{}^{i}w_{i+1} = {}^{i+1}_{i}R\,{}^{i}w_i + {}^{i+1}_{i}R\,{}_{i+1}^{\;\;i}R\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{1+1}$$
$$^{i+1}w_{i+1} = {}^{i+1}_{i}R\,{}^{i}w_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{1+1}$$

Velocity propagation



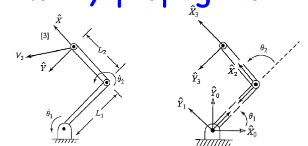
Lineal velocities **Angular velocities**

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_i + {}^{i}w_i \times {}^{i}P_{i+1}\right) \qquad {}^{i+1}w_{i+1} = {}^{i+1}_{i}R\,{}^{i}w_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{1+1}$$

$$\underbrace{\begin{bmatrix} ^{B}v_B \\ ^{B}w^{-}_{B} \end{bmatrix}}_{^{B}v_B} = \underbrace{\begin{bmatrix} ^{B}_{A}R & -{}^{B}_{A}R\,{}^{A}P_{BORG}\times \\ 0 & ^{B}_{A}R \end{bmatrix}}_{\substack{^{B}_{A}T_v;\,Tranformation: A\to B \\ Here\ \dot{\theta}_{i+1}=0}} \underbrace{\begin{bmatrix} ^{A}v_A \\ ^{A}w_A \end{bmatrix}}_{^{B}v_B}$$

$$\underbrace{\begin{bmatrix} ^{A}v_A \\ ^{A}w^{-}_{A} \end{bmatrix}}_{^{A}v_A} = \underbrace{\begin{bmatrix} ^{A}_{B}R & ^{A}P_{BORG}\times {}^{B}_{A}R^{A} \\ 0 & ^{A}_{B}R \end{bmatrix}}_{\substack{^{A}_{B}T_v;\,Tranformation: B\to A \\ Here\ \dot{\theta}_{i+1}=0}} \underbrace{\begin{bmatrix} ^{B}v_B \\ ^{B}w_B \end{bmatrix}}_{^{B}v_B}$$

Velocity propagation: Example 1



$$^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
$$^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
$$^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$^{1}\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \qquad {}^{1}\upsilon_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

Use:

$$\begin{bmatrix} ^{B}v_B \\ ^{B}w^{-}_{B} \end{bmatrix} = \begin{bmatrix} ^{B}_{A}R & -{}^{B}_{A}R\,{}^{A}P_{BORG}\times \\ 0 & ^{B}_{A}R \end{bmatrix} \begin{bmatrix} ^{A}v_A \\ ^{A}w_A \end{bmatrix}$$

$$^{0}_{3}R = {}^{0}_{1}R\;{}^{1}_{2}R\;{}^{2}_{3}R = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$^{2}\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix},$$

$$^{2}\upsilon_2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ l_1\dot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 \\ 0 \end{bmatrix},$$

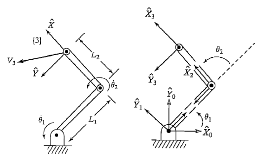
$$^{0}\upsilon_3 = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}.$$

$$^{3}\omega_3 = {}^{2}\omega_2,$$

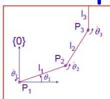
$$^{3}\upsilon_3 = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}. \longrightarrow {}^{3}J(\Theta) = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix},$$

$$^{0}\upsilon_3 = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}. \longrightarrow {}^{0}J(\Theta) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}.$$

Velocity propagation: Example 2



$$v_{i+1} = v_i + \omega_i \times P_{i+1}$$



- $v_{P_1} = 0$ $\quad {}^{0}\omega_1 = \dot{\theta}_1.{}^{0}Z_1$
- $v_{P_2} = v_{P_1} + \omega_1 \times P_2$
- $v_{P_3} = v_{P_2} + \omega_2 \times P_3$

$$^{0}v_{P_2} = 0 + \begin{bmatrix} 0 & -\dot{\theta}_1 & 0 \\ \dot{\theta}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} l_1.c_1 \\ l_1.s_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1.s_1 \\ l_1.c_1 \\ 0 \end{bmatrix} . \dot{\theta}_1$$

$$^{0}v_{P_3} = {}^{0}v_{P_2} + {}^{0}\omega_2 \times {}^{0}P_3$$

$$^{0}v_{P_3} = \begin{bmatrix} -l_1.s_1 \\ l_1.c_1 \\ 0 \end{bmatrix} . \dot{\theta}_1 + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . (\dot{\theta}_1 + \dot{\theta}_2) . {}^{0}P_3 \qquad \begin{bmatrix} l_2.c_{12} \\ l_2.s_{12} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -l_1.s_1 \\ l_1.c_1 \\ 0 \end{bmatrix} . \dot{\theta}_1 + \begin{bmatrix} -l_2.s_{12} \\ l_2.c_{12} \\ 0 \end{bmatrix} . (\dot{\theta}_1 + \dot{\theta}_2)$$

$$^{0}\omega_3 = (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) . {}^{0}Z_0$$

$$^{0}v_{P_3} = \underbrace{\begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{J_v} . \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

$$^{0}\omega_3 = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{J_\omega} . \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = J . \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix}$$

Changing Jacobian's Reference Frames

$$^{0}\upsilon = \begin{bmatrix} ^{0}v \\ ^{0}w \end{bmatrix} = \underbrace{{}^{0}J(\theta)}_{At\ EE\ pose}\dot{\theta};\ the\ resulting\ Cartesian\ velocity\ will\ be\ in\ \{0\}$$

$$^{B}\upsilon = \begin{bmatrix} ^{B}v \\ ^{B}w \end{bmatrix} = \underbrace{{}^{B}J(\theta)}_{At\ B\ pose}\dot{\theta};\ the\ resulting\ Cartesian\ velocity\ will\ be\ in\ \{B\}$$

In general:

$$\begin{bmatrix} ^{A}v \\ ^{A}w \end{bmatrix} = {}^{A}J(\theta)\dot{\theta} = \begin{bmatrix} ^{A}_{B}R & 0_{3x3} \\ 0_{3x3} & ^{A}_{B}R \end{bmatrix} \begin{bmatrix} ^{B}v \\ ^{B}w \end{bmatrix} = \begin{bmatrix} ^{A}_{B}R & 0_{3x3} \\ 0_{3x3} & ^{A}_{B}R \end{bmatrix} \underbrace{{}^{B}J(\theta)\dot{\theta}}_{At\ B\ pose}$$

$$^{A}J(\theta) = \begin{bmatrix} ^{A}_{B}R & 0_{3x3} \\ 0_{3x3} & ^{A}_{B}R \end{bmatrix} {}^{B}J(\theta)$$

Changing Jacobian's Reference Frames

From previous exercises

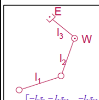
$$^{3}J_v = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix}$$

$$^{0}_{3}R = \begin{bmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{bmatrix}$$

$c_{12} = c_1c_2 - s_1s_2$
$s_{12} = c_1s_2 + s_1c_2$

$$^{0}J_v = {}^{0}_{3}R\,{}^{3}J_v = \begin{bmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{bmatrix} \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

Extending the Jacobians



Wrist Point

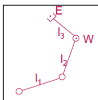
$$x = l_1c_1 + l_2c_{12}$$
$$y = l_1s_1 + l_2s_{12}$$

End-Effector Point

$$x = l_1c_1 + l_2c_{12} + l_3c_{123}$$
$$y = l_1s_1 + l_2s_{12} + l_3s_{123}$$

$$J_W = \begin{bmatrix} -l_1s_1 - l_2s_{12} & -l_2s_{12} & 0 \\ l_1c_1 + l_2c_{12} & l_2c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad {}^0J_E = \begin{pmatrix} I & -{}^0\hat{P}_{WE} \\ 0 & I \end{pmatrix} {}^0J_W$$

$${}^0P_{WE} = \begin{bmatrix} l_3c_{123} \\ l_3s_{123} \\ 0 \end{bmatrix} \Rightarrow {}^0\hat{P}_{WE} = \begin{pmatrix} 0 & 0 & l_3s_{123} \\ 0 & 0 & -l_3c_{123} \\ -l_3s_{123} & l_3c_{123} & 0 \end{pmatrix}$$



Wrist Point

$$x = l_1c_1 + l_2c_{12}$$
$$y = l_1s_1 + l_2s_{12}$$

End-Effector Point

$$x = l_1c_1 + l_2c_{12} + l_3c_{123}$$
$$y = l_1s_1 + l_2s_{12} + l_3s_{123}$$

$$J_W = \begin{bmatrix} -l_1s_1 - l_2s_{12} & -l_2s_{12} & 0 \\ l_1c_1 + l_2c_{12} & l_2c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad {}^0J_E = \begin{bmatrix} -l_1s_1 - l_2s_{12} - l_3s_{123} & -l_2s_{12} - l_3s_{123} & -l_3s_{123} \\ l_1c_1 + l_2c_{12} + l_3c_{123} & l_2c_{12} + l_3c_{123} & l_3c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

The Forward Kinematics for 3R Manipulator

$$\begin{pmatrix} x_E \\ y_E \end{pmatrix}_{\{EE\}} = \begin{pmatrix} l_1c_1 + l_2c_{12} + l_3c_{123} \\ l_1s_1 + l_2s_{12} + l_3s_{123} \end{pmatrix}$$

$${}^0J_E = \begin{pmatrix} -l_1s_1 - l_2s_{12} - l_3s_{123} & -l_2s_{12} - l_3s_{123} & l_3s_{123} \\ -l_1c_1 - l_2c_{12} - l_3c_{123} & l_2c_{12} - l_3c_{123} & l_3c_{123} \end{pmatrix}$$

Kinematic Singularity

The Effector Locality loses the ability to move in a direction or to rotate about a direction - singular direction

$$\boxed{\delta x = J(\theta)\delta\theta} \qquad J \equiv \begin{pmatrix} \dfrac{\partial x}{\partial \theta_1} & \dfrac{\partial x}{\partial \theta_2} \\ \dfrac{\partial y}{\partial \theta_1} & \dfrac{\partial y}{\partial \theta_2} \end{pmatrix} = \begin{bmatrix} -y & -l_2s_{12} \\ x & l_2c_{12} \end{bmatrix}$$

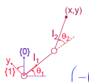
$$\dot{x} = J(\theta)\dot{\theta}$$

$$\dot{\theta} = J^{-1}(\theta)\dot{X};\ a\ problem\ occurs\ if \det(J(\theta)) = 0$$

Singularities occur when one or more axes become aligned resulting in the loss of degrees of freedom – the gimbal lock problem again

Kinematic Singularity

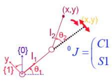
Example (Kinematic Singularities)



$$x = l_1C1 + l_2C12$$
$$y = l_1S1 + l_2S12$$
$$J = \begin{pmatrix} -(l_1S1 + l_2S12) & -l_2S12 \\ l_1C1 + l_2C12 & l_2C12 \end{pmatrix}$$
$$\det(J) = l_1l_2S2$$

Singularity at $q_2 = k\pi$

Example (Kinematic Singularities)

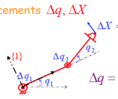


$${}^1J = {}^1_0R\ {}^0J$$
$${}^0J = \begin{pmatrix} C1 & -S1 \\ S1 & C1 \end{pmatrix} \begin{pmatrix} -l_2S2 & -l_2S2 \\ l_1 + l_2C2 & l_2C2 \end{pmatrix}$$

At Singularity $\quad {}^1J = \begin{pmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{pmatrix}$

$$\begin{bmatrix} {}^1\delta x = 0 \\ {}^1\delta y = (l_1 + l_2)\delta\theta_1 + l_2\delta\theta_2 \end{bmatrix}$$

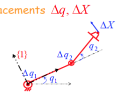
Small Displacements $\Delta q, \Delta X$



$$\Delta X = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$
$$\Delta q = J^{-1}\Delta X$$

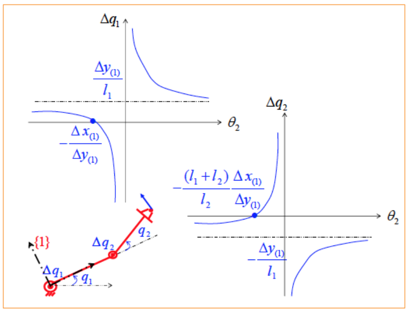
small θ_2 $\quad J_{(1)}^{-1} \cong \begin{pmatrix} \dfrac{1}{l_1\theta_2} & \dfrac{1}{l_1} \\ -\dfrac{l_1 + l_2}{l_1l_2\theta_2} & -\dfrac{1}{l_1} \end{pmatrix}$

Small Displacements $\Delta q, \Delta X$

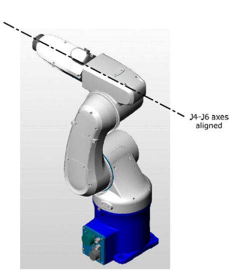


$$\Delta X = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$
$$\Delta q_1 = \frac{\Delta x_{(1)}}{l_1} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$
$$\Delta q_2 = \frac{(l_1 + l_2)\Delta x_{(1)}}{l_1l_2} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$

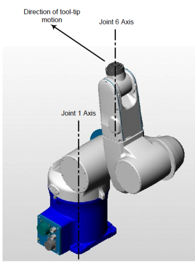
Kinematic Singularity



Robot configutarion with Singularity

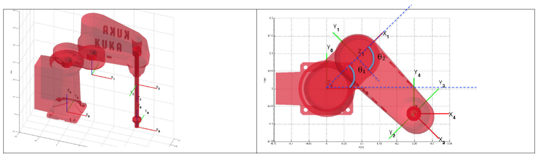


Wrist Singularity



Alignment Singularity

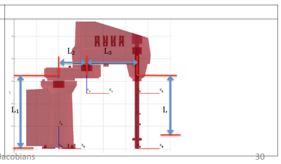
Jacobian: Scara Robot



Standard DH parameters

	θ	d	a	α
1	θ_1	L_1	L_2	0
2	θ_2	0	L_3	π
3	0	L	0	0
4	θ_4	0	0	0

	θ	d	a	α
1	θ_1	0.7	0.5	0
2	θ_2	0	0.6	π
3	0	L	0	0
4	θ_4	0	0	0



Jacobian: Scara Robot

Involved equations

$$
{}^0_6T = \begin{bmatrix} c_{12} & s_{12} & 0 & L_3c_{12}+L_2c_1 \\ s_{12} & -c_{12} & 0 & L_3s_{12}+L_2s_1 \\ 0 & 0 & -1 & L_1-L \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} L_3c_{12}+L_2c_1 \\ L_3s_{12}+L_2s_1 \\ L_1-L \end{bmatrix}
$$

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \frac{\partial x}{\partial L} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \frac{\partial y}{\partial L} \\ \frac{\partial z}{\partial \theta_1} & \frac{\partial z}{\partial \theta_2} & \frac{\partial z}{\partial L} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{L} \end{bmatrix} \rightarrow \dot{X} = J\dot{\theta}
$$

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -L_3s_{12}-L_2s_1 & -L_3s_{12} & 0 \\ L_3c_{12}+L_2c_1 & L_3c_{12} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{L} \end{bmatrix} \rightarrow J = \begin{bmatrix} -L_3s_{12}-L_2s_1 & -L_3s_{12} & 0 \\ L_3c_{12}+L_2c_1 & L_3c_{12} & 0 \\ 0 & 0 & -1 \end{bmatrix}
$$

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Jacobian: Scara Robot

Pseudo-inversa

$\dot{X} = J\dot{\theta}$

$if\ J_{NxM}; M > N$

$J^T\dot{X} = J^TJ\dot{\theta} \rightarrow \dot{\theta} = \left(J^TJ\right)^{-1} J^T\dot{X}$

$\left(J^TJ\right)^{-1}_{NxN} \approx Pseudo-inversa \rightarrow is\ square$

Jacobian singularities

$$
J = \begin{bmatrix} -L_3s_{12}-L_2s_1 & -L_3s_{12} & 0 \\ L_3c_{12}+L_2c_1 & L_3c_{12} & 0 \\ 0 & 0 & -1 \end{bmatrix}
$$

$\det(J) = L_3c_{12}\left[L_3s_{12}+L_2s_1\right] - L_3s_{12}\left[L_3c_{12}+L_2c_1\right] = 0$

$c_{12}s_1 - s_{12}c_1 = 0 \rightarrow s_2 = 0 \rightarrow \theta_2 = 0$

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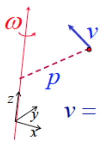
Static Forces

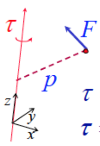
Angular/Linear – Velocities/Forces



$v = \omega \times p$

$v = -\hat{p}\ \omega$

$$
\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -p_y \\ p_x \end{pmatrix} \dot{\theta}
$$



$\tau = p \times F$

$\tau = \hat{p}\ F$

$\tau = (-\hat{p})^T\ F$

$$
\tau = \begin{pmatrix} -p_y & p_x \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}
$$

$\boxed{v = J\ \dot{\theta}} \longleftrightarrow \boxed{\tau = J^T F}$

Velocity / Force duality

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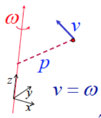
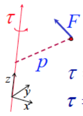
Static Forces

Angular/Linear – Velocities/Forces



$v = \omega \times p$

$v = -\hat{p}\ \omega$



$\tau = p \times F$

$\tau = \hat{p}\ F$

$\tau = (-\hat{p})^T\ F$

Jacobians in Force Domain

$Assu\min g\ Virtual\ Work\ at\ static \rightarrow \delta x \approx Infenitesimal$

$\Sigma F's = \Sigma\tau's;\ Force\,\&\,Torque\ balance$

$F_{6x1}\delta x = \tau_{6x1}\delta\theta; F: force-moments\ vector$

$F^T\delta x = \tau^T\delta\theta; by\ definition: \delta x = J\delta\theta$

$F^TJ = \tau^T \rightarrow \tau = J^TF$

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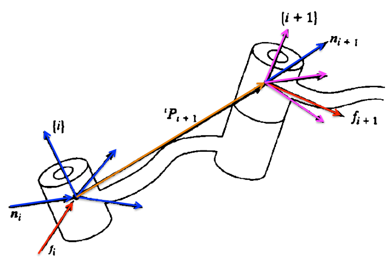
Cartesian transformation



$$
\underbrace{\begin{bmatrix} {}^AF_A \\ {}^AN_A \end{bmatrix}}_{{}^A\mathcal{F}_A} = \underbrace{\begin{bmatrix} {}^A_BR & 0 \\ P_{BORG} \times {}^A_BR & {}^A_BR \end{bmatrix}}_{{}^A_BT_f} \underbrace{\begin{bmatrix} {}^BF_B \\ {}^BN_B \end{bmatrix}}_{{}^B\mathcal{F}_B}
$$

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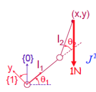
Static Force-moment: Example

Example (Static Forces)



$$
J = \begin{pmatrix} -(l_1S1+l_2S12) & -l_2S12 \\ l_1C1+l_2C12 & l_2C12 \end{pmatrix}
$$

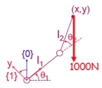
$$
J^T = \begin{pmatrix} -(l_1S1+l_2S12) & l_1C1+l_2C12 \\ -l_2S12 & l_2C12 \end{pmatrix}
$$

$\boxed{\tau = J^TF}$

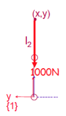
$l_1 = l_2 = 1;\quad \theta_1 = 0;\ \theta_2 = 60^\circ$

$$
\tau = \begin{pmatrix} -(l_1S1+l_2S12) & l_1C1+l_2C12 \\ -l_2S12 & l_2C12 \end{pmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\begin{bmatrix} l_1C1+l_2C12 \\ l_2C12 \end{bmatrix} = -\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}
$$

Example (Static Forces)



$\boxed{\tau = J^TF}$



$$
\tau = \begin{pmatrix} -(l_1S1+l_2S12) & l_1C1+l_2C12 \\ -l_2S12 & l_2C12 \end{pmatrix} \begin{bmatrix} 0 \\ -1K \end{bmatrix} = \begin{bmatrix} l_1C1+l_2C12 \\ l_2C12 \end{bmatrix} (-1K) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

$l_1 = l_2 = 1;\quad \theta_1 = 90; \theta_2 = 0^\circ$

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