Prerequisites from Linear Algebra and Matrix Theory

§1 Finite-Dimensional Complex Linear Spaces and Matrices.

1.1 Complex Linear n-space.

We define an n-vector x to be an ordered n-tuple of complex numbers $\xi_1, \xi_2, \ldots \xi_n$ which we write as a column:

$$x = \begin{bmatrix} \xi_1 \\ \cdots \\ \xi_n \end{bmatrix}$$

The set of all such vectors is denoted by $\mathbb{C}^n$. We denote complex conjugation by a bar:($\bar{z} = a - ib, z = a + ib$)

$$\bar{x} = \begin{bmatrix} \bar{\xi}_1 \\ \cdots \\ \bar{\xi}_n \end{bmatrix}$$

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The vector with zero components will be denoted by 0. The transpose of an n-vector x is an ordered n-tuple of complex numbers written as a row:

$$x^t = (\xi_1, \xi_2, \dots, \xi_n)$$

The Hermitian conjugate of x is defined by

$$x^* = \overline{x^t} = (\bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_n)$$

The vectors $x + y$ and λx, where λ is a complex number are defined by

$$\begin{bmatrix} \xi_1 + \eta_1 \\ \dots \\ \xi_n + \eta_n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda\eta_1 \\ \dots \\ \lambda\eta_n \end{bmatrix}$$

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Addition of vectors and multiplication of vectors by complex numbers satisfy the usual rules:

$$x + 0 = x, x + y = y + x, x + (y + z) = (x + y) + z, 1 \times x = x$$
$$\alpha(x + y) = \alpha x + \alpha y, (\alpha + \beta)x = \alpha x + \beta x, \alpha(\beta x) = (\alpha\beta)x$$

As usual $(x, y) = y^* x$ is the scalar product

$$(x, y) = y^* x = \xi_1 \bar{\eta}_1 + \ldots \xi_n \bar{\eta}_n \tag{1}$$

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It follows from the definition (1) that

$$
\begin{aligned}
(y,x) = \overline{(x,y)}, (\alpha x, \beta y) = \alpha\bar{\beta}(x,y) \\
(x+y,z) = (x,z) + (y,z), (x, y+z) = (x,y) + (x,z) \\
(x,x) = |\xi_1|^2 + \ldots + |\xi_n|^2 > 0, x \neq 0
\end{aligned}
$$

The norm (modulus, length) of a vector $x \in \mathbb{C}^n$ is denoted by $|x|$:

$$|x| = \sqrt{(x,x)} = \sqrt{|\xi_1|^2 + \ldots + |\xi_n|^2} \tag{2}$$

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The norm has the following properties:

1. $|x| > 0$ if $x \neq 0$
2. $|\bar{x}| = |x|$
3. $|\lambda x| = |\lambda||x|$ where $\lambda \in \mathbb{C}$ is a scalar
4. $|x + y| \leq |x| + |y|$ (triangle inequality)
5. $|(x, y)| \leq |x||y|$ (Cauchy-Schwartz-Bunyakovskii inequality)

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In general let V a vector space on the field $\mathbb{K}$. ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). We have the following:

Definition

A norm on V is an application $x \mapsto |x|$ from V to $[0, +\infty[$ such that for any vectors $x, y \in V$ and for every scalar $\lambda \in \mathbb{K}$ we have:

- $|x| = 0 \Rightarrow x = 0$
- $|\lambda x| = |\lambda||x|$ homogeneity
- $|x + y| \leq |x| + |y|$ triangle inequality

We then say that V is a **normed** space on $\mathbb{K}$.

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Let $p \in \mathbb{N}$ be a natural number ≥ 1. For $z \in \mathbb{C}^n$ the following also define norms on $\mathbb{C}^n$:

$$|z|_\infty = \max_{j=1}^{n} |z_j|, \qquad |z|_p = \left(\sum_{j=1}^{n} |z_j|^p \right)^{\frac{1}{p}}$$

$|\bullet|_\infty$ is called the *maximum norm*, $|\bullet|_p$ is called the p-norm. The norm in (2) is the 2- norm. All norms in $\mathbb{C}^n$ define the same topology.This is a consequence of the fact that, as we will show now, in finite dimensional space all norms are equivalent.

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Definition

Two norms N_1, N_2 on a vector space V are called equivalent, if there are constants $c, c' > 0$ such that

$$cN_1(x) \leq N_2(x) \leq c'N_1(x) \quad \text{for all} \quad x \in V$$

Proposition

On a finite-dimensional vector space V (over $\mathbb{R}$ or $\mathbb{C}$) all norms are equivalent.

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