Chapter 1

Matlab laboratories

Laboratory 1

Drawing realizations from a Gaussian process

Let $\boldsymbol{g} = \{\boldsymbol{g}(u),\ u \in \mathbb{R}^N\}$, with $N = 100$, be a Gaussian process with zero mean and covariance function

$$\mathbb{E}[\boldsymbol{g}(\tilde{u})\boldsymbol{g}(u)] = \lambda K_G(\tilde{u}, u)$$

where K_G denotes the Gaussian kernel

$$[K_G(\tilde{u}, u)]_{t,s} = \exp\left(-\frac{(\tilde{u}(t) - u(s))^2}{2\beta}\right). \tag{1.1}$$

We are interested in drawing realizations of $\boldsymbol{g}(u)$ with $u = [1 \dots N]^T$. In order to do that, we construct the kernel matrix $K = \lambda K_G(u, u)$ where $K_G(u, u)$ is obtained calling the function

```
Gaussian_kernel(u, ut,beta)
```

where the input arguments correspond to $u, \tilde{u}, \beta$, respectively, and the output is $K_G(u, \tilde{u})$. Then, we compute the squared root matrix L of K through the instruction `chol(K,'lower')` which is the lower triangular Cholesky factor of matrix K, i.e. $K = LL^T$. Then, a realization of $\boldsymbol{g}(u)$ is obtained by `L*e` where `e` is a realization of a Gaussian r.ve. of dimension N with zero mean and covariance matrix equal to the identity. The latter can be obtained using the Matlab instruction `randn`. Draw ten realizations of $\boldsymbol{g}(u)$ using the hyperparameters: i) $\lambda = 1$, $\beta = 20$; ii) $\lambda = 1$, $\beta = 1$; iii) $\lambda = 1$, $\beta = 150$, iv) $\lambda = 10$, $\beta = 20$.

Question 1: According to the results you found, which type of a priori information the kernel function K imposes? Motivate the answer.

Nonparametric system identification of a single-link manipulator

System and data description. We consider the single-link manipulator depicted in Figure 1.1: $\tau(t)$ is the torque generated by the actuator and $q_m(t)$ is the resulting angular position of the rotor of the actuator. We have two data sets collected from this manipulator. More precisely,

- First dataset:
  - inputs: $u1 = [q_m(1) \dots q_m(N)]^T$, $u1d = [\dot{q}_m(1) \dots \dot{q}_m(N)]^T$;
  - output: $y1 = [\tau(1) \dots \tau(N)]^T$.
- Second dataset:
  - inputs: $u2 = [q_m(1) \dots q_m(N)]^T$, $u2d = [\dot{q}_m(1) \dots \dot{q}_m(N)]^T$;
  - output: $y2 = [\tau(1) \dots \tau(N)]^T$.

Here, $N = 201$ and the sampling time is $T_s = 0.1\,s$. These data are stored in the file `manipulator.mat`.

Data plotting. Plot the inputs and the output of the first and of the second data set.

Computation of the acceleration. Each data set provides only the angular position and velocity of the rotor of the actuator, however, in what follows we will need also the acceleration. The latter can be estimated

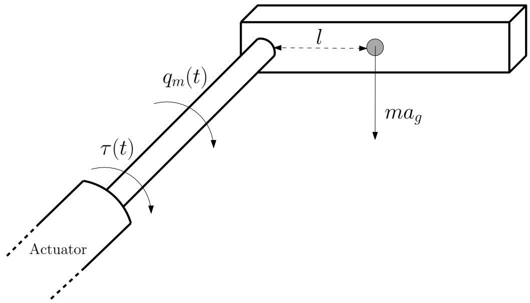

Figure 1.1: Schematic diagram of a single-link manipulator.

using the Matlab function `ud_dot=derivative(ud,Ts)` where `ud` denotes the velocity vector, `ud_dot` its derivative vector, and `Ts` the sampling time. This function computes the derivative using the backward Euler method. In addition, to reduce the effect of the noise, the latter is filtered with a first-order low pass filter.

Inverse dynamic. We want to estimate the inverse dynamic of the manipulator, that is to find a mathematical model with input q_m and with output τ, from the first data set. From physics, we know that the nonlinear dynamic relation between q_m and τ is given by

$$\tau(t) = J\ddot{q}_m(t) - 2lma_g \sin(q_m(t)) + \tau_f(t) \tag{1.2}$$

where J is the moment of inertia of the link; $2l$ and m are the length and the mass, respectively, of the link; a_g is the gravitational acceleration; τ_f is the friction (composed by stiction, viscous friction and stribeck effect) acting on the manipulator. Adopting the nonparametric Bayesian viewpoint, and in view of (1.2), one single measurement is modeled as

$$\boldsymbol{y}_1(t) = \boldsymbol{h}(x_1(t)) + \boldsymbol{w}(t), \quad t = 1 \dots N$$

where

$$x_1(t) = \begin{bmatrix} q_m(t) \\ \dot{q}_m(t) \\ \ddot{q}_m(t) \end{bmatrix} \in \mathbb{R}^3$$

is called input location at time t and the subscript 1 stresses the fact that $q_m(t)$, $\dot{q}_m(t)$ and $\ddot{q}_m(t)$ are taken from the first dataset; $\boldsymbol{h} = \{\boldsymbol{h}(u),\ u \in \mathbb{R}^3\}$ is a stochastic Gaussian process taking values in $\mathbb{R}$ which will be specified later; $\boldsymbol{y}_1(t)$ models the measured torque at time t in the first dataset. Moreover, we assume that $\boldsymbol{w}(t) \sim \mathcal{N}(0, \sigma^2)$ with $\sigma^2 = 4.2$, $\mathbb{E}[\boldsymbol{w}(t)\boldsymbol{w}(s)] = 0$ for any $t \neq s$. Stacking the N measurements we obtain

$$\boldsymbol{y}_1 = \boldsymbol{g}(x_1) + \boldsymbol{w}$$

where

$$\boldsymbol{g}(x_1) = \begin{bmatrix} \boldsymbol{h}(x_1(1)) \\ \vdots \\ \boldsymbol{h}(x_1(N)) \end{bmatrix}, \; x_1 = \begin{bmatrix} x_1^T(1) \\ \vdots \\ x_1^T(N) \end{bmatrix}, \; \boldsymbol{y}_1 = \begin{bmatrix} \boldsymbol{y}_1(1) \\ \vdots \\ \boldsymbol{y}_1(N) \end{bmatrix}, \; \boldsymbol{w} = \begin{bmatrix} \boldsymbol{w}(1) \\ \vdots \\ \boldsymbol{w}(N) \end{bmatrix}$$

and $\boldsymbol{w} \sim \mathcal{N}(0, \sigma^2 I_N)$. $\boldsymbol{g} = \{\boldsymbol{g}(x),\ x \in \mathbb{R}^{N\times 3}\}$ is a Gaussian process taking values in $\mathbb{R}^N$ with zero mean and such that

$$\mathbb{E}[\boldsymbol{g}(x_1)\boldsymbol{g}(x_2)^T] = \lambda K_G(x_1, x_2)$$

with $\lambda > 0$. Moreover, $\boldsymbol{w}$ is independent from $\boldsymbol{g}$. In what follows, we consider the Gaussian kernel defined

$$[K_G(x_1, x_2)]_{t,s} = \exp\left(-\frac{\|x_1(t) - x_2(s)\|^2}{2\beta}\right) \tag{1.3}$$

with $x_1(t), x_2(t) \in \mathbb{R}^3$ and $t, s = 1 \dots N$. Note that, the latter is the generalized version of the one in (1.1) (indeed, in this case $\boldsymbol{h}$ is indexed in a 3-dimensional space). Then, the MAP estimate of $\boldsymbol{g}$ is:

$$\hat{g}_{MAP}(\cdot) = \lambda K_G(\cdot, x_1)(\lambda K_G(x_1, x_1) + \sigma^2 I_N)^{-1} y_1$$

and the resulting model is

$$\boldsymbol{y}(t) = \hat{h}_{MAP}(x(t)) + \boldsymbol{w}(t). \tag{1.4}$$

Validation step. We use the second dataset to check whether model (1.4) is good or not. Let $y_2 = [\,y_2(1)\ \dots\ y_2(N)\,]^T$ be the vector containing the values of the applied torque in the second dataset. The prediction of y_2 from model (1.4) is:

$$\begin{aligned} \hat{y}_2 &= \hat{g}_{MAP}(x_2) \\ &= \lambda K_G(x_2, x_1)(\lambda K_G(x_1, x_1) + \sigma^2 I_N)^{-1} y_1 \end{aligned} \tag{1.5}$$

where $x_2(t) = [\,q_m(t)\ \dot{q}_m(t)\ \ddot{q}_m(t)\,]^T$ is the input location at time t with angular position, velocity and acceleration of the second dataset and $x_2 = [\,x_2(1) \dots x_2(N)\,]^T$. Compute the prediction (1.5) using the Gaussian kernel in (1.3) with $\lambda = 10^4$ and $\beta = 560$. The corresponding kernel matrices $K_G(x_1, x_1)$ and $K_G(x_2, x_1)$ are computed by calling the function

```
Gaussian_kernel(x,xt,beta);
```

the input arguments correspond to $x, \tilde{x}, \beta$, respectively, and the output is $K_G(x, \tilde{x})$.

Question 2: According to the results you found, does the estimated model describe well the manipulator system? Motivate the answer.

Validation with other values of λ and β. Do the same as in the previous point but using now the following values for λ and β:

	λ	β
Case 2	1	560
Case 3	10^4	50

Question 3: According to the results you found, which is the role of λ and β?