University of Colorado at Boulder

ASEN 5044 - Statistical State Estimation for Dynamical Systems

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Professor: Dr. Nisar Ahmed

Final Project
(Cooperative Air-Ground Robot Localization)

Team Members:

Nicholas Martinez

Whit Whittall

Michael Bernabei

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Part I: Deterministic System Analysis

Part 1.

We are given the Equation Of Motion (EOM) for the Unmanned Ground Vehicle (UGV). The EOM is,

$$\begin{aligned}\dot{\xi}_g &= v_g \cos\theta_g + \tilde{w}_{x,g} \\ \dot{\eta}_g &= v_g \sin\theta_g + \tilde{w}_{y,g} \\ \dot{\theta}_g &= \frac{v_g}{L}\tan\phi_g + \tilde{w}_{\omega,g}\end{aligned}$$

and for the Unmanned Aerial Vehicle (UAV) we have the following EOM,

$$\begin{aligned}\dot{\xi}_a &= v_a \cos\theta_a + \tilde{w}_{x,a} \\ \dot{\eta}_a &= v_a \sin\theta_a + \tilde{w}_{y,a} \\ \dot{\theta}_a &= \frac{v_a}{L}\tan\phi_a + \tilde{w}_{\omega,a}\end{aligned}$$

where $\tilde{w}_a = [\tilde{w}_{x,a}, \tilde{w}_{y,a}, \tilde{w}_{\omega,a}]^T$ and $\tilde{w}_g = [\tilde{w}_{x,g}, \tilde{w}_{y,g}, \tilde{w}_{\omega,g}]^T$ are the process noise for the UAV And UGV respectively. We are also given the following sensing model,

$$y(t) = \begin{bmatrix} \arctan\left(\frac{\eta_a - \eta_g}{\xi_a - \xi_g}\right) - \theta_g \\ \sqrt{(\xi_g - \xi_a)^2 + (\eta_g - \eta_a)^2} \\ \arctan\left(\frac{\eta_g - \eta_a}{\xi_g - \xi_a}\right) - \theta_a \\ \xi_a \\ \eta_a \end{bmatrix} + \tilde{\mathbf{v}}(t)$$

where $\tilde{\mathbf{v}}(t) \in \mathbb{R}^5$ is the sensor error vector. Finally, we are given the combined states, control inputs, and disturbance inputs as,

$$\begin{aligned}\mathbf{x}(t) &= [\xi_g \quad \eta_g \quad \theta_g \quad \xi_a \quad \eta_a \quad \theta_a]^T, \\ \mathbf{u}(t) &= [\mathbf{u}_g \;\; \mathbf{u}_a]^T, \\ \tilde{\mathbf{w}}(t) &= [\tilde{\mathbf{w}}_{\mathbf{g}} \;\; \tilde{\mathbf{w}}_{\mathbf{a}}]^T\end{aligned}$$

The state is $x = [\xi_g \;\, \eta_g \;\, \theta_g \;\, \xi_a \;\, \eta_a \;\, \theta_a]^T = [x_1 \;\, x_2 \, x_3 \;\, x_4 \;\, x_5 \;\, x_6]^T$ and our inputs $u = [\mathbf{u}_g \;\, \mathbf{u}_a]^T = [v_g \;\; \phi_g \;\; v_a \;\; \phi_a]^T = [u_1 \;\; u_2 \;\; u_3 \;\; u_4]^T$. We then have the following after substituting in our state and input variables,

$$\dot{x} = \begin{bmatrix} \dot{\xi}_g \\ \dot{\eta}_g \\ \dot{\theta}_g \\ \dot{\xi}_a \\ \dot{\eta}_a \\ \dot{\theta}_a \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} \mathcal{F}_1(x,u) \\ \mathcal{F}_2(x,u) \\ \mathcal{F}_3(x,u) \\ \mathcal{F}_4(x,u) \\ \mathcal{F}_5(x,u) \\ \mathcal{F}_6(x,u) \end{bmatrix} = \begin{bmatrix} u_1 \cos x_3 \\ u_1 \sin x_3 \\ \frac{u_1}{L} \tan u_2 \\ u_3 \cos x_6 \\ u_3 \sin x_6 \\ \frac{u_3}{L} \tan u_4 \end{bmatrix}$$

$$y = \begin{bmatrix} \arctan\left(\frac{x_5 - x_2}{x_4 - x_1}\right) - x_3 \\ \sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2} \\ \arctan\left(\frac{x_2 - x_5}{x_1 - x_4}\right) - x_6 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \mathcal{H}_1(x,u) \\ \mathcal{H}_2(x,u) \\ \mathcal{H}_3(x,u) \\ \mathcal{H}_4(x,u) \\ \mathcal{H}_5(x,u) \end{bmatrix}$$

We now need to compute the partials of $\mathcal{F}_{1\ldots6}$ with respect to x,

$$\begin{array}{llllll}
\frac{\partial \mathcal{F}_1}{\partial x_1} = 0 & \frac{\partial \mathcal{F}_2}{\partial x_1} = 0 & \frac{\partial \mathcal{F}_3}{\partial x_1} = 0 & \frac{\partial \mathcal{F}_4}{\partial x_1} = 0 & \frac{\partial \mathcal{F}_5}{\partial x_1} = 0 & \frac{\partial \mathcal{F}_6}{\partial x_1} = 0 \\
\frac{\partial \mathcal{F}_1}{\partial x_2} = 0 & \frac{\partial \mathcal{F}_2}{\partial x_2} = 0 & \frac{\partial \mathcal{F}_3}{\partial x_2} = 0 & \frac{\partial \mathcal{F}_4}{\partial x_2} = 0 & \frac{\partial \mathcal{F}_5}{\partial x_2} = 0 & \frac{\partial \mathcal{F}_6}{\partial x_2} = 0 \\
\frac{\partial \mathcal{F}_1}{\partial x_3} = -u_1 \sin x_3 & \frac{\partial \mathcal{F}_2}{\partial x_3} = u_1 \cos x_3 & \frac{\partial \mathcal{F}_3}{\partial x_3} = 0 & \frac{\partial \mathcal{F}_4}{\partial x_3} = 0 & \frac{\partial \mathcal{F}_5}{\partial x_3} = 0 & \frac{\partial \mathcal{F}_6}{\partial x_3} = 0 \\
\frac{\partial \mathcal{F}_1}{\partial x_4} = 0 & \frac{\partial \mathcal{F}_2}{\partial x_4} = 0 & \frac{\partial \mathcal{F}_3}{\partial x_4} = 0 & \frac{\partial \mathcal{F}_4}{\partial x_4} = 0 & \frac{\partial \mathcal{F}_5}{\partial x_4} = 0 & \frac{\partial \mathcal{F}_6}{\partial x_4} = 0 \\
\frac{\partial \mathcal{F}_1}{\partial x_5} = 0 & \frac{\partial \mathcal{F}_2}{\partial x_5} = 0 & \frac{\partial \mathcal{F}_3}{\partial x_5} = 0 & \frac{\partial \mathcal{F}_4}{\partial x_5} = 0 & \frac{\partial \mathcal{F}_5}{\partial x_5} = 0 & \frac{\partial \mathcal{F}_6}{\partial x_5} = 0 \\
\frac{\partial \mathcal{F}_1}{\partial x_6} = 0 & \frac{\partial \mathcal{F}_2}{\partial x_6} = 0 & \frac{\partial \mathcal{F}_3}{\partial x_6} = 0 & \frac{\partial \mathcal{F}_4}{\partial x_6} = -u_3 \sin x_6 & \frac{\partial \mathcal{F}_5}{\partial x_6} = u_3 \cos x_6 & \frac{\partial \mathcal{F}_6}{\partial x_6} = 0
\end{array}$$

and with respect to u,

$$\begin{array}{llllll}
\frac{\partial \mathcal{F}_1}{\partial u_1} = \cos x_3 & \frac{\partial \mathcal{F}_2}{\partial u_1} = \sin x_3 & \frac{\partial \mathcal{F}_3}{\partial u_1} = \frac{\tan u_2}{L} & \frac{\partial \mathcal{F}_4}{\partial u_1} = 0 & \frac{\partial \mathcal{F}_5}{\partial u_1} = 0 & \frac{\partial \mathcal{F}_6}{\partial u_1} = 0 \\
\frac{\partial \mathcal{F}_1}{\partial u_2} = 0 & \frac{\partial \mathcal{F}_2}{\partial u_2} = 0 & \frac{\partial \mathcal{F}_3}{\partial u_2} = \frac{u_1}{L} \sec^2 u_2 & \frac{\partial \mathcal{F}_4}{\partial u_2} = 0 & \frac{\partial \mathcal{F}_5}{\partial u_2} = 0 & \frac{\partial \mathcal{F}_6}{\partial u_2} = 0 \\
\frac{\partial \mathcal{F}_1}{\partial u_3} = 0 & \frac{\partial \mathcal{F}_2}{\partial u_3} = 0 & \frac{\partial \mathcal{F}_3}{\partial u_3} = 0 & \frac{\partial \mathcal{F}_4}{\partial u_3} = \cos x_6 & \frac{\partial \mathcal{F}_5}{\partial u_3} = \sin x_6 & \frac{\partial \mathcal{F}_6}{\partial u_3} = \frac{\tan u_4}{L} \\
\frac{\partial \mathcal{F}_1}{\partial u_4} = 0 & \frac{\partial \mathcal{F}_2}{\partial u_4} = 0 & \frac{\partial \mathcal{F}_3}{\partial u_4} = 0 & \frac{\partial \mathcal{F}_4}{\partial u_4} = 0 & \frac{\partial \mathcal{F}_5}{\partial u_4} = 0 & \frac{\partial \mathcal{F}_6}{\partial u_4} = \frac{u_3}{L} \sec^2 u_4
\end{array}$$

finally we compute $\mathcal{H}_{1\ldots5}$ with respect to x. In the following we show the two most complex partial derivative computations. The remaining partials were computed using similar techniques and therefore we omit them for brevity.

Utilize the chain rule with $u = (\frac{x_5 - x_2}{x_4 - x_1})$, then,

$$
\begin{aligned}
\frac{\partial \mathcal{H}_1}{\partial x_1} &= \frac{\partial \mathcal{H}_1}{\partial u} \times \frac{\partial u}{\partial x_1} \\
&= \left(\frac{1}{(\frac{x_5 - x_2}{x_4 - x_1})^2 + 1} \right) \times \left(\frac{0 \times (x_4 - x_1) - (x_5 - x_2) \times -1}{(x_4 - x_1)^2} \right) \\
&= \left(\frac{1}{(\frac{x_5 - x_2}{x_4 - x_1})^2 + 1} \right) \times \left(\frac{(x_5 - x_2)}{(x_4 - x_1)^2} \right) \\
&= \left(\frac{1}{\frac{(x_5 - x_2)^2}{(x_4 - x_1)^2} + 1} \right) \times \left(\frac{(x_5 - x_2)}{(x_4 - x_1)^2} \right) \\
&= \left(\frac{1}{\frac{(x_5 - x_2)^2 + (x_4 - x_1)^2}{(x_4 - x_1)^2}} \right) \times \left(\frac{(x_5 - x_2)}{(x_4 - x_1)^2} \right) \\
&= \left(\frac{(x_4 - x_1)^2}{(x_5 - x_2)^2 + (x_4 - x_1)^2} \right) \times \left(\frac{(x_5 - x_2)}{(x_4 - x_1)^2} \right) \\
&= \frac{x_5 - x_2}{(x_5 - x_2)^2 + (x_4 - x_1)^2}
\end{aligned}
$$

and we show the case when the partial we are computing is in the numerator,

$$
\begin{aligned}
\frac{\partial \mathcal{H}_1}{\partial x_2} &= \frac{\partial \mathcal{H}_1}{\partial u} \times \frac{\partial u}{\partial x_2} \\
&= \left(\frac{1}{(\frac{x_5 - x_2}{x_4 - x_1})^2 + 1} \right) \times \left(\frac{-1 \times (x_4 - x_1) - (x_5 - x_2) \times 0}{(x_4 - x_1)^2} \right) \\
&= \left(\frac{1}{(\frac{x_5 - x_2}{x_4 - x_1})^2 + 1} \right) \times \left(-\frac{(x_4 - x_1)}{(x_4 - x_1)^2} \right) \\
&= \left(\frac{1}{\frac{(x_5 - x_2)^2}{(x_4 - x_1)^2} + 1} \right) \times \left(-\frac{(x_4 - x_1)}{(x_4 - x_1)^2} \right) \\
&= \left(\frac{1}{\frac{(x_5 - x_2)^2 + (x_4 - x_1)^2}{(x_4 - x_1)^2}} \right) \times \left(-\frac{(x_4 - x_1)}{(x_4 - x_1)^2} \right) \\
&= \left(\frac{(x_4 - x_1)^2}{(x_5 - x_2)^2 + (x_4 - x_1)^2} \right) \times \left(-\frac{(x_4 - x_1)}{(x_4 - x_1)^2} \right) \\
&= -\frac{x_4 - x_1}{(x_5 - x_2)^2 + (x_4 - x_1)^2}
\end{aligned}
$$

Now, let $u = (x_1 - x_4)^2 + (x_2 - x_5)^2$, $v = (x_1 - x_4)^2$, and $w = (x_1 - x_4)$, then for our next complex partial we have,

$$
\begin{aligned}
\frac{\partial \mathcal{H}_1}{\partial x_1} &= \frac{\partial \mathcal{H}_1}{\partial u} \times \frac{\partial u}{\partial v} \times \frac{\partial v}{\partial w} \times \frac{\partial w}{\partial x_1} \\
&= \left(\frac{1}{2} \times \frac{1}{\sqrt{u}}\right) \times (1) \times \Big(2(w)\Big) \times (1) \\
&= \left(\frac{1}{2} \times \frac{1}{\sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2}}\right) \times (1) \times \Big(2(x_1 - x_4)\Big) \times (1) \\
&= \frac{x_1 - x_4}{\sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2}}
\end{aligned}
$$

therefore we have the following Jacobians,

$$
\frac{\partial f}{\partial x} = \begin{bmatrix}
0 & 0 & -u_1 \sin x_3 & 0 & 0 & 0 \\
0 & 0 & u_1 \cos x_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & u_3 \sin x_6 \\
0 & 0 & 0 & 0 & 0 & -u_3 \cos x_6 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\frac{\partial f}{\partial u} = \begin{bmatrix}
\cos x_3 & 0 & 0 & 0 \\
\sin x_3 & 0 & 0 & 0 \\
\frac{1}{L} \tan u_2 & \frac{u_1}{L} \sec^2 u_2 & 0 & 0 \\
0 & 0 & \cos x_6 & 0 \\
0 & 0 & \sin x_6 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\frac{\partial h}{\partial x} = \begin{bmatrix}
\frac{x_5 - x_2}{(x_4 - x_1)^2 + (x_2 - x_5)^2} & -\frac{x_4 - x_1}{(x_4 - x_1)^2 + (x_2 - x_5)^2} & -1 & -\frac{x_5 - x_2}{(x_4 - x_1)^2 + (x_5 - x_2)^2} & \frac{x_4 - x_1}{(x_4 - x_1)^2 + (x_5 - x_2)^2} & 0 \\
\frac{x_1 + x_4}{\sqrt{(x_1 + x_4)^2 + (x_2 + x_5)^2}} & \frac{x_2 + x_5}{\sqrt{(x_1 + x_4)^2 + (x_2 + x_5)^2}} & 0 & \frac{x_1 + x_4}{\sqrt{(x_1 + x_4)^2 + (x_2 + x_5)^2}} & \frac{x_2 + x_5}{\sqrt{(x_1 + x_4)^2 + (x_2 + x_5)^2}} & 0 \\
-\frac{x_2 - x_5}{(x_1 - x_4)^2 + (x_2 - x_5)^2} & \frac{x_1 - x_4}{(x_1 - x_4)^2 + (x_2 - x_5)^2} & 0 & \frac{x_2 - x_5}{(x_1 - x_4)^2 + (x_2 - x_5)^2} & -\frac{x_1 - x_4}{(x_1 - x_4)^2 + (x_2 - x_5)^2} & -1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
$$

$$
\frac{\partial h}{\partial u} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

Part II: Stochastic Nonlinear Filtering