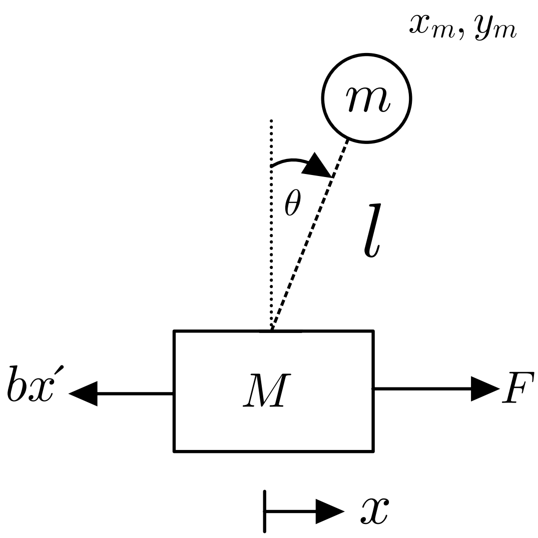


$$x_m = x + l\sin(\theta) \qquad x'_m = x' + l\theta'\cos(\theta)$$
$$y_m = l\cos(\theta) \qquad y'_m = -l\theta'\sin(\theta)$$

$$KE = \frac{1}{2}M(x')^2 + \frac{1}{2}m\left[(x'_m)^2 + (y'_m)^2\right]$$
$$= \frac{1}{2}(M+m)(x')^2 + \frac{1}{2}ml^2(\theta')^2 + ml\theta'x'\cos(\theta)$$
$$PE = mgl\cos(\theta)$$
$$R = \frac{1}{2}b(x')^2$$

$$L = KE - PE = \frac{1}{2}(M+m)(x')^2 + \frac{1}{2}ml^2(\theta')^2 + ml\theta'x'\cos(\theta) - mgl\cos(\theta)$$

for x

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x'}\right) - \frac{\partial L}{\partial x} + \frac{2R}{\partial x'} = F$$

$$\frac{d}{dt}\left[(M+m)x' + ml\theta'\cos(\theta)\right] + bx' = F$$

$$(M+m)x'' + ml\theta''\cos(\theta) - ml(\theta')^2\sin(\theta) + bx' = F \qquad 1$$

for θ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta'}\right) - \frac{\partial L}{\partial \theta} + \frac{2R}{\partial \theta'} = 0$$

$$\frac{d}{dt}\left(ml^2\theta' + mlx'\cos(\theta)\right) + ml\theta'x'\sin(\theta) - mgl\sin(\theta) = 0$$

$$ml\left(l\theta'' + x''\cos(\theta) - x'\theta'\sin(\theta)\right) + ml\theta'x'\sin(\theta) - mgl\sin(\theta) = 0$$

$$l\theta'' + x''\cos(\theta) - g\sin(\theta) = 0 \qquad 2$$

Non-linear model

solve coupled system (equations 1 & 2) for $x'' \ \& \ \theta''$

$$x'' = \frac{ml\left(\theta'\right)^2 \sin\left(\theta\right) - gm\cos(\theta)\sin\left(\theta\right) - bx' + F}{(m+M) - m\cos^2\left(\theta\right)}$$

$$\theta'' = \frac{(M+m)\,g\sin\left(\theta\right) - ml\left(\theta'\right)^2\cos\left(\theta\right)\sin\left(\theta\right) - F\cos\left(\theta\right) + bx'\cos\left(\theta\right)}{l\left(M+m\right) - ml\cos^2\left(\theta\right)}$$

$$\frac{d}{dt}\begin{bmatrix} x \\ x' \\ \theta \\ \theta' \end{bmatrix} = \begin{bmatrix} x' \\ \dfrac{ml\left(\theta'\right)^2 \sin\left(\theta\right) - gm\cos(\theta)\sin\left(\theta\right) - bx' + F}{(m+M) - m\cos^2\left(\theta\right)} \\ \theta' \\ \dfrac{(M+m)\,g\sin\left(\theta\right) - ml\left(\theta'\right)^2\cos\left(\theta\right)\sin\left(\theta\right) - F\cos\left(\theta\right) + bx'\cos\left(\theta\right)}{l\left(M+m\right) - ml\cos^2\left(\theta\right)} \end{bmatrix}$$

Linear model for controller design:

Linearized around: $\begin{aligned} \theta^* &= 0 \\ x^* &= 0 \end{aligned}$

$$l\theta'' + x'' - g(\theta) = 0$$

$$(M+m)\,x'' + ml\theta'' + bx' = F$$

$$\frac{d}{dt}\begin{bmatrix} x \\ x' \\ \theta \\ \theta' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\dfrac{b}{M} & \dfrac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{b}{Ml} & \dfrac{(M+m)g}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x \\ x' \\ \theta \\ \theta' \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ -\dfrac{1}{Ml} \end{bmatrix} F$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{b}{M} & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{b}{Ml} & \frac{(M+m)g}{Ml} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{Ml} \end{bmatrix}$$

$$Q = \begin{bmatrix} q_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & q_\theta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} r_{cost} \end{bmatrix}$$

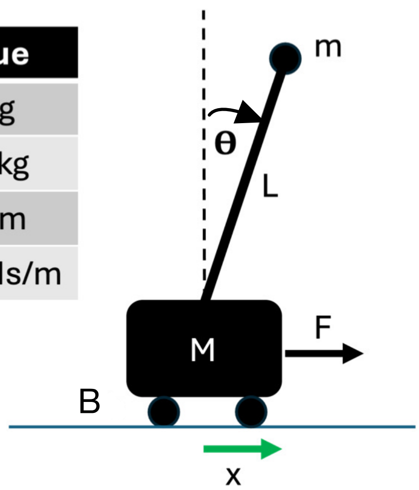
LQR:

$$\min \left\{ \int_0^\infty (x'Qx + u'Ru)\, dt \right\} \longrightarrow k$$

such that:

$$u = F = -kx$$

Parameter	Value
M	1 kg
m	0.5 kg
L	0.4 m
B	0.01 Ns/m



$$q_x = 1500$$
$$q_\theta = 1200$$
$$r_{cost} = 0.1$$

$$k = \begin{bmatrix} -1225 & -75.5 & -242.9 & -42.6 \end{bmatrix}$$