UNIVERSITY OF COLORADO AT BOULDER

ASEN 5044 - STATISTICAL STATE ESTIMATION FOR DYNAMICAL SYSTEMS

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Project Report 1 (Cooperative Air-Ground Robot Localization)

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Part 0: Introduction

Team Member Contributions
Nicholas Martinez:

1. Nonlinear System Modeling with ODE45

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1. Linearized DT LTV System Modeling

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1. Computation of CT Jacobian Matrices

Our group is nearly complete with Part I Deterministic System Analysis. At this time, we've computed the CT Jacobian matrices. This yielded a time varying system, so we have skipped observability and stability analysis. We performed a full nonlinear simulation of the system dynamics. We simulated the linearized DT dynamics and validated these results against the nonlinear simulation and the posted "solution sketches" on canvas. Our results match the posted results on canvas. The only thing we have not done at this stage is simulated the measurement models.

Part I: Deterministic System Analysis

Part 1.

We are given the Equation Of Motion (EOM) for the Unmanned Ground Vehicle (UGV).
The EOM is,

$$
\begin{aligned}
\dot{\xi}_g &= v_g \cos\theta_g + \tilde{w}_{x,g} \\
\dot{\eta}_g &= v_g \sin\theta_g + \tilde{w}_{y,g} \\
\dot{\theta}_g &= \frac{v_g}{L}\tan\phi_g + \tilde{w}_{\omega,g}
\end{aligned}
$$

and for the Unmanned Aerial Vehicle (UAV) we have the following EOM,

$$
\begin{aligned}
\dot{\xi}_a &= v_a \cos\theta_a + \tilde{w}_{x,a} \\
\dot{\eta}_a &= v_a \sin\theta_a + \tilde{w}_{y,a} \\
\dot{\theta}_a &= \omega_a + \tilde{w}_{\omega,a}
\end{aligned}
$$

where $\tilde{w}_a = [\tilde{w}_{x,a}, \tilde{w}_{y,a}, \tilde{w}_{\omega,a}]^T$ and $\tilde{w}_g = [\tilde{w}_{x,g}, \tilde{w}_{y,g}, \tilde{w}_{\omega,g}]^T$ are the process noise for the UAV And UGV respectively. We are also given the following sensing model,

$$y(t) = \begin{bmatrix} \arctan\left(\frac{\eta_a - \eta_g}{\xi_a - \xi_g}\right) - \theta_g \\ \sqrt{(\xi_g - \xi_a)^2 + (\eta_g - \eta_a)^2} \\ \arctan\left(\frac{\eta_g - \eta_a}{\xi_g - \xi_a}\right) - \theta_a \\ \xi_a \\ \eta_a \end{bmatrix} + \tilde{\mathbf{v}}(t)$$

where $\tilde{\mathbf{v}}(t) \in \mathbb{R}^5$ is the sensor error vector. Finally, we are given the combined states, control inputs, and disturbance inputs as,

$$\begin{aligned} \mathbf{x}(t) &= [\xi_g \quad \eta_g \quad \theta_g \quad \xi_a \quad \eta_a \quad \theta_a]^T, \\ \mathbf{u}(t) &= [\mathbf{u}_g \ \ \mathbf{u}_a]^T, \\ \tilde{\mathbf{w}}(t) &= [\tilde{\mathbf{w}}_\mathbf{g} \ \ \tilde{\mathbf{w}}_\mathbf{a}]^T \end{aligned}$$

The state is $x = [\xi_g \ \eta_g \ \theta_g \ \xi_a \ \eta_a \ \theta_a]^T = [x_1 \ x_2 \, x_3 \ x_4 \ x_5 \ x_6]^T$ and our inputs $u = [\mathbf{u}_g \ \mathbf{u}_a]^T = [v_g \ \ \phi_g \ \ v_a \ \ \phi_a]^T = [u_1 \ \ u_2 \ \ u_3 \ \ u_4]^T$. We then have the following after substituting in our state and input variables,

$$\dot{x} = \begin{bmatrix} \dot{\xi}_g \\ \dot{\eta}_g \\ \dot{\theta}_g \\ \dot{\xi}_a \\ \dot{\eta}_a \\ \dot{\theta}_a \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} \mathcal{F}_1(x,u) \\ \mathcal{F}_2(x,u) \\ \mathcal{F}_3(x,u) \\ \mathcal{F}_4(x,u) \\ \mathcal{F}_5(x,u) \\ \mathcal{F}_6(x,u) \end{bmatrix} = \begin{bmatrix} u_1 \cos x_3 \\ u_1 \sin x_3 \\ \frac{u_1}{L} \tan u_2 \\ u_3 \cos x_6 \\ u_3 \sin x_6 \\ u_4 \end{bmatrix}$$

$$y = \begin{bmatrix} \arctan\left(\frac{x_5 - x_2}{x_4 - x_1}\right) - x_3 \\ \sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2} \\ \arctan\left(\frac{x_2 - x_5}{x_1 - x_4}\right) - x_6 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \mathcal{H}_1(x,u) \\ \mathcal{H}_2(x,u) \\ \mathcal{H}_3(x,u) \\ \mathcal{H}_4(x,u) \\ \mathcal{H}_5(x,u) \end{bmatrix}$$

We now need to compute the partials of $\mathcal{F}_{1\ldots6}$ with respect to x,

$\frac{\partial \mathcal{F}_1}{\partial x_1} = 0$ $\quad \frac{\partial \mathcal{F}_2}{\partial x_1} = 0$ $\quad \frac{\partial \mathcal{F}_3}{\partial x_1} = 0$ $\quad \frac{\partial \mathcal{F}_4}{\partial x_1} = 0$ $\quad \frac{\partial \mathcal{F}_5}{\partial x_1} = 0$ $\quad \frac{\partial \mathcal{F}_6}{\partial x_1} = 0$

$\frac{\partial \mathcal{F}_1}{\partial x_2} = 0$ $\quad \frac{\partial \mathcal{F}_2}{\partial x_2} = 0$ $\quad \frac{\partial \mathcal{F}_3}{\partial x_2} = 0$ $\quad \frac{\partial \mathcal{F}_4}{\partial x_2} = 0$ $\quad \frac{\partial \mathcal{F}_5}{\partial x_2} = 0$ $\quad \frac{\partial \mathcal{F}_6}{\partial x_2} = 0$

$\frac{\partial \mathcal{F}_1}{\partial x_3} = -u_1 \sin x_3$ $\quad \frac{\partial \mathcal{F}_2}{\partial x_3} = u_1 \cos x_3$ $\quad \frac{\partial \mathcal{F}_3}{\partial x_3} = 0$ $\quad \frac{\partial \mathcal{F}_4}{\partial x_3} = 0$ $\quad \frac{\partial \mathcal{F}_5}{\partial x_3} = 0$ $\quad \frac{\partial \mathcal{F}_6}{\partial x_3} = 0$

$\frac{\partial \mathcal{F}_1}{\partial x_4} = 0$ $\quad \frac{\partial \mathcal{F}_2}{\partial x_4} = 0$ $\quad \frac{\partial \mathcal{F}_3}{\partial x_4} = 0$ $\quad \frac{\partial \mathcal{F}_4}{\partial x_4} = 0$ $\quad \frac{\partial \mathcal{F}_5}{\partial x_4} = 0$ $\quad \frac{\partial \mathcal{F}_6}{\partial x_4} = 0$

$\frac{\partial \mathcal{F}_1}{\partial x_5} = 0$ $\quad \frac{\partial \mathcal{F}_2}{\partial x_5} = 0$ $\quad \frac{\partial \mathcal{F}_3}{\partial x_5} = 0$ $\quad \frac{\partial \mathcal{F}_4}{\partial x_5} = 0$ $\quad \frac{\partial \mathcal{F}_5}{\partial x_5} = 0$ $\quad \frac{\partial \mathcal{F}_6}{\partial x_5} = 0$

$\frac{\partial \mathcal{F}_1}{\partial x_6} = 0$ $\quad \frac{\partial \mathcal{F}_2}{\partial x_6} = 0$ $\quad \frac{\partial \mathcal{F}_3}{\partial x_6} = 0$ $\quad \frac{\partial \mathcal{F}_4}{\partial x_6} = -u_3 \sin x_6$ $\quad \frac{\partial \mathcal{F}_5}{\partial x_6} = u_3 \cos x_6$ $\quad \frac{\partial \mathcal{F}_6}{\partial x_6} = 0$

and with respect to u,

$$
\begin{array}{llllll}
\frac{\partial \mathcal{F}_1}{\partial u_1} = \cos x_3 & \frac{\partial \mathcal{F}_2}{\partial u_1} = \sin x_3 & \frac{\partial \mathcal{F}_3}{\partial u_1} = \frac{\tan u_2}{L} & \frac{\partial \mathcal{F}_4}{\partial u_1} = 0 & \frac{\partial \mathcal{F}_5}{\partial u_1} = 0 & \frac{\partial \mathcal{F}_6}{\partial u_1} = 0 \\
\frac{\partial \mathcal{F}_1}{\partial u_2} = 0 & \frac{\partial \mathcal{F}_2}{\partial u_2} = 0 & \frac{\partial \mathcal{F}_3}{\partial u_2} = \frac{u_1}{L}\sec^2 u_2 & \frac{\partial \mathcal{F}_4}{\partial u_2} = 0 & \frac{\partial \mathcal{F}_5}{\partial u_2} = 0 & \frac{\partial \mathcal{F}_6}{\partial u_2} = 0 \\
\frac{\partial \mathcal{F}_1}{\partial u_3} = 0 & \frac{\partial \mathcal{F}_2}{\partial u_3} = 0 & \frac{\partial \mathcal{F}_3}{\partial u_3} = 0 & \frac{\partial \mathcal{F}_4}{\partial u_3} = \cos x_6 & \frac{\partial \mathcal{F}_5}{\partial u_3} = \sin x_6 & \frac{\partial \mathcal{F}_6}{\partial u_3} = 0 \\
\frac{\partial \mathcal{F}_1}{\partial u_4} = 0 & \frac{\partial \mathcal{F}_2}{\partial u_4} = 0 & \frac{\partial \mathcal{F}_3}{\partial u_4} = 0 & \frac{\partial \mathcal{F}_4}{\partial u_4} = 0 & \frac{\partial \mathcal{F}_5}{\partial u_4} = 0 & \frac{\partial \mathcal{F}_6}{\partial u_4} = 1
\end{array}
$$

finally we compute $\mathcal{H}_{1\ldots5}$ with respect to x. In the following we show the two most complex partial derivative computations. The remaining partials were computed using similar techniques and therefore we omit them for brevity.

Utilize the chain rule with $u = (\frac{x_5 - x_2}{x_4 - x_1})$, then,

$$
\begin{aligned}
\frac{\partial \mathcal{H}_1}{\partial x_1} &= \frac{\partial \mathcal{H}_1}{\partial u} \times \frac{\partial u}{\partial x_1} \\
&= \left(\frac{1}{(\frac{x_5-x_2}{x_4-x_1})^2 + 1}\right) \times \left(\frac{0 \times (x_4 - x_1) - (x_5 - x_2) \times -1}{(x_4 - x_1)^2}\right) \\
&= \left(\frac{1}{(\frac{x_5-x_2}{x_4-x_1})^2 + 1}\right) \times \left(\frac{(x_5 - x_2)}{(x_4 - x_1)^2}\right) \\
&= \left(\frac{1}{\frac{(x_5-x_2)^2}{(x_4-x_1)^2} + 1}\right) \times \left(\frac{(x_5 - x_2)}{(x_4 - x_1)^2}\right) \\
&= \left(\frac{1}{\frac{(x_5-x_2)^2+(x_4-x_1)^2}{(x_4-x_1)^2}}\right) \times \left(\frac{(x_5 - x_2)}{(x_4 - x_1)^2}\right) \\
&= \left(\frac{(x_4 - x_1)^2}{(x_5 - x_2)^2 + (x_4 - x_1)^2}\right) \times \left(\frac{(x_5 - x_2)}{(x_4 - x_1)^2}\right) \\
&= \frac{x_5 - x_2}{(x_5 - x_2)^2 + (x_4 - x_1)^2}
\end{aligned}
$$

and we show the case when the partial we are computing is in the numerator,

$$
\begin{aligned}
\frac{\partial \mathcal{H}_1}{\partial x_2} &= \frac{\partial \mathcal{H}_1}{\partial u} \times \frac{\partial u}{\partial x_2} \\
&= \left(\frac{1}{(\frac{x_5 - x_2}{x_4 - x_1})^2 + 1} \right) \times \left(\frac{-1 \times (x_4 - x_1) - (x_5 - x_2) \times 0}{(x_4 - x_1)^2} \right) \\
&= \left(\frac{1}{(\frac{x_5 - x_2}{x_4 - x_1})^2 + 1} \right) \times \left(- \frac{(x_4 - x_1)}{(x_4 - x_1)^2} \right) \\
&= \left(\frac{1}{\frac{(x_5 - x_2)^2}{(x_4 - x_1)^2} + 1} \right) \times \left(- \frac{(x_4 - x_1)}{(x_4 - x_1)^2} \right) \\
&= \left(\frac{1}{\frac{(x_5 - x_2)^2 + (x_4 - x_1)^2}{(x_4 - x_1)^2}} \right) \times \left(- \frac{(x_4 - x_1)}{(x_4 - x_1)^2} \right) \\
&= \left(\frac{(x_4 - x_1)^2}{(x_5 - x_2)^2 + (x_4 - x_1)^2} \right) \times \left(- \frac{(x_4 - x_1)}{(x_4 - x_1)^2} \right) \\
&= - \frac{x_4 - x_1}{(x_5 - x_2)^2 + (x_4 - x_1)^2}
\end{aligned}
$$

Now, let $u = (x_1 - x_4)^2 + (x_2 - x_5)^2$, $v = (x_1 - x_4)^2$, and $w = (x_1 - x_4)$, then for our next complex partial we have,

$$
\begin{aligned}
\frac{\partial \mathcal{H}_1}{\partial x_1} &= \frac{\partial \mathcal{H}_1}{\partial u} \times \frac{\partial u}{\partial v} \times \frac{\partial v}{\partial w} \times \frac{\partial w}{\partial x_1} \\
&= \left(\frac{1}{2} \times \frac{1}{\sqrt{u}} \right) \times (1) \times \left(2(w) \right) \times (1) \\
&= \left(\frac{1}{2} \times \frac{1}{\sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2}} \right) \times (1) \times \left(2(x_1 - x_4) \right) \times (1) \\
&= \frac{x_1 - x_4}{\sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2}}
\end{aligned}
$$

therefore we have the following Jacobians,

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 & -u_1 \sin x_3 & 0 & 0 & 0 \\ 0 & 0 & u_1 \cos x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_3 \sin x_6 \\ 0 & 0 & 0 & 0 & 0 & -u_3 \cos x_6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} \cos x_3 & 0 & 0 & 0 \\ \sin x_3 & 0 & 0 & 0 \\ \frac{\tan u_2}{L} & \frac{u_1 \sec^2 u_2}{L} & 0 & 0 \\ 0 & 0 & \cos x_6 & 0 \\ 0 & 0 & \sin x_6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial h}{\partial x} = \begin{bmatrix} \frac{x_5 - x_2}{(x_4-x_1)^2+(x_2-x_5)^2} & -\frac{x_4-x_1}{(x_4-x_1)^2+(x_2-x_5)^2} & -1 & -\frac{x_5-x_2}{(x_4-x_1)^2+(x_5-x_2)^2} & \frac{x_4-x_1}{(x_4-x_1)^2+(x_5-x_2)^2} & 0 \\ \frac{x_1+x_4}{\sqrt{(x_1+x_4)^2+(x_2+x_5)^2}} & \frac{x_2+x_5}{\sqrt{(x_1+x_4)^2+(x_2+x_5)^2}} & 0 & \frac{x_1+x_4}{\sqrt{(x_1+x_4)^2+(x_2+x_5)^2}} & \frac{x_2+x_5}{\sqrt{(x_1+x_4)^2+(x_2+x_5)^2}} & 0 \\ -\frac{x_2-x_5}{(x_1-x_4)^2+(x_2-x_5)^2} & \frac{x_1-x_4}{(x_1-x_4)^2+(x_2-x_5)^2} & 0 & \frac{x_2-x_5}{(x_1-x_4)^2+(x_2-x_5)^2} & -\frac{x_1-x_4}{(x_1-x_4)^2+(x_2-x_5)^2} & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\frac{\partial h}{\partial u} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Part 2.

The CT Jacobian matrices show that the cooperative air-ground localization system is time varying. For this reason, we skipped observability and stability analysis. Furthermore, it is dependent on the state and inputs. In order simulate the DT LTV system, we linearize around a known nominal trajectory. We find this trajectory by solving the nonlinear ODEs with inital state $x = [10, 0, \pi/2, -60, 0, -\pi/2]^T$ and inputs $u = [2, -\pi/18, 12, \pi/2]^T$. We define "eulerized" DT Jacobians,

$$\tilde{F}_k = \Delta T * \tilde{A}_{nom[k]}$$

$$\tilde{G}_k = \Delta T * \tilde{B}_{nom[k]}$$

Where,

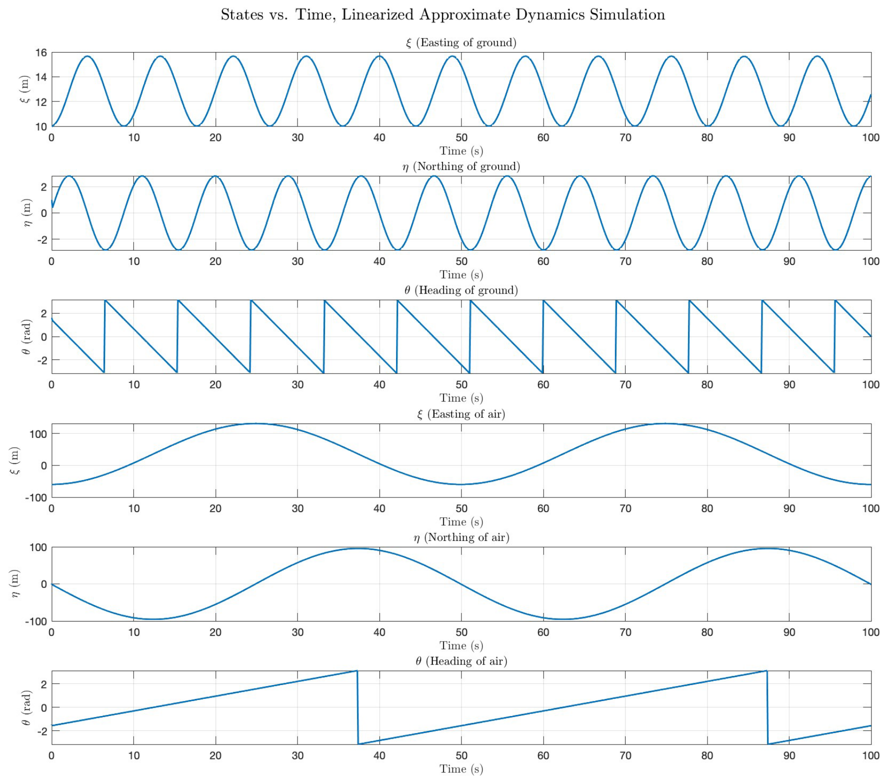
$$\tilde{A}_{nom[k]} = \frac{\partial f}{\partial x}\Big|_{t=t_k, nom[k]}$$

$$\tilde{B}_{nom[k]} = \frac{\partial f}{\partial u}\bigg|_{t=t_k, nom[k]}$$

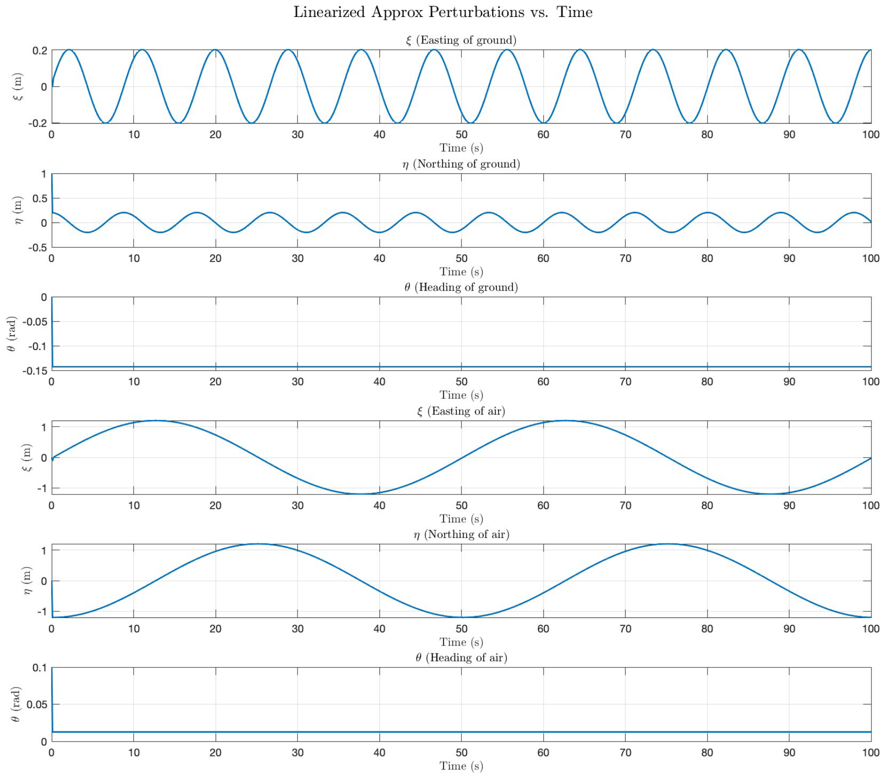
With these DT jacobians, we can model the system as,

$$x(k+1) = x_{nom,k} + \tilde{F}_k \delta x_k + \tilde{G}_k \delta u_k$$

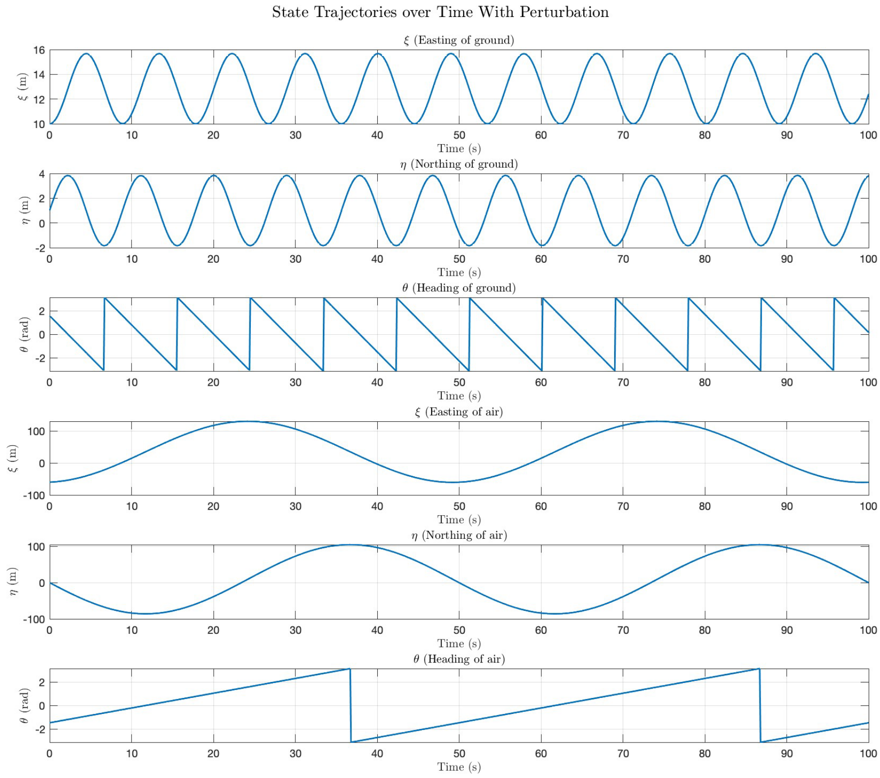
The full state results of modeling using initial perturbation state conditions of $\delta x = [0, 1, 0, 0, 0, 0.1]$ are shown below.



The perturbation states from the same simulation are shown below.



Part 3.
To validate the DT LTV model we simulated in part 2, we performed a full nonlinear simulation of the system with the same initial perturbation state conditions of $\delta x = [0, 1, 0, 0, 0, 0.1]$. We performed this simulation using ODE45 in MATLAB. The results of the simulation are shown below.



The linearized approximate dynamics simulation results are very close to that of the nonlinear simulation. By closely comparing the two plots, we were able to observe a very slight phase shift between the states of the two simulations, but the linearized simulation matches the nonlinear simulation very well.

Appendix

We have attached a link to our team GitHub below. All code used to simulate the system and generate plots can be found at:
`https://github.com/bernabei24/final_project_asen_5044_f24/tree/main`