Numerical solution of the statics problem for nonlinear elastic truss structures

Bernardo Ardini

Padova, 16 dicembre 2024

The truss structure

Vertices

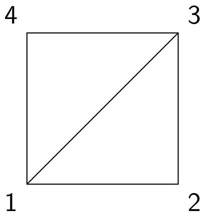
$U = \{1, \ldots, n\}$, vertex $v \in U$ occupies a position $x^{(v)} \in \mathbb{R}^3$

Elements

$\mathcal{E} \subset \{(v, w) \mid v, w \in U \text{ and } v < w\}$ element (v, w) connect v with w

Example

Vertices $U = \{1, 2, 3, 4\}$
Elements $\mathcal{E} = \{(1, 2), (2, 3), (3, 4), (1, 3)\}$



Configuration and reference configuration

- Configuration fully described by $\{x^{(v)}\}_{v \in U}$
- We call $\{X^{(v)}\}_{v \in U}$ the reference configuration of the structure

Tensions and constitutive relations

Fixed a configuration $\{x^{(v)}\}_{v \in U}$

Element tension

To each element $(v, w) \in \mathcal{E}$ we assign a tension $T^{(v,w)}$

Constitutive relation

Describe the particular behavior of the material

$$T^{(v,w)} = \hat{T}^{(v,w)}(x^{(w)} - x^{(v)})$$

Example

We will use

$$\hat{T}(\Delta x) = EV\epsilon \frac{\Delta x}{|\Delta x|^2}$$

where

- V referential volume of element
- E Young modulus
- $\epsilon = \log \frac{|\Delta x|}{L}$ logarithmic strain
- L referential length of element

Tensions and constitutive relations

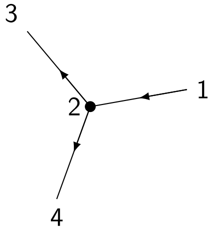
Vertex tension

For each vertex $v \in U$ we define

$$T^{(v)} = \sum_{e \in \mathcal{E}} s(e,v)\, T^{(e)}$$

where $s(e,v) \in \{0, \pm 1\}$ take into account whether v is e_1 or e_2 with $e = (e_1, e_2)$

$$s(e,v) = \begin{cases} 1 & \text{if } v = e_1 \\ -1 & \text{if } v = e_2 \\ 0 & \text{otherwise} \end{cases}$$



The statics problem

Some definitions

- $W = \{1, \dots, m\} \subset U$ mobile vertices
- $V = U \setminus W$ fixed vertices
- $\{X^{(v)}\}_{v \in U}$ reference configuration
- $X = \{X^{(w)}\}_{w \in W} \in \mathbb{R}^{3m}$ reference positions of mobile vertices
- $x = \{x^{(w)}\}_{w \in W} \in \mathbb{R}^{3m}$ positions of mobile vertices
- $T = T(x) = \{T^{(w)}\}_{w \in W} \in \mathbb{R}^{3m}$ tensions on mobile vertices
- $F = \{F^{(w)}\}_{w \in W} \in \mathbb{R}^{3m}$ external forces on mobile vertices

Problem

To find $x \in \mathbb{R}^{3m}$ such that

$$T(x) + F = 0$$

The statics problem

Stiffness matrix

Defined as the Jacobian matrix $K(x) = DT(x)$
It tell us how much the structure if stiff at configuration x

Theorem (local existence and uniqueness result)

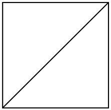
Let $T \in C^1(\Omega)$ with $\Omega \subset \mathbb{R}^{3m}$ open. Let $x_0 \in \mathbb{R}^{3m}$ and $F_0 \in \mathbb{R}^{3m}$ be such that

1. $\det DT(x_0) \neq 0$,
2. $T(x_0) + F_0 = 0$.

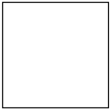
Then there exists $\delta > 0$ and $\epsilon > 0$ such that for every $F \in B(F_0, \delta)$ there exists a unique $x \in B(x_0, \epsilon)$ solution of $T(x) + F = 0$.

Proof.

Simple application of Inverse Function Theorem □



Admissible



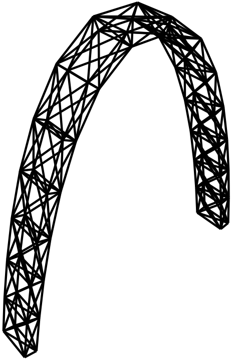
Not admissible

Newton-Raphson method

Idea

We want to find the equilibrium configuration of the structure given an external load F

1. Start in reference configuration with zero tensions and $F_0 = 0$ external forces

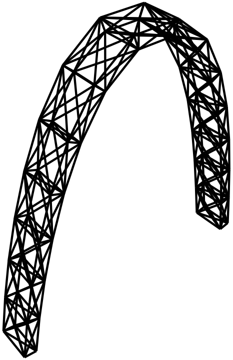


Newton-Raphson method

Idea

We want to find the equilibrium configuration of the structure given an external load F

1. Start in reference configuration with zero tensions and $F_0 = 0$ external forces
2. Increment the external load of $\delta F = \frac{F}{N}$ and find the local solution via Newton method

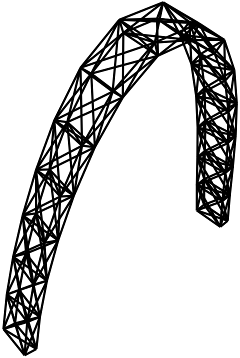


Newton-Raphson method

Idea

We want to find the equilibrium configuration of the structure given an external load F

1. Start in reference configuration with zero tensions and $F_0 = 0$ external forces
2. Increment the external load of $\delta F = \frac{F}{N}$ and find the local solution via Newton method
3. repeat previous step until F is reached

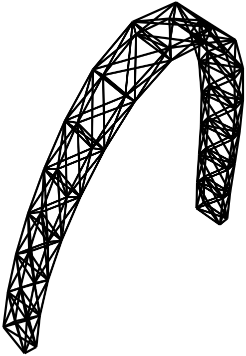


Newton-Raphson method

Idea

We want to find the equilibrium configuration of the structure given an external load F

1. Start in reference configuration with zero tensions and $F_0 = 0$ external forces
2. Increment the external load of $\delta F = \frac{F}{N}$ and find the local solution via Newton method
3. repeat previous step until F is reached

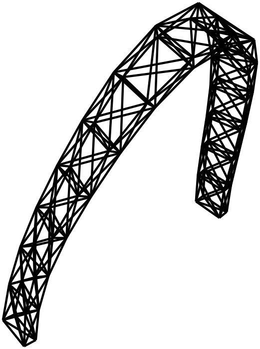


Newton-Raphson method

Idea

We want to find the equilibrium configuration of the structure given an external load F

1. Start in reference configuration with zero tensions and $F_0 = 0$ external forces
2. Increment the external load of $\delta F = \frac{F}{N}$ and find the local solution via Newton method
3. repeat previous step until F is reached



Newton-Raphson method

Newton method

If we want so solve

$$T(x + \delta x) + F = 0,$$

we make the linearization

$$T(x+\delta x) + F \simeq R + K(x)\delta x.$$

and then

$$\delta x = -K^{-1}R.$$

Repeat until tolerance is reached

Algorithm

Input: $\{X^{(v)}\}_{v \in U}$, $\{F_j\}_{j=0,\dots,N}$

- $x = X$
- For j from 1 to N do
  - $F = F_j$
  - $T = T(x)$
  - $K = K(x)$
  - $R = T(x) + F$
  - While $\frac{R}{F} > \text{tol}$
    1. $\delta x = -K^{-1}R$
    2. $x = x + \delta x$
    3. $T = T(x)$
    4. $K = K(x)$
    5. $R = T(x) + F$
  - End do
- End do

Newton-Raphson method

Solving the linear system

To compute $\delta x = -K^{-1}R$ we perform LU factorization

1. compute L and U such that $K = LU$
2. compute $y = L^{-1}R$ by forward substitution
3. compute $\delta x = -U^{-1}y$ by backward substitution

Algorithm

Input: $\{X^{(v)}\}_{v\in U}$, $\{F_j\}_{j=0,\dots,N}$

- $x = X$
- For j from 1 to N do
    - $F = F_j$
    - $T = T(x)$
    - $K = K(x)$
    - $R = T(x) + F$
    - While $\frac{R}{F} >$ tol
        1. $\delta x = -K^{-1}R$
        2. $x = x + \delta x$
        3. $T = T(x)$
        4. $K = K(x)$
        5. $R = T(x) + F$
    - End do
- End do

Newton-Raphson method

What we can make parallel?

- computation of $T(x)$ and $K(x)$
- LU factorization

Algorithm

Input: $\{X^{(v)}\}_{v \in U}$, $\{F_j\}_{j=0,\dots,N}$

- $x = X$
- For j from 1 to N do
  - $F = F_j$
  - $T = T(x)$
  - $K = K(x)$
  - $R = T(x) + F$
  - While $\frac{R}{F} >$ tol
    1. $\delta x = -K^{-1}R$
    2. $x = x + \delta x$
    3. $T = T(x)$
    4. $K = K(x)$
    5. $R = T(x) + F$
  - End do
- End do

Computation of $T(x)$ and $K(x)$

We consider only $T(x)$ for simplicity

Irregular parallelism

- we parallelize the external loop
- when $s(e, w) = 0$ we do not need to compute anything
- we use tasks

Algorithm

Input: x, X
$T = 0$
For $w \in W$ do
    For $e \in \mathcal{E}$ do
        If $s(e, w) \neq 0$ do
            $T^{(e)} = \hat{T}^{(e)}(x^{(e_2)} - x^{(e_1)})$
            $T^{(w)} = T^{(w)} + s(e, w)$
        End do
    End do
End do

LU factorization

Problem

Given suitable $A \in \mathbb{R}^{n \otimes n}$ we have to compute L and U such that $A = LU$

Idea

We decompose each matrix in four blocks with $A_{11} \in \mathbb{R}^{m \otimes m}$ where $m < n$ (e.g. $n = 600$ and $m = 70$)

$$A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \qquad L = \left(\begin{array}{c|c} L_{11} & 0 \\ \hline L_{21} & L_{22} \end{array}\right) \qquad U = \left(\begin{array}{c|c} U_{11} & U_{12} \\ \hline 0 & U_{22} \end{array}\right)$$

New problem

To find L_{ij} and U_{ij} such that

1. $A_{11} = L_{11} U_{11}$
2. $A_{12} = L_{11} U_{12}$
3. $A_{21} = L_{21} U_{11}$
4. $A_{22} = L_{22} U_{22}$

Solution

1. find L_{11} and U_{11} with serial algorithm (Gauss elimination)
2. find U_{12} with forward substitution
3. find L_{21} with backward substitution
4. find L_{22} and U_{22} recursively

LU factorization

New problem

To find L_{ij} and U_{ij} such that

1. $A_{11} = L_{11} U_{11}$
2. $A_{12} = L_{11} U_{12}$
3. $A_{21} = L_{21} U_{11}$
4. $A_{22} = L_{22} U_{22}$

Solution

1. find L_{11} and U_{11} with serial algorithm (Gauss elimination)
2. find U_{12} with forward substitution
3. find L_{21} with backward substitution
4. find L_{22} and U_{22} recursively

LU factorization

New problem

To find L_{ij} and U_{ij} such that

1. $A_{11} = L_{11} U_{11}$
2. $A_{12} = L_{11} U_{12}$
3. $A_{21} = L_{21} U_{11}$
4. $A_{22} = L_{22} U_{22}$

Solution

1. find L_{11} and U_{11} with serial algorithm (Gauss elimination)
2. find U_{12} with forward substitution
3. find L_{21} with backward substitution
4. find L_{22} and U_{22} recursively

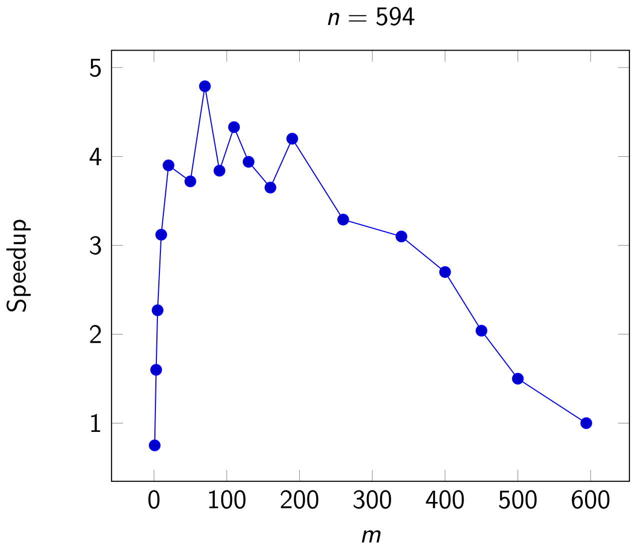
Parallelization

Both step (2) and (3) can be performed in parallel

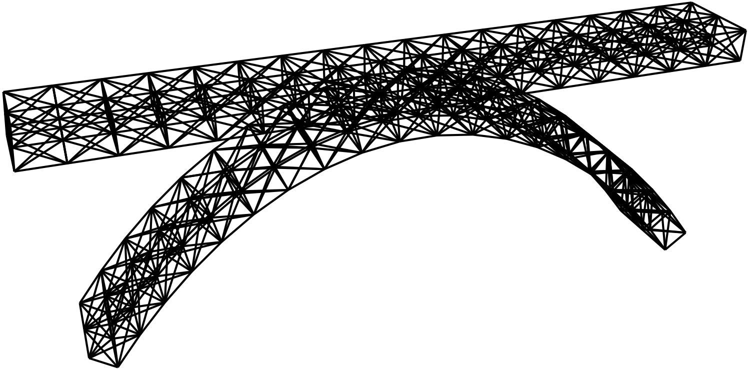
For example for step (2)

If we want to find U_{12} from $A_{12} = L_{11} U_{12}$ we write $U_{12} = (b_1 | \dots | b_{n-m})$ and $A_{12} = (a_1 | \dots | a_{n-m})$ and we solve in parallel $b_i = L_{11}^{-1} a_i$

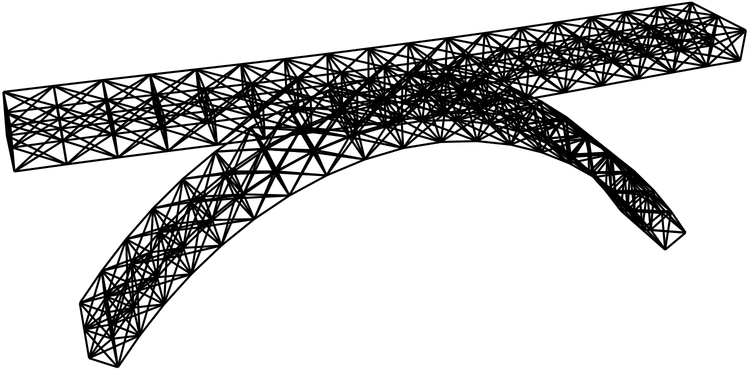
Speedup



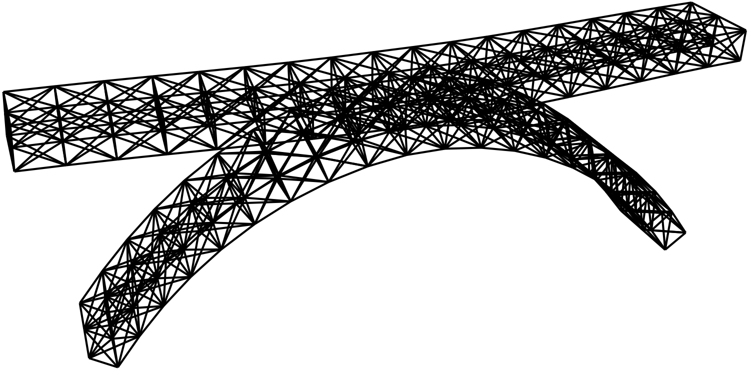
The bridge



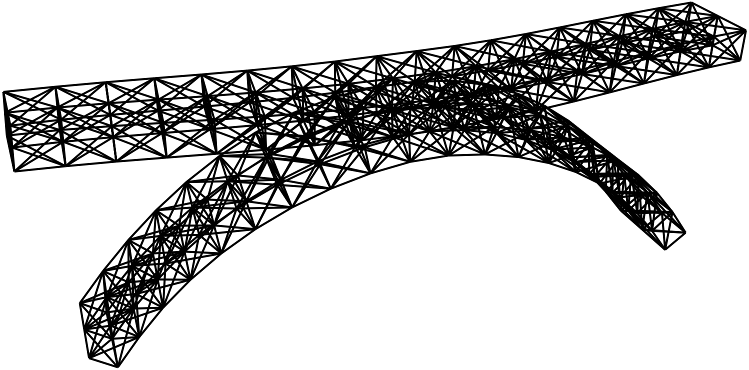
The bridge



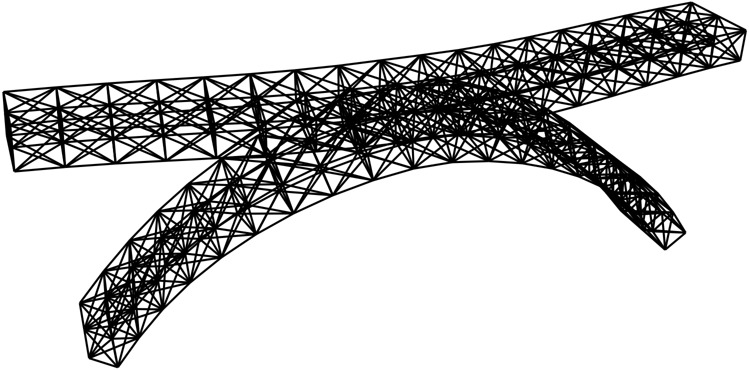
The bridge



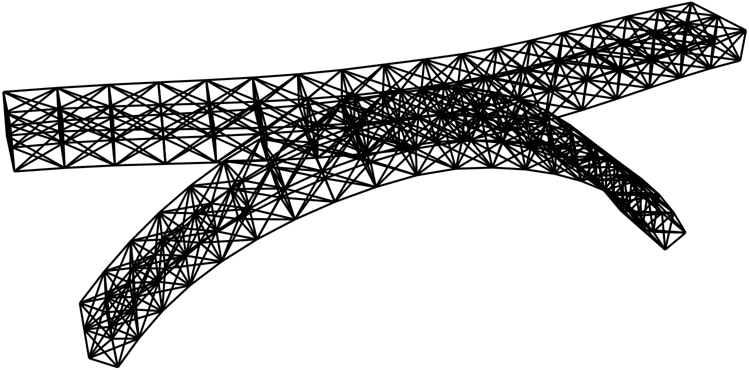
The bridge



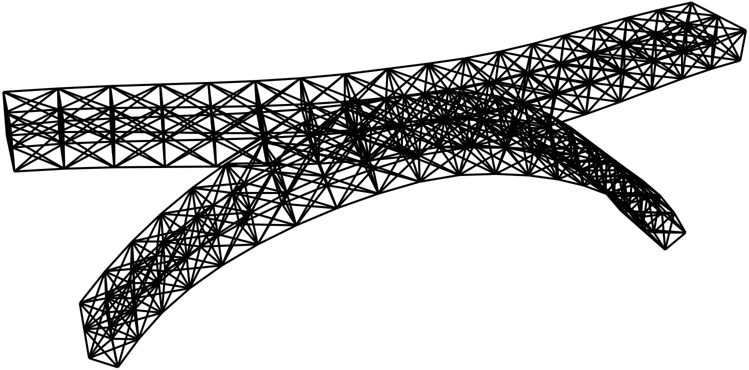
The bridge



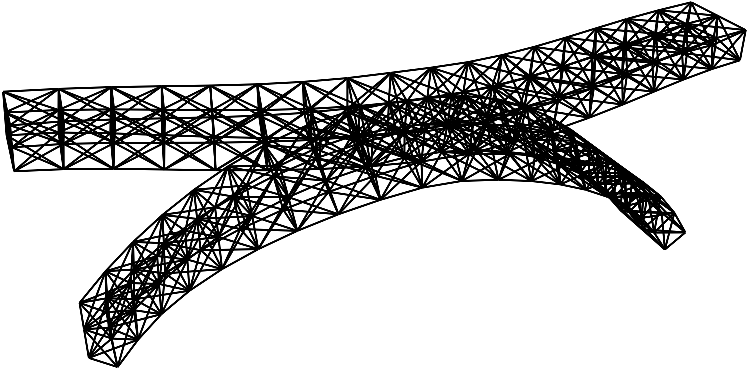
The bridge



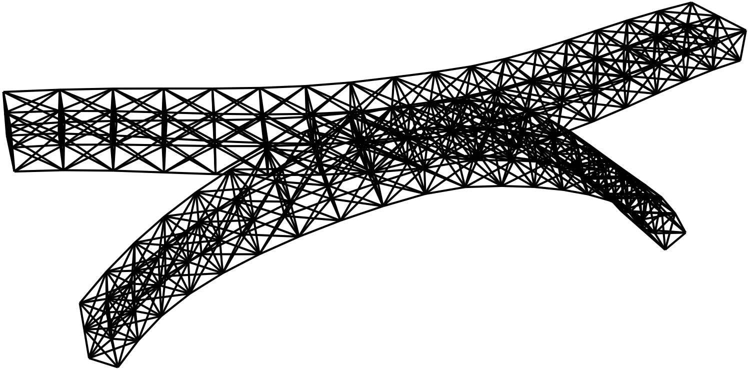
The bridge



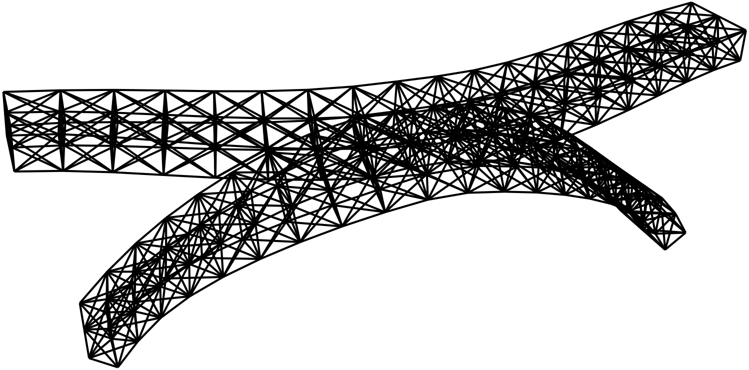
The bridge



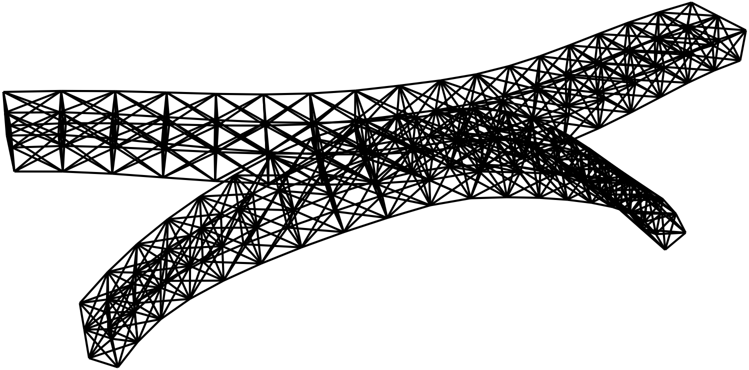
The bridge



The bridge



The bridge



References

[1] B. Ardini. "The mechanics and thermodynamics of truss structures". 2024.

[2] J. Bonet and R.D. Wood. *Nonlinear Continuum Mechanics for Finite Element Analysis*. Cambridge University Press, 2008.

[3] M.E. Gurtin, E. Fried, and L. Anand. *The Mechanics and Thermodynamics of Continua*. Cambridge University Press, 2010.

[4] A. Quarteroni, R. Sacco, and F. Saleri. *Numerical Mathematics*. Springer, 1991.