The application of bifurcation theory to the buckling of nonlinear elastic structures

Bernardo Ardini [2146900]

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1 Notation and basic notions

Operators in Banach spaces Given two Banach space X and E we denote by $\mathscr{L}(X,E)$ the set of linear and continuous maps from X into E. The set of compact linear maps from X into E is denoted by $\mathscr{K}(X,E)$. Given $A \in \mathscr{L}(X,E)$ we call $N(A)$ and $R(A)$ its null-space and range respectively. We say that $A \in \mathrm{Iso}(X,E)$ if $N(A)=\{0\}$ and $R(A)=E$. By closed graph theorem this implies that $A^{-1} \in \mathscr{L}(E,X)$.

Duality The symbol $(\cdot,\cdot)$ always denotes the inner product in L^2. The duality between a Banach space V and its dual V' is denoted by $\langle f, v\rangle$ for $v \in V$ and $f \in V'$. Sometimes we will write simply fv instead. In particular if $L \in \mathscr{L}(V, V')$ then $Lvw = \langle Lv, w\rangle$ for $v, w \in V$. We will use also notations like $Lv^2 = Lvv$ for $v \in V$.

Fréchet derivatives Given a C^1 map F from an open subset U of Banach space X into another Banach space E, its Fréchet derivative is a map $DF = F' \in C^0(U, \mathscr{L}(X, E))$. In particular if Π is a C^2 map from a Banach space V into $\mathbb{R}$, its Fréchet derivative is a map $\Pi' \in C^1(V, V')$. The double Fréchet derivative is a map $\Pi'' \in C^0(V, \mathscr{L}(V, V'))$. By Schwartz theorem $\Pi''(u)vw = \Pi''(u)wv$ for all $u \in V$ and $v, w \in V$. Similar considerations holds for higher order Fréchet derivatives.

2 Preliminary results

In this Section are collected some results on Banach spaces that will be useful later.

2.1 Orthogonal spaces

Let V be Banach space.

Definition 1. If $N \subset V$ and $R \subset V'$ are linear subspaces, we define their orthogonal spaces as

$$N^\perp = \{f \in V' \mid \langle f, v\rangle = 0\ \forall v \in N\} \quad \text{and} \quad R^\perp = \{v \in V \mid \langle f, v\rangle = 0\ \forall f \in R\}.$$

Remark 2. It is clear that $N^\perp$ and $R^\perp$ are closed respectively in V' and V.

Proposition 3. If $N \subset V$ and $R \subset V'$ are linear subspaces then $(N^\perp)^\perp = \overline{N}$ and $(R^\perp)^\perp \supset \overline{R}$. Moreover if V is reflexive then $(R^\perp)^\perp = \overline{R}$

Proof. It is obvious that $(N^\perp)^\perp \supset \overline{N}$ and $(R^\perp)^\perp \supset \overline{R}$.

By contradiction let $v_0 \in (N^\perp)^\perp$ such that $v_0 \notin \overline{N}$. Then by Hahn–Banach $\{v_0\}$ and $\overline{N}$ are strictly separated, so there exists $f \in V'$ and $\alpha \in \mathbb{R}$ such that

$$\langle f, v\rangle < \alpha < \langle f, v_0\rangle \quad \text{for all } v \in N.$$

Since N is a linear subspace, $\langle f, v\rangle = 0$ for all $v \in N$. Then it must be $\langle f, v_0\rangle > 0$. Moreover $f \in N^\perp$ so $\langle f, v_0\rangle = 0$. Contradiction.

Now assume that V is reflexive. By contradiction let $f_0 \in (R^\perp)^\perp$ such that $f_0 \notin \overline{R}$. Then by Hahn-Banach $\{f_0\}$ and $\overline{R}$ are strictly separated, so there exists $\xi \in V''$ and $\alpha \in \mathbb{R}$ such that

$$\langle \xi, f\rangle < \alpha < \langle \xi, f_0\rangle \quad \text{for all } f \in R.$$

Since R is a linear subspace, $\langle \xi, f\rangle = 0$ for all $f \in R$. Then it must be $\langle \xi, f_0\rangle > 0$. Since V is reflexive there exists $v \in V$ such that $\langle \xi, f\rangle = \langle f, v\rangle$ for all $f \in V'$. Then $v \in R^\perp$ so $\langle \xi, f_0\rangle = \langle f_0, v\rangle = 0$. Contradiction. □

2.2 Self-adjoint operators

Let V be a reflexive Banach space and $L \in \mathscr{L}(V, V')$.

Definition 4. We say that L is self-adjoint if $\langle Lv, w\rangle = \langle Lw, v\rangle$ for all $v, w \in V$.

Proposition 5. If L is self-adjoint and $R(L)$ is closed then $R(L) = N(L)^\perp$.

Proof. First we show that $R(L)^\perp = N(L)$. Indeed $w \in R(L)^\perp$ if and only if $\langle Lv, w\rangle = \langle Lw, v\rangle = 0$ for all $v \in V$ and this is equivalent to $Lw = 0$.

Then $(R(L)^\perp)^\perp = N(L)^\perp$ and the thesis follows by Proposition 3 and the fact that $R(L)$ is closed. □

2.3 Complemented spaces and codimension

Let E be a Banach space.

Definition 6. A closed subspace R of E is complemented if there exists a closed complement C, i.e., a closed subspace of E such that $E = R \oplus C$. The codimension of R is the dimension of C (it is obvious that it does not depend on C).

Remark 7. Every finite dimensional space R is complemented. Indeed take a basis $\{f_i\}_{i=1}^n$ of R and consider the functionals $\xi_i \in R'$ that send an $f \in R$ into its components along vector f_i. Then extend ξ_i as a functional in E' by Hahn–Banach theorem. Now a complement of R is simply

$$C = \{f \in E \mid \langle \xi_i, f\rangle = 0 \quad \forall i = 1, \ldots, n\}.$$

Indeed every $f \in E$ decompose uniquely as $f = \sum_{i=1}^n \langle \xi_i, f\rangle f_i + (f - \sum_{i=1}^n \langle \xi_i, f\rangle f_i)$.

We give below a necessary and sufficient condition for R to be complemented and of finite codimension.

Proposition 8. Let R be a closed subspace of E. Then R is complemented and of finite codimension n if and only if there exist $\{\xi_i\}_{i=1}^n \subset E'$ linearly independent such that

$$R = \{f \in E \mid \langle \xi_i, f\rangle = 0 \quad \forall i = 1, \ldots, n\}. \tag{1}$$

Proof. Assume that R has a complement C of finite dimension n. Let $\{f_i\}_{i=1}^n$ be a basis of C. Now given any $f \in E$, decompose it uniquely as $f = r + c$, where $r \in R$ and $c = \sum_{i=1}^n c_i f_i \in C$. We define $\{\xi_i\}_{i=1}^n \subset E'$ by $\langle \xi_i, f\rangle = c_i$. It is clear that (1) holds.

Conversely assume that there exist $\{\xi_i\}_{i=1}^n$ such that (1) holds. Then let $f_1 \in E$ such that $\langle \xi_1, f_1\rangle = 1$. Define

$$R_1 = \{f \in E \mid \langle \xi_1, f\rangle = 0\} \quad \text{and} \quad C_1 = \mathrm{span}(f_1).$$

It is clear that $E = R_1 \oplus C_1$. Indeed given $f \in E$ we write it uniquely as $f = (f - \langle \xi_1, f\rangle f_1) + \langle \xi_1, f\rangle f_1$. Now define $\tilde{\xi}_2 = \xi_2 - \langle \xi_2, f_1\rangle \xi_1$ so that $\langle \tilde{\xi}_2, f_1\rangle = 0$. Let $f_2 \in R_1$ such that $\langle \tilde{\xi}_2, f_2\rangle = 1$. If we set

$$R_2 = \{f \in E \mid \langle \xi_1, f\rangle = 0 \text{ and } \langle \tilde{\xi}_2, f\rangle = 0\} \quad \text{and} \quad C_2 = \mathrm{span}(f_1, f_2),$$

it is easy to prove that $E = R_2 \oplus C_2$. By induction we construct $\{\tilde{\xi}_i\}_{i=1}^n$, with $\tilde{\xi}_1 = \xi_1$, and $\{f_i\}_{i=1}^n$ such that if

$$R_n = \{f \in E \mid \langle \tilde{\xi}_i, f\rangle = 0 \quad \forall i = 1, \ldots, n\}. \quad \text{and} \quad C_n = \mathrm{span}(\{f_i\}_{i=1}^n),$$

then $E = R_n \oplus C_n$. The span of $\{\tilde{\xi}_i\}_{i=1}^n$ coincides with the span of $(\{\xi_i\}_{i=1}^n)$ so $R = R_n$. Moreover $\{f_i\}_{i=1}^n$ are linearly independent, so $C = C_n$ is a complement of R of dimension n. □

We get the following corollary that relates the dimension of the nullspace of an adjoint operator with the codimension of its range.

Corollary 9. Let V a reflexive Banach space and $L \in \mathscr{L}(V, V')$ a self-adjoint operator with $R(L)$ closed and $N(L)$ of finite dimension. Then $R(L)$ is complemented and $\operatorname{codim}(R(L)) = \dim(N(L))$.

Proof. Let $\{v_i\}_{i=1}^n$ be a basis of $N(L)$. Since V is reflexive there exist unique $\{\xi_i\}_{i=1}^n \subset V''$ such that $\langle \xi_i, f\rangle = \langle f, v_i\rangle$ for any $f \in V'$. By Proposition 5 we have that

$$R(L) = N(L)^\perp = \{f \in V' \mid \langle \xi_i, f\rangle = 0 \quad \forall i = 1, \dots, n\}.$$

The thesis follows by Proposition 8. □

3 Bifurcation theory in Banach spaces

Let X and E be Banach spaces. Consider a map F of class C^r, with $r \geq 1$, from a neighborhood of $x_0 \in X$ into E such that $F(x_0) = 0$. We are concerned with the equation

$$F(x) = 0$$

when x is near x_0. By making a change of variable we can always assume that $x_0 = 0$. When the inverse function theorem applies the set of solutions around the origin is a regular manifold. When it does not apply it can be more complicated and bifurcations can occur.

In this Section we first recall the Implicit Function Theorem. Then we present the Lyapunov-Schmidt procedure, a tool allowing us to reduce the study of the set of solutions to the equation $F(x) = 0$ to a simpler problem involving only finite dimensional spaces. Then we will prove a theorem that gives sufficient conditions for the existence of a bifurcation. All the results of this Section are taken from [7].

3.1 The Implicit Function Theorem

The following theorem is a well known form of the Implicit Function Theorem.

Theorem 10. Let V, Λ and E be Banach spaces and $U \subset V \times \Lambda$ open. Let $F = F(u, \lambda) \in C^r(U, E)$ with $r \geq 1$. Let $(u_0, \lambda_0) \in U$ such that $F(u_0, \lambda_0) = 0$ and $D_u F(u_0, \lambda_0) \in \operatorname{Iso}(V, E)$. Then there exists $\delta > 0$ and a unique $u \in C^r(B(\lambda_0, \delta), V)$ such that $u(\lambda_0) = u_0$ and

$$F(u(\lambda), \lambda) = 0$$

for all $\lambda \in B(\lambda_0, \delta)$.

We report below an alternative form of the Implicit Function Theorem which is more suitable for treating the problem described at the beginning of this Section.

Corollary 11. Let X and E be a Banach spaces and $U \subset X$ open. Let $F \in C^r(U, E)$ with $r \geq 1$. Let $0 \in U$ and assume that

(i) $F(0) = 0$,

(ii) $N(DF(0)) = X_1$ has a complement X_2,

(iii) $R(DF(0)) = E$.

Then there exists $\delta > 0$ such that for all $x_1 \in X_1$ with $\|x_1\| \leq \delta$ there exists a unique $x_2 = \chi(x_1) \in X_2$ such that

$$F(x_1 + x_2) = 0.$$

Moreover $\chi(0) = 0$ and χ is a C^r map from $B(0, \delta) \subset X_1$ into X_2.

Proof. Define $G(x_2, x_1) = F(x_1 + x_2)$ for $x_1 \in X_1$ and $x_2 \in X_2$ with $x_1 + x_2 \in U$. The map G is from a neighborhood of $(0,0)$ in $X_2 \times X_1$ into E. By (i) we have $G(0,0) = 0$. Moreover $D_{x_2}G(0,0) = DF(0,0) \in \mathrm{Iso}(X_2, E)$ because of (ii) and (iii).

Then by Theorem 10 with X_2 and X_1 instead of V and Λ respectively, we have the thesis. □

3.2 The Lyapunov–Schmidt procedure

Let X and E be Banach spaces and F a map of class C^r, with $r \geq 1$, from a neighborhood of the origin in X into E such that $F(0) = 0$. Assume that $DF(0)$ is Fredholm, i.e.,

(a) $N(DF(0)) = X_1$ has finite dimension n and

(b) $R(DF(0)) = E_1$ is a closed subspace of finite codimension.

Let X_2 and E_2 some complements of X_1 and E_1 respectively. Then consider the projection Q into E_1 associated to the decomposition $E = E_1 \otimes E_2$. Now equation $F(x) = 0$ can be written as

$$QF(x) = 0 \quad \text{and} \quad (I - Q)F(x) = 0.$$

Let $G(x) = QF(x)$ where G is defined in a neighborhood of 0 into E_1. Now by applying Corollary 11 we have that there exists χ of class C^r from a neighborhood of the origin in X_1 into X_2 such that $\chi(0) = 0$ and

$$QF(x_1 + \chi(x_1)) = 0.$$

Then it remains only to solve the following *bifurcation equation*:

$$(I - Q)F(x_1 + \chi(x_1)) = 0 \quad \text{for } x_1 \in X_1 \text{ near the origin.} \tag{2}$$

Since X_1 and E_2 are finite dimensional it is a fully finite dimensional problem.

If $E_1 = E$ then $E_2 = 0$ and equation (2) is trivially satisfied. In this case the solutions of $F(x) = 0$ around 0 are of the form $x_1 + \chi(x_1)$ so they form a C^r manifold of dimension n.

When $R(DF(0)) \neq E$ bifurcations can occur. We confine ourself with the case in which $\mathrm{codim}(E_1) = \dim(E_2) = 1$. In this case equation (2) is actually a scalar equation. Indeed by Proposition 8 there exists $\xi \in E'$ such that $E_1 = \{f \in E \mid \langle \xi, f \rangle = 0\}$, so the bifurcation equation can be rewritten as

$$G(x_1) = \langle \xi, F(x_1 + \chi(x_1)) \rangle = 0.$$

By choosing a basis in X_1, the function G can be understood as a function from a neighborhood of the origin in $\mathbb{R}^n$ into $\mathbb{R}$.

3.3 A local bifurcation theorem

We now want to understand under which conditions from equation $G(x_1) = 0$ arise a bifurcation. The following result will be useful.

Lemma 12 (Morse lemma)**.** Let G a C^r map, with $r \geq 2$, from a neighborhood of the origin in $\mathbb{R}^n$ into $\mathbb{R}$. Assume that

(i) $G(0) = 0$,

(ii) $DG(0) = 0$ and

(iii) $D^2G(0)$ is a nonsingular n-by-n matrix.

Then there exists a C^{r-2} change of variable $y = y(x)$ around the origin such that $y(0) = 0$ and
$$G(x) = \frac{1}{2} y(x)^T D^2 G(0) y(x).$$
Moreover if $r \geq 3$ then $Dy(0) = I$.

We get immediately the following corollary.

Corollary 13. Assume that the hypotheses of Morse lemma holds. Assume also that $n = 2$ and that $D^2G(0)$ is indefinite. Then the set of solutions of the equation $G(x) = 0$ near the origin is made by two C^{r-2} curves intersecting only at the origin. Moreover if $r \geq 3$ they intersect transversally.

Proof. The equation $\frac{1}{2} y^T D^2 G(0) y$ admits as a solution set two distinct lines intersecting at the origin. Indeed we can assume that
$$D^2G(0) = \begin{pmatrix} \frac{1}{a_1^2} & 0 \\ 0 & -\frac{1}{a_2^2} \end{pmatrix}$$
so we have
$$\frac{y_1^2}{a_1^2} - \frac{y_2^2}{a_2^2} = 0$$
which is exactly the equation of two lines intersecting at the origin.

Applying the change of variable, we find that the set of solutions to $G(x) = 0$ is a deformation of these two lines. When $r \geq 3$ then the change of variable is at least C^1 and the Jacobian matrix is the identity at the origin so the two curves still intersect transversally. □

Now we can obtain the main result of this Section.

Theorem 14. Let X and E be Banach spaces and F a map of class C^r, with $r \geq 2$, from a neighborhood of the origin in X into E such that $F(0) = 0$. Assume that

(i) $DF(0)$ is Fredholm,

(ii) $N(DF(0))$ has dimension 2,

(iii) $R(DF(0))$ is closed and has codimension 1.

Let $\xi \in E'$ such that $R(DF(0)) = \{f \in E \mid \langle \xi, f \rangle = 0\}$. Define a 2-by-2 matrix M as the Hessian of the map
$$\begin{aligned} N(DF(0)) &\to \mathbb{R} \\ x &\mapsto \langle \xi, F(x) \rangle, \end{aligned}$$
computed at the origin and expressed with respect to some basis in $N(DF(0))$. If M is nonsingular and indefinite then the set of solution to $F(x) = 0$ near the origin is made of two C^{r-2} curves intersecting only at the origin. If $r \geq 3$ then they intersect transversally.

Proof. We have only to show that $DG(0) = 0$ and $D^2G(0) = M$. Indeed by applying Corollary 13 we would get the thesis.

Step 1. For $v \in N(DF(0))$
$$DG(x)v = \langle \xi, DF(x + \chi(x))(v + D\chi(x)v) \rangle \quad \text{so} \quad DG(0)v = \langle \xi, DF(0)(v + D\chi(0)v) \rangle = 0,$$

where last equality holds by definition of ξ.

Step 2. From

$$F(x + \chi(x)) = 0 \quad \text{when } x \in N(DF(0)) \text{ is near } 0,$$

we get by differentiating that

$$DF(0)v + DF(0)D\chi(0)v = 0$$

for all $v \in N(DF(0))$. But $DF(0)v = 0$ so $DF(0)D\chi(0)v = 0$. It follows that $D\chi(0)v \in N(DF(0)) \cap X_2$ so $D\chi(0) = 0$.

Step 3. For $v, w \in N(DF(0))$ we have

$$\begin{aligned} D^2G(0)vw &= \langle \xi, D^2F(0)(v + D\chi(0)v)(w + D\chi(0)w)\rangle + \langle \xi, DF(0)D^2\chi(0)vw\rangle = \\ &= \langle \xi, D^2F(0)vw\rangle = Mv \cdot w. \end{aligned}$$

where we have used the definition of ξ and the previous step to say that $D\chi(0) = 0$. □

4 Buckling of general nonlinear elastic structures

Consider an elastic structure whose kinematically admissible configurations are described by the elements u of a reflexive Banach space V. Assume that the energy of the structure comprehensive of both the elastic energy and the energy of the loads is

$$\Pi \in C^{r+1}(V \times \mathbb{R}) \quad \text{with } r \geq 2, \qquad (u, \lambda) \mapsto \Pi(u, \lambda),$$

where $\lambda \in \mathbb{R}$ describes the intensity of the loads. We will see examples of such structures in the following sections.

Remark 15. In what follows we will denote by $\cdot'$ the Fréchet derivatives with respect to u and by $\hat{\cdot}$ the derivatives with respect to λ. The Fréchet derivative with respect to (u, λ) is denoted by $D\cdot$.

The out of balance force acting on the structure is defined as $F = \Pi' \in C^2(V \times \mathbb{R}, V')$. Hence we say that $u \in V$ is an equilibrium configuration corresponding to a load $\lambda \in \mathbb{R}$ if $F(u, \lambda) = 0$. We are interested in the study of the solution set of the equilibrium equation

$$F(u, \lambda) = 0.$$

Remark 16. By Schwartz theorem $F'(u, \lambda) = \Pi''(u, \lambda)$ is a self-adjoint operator in $\mathscr{L}(V, V')$ for all $(u, \lambda) \in V \times \mathbb{R}$. Indeed $\Pi''(u, \lambda)vw = \Pi''(u, \lambda)wv$ for $v, w \in V$. Moreover in concrete problems it happens that $R(F'(u, \lambda))$ is closed for all $(u, \lambda) \in V \times \mathbb{R}$. Then by Proposition 5 we have $R(F'(u, \lambda)) = N(F'(u, \lambda))^{\perp}$.

4.1 Critical points

Let $(u_0, \lambda_0) \in V \times \mathbb{R}$ such that $N(F'(u_0, \lambda_0))) = \{0\}$. Then by Remark 16 we have that $R(F'(u_0, \lambda_0)) = V'$. This means that $F'(u_0, \lambda_0) \in \mathrm{Iso}(V, V')$. By the Implicit Function Theorem then the solutions near (u_0, λ_0) lies on a curve of the form $\lambda \mapsto (u(\lambda), \lambda)$ where u is a C^2 map defined in a neighborhood of λ_0 into V.

The structure of the solution set can be more complicated near a critical point.

Definition 17. A point $(u_c, \lambda_c) \in V \times \mathbb{R}$ is called a *critical point* if $N(F'(u_c, \lambda_c)) \neq \{0\}$.

Remark 18. For simplify the notation we will simply write a subscript c to denote quantities computed at (u_c, λ_c).

We will confine ourself to the case in which $N(F'_c) = \mathrm{span}(v_c)$ has dimension one. In this case we distinguish two different types of critical points.

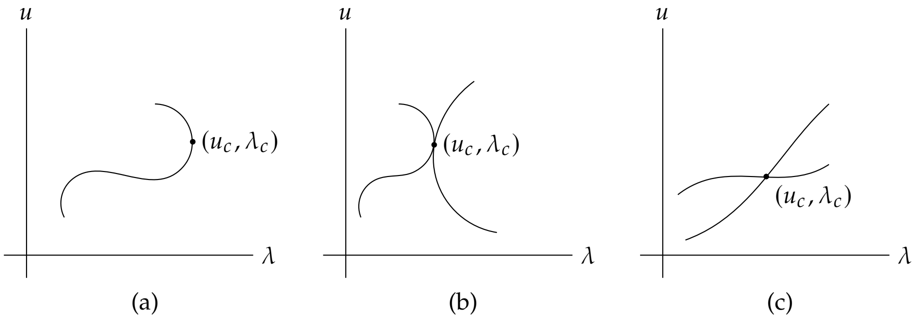

Figure 1: Examples of critical points.

Limit points If $\hat{F}_c v_c \neq 0$ then (λ_c, u_c) is called a *limit point*. Suppose there exists a C^1 curve of solutions $t \mapsto (u(t), \lambda(t))$ defined in a neighborhood of $t = 0$ such that $(u_0, \lambda_0) = (u_c, \lambda_c)$. (We mark with a 0 subscript quantities computed at $t = 0$). From $0 = F(u, \lambda)$, by differentiating with respect to t we get

$$0 = F'_c \dot{u}_0 + \dot{\lambda}_0 \hat{F}_c. \tag{3}$$

Acting with this equality on v_c and using that F'_c is self-adjoint we have

$$0 = F'_c \dot{u}_0 v_c + \dot{\lambda}_0 \hat{F}_c v_c = F'_c v_c \dot{u}_0 + \dot{\lambda}_0 \hat{F}_c v_c = \dot{\lambda}_0 \hat{F}_c v_c.$$

Since $\hat{F}_c v_c \neq 0$ it follows that $\dot{\lambda}_0 = 0$ and substituting back into (3) we have $\dot{u}_0 \in N(F'_c)$. The typical situation is depicted in Figure 1 (a).

Remark 19. Clearly there can be more then one curve passing through (u_c, λ_c). In this case all this curves would intersect tangentially but (u_c, λ_c) would not a limit point but actually a bifurcation point. See Figure 1 (b). Anyway we will not deal with such situation.

Bifurcation points If $\hat{F}_c v_c = 0$ then (λ_c, u_c) is called a *bifurcation point*. To motivate such denomination lets argue as follows.

Step 1. Assume there exists a C^2 curve $\lambda \mapsto (u^*(\lambda), \lambda)$ defined in a neighborhood of λ_c and passing through (u_c, λ_c). We refer to it as the fundamental path.

Step 2. We look for others C^2 curves of solutions of the form

$$t \mapsto (u(t), \lambda(t)) = (u^*(\lambda(t)) + w(t), \lambda(t)),$$

where $\lambda_0 = \lambda_c$, $w_0 = 0$ and t lies in a neighborhood of 0.

Step 3. By differentiating $0 = F(u, \lambda)$ with respect to t we have

$$0 = F'_c \dot{u}_0 + \dot{\lambda}_0 \hat{F}_c = \dot{\lambda}_0 (F'_c \hat{u}^*_c + \hat{F}_c) + F'_c \dot{w}_0 = F'_c \dot{w}_0.$$

We used that also (u^*, λ) is a curve of solutions. It follows that $\dot{w}_0 = \alpha v_c$ for some $\alpha \in \mathbb{R}$.

Step 3. Differentiating again with respect to t we get

$$0 = F'_c \ddot{u}_0 + F''_c \dot{u}_0^2 + 2\dot{\lambda}_0 \hat{F}'_c \dot{u}_0 + \ddot{\lambda}_0 \hat{F}_c + \dot{\lambda}_0^2 \hat{\hat{F}}_c. \tag{4}$$

We act with this equality on v_c so, using $F'_c v_c = 0$ and $\hat{F}_c v_c = 0$, we have

$$0 = F''_c \dot{u}_0^2 v_c + 2\dot{\lambda}_0 \hat{F}'_c \dot{u}_0 v_c + \dot{\lambda}_0^2 \hat{\hat{F}}_c v_c.$$

Hence

$$0 = \dot{\lambda}_0^2 (F''_c (\hat{u}^*_c)^2 v_c + 2\hat{F}' \hat{u}^*_c v_c + \hat{\hat{F}}_c v_c) + 2\dot{\lambda}_0 F''_c \hat{u}^*_c \dot{w}_0 v_c + F''_c \dot{w}_0^2 v_c + 2\dot{\lambda}_0 \hat{F}'_c \dot{w}_0 v_c.$$

Using that (u^*, λ) is a curve of solutions the first term vanish and we are left with

$$0 = F_c'' v_c^3 \alpha^2 + 2\dot{\lambda}_0 (F_c'' \hat{u}_c^* v_c^2 + \hat{F}_c' v_c^2)\alpha. \tag{5}$$

The solution $\alpha = 0$ correspond to the fundamental path. Suppose $F_c'' v_c^3 \neq 0$. If $F_c'' \hat{u}_c^* v_c^2 + \hat{F}_c' v_c^2 = 0$ then the only solution is $\alpha = 0$ and no bifurcation occur. (Actually there could be a bifurcation if another curve of solution intersect the fundamental path tangentially as in Figure 1 (b)). If instead $F_c'' \hat{u}_c^* v_c^2 + \hat{F}_c' v_c^2 \neq 0$ a solution with $\alpha \neq 0$ (and $\dot{\lambda}_0 \neq 0$) is admitted. This suggest that another curve of solutions intersect the fundamental path at (u_c, λ_c) transversally. The situation is represented in Figure 1 (c).

4.2 A local bifurcation theorem suitable for the elastic buckling analysis

We now want to understand under which conditions at a bifurcation point, as defined in the previous paragraph, a bifurcation actually occurs. Thanks to the insight just gained by the previous computations, we claim the following theorem. It is the main theorem of this work. We obtained the proof by generalizing that of [7, Theorem 3.3.3].

Theorem 20. Let V be a reflexive Banach space and F a C^r map, with $r \geq 2$, from a neighborhood of $(u_c, \lambda_c) \in V \times \mathbb{R}$ into V'. Suppose

(i) $F_c = 0$,

(ii) $F_c' \in \mathscr{L}(V, V')$ is self-adjoint and $R(F_c')$ is closed,

(iii) $N(F_c') = \text{span}(v_c)$ is one dimensional.

Suppose also that there exists $(\hat{u}_c^*, 1) \in V \times \mathbb{R}$ such that

(iv) $F_c' \hat{u}_c^* + \hat{F}_c = 0$,

(v) $F_c'' (\hat{u}_c^*)^2 v_c + 2\hat{F}_c' \hat{u}_c^* v_c + \hat{\hat{F}}_c v_c = 0$ and

(vi) $F_c'' \hat{u}_c^* v_c^2 + \hat{F}_c' v_c^2 \neq 0$.

Then at (u_c, λ_c) a bifurcation occur. In fact the set of solutions of $F(u, \lambda) = 0$ near (u_c, λ_c) consists of two C^{r-2} curves intersecting only at (u_c, λ_c). If $r \geq 3$ they intersect transversally.

Proof. We want to apply Theorem 14 to map F. The proof if complete if we verify the hypotheses of that theorem. This is done in the steps below.

Step 1. We prove that $N(DF_c)$ is of dimension 2 and spanned by $(v_c, 0)$ and $(\hat{u}_c^*, 1)$. By (ii) and (iv) it is clear that these two vectors both belong to $N(DF_c)$ and are linearly independent. Let $(\dot{u}_0, \dot{\lambda}_0) \in N(DF_c)$. We need only to prove that it is a linear combination of $(v_c, 0)$ and $(\hat{u}_c^*, 1)$. We have

$$\dot{\lambda}_0 F_c' \hat{u}_c^* + \dot{\lambda}_0 \hat{F}_c = 0 \quad \text{and} \quad F_c' \dot{u}_0 + \dot{\lambda}_0 \hat{F}_c = 0.$$

By making the difference

$$F_c' (\dot{u}_0 - \dot{\lambda}_0 \hat{u}_c^*) = 0,$$

so there exists $\beta \in \mathbb{R}$ such that $\dot{u}_0 = \dot{\lambda}_0 \hat{u}_c^* + \beta v_c$. Hence

$$(\dot{u}_0, \dot{\lambda}_0) = \dot{\lambda}_0 (\hat{u}_c^*, 1) + \beta (v_c, 0) \in \text{span}\left(\{(v_c, 0), (\hat{u}_c^*, 1)\}\right).$$

Step 2. We prove that $R(DF_c) = R(F_c')$. From this would follow that $R(DF_c)$ is closed and of codimension 1 (see Remark 16). The general element of $f \in R(DF_c)$ can be written as

$$f = F_c' v + \beta \hat{F}_c \quad \text{for } v \in V \text{ and } \beta \in \mathbb{R}.$$

Now from hypothesis (iv) we have that $\hat{F}_c = -F'_c \hat{u}^*_c$. This means that actually $f \in R(F'_c)$.

Step 3. From Remark 16 we know that

$$R(F'_c) = \{f \in V' \mid \langle f, v_c \rangle = 0\} = \{f \in V' \mid \langle \xi_c, f \rangle = 0\},$$

where $\xi_c \in V''$ is the element corresponding to v_c. We now compute the matrix M defined as the Hessian of the map

$$\begin{aligned} &N(DF(0)) \to \mathbb{R} \\ &\qquad (w, \mu) \mapsto \langle \xi_c, F(w, \mu) \rangle = \langle F(w, \mu), v_c \rangle. \end{aligned}$$

By using $(v_c, 0)$ and $(\hat{u}^*_c, 1)$ as a basis of $N(DF_c)$, after some computations we get

$$M = \begin{pmatrix} F''_c v_c^3 & F''_c \hat{u}^*_c v_c^2 + \hat{F}'_c v_c^2 \\ F''_c \hat{u}^*_c v_c^2 + \hat{F}'_c v_c^2 & F''_c (\hat{u}^*_c)^2 v_c + 2\hat{F}'_c \hat{u}^*_c v_c + \hat{\hat{F}}_c v_c. \end{pmatrix}$$

Notice that by (v) and (vi) it follows that $\det(Q) < 0$. So Q is nonsingular and indefinite. □

Remark 21. In the discussion of previous Subsection we supposed that $F''_c v_c^3 \neq 0$. The theorem ensure the existence of a bifurcating curve intersecting transversally the fundamental path even without this additional assumption.

4.3 Koiter asymptotic analysis

Theorem 20 make rigorous the asymptotic analysis of buckled structures introduced by Koiter [6] in 1942. It consists in the asymptotic approximation of the bifurcating curve near the bifurcation point. We will give now a brief description of it. In the following discussion we assume that $r \geq 6$.

Fundamental path Let $\lambda \mapsto (u^*(\lambda), \lambda)$ be a C^{r-2} curve of solutions defined for $\lambda \in [0, \Lambda]$. We call it the *fundamental path* and we assume it is known. In practice this path describes the deformation of the structure before the onset of instability, starting from its reference configuration $u = 0$, when $\lambda = 0$, and then increasing the load.

Critical load Let $\lambda_c \in (0, \Lambda)$ be the *critical load*, i.e., the smallest load such that $(u^*_c, \lambda_c) = (u_c, \lambda_c)$ is a critical point. Assume that all the hypotheses of Theorem 20 are satisfied. Hence there exists a C^{r-2} curve $t \mapsto (u(t), \lambda(t))$ intersecting the fundamental path transversally at (u_c, λ_c). Assume that $(u_0, \lambda_0) = (u_c, \lambda_c)$.

Elastic buckling In practice the equilibrium solutions $(u^*(\lambda), \lambda)$ for $\lambda > \lambda_c$ are unstable. This means that when the critical load is reached the fundamental path is abandoned and the equilibrium is established on the bifurcating curve which we will call *buckling path*. This phenomenon is indeed called *elastic buckling*.

Remark 22. In this work we will not deal with the stability of elastic equilibrium so we accept the previous consideration as granted. For possible definitions of stability of elastic equilibrium see [6, Chapter 2].

First order approximation The computations made in Subsection 4.1 when the bifurcations points where introduced, are completely justified by Theorem 20. So we can write the buckling path as

$$t \mapsto (u(t), \lambda(t)) = (u^*(\lambda(t)) + w(t), \lambda(t)),$$

where $w(\cdot)$ and $\lambda(\cdot)$ are C^{r-2} curves vanishing at $t = 0$. We choose $\alpha = 1$ so $\dot{w}_0 = v_c$. A different choice corresponds to a different parametrization of the curve. From equation (5) it follows that

$$\dot{\lambda}_0 = -\frac{1}{2}\frac{F_c'' v_c^3}{F_c'' \hat{u}_c^* v_c^2 + \hat{F}_c' v_c^2}. \tag{6}$$

In this way we obtain the first order approximation of the buckling path:

$$(u(t), \lambda(t)) = (u^*(\lambda_c + \dot{\lambda}_0 t) + v_c t, \lambda_c + \dot{\lambda}_0 t) + \mathcal{O}(t^2). \tag{7}$$

Second order approximation We start from equation (4). Substituting onto it $\dot{u}_0 = \dot{\lambda}_0 \hat{u}_c^* + v_c$ and $\ddot{u}_0 = \ddot{\lambda}_0 \hat{u}_c^* + \dot{\lambda}_0^2 \hat{\hat{u}}_c^* + \ddot{w}_0$ we have

$$0 = \ddot{\lambda}_0 (F_c' \hat{u}_c^* + \hat{F}_c) + \dot{\lambda}_0^2 (F_c' \hat{\hat{u}}_c^* + F_c'' \hat{u}_c^* + 2F_c' \hat{u}_c^* + \hat{\hat{F}}_c) + F_c' \ddot{w}_0 + F_c'' v_c^2 + 2\dot{\lambda}_0 (F_c'' \hat{u}_c^* v_c + \hat{F}_c' v_c).$$

The first two term vanish so we obtain the following equation for $\ddot{w}_0$:

$$F_c' \ddot{w}_0 = -F_c'' v_c^2 - 2\dot{\lambda}_0 (F_c'' \hat{u}_c^* v_c + \hat{F}_c' v_c). \tag{8}$$

This equation does not have a unique solution since $N(F_c') \neq \{0\}$. (From Theorem 20 we know that a solution must exists. This can also be verified by noticing that the right hand side lies in $N(F_c')^\perp = R(F_c')$ by using the expression for $\dot{\lambda}_0$ above.) So we choose one possible solution $\ddot{w}_0$. A different choice correspond to a different parametrization of the buckling curve. After some tedious calculations we obtain

$$\begin{aligned}\ddot{\lambda}_0 = &- \frac{F_c''(v_c^2 \ddot{w}_0 + \dot{\lambda}_0 v_c \ddot{w}_0 \hat{u}_c^* + \dot{\lambda}_0^2 v_c^2 \hat{\hat{u}}_c^*)}{\hat{F}_c' v_c^2 + F_c'' v_c^2 \hat{u}_c^*} - \\ &- \frac{F_c''' \left(\frac{1}{3} v_c^4 + \dot{\lambda}_0 v_c^3 \hat{u}_c^* + \dot{\lambda}_0^2 v_c^2 (\hat{u}_c^*)^2\right)}{\hat{F}_c' v_c^2 + F_c'' v_c^2 \hat{u}_c^*} - \\ &- \frac{\dot{\lambda}_0 (\hat{F}_c' v_c \ddot{w}_0 + \hat{F}_c'' v_c^3) + \dot{\lambda}_0^2 (2\hat{F}_c'' v_c^2 \hat{u}_c^* + \hat{\hat{F}}_c' v_c^2)}{\hat{F}_c' v_c^2 + F_c'' v_c^2 \hat{u}_c^*}.\end{aligned} \tag{9}$$

In this way the second order approximation of the buckling path is obtained:

$$(u(t), \lambda(t)) = \left(u^*\left(\lambda_c + \dot{\lambda}_0 t + \frac{1}{2}\ddot{\lambda}_0 t^2\right) + v_c t + \frac{1}{2}\ddot{w}_0 t^2, \lambda_c + \dot{\lambda}_0 t + \frac{1}{2}\ddot{\lambda}_0 t^2\right) + \mathcal{O}(t^3).$$

5 Buckling of Euler beam under vertical loads

In this Section we apply the theory developed in the previous one, to the classical model of Euler beam or "elastica". This models was proposed by Euler in 1744 in the appendix *De Curvis Elasticis* of his work *Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes*.

5.1 The Euler beam model

Kinematics Consider an elastic beam in the plane. Its reference configuration be the segment $\{0\} \times [0, L]$. The generic deformed configuration is described by the map $p\colon [0, L] \to \mathbb{R}^2$ where $p(x)$ represent the position on the plane of the point initially at position $(0, x)$. We assume that the beam is inextensible and that it is stuck at one end so $p(0) = 0$. Then p can be parametrized as

$$p(x) = \int_0^x \begin{pmatrix} \sin u(y) \\ \cos u(y) \end{pmatrix} dy$$

where $u\colon [0, L] \to \mathbb{R}$ and $u(x)$ represent the angle between the vertical direction and the tangent unit vector to the beam at position x.

Statics Consider the portion of the beam corresponding to positions in the interval $[x_0, x]$. Let $n^+(x)$ and $n^-(x_0)$ the internal forces acting on this portion in its extremities. Let also $f(y)$ be the external force per unit length acting at position y. Then the balance of forces reads

$$n^+(x) - n^-(x_0) + \int_{x_0}^x f(y)dy = 0.$$

By choosing $x_0 = x$ and supposing f to be absolutely continuous with respect to Lebesgue measure, it follows that $n^+(x) = -n^-(x)$ which we call simply $n(x)$. By differentiating with respect to x the above balance of forces we obtain its equivalent local version

$$n' + f = 0.$$

Now call $m^+(x)$ and $m^-(x_0)$ the internal torques acting at the extremities of the portion and let $g(y)$ be the torque per unit length acting at position y. Then the balance of torques is

$$m^+(x) + m^-(x_0) + p(x) \times n(x) - p(x_0) \times n(x_0) + \int_{x_0}^x (g(y) + p(y) \times f(y))\, dy.$$

Assuming g absolutely continuous we get $m^+(x) = -m^-(x) = m(x)$ and the local form of the balance of torques is

$$m' + p' \times n + p \times n' + g + p \times f = 0,$$

or equivalently

$$m' + p' \times n + g = 0.$$

Remark 23. All the torques are orthogonal to the plane of the beam. In particular we can write $m = Me_3$ and $g = Ge_3$ where e_3 is the unit vector orthogonal to the plane.

Euler's constitutive assumption The elastica theory is characterized by the following constitutive assumption. That the internal moment at position x is given by

$$M(x) = -EI(x)u'(x),$$

where $EI(x) > 0$ is a measure of the local stiffness of the beam and $u'(x)$ is the local curvature. The mechanical meaning of the minus sign can be easily understood.

Remark 24. There are no constitutive assumptions on the internal force n. Indeed since the beam is inextensible and inshearable the the tension and shear forces act as Lagrange multipliers to ensure that constraints.

5.2 Formulation of the problem

Assume that the beam is constrained such as $u(0) = 0$. Assume also that at position $x = L$ is applied a vertical compressive load $n(L) = -\lambda e_2$ which exerts no moment so $M(L) = 0$. Assume that on the beam does act any distributed force or torque so $f = 0$ and $g = 0$.

Then the local balance of forces gives that $n = -\lambda e_2$ is constant. The local balance of torques can be written as

$$M' - \lambda \sin u = 0.$$

In conclusion u must satisfy the following boundary value problem in strong form:

$$\begin{cases} -(EIu')' - \lambda \sin u = 0 & \text{in } \Omega = (0, L) \\ u(0) = 0 \\ u'(L) = 0. \end{cases} \tag{10}$$

We now derive the weak formulation of this problem. Define the spaces $H = L^2(\Omega)$ and $V = \{v \in H^1(\Omega) \mid v(0) = 0\}$. As usual we see V as a subspace of H, we identify H with H', and we see H' as a subspace of V':

$$V \subset H = H' \subset V'.$$

Assume that $EI \in L^\infty(\Omega)$ and $\operatorname{essinf}_\Omega EI > 0$. We introduce the continuous and coercive bilinear form on V defined by

$$a(u, v) = \int_\Omega EIu'v' d\mathscr{L}.$$

Now problem (10) in weak form becomes: find $u \in V$ such that

$$a(u, v) - \lambda(\sin u, v) = 0 \quad \text{for all } v \in V.$$

Let $A \in \mathscr{L}(V, V')$ the operator defined by $\langle Au, v\rangle = a(u, v)$. Define $F \colon V \times \mathbb{R} \to V'$ by

$$F(u, \lambda) = Au - \lambda \sin u,$$

where $\sin u \in V'$ acts on $v \in V$ by $(\cdot, \cdot)$. The problem can be rewritten as $F(u, \lambda) = 0$.

Remark 25. The energy associated to this model is

$$\Pi \colon V \times \mathbb{R} \to \mathbb{R} \quad \text{with} \quad \Pi(u, \lambda) = \frac{1}{2} \int_\Omega EI|u'|^2 d\mathscr{L} + \lambda \int_\Omega \cos u \, d\mathscr{L}.$$

The first term represent the elastic energy, the second one the energy of the loads. It is easy to see that $\Pi \in C^\infty(V, \mathbb{R})$ and that $\Pi' = F$. So this model is completely included in the theory developed in the previous Section.

5.3 A technical lemma

The Fréchet derivative of F with respect to u is given by

$$F'(u, \lambda)vw = a(v, w) - \lambda(v \cos u, w).$$

In particular

$$F'(0, \lambda)vw = a(v, w) - \lambda(v, w) = \langle Lv, w\rangle,$$

where $L = A - \lambda T \in \mathscr{L}(V, V')$. Here $T \in \mathscr{L}(V, H)$ is the embedding of V into H. The following technical result holds, allowing us to apply Proposition 5.

Lemma 26. The operator L is self adjoint and $R(L)$ is closed.

Proof. It is clear that L is self-adjoint since a is symmetric.

We know that $T \in \mathscr{K}(V,H)$ by Rellich-Kondrachov theorem. Now A is an elliptic operator, so is an isomorphism because it maps $f \in V'$ to the unique solution $u = A^{-1}f$ of the elliptic problem $Au = f$. Then we can write $L = (I - \lambda T A^{-1})A$ and $R(L) = R(I - \lambda T A^{-1})$. We have to prove that $R(I - \lambda T A^{-1})$ is closed. By Fredholm Alternative Theorem it is sufficient to show that $TA^{-1} \in \mathscr{L}(V')$ is compact.

Let $\{f_n\}$ be a bounded sequence in V. Then $\{A^{-1}f_n\}$ is bounded in V by coercivity of a. But since T is compact, up to a subsequence $\{TA^{-1}f_n\}$ converges in H and so also in V'. This means exactly that TA^{-1} is compact in V'. □

5.4 Application of the local bifurcation theorem

We immediatly find the fundamental path of trivial equilibrium solutions

$$\lambda \mapsto (\hat{u}^*(\lambda), \lambda) = (0, \lambda),$$

corresponding to straight configurations of the beam. To find the critical load we have to solve the following eigenvalue problem: find $v \neq 0 \in V$ and $\lambda \in \mathbb{R}$ such that

$$F'(0,\lambda)v = 0 \iff \begin{cases} -(EIv')' - \lambda v = 0 & \text{in } (0,L) \\ v(0) = 0 \\ v'(L) = 0. \end{cases}$$

Assume for simplicity that $EI > 0$ is constant. Now the problem can be solved explicitly and the eigenpairs are of the form

$$(v_n, \lambda_n) \quad \text{with} \quad v_n = \sin\left(\frac{n\pi x}{2L}\right) \quad \text{and} \quad \lambda_n = \frac{n^2\pi^2 EI}{4L^2},$$

for $n \in \mathbb{Z}_{\geq 1}$. The critical load is the minimum eigenvalue so

$$\lambda_c = \frac{\pi^2 EI}{4L^2} \quad \text{and} \quad v_c(x) = \sin\left(\frac{\pi x}{2L}\right).$$

In particular $N(F_c')$ has dimension one. By applying also Lemma 26 we immediately find that hypothesis (i), (ii), (iii) of Theorem 20 are satisfied. Moreover $(\dot{\hat{u}}_c^*, \dot{\lambda}_c^*) = (0,1) \in N(DF_c)$, $\hat{\dot{F}}_c = 0$ and since $\hat{F}_c' vw = -(v,w)$ we have $\hat{F}_c' v_c^2 = -\|v_c\|_H^2 \neq 0$. Then also hypotheses (iv), (v) and (vii) are satisfied.

The bifurcation theorem applies. So near $(0, \lambda_c)$ the solution set consists of the trivial solutions and of the buckling path intersecting it transversally. The bifurcation diagram can be schematically represented as in Figure 2 (a).

5.5 Application of Koiter asymptotic analysis

We choose $\dot{w}_0 = v_c$. After some computation one finds that $F''(u,\lambda)vwz = \lambda(v \sin u, w)$. Then $F_c'' = 0$. From equation (6) it follows that $\dot{\lambda}_0 = 0$. Moreover from equation (8) we have $F'c\ddot{w}_0 = 0$, so we choose $\ddot{w}_0 = v_c$. From equation (9)

$$\ddot{\lambda}_0 = -\frac{F_c''' v_c^4}{3\hat{F}_c' v_c^2} = \frac{\lambda_c \int_0^L \sin\left(\frac{\pi x}{2L}\right)^4 dx}{3\int_0^L \sin\left(\frac{\pi x}{2L}\right)^2 dx} = \frac{\pi^2 EI}{16L^2}.$$

So we approximate the buckling path as

$$u(t) = \left(t + \frac{t^2}{2}\right)\sin\left(\frac{\pi x}{2L}\right) + \mathcal{O}(t^3) \quad \text{and} \quad \lambda(t) = \left(1 + \frac{t^2}{8}\right)\frac{\pi^2 EI}{4L^2} + \mathcal{O}(t^3)$$

The result of this analysis are in Figure 2 (b).

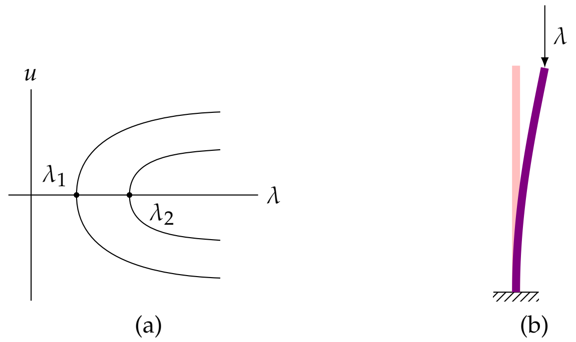

Figure 2: (a) Schematic representation of the bifurcation diagram. (b) Buckled configuration of Euler beam calculated using Koiter asymptotic analysis.

6 Buckling of hyperelastic continuous bodies

In this Section we apply the theory of elastic buckling to hyperelastic continuous bodies.

6.1 The theory of nonlinear hyperelastic materials

We start with a brief introduction to the theory of nonlinear hyperelastic materials. More details can be found in [5] and [2].

Reference configuration and deformed configuration Let Ω be an open bounded subset of $\mathbb{R}^2$, with Lipschitz boundary. Consider an elastic body whose reference configuration is Ω. A deformed configuration of the body is described by the placement $p\colon \Omega \longrightarrow \mathbb{R}^2$, so $p(x)$ is the position occupied by material point initially at x. We assume $p \in W^{1,p}(\Omega)^2$ for some $p \in (1, \infty)$.

Remark 27. With that choice of p the space $W^{1,p}(\Omega)^2$ is a reflexive Banach space.

Dirichlet boundary conditions Let Γ_D be a relatively open subset of $\partial\Omega$ with strictly positive Hausdorff measure. Suppose that the body is constrained so that $p = g$ in Γ_D in the sense of trace, where $g \in W^{1,p}(\Omega)^2$ is fixed. We introduce the space

$$V = \{v \in W^{1,p}(\Omega)^2 \mid v = 0 \text{ in } \Gamma_D \text{ in the sense of trace}\}.$$

Then we can write $p = g + u$ where the displacement $u \in V$ can be considered as a new descriptor of the configuration of the body. In what follows we assume that $g(x) = x$ a.e. in Ω. This means that we are simply keeping fixed the portion Γ_D of the boundary. In this case $p(x) = x + u(x)$ and $u = 0$ correspond to the reference configuration.

Measures of strain The first measure of strain that we consider is the deformation gradient is defined as $F = Du = I + H$ where $H = Du$. Another measure of strain which is rotational invariant is the Green–St. Venant strain tensor defined as

$$E = \frac{F^T F - I}{2} = \frac{H + H^T + H^T H}{2}.$$

Elastic energy density The basis assumption of hyperelasticity is that the free (or elastic) energy density W depends locally on the strain through a constitutive relation. Assuming the material to be homogeneous, this means that $W(x) = \hat{W}(F(x)) = \overline{W}(E(x))$ where

$\hat{W}\colon \mathrm{Lin} \to \mathbb{R}$ and $\overline{W}\colon \mathrm{Sym} \to \mathbb{R}$. We will assume that $\overline{W}$ is the one given by the Green–Kirchhoff constitutive theory (see [8]), i.e.,

$$\overline{W}(E) = \frac{\gamma}{2}(\mathrm{tr}\, E)^2 + \mu|E|^2,$$

where γ and μ are positive constants.

Remark 28. The corresponding function $\hat{W}$ is

$$\hat{W}(F) = \frac{\gamma}{8}\left(|F|^2 - 2\right)^2 + \frac{\mu}{4}\left|F^T F - I\right|^2.$$

We also report below its gradient and hessian because they will be useful later.

$$\begin{aligned}
\nabla\hat{W}(F) &= \lambda\frac{|F|^2-2}{2}F + \mu(FF^TF - F),\\
D^2\hat{W}(F)KH &= \lambda\left((F\cdot H)(F\cdot K) + \frac{|F|^2-2}{2}K\cdot H\right) + \\
&\quad + \mu\left((KF^T)\cdot(HF^T) + (K^TF)\cdot(F^TH) + (F^TK)\cdot(F^TH) - K\cdot H\right).
\end{aligned} \tag{11}$$

Elastic energy of the body The total elastic energy associated to a displacement $u \in V$ is obtained by integrating the elastic energy density over the whole body:

$$\Phi(u) = \int_\Omega \hat{W}(I + Du)d\mathscr{L} = \int_\Omega \overline{W}\left(\frac{Du + (Du)^T + (Du)^T Du}{2}\right)d\mathscr{L}.$$

This energy is well defined if we choose $p = 4$. In this case $\Phi \in C^\infty(V)$.

Energy of the loads Suppose that on $\Gamma_N \subset \partial\Omega$, relatively open and disjoint from Γ_D, acts a traction $h \in L^{p'}(\Gamma_N)^2$. Then the energy associated to that load is

$$-fu = -\int_{\Gamma_N} h\cdot u d\mathscr{H}, \qquad f \in V'.$$

Total energy of the body The total energy is given by

$$\Pi(u,\lambda) = \Phi(u) - \lambda f u,$$

where we have introduced the multiplicative parameter $\lambda \in \mathbb{R}$ that represent the intensity of the traction. It is clear that $\Pi \in C^\infty(V \times \mathbb{R})$.

Equilibrium equation We now show that the equilibrium equation associated to the energy Π corresponds to the weak form of the local balance of forces of continuum solid mechanics. Indeed we have that

$$F(u,\lambda)v = \Pi(u,\lambda)v = \int_\Omega \nabla\hat{W}(I + Du)\cdot Dv d\mathscr{L} - \lambda\int_{\Gamma_N} h\cdot v d\mathscr{H},$$

and by formally applying Green's theorem, we find that the strong form of the equilibrium equation $F(u,\lambda) = 0$ is

$$\begin{cases}
\mathrm{div}(\nabla\hat{W}(I + Du)) = 0 & \text{in } \Omega\\
u = 0 & \text{in } \Gamma_D\\
\nabla\hat{W}(I + Du)\nu = \lambda h & \text{in } \Gamma_N.
\end{cases}$$

This is indeed the local balance of forces: we recognize the Piola stress tensor $S = \nabla\hat{W}(F)$.

6.2 On the applicability of the local bifurcation theorem

We start with the following technical lemma.

Lemma 29. For all $u \in V$ and $\lambda \in \mathbb{R}$ the operator $F'(u, \lambda) \in \mathscr{L}(V, V')$ is coercive, i.e., there exists $\alpha > 0$ such that $F'(u, \lambda)v^2 \geq \alpha \|v\|_V^4$ for all $v \in V$.

Sketch of the proof. By using (11) and Young inequality can be easily shown that

$$D^2 \hat{W}(F)H^2 \geq \mathrm{c}|H|^4 \quad \text{for all } H \in \mathrm{Lin},$$

where c is a positive constant. Then

$$F'(u, \lambda)v^2 = \int_\Omega D^2 \hat{W}(I + Du)(Dv)^2 d\mathscr{L} \geq c|\Omega|\|Dv\|_{L^4}^4.$$

By Poincarè inequality there exists $C > 0$ such that $\|u\|_{L^4} \leq \|Du\|_{L^4}$ and the thesis follows. □

We know prove that $F'(u, \lambda)$ has closed range. Since we know also that $F'(u, \lambda) = \Pi''(u, \lambda)$ is self-adjoint, this result will allow us to apply Proposition 5.

Lemma 30. For all $u \in V$ and $\lambda \in \mathbb{R}$ the operator $F'(u, \lambda) \in \mathscr{L}(V, V')$ has closed range.

Proof. Call $L = F'(u, \lambda)$. Let $\{f_n\} \subset R(L)$, so $f_n = Lv_n$ for some $\{v_n\} \subset V$. Assume that $f_n \to f$ in V'. We have to show that there exists $v \in V$ such that $f = Lv$.

Now from previous lemma

$$\alpha \|v_n - v_m\|_V^4 \leq L(v_n - v_m)^2 = \langle f_n - f_m, v_n - v_m \rangle \leq \|f_n - f_m\|_{V'} \|v_n - v_m\|_V.$$

Then

$$\alpha \|v_n - v_m\|_V^3 \leq \|f_n - f_m\|_{V'}.$$

Since $\{f_n\}$ is a Cauchy sequence, also $\{v_n\}$ is a Cauchy sequence. Then $v_n \to v$ in V to some v. By continuity $f = Lv$. □

To apply the local bifurcation theorem we would first like to prove the existence of a fundamental path of class C^2 passing through a critical point. Since this seems a nontrivial task, we assume that a C^2 fundamental path $\lambda \mapsto (u^*(\lambda), \lambda)$, with $\lambda \in [0, \Lambda]$, exists. We also assume that there exists a minimum $\lambda_c \in (0, \Lambda)$ such that $(u_c, \lambda_c) = (u_c^*, \lambda_c)$ is a critical point.

Then Theorem 20 applies if $F_c'' \hat{u}_c^* v_c^2 + \hat{F}_c' v_c^2 \neq 0$. Clearly this condition must be checked once the critical point has been determined.

6.3 Finite element implementation of Koiter asymptotic analysis

We know present a finite element implementation of Koiter asymptotic analysis based on [4].

Finite element approximation Suppose Ω be polygonal and let $\mathscr{T}_h$ be a triangulation of it. We approximate the space V with the continuous $\mathbb{P}_1$ finite element space V_h of piecewise linear function on $\mathscr{T}_h$.

Determination of the fundamental path In principle to determine numerically the fundamental path one should solve a nonlinear equation. This is usually done with Newton–Raphson method. Anyway, since the effect of nonlinearities can be neglected before buckling occurs, we can determine the fundamental path by a linear analysis. In particular first suppose that u^* is linear in λ so, $u^*(\lambda) = \lambda \hat{u}_0^*$. Then we linearize the equilibrium equation

$$F(\lambda \hat{u}_0^*, \lambda) = 0,$$

obtaining

$$F_0 + \lambda F_0' \hat{u}_0^* + \lambda \hat{F}_0 = 0 \iff \Phi_0'' \hat{u}_0^* = f.$$

In the finite element discretization this reduces to the solution of a linear system.

Determination of the critical load To determine the critical load along the fundamental path one should solve the nonlinear eigenvalue problem

$$F'(\lambda \hat{u}_0^*, \lambda) v = 0,$$

for $v \in V$ and $\lambda \in \mathbb{R}$. By linearization this becomes

$$(F_0' + \lambda F_0'' \hat{u}_0^* + \lambda \hat{F}_0') v = 0 \iff \Phi_0'' v = -\lambda (\Phi_0''' \hat{u}_0^*) v,$$

which is a generalized eigenvalue problem that can be easily solved numerically. The smallest eigenvalue is the critical load λ_c and the corresponding eigenvector is v_c. In this way we have found also $u_c = \lambda_c \hat{u}_0^*$.

Second order approximation of the buckling path Now by using (6), (7), (8) and (9), we can approximate the buckling path up to first order.

Some examples In the figures below are reported some examples. On the left is reported the body in its reference configuration. In the center is reported the body in the pre-buckling configuration. On the right is the first order approximation of an equilibrium configuration along the buckling path.

References

[1] S. Antman. *Nonlinear Problems of Elasticity*. Springer, 2005.

[2] J.M. Ball. "Convexity conditions and existence theorems in nonlinear elasticity". In: *Arch. Rational Mech. Anal.* 63 (1976), pp. 337–403.

[3] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, 2011.

[4] R. Casciaro. "Computational asymptotic post-buckling analysis of slender elastic structures". In: *Phenomenological and Mathematical Modelling of Structural Instabilities*. Springer, 2005, pp. 195–276.

[5] M.E. Gurtin, E. Fried, and L. Anand. *The Mechanics and Thermodynamics of Continua*. Cambridge University Press, 2010.

[6] W.T. Koiter. *The Stability of Elastic Equilibrium*. Air Force Flight Dynamics Laboratory, 1945.

[7] L. Niremberg. *Topics in Nonlinear Functional Analysis*. AMS/Courant Institute of Mathematical Sciences, 2001.

[8] C. Truesdell and W. Noll. *The Non-Linear Field Theories of Mechanics*. Springer, 2004.

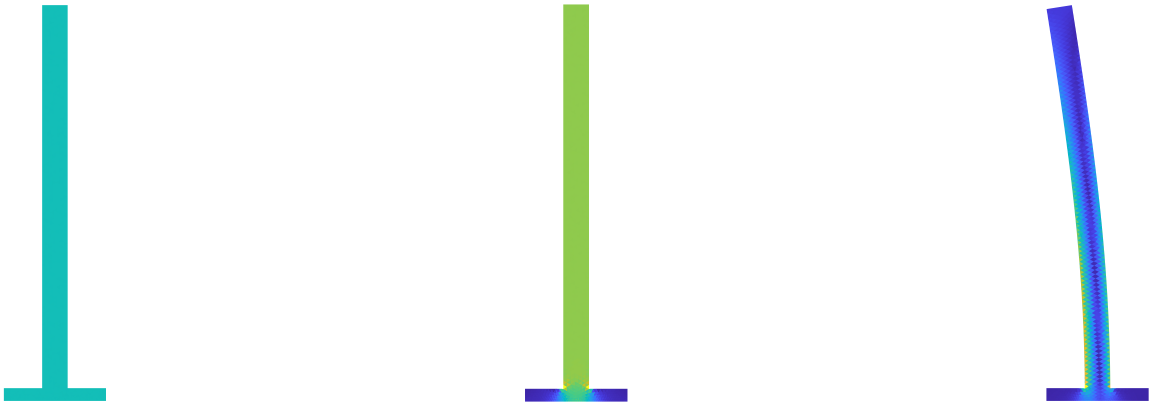

Figure 3: Beam under a vertical load uniformly distributed on the top.

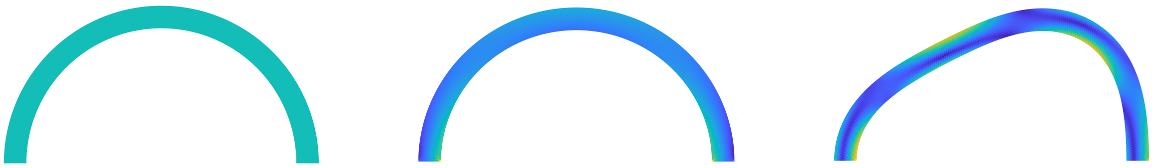

Figure 4: Half ring under uniform pressure on the outer boundary.

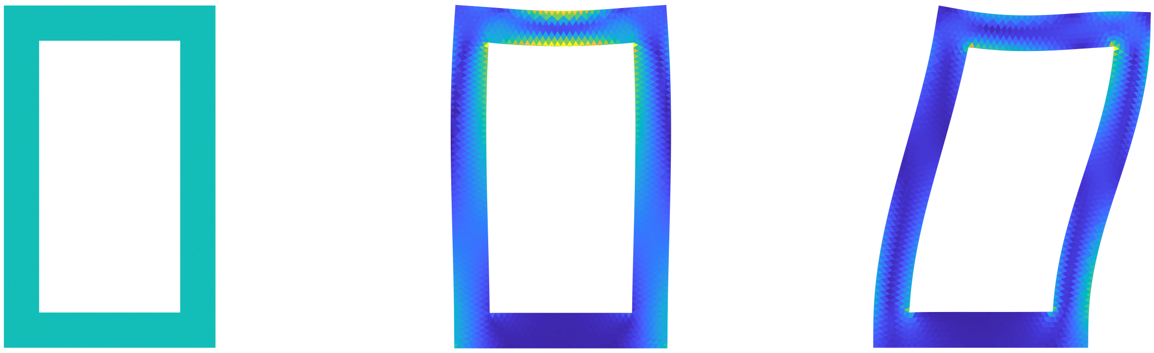

Figure 5: Box structure under a vertical load uniformly distributed on the top.

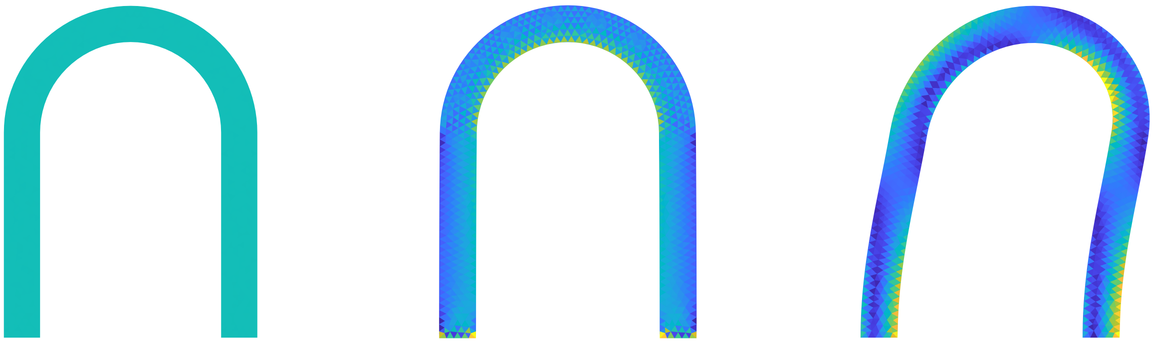

Figure 6: Round arch under uniform pressure on the top.