

Numerical solution of the problem of static for nonlinear elastic truss structures

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The truss structure

Vertices

$U = \{1, \dots, n\}$, vertex $v \in U$ occupies a position $x^{(v)} \in \mathbb{R}^3$

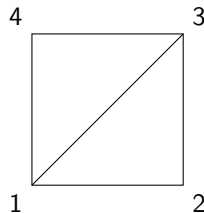
Elements

$\mathcal{E} \subset \{(v, w) \mid v, w \in U \text{ and } v < w\}$
element (v, w) connect v with w

Example

Vertices $U = \{1, 2, 3, 4\}$

Elements $\mathcal{E} = \{(1, 2), (2, 3), (3, 4), (1, 3)\}$



Configuration and reference configuration

- Configuration fully described by $\{x^{(v)}\}_{v \in U}$
- We call $\{X^{(v)}\}_{v \in U}$ the reference configuration of the structure

Tensions and constitutive relations

Fixed a configuration $\{x^{(v)}\}_{v \in U}$

Element tension

To each element $(v, w) \in \mathcal{E}$ we assign a tension $T^{(v, w)}$

Constitutive relation

Describe the particular behavior of the material

$$T^{(v, w)} = \hat{T}^{(v, w)}(x^{(w)} - x^{(v)})$$

Example

We will use

$$\hat{T}(\Delta x) = EV\epsilon \frac{\Delta x}{|\Delta x|^2}$$

where

- V referential volume of element
- E Young modulus
- $\epsilon = \log \frac{|\Delta x|}{L}$ logarithmic strain
- L referential length of element

Tensions and constitutive relations

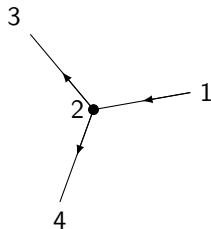
Vertex tension

For each vertex $v \in U$ we define

$$T^{(v)} = \sum_{e \in \mathcal{E}} s(e, v) T^{(e)}$$

where $s(e, v) \in \{0, \pm 1\}$ take into account whether v is e_1 or e_2 with $e = (e_1, e_2)$

$$s(e, v) = \begin{cases} 1 & \text{if } v = e_1 \\ -1 & \text{if } v = e_2 \\ 0 & \text{otherwise} \end{cases}$$



The problem of static

Some definitions

- $W = \{1, \dots, m\} \subset U$ mobile vertices
- $V = U \setminus W$ fixed vertices
- $\{X^{(v)}\}_{v \in U}$ reference configuration
- $X = \{X^{(w)}\}_{w \in W} \in \mathbb{R}^{3m}$ reference positions of mobile vertices
- $x = \{x^{(w)}\}_{w \in W} \in \mathbb{R}^{3m}$ positions of mobile vertices
- $T = T(x) = \{T^{(w)}\}_{w \in W} \in \mathbb{R}^{3m}$ tensions on mobile vertices
- $F = \{F^{(w)}\}_{w \in W} \in \mathbb{R}^{3m}$ external forces on mobile vertices

Problem

To find $x \in \mathbb{R}^{3m}$ such that

$$T(x) + F = 0$$

The problem of static

Stiffness matrix

Defined as the Jacobian matrix $K(x) = DT(x)$

It tell us how much the structure if stiff at configuration x

Theorem (local existence and uniqueness result)

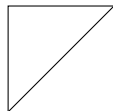
Let $T \in C^1(\Omega)$ with $\Omega \subset \mathbb{R}^{3m}$ open. Let $x_0 \in \mathbb{R}^{3m}$ and $F_0 \in \mathbb{R}^{3m}$ be such that

- ① $\det DT(x_0) \neq 0$,
- ② $T(x_0) + F_0 = 0$.

Then there exists $\delta > 0$ and $\epsilon > 0$ such that for every $F \in B(F_0, \delta)$ there exists a unique $x \in \mathbb{R}^{3m}$ solution of $T(x) + F = 0$.

Proof.

Simple application of Inverse Function Theorem



Admissible



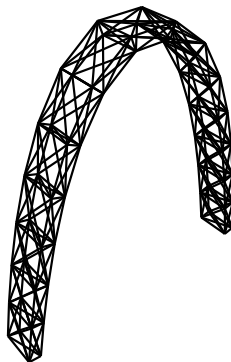
Not admissible

Newton-Raphson method

Idea

We want to find the equilibrium configuration of the structure given an external load F

- 1 Start in reference configuration with zero tensions and $F_0 = 0$ external forces

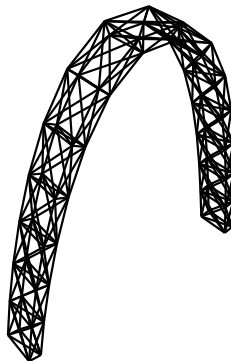


Newton-Raphson method

Idea

We want to find the equilibrium configuration of the structure given an external load F

- 1 Start in reference configuration with zero tensions and $F_0 = 0$ external forces
- 2 Increment the external load of $\delta F = \frac{F}{N}$ and find the local solution via Newton method

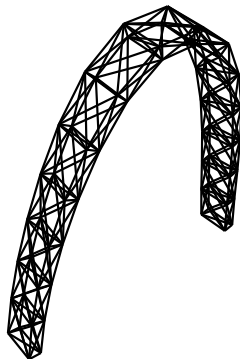


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- 3 repeat previous step until F is reached

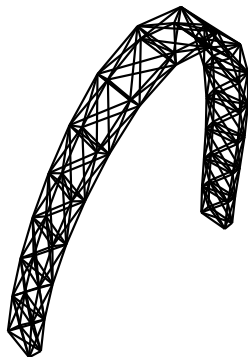


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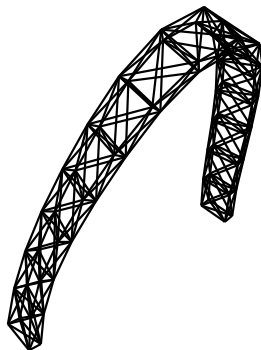


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Newton-Raphson method

Newton method

If we want so solve

$$T(x + \delta x) + F = 0,$$

we make the linearization

$$T(x + \delta x) + F \simeq R + K(x)\delta x.$$

and then

$$\delta x = -K^{-1}R.$$

Repeat until tolerance is reached

Algorithm

Input: $\{X^{(v)}\}_{v \in U}$, $\{F_j\}_{j=0, \dots, N}$

- $x = X$
- For j from 1 to N do
 - $F = F_j$
 - $T = T(x)$
 - $K = K(x)$
 - $R = T(x) + F$
 - While $\frac{R}{F} > \text{tol}$
 - ① $\delta x = -K^{-1}R$
 - ② $x = x + \delta x$
 - ③ $T = T(x)$
 - ④ $K = K(x)$
 - ⑤ $R = T(x) + F$
 - End do
 - End do

Newton-Raphson method

Solving the linear system

To compute $\delta x = -K^{-1}R$ we perform LU factorization

- ① compute L and U such that $K = LU$
- ② compute $y = L^{-1}R$ by forward substitution
- ③ compute $\delta x = -U^{-1}y$ by backward substitution

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 - End do
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Newton-Raphson method

What we can make parallel?

- computation of $T(x)$ and $K(x)$
- LU factorization

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Input: $\{X^{(v)}\}_{v \in U}$, $\{F_j\}_{j=0, \dots, N}$

- $x = X$
- For j from 1 to N do
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 - End do
- End do

Computation of $T(x)$ and $K(x)$

We consider only $T(x)$ for simplicity

Irregular parallelism

- we parallelize the external loop
- when $s(e, w) = 0$ we do not need to compute anything
- we use tasks

Algorithm

Input: x, X

$T = 0$

For $w \in W$ do

For $e \in \mathcal{E}$ do

If $s(e, w) \neq 0$ do

$$T^{(e)} = \hat{T}^{(e)}(x^{(e_2)} - x^{(e_1)})$$

$$T^{(w)} = T^{(w)} + s(e, w)$$

End do

End do

End do

LU factorization

Problem

Given suitable $A \in \mathbb{R}^{n \times n}$ we have to compute L and U such that $A = LU$

Idea

We decompose each matrix in four blocks with $A_{11} \in \mathbb{R}^{m \times m}$ where $m < n$ (e.g. $n = 600$ and $m = 70$)

$$A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right) \quad L = \left(\begin{array}{c|c} L_{11} & 0 \\ \hline L_{21} & L_{22} \end{array} \right) \quad U = \left(\begin{array}{c|c} U_{11} & U_{12} \\ \hline 0 & U_{22} \end{array} \right)$$

New problem

To find L_{ij} and U_{ij} such that

- ① $A_{11} = L_{11} U_{11}$
- ② $A_{12} = L_{11} U_{12}$
- ③ $A_{21} = L_{21} U_{11}$
- ④ $A_{22} = L_{22} U_{22}$

Solution

- ① find L_{11} and U_{11} with serial algorithm (Gauss elimination)
- ② find U_{12} with forward substitution
- ③ find L_{21} with backward substitution
- ④ find L_{22} and U_{22} recursively

LU factorization

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To find L_{ij} and U_{ij} such that

$$\textcircled{1} A_{11} = L_{11} U_{11}$$

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Solution

- $\textcircled{1}$ find L_{11} and U_{11} with serial algorithm (Gauss elimination)
- $\textcircled{2}$ find U_{12} with forward substitution
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LU factorization

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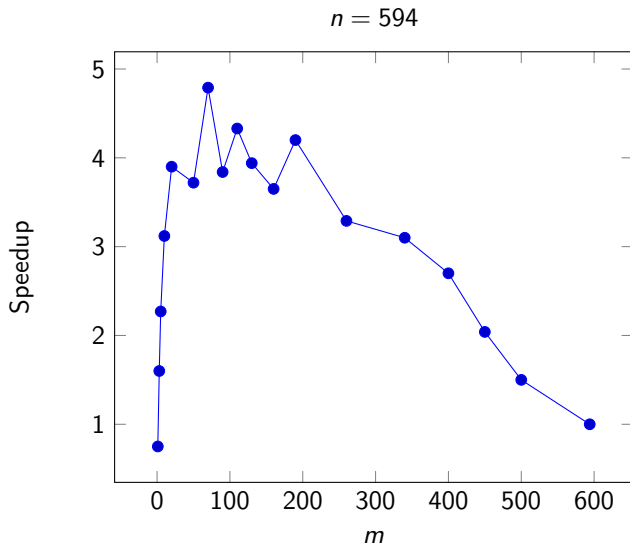
Parallelization

Both step (2) and (3) can be performed in parallel

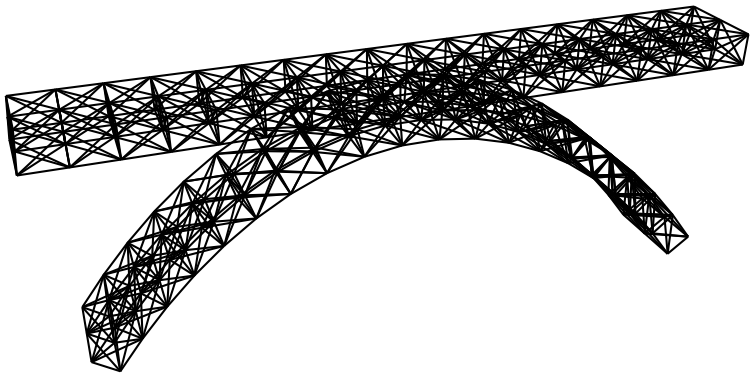
For example for step (2)

If we want to find U_{12} from $A_{12} = L_{11} U_{12}$ we write $U_{12} = (b_1 | \dots | b_{n-m})$ and $A_{12} = (a_1 | \dots | a_{n-m})$ and we solve in parallel $b_i = L_{11}^{-1} a_i$

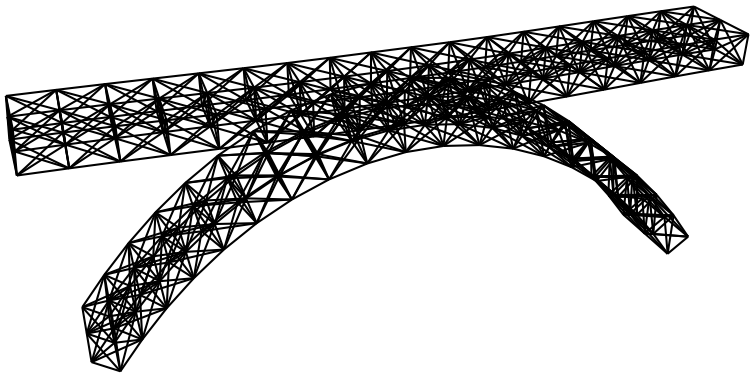
Speedup



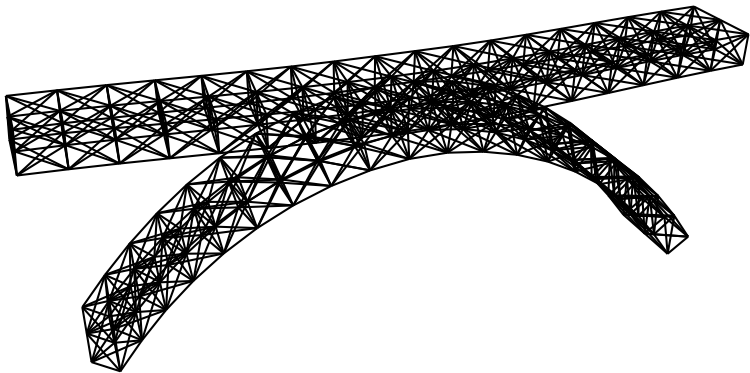
The bridge



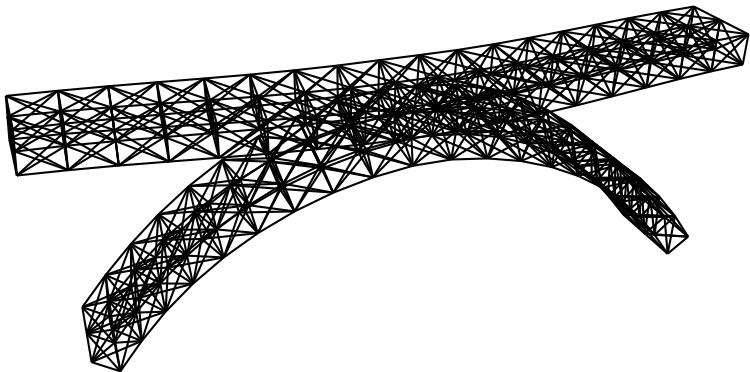
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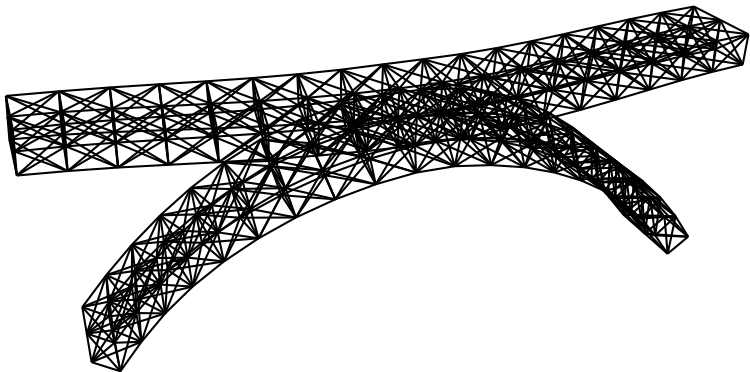
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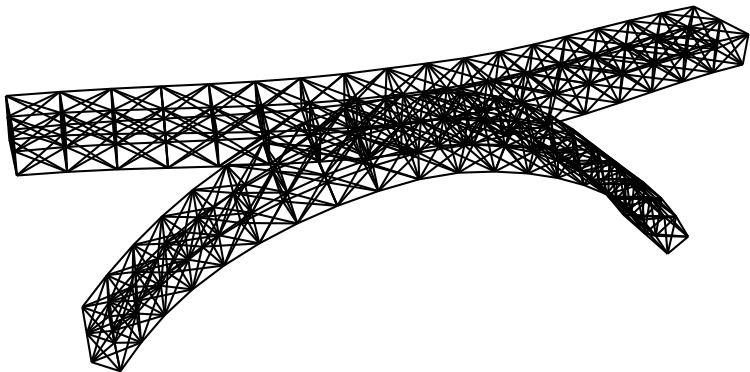
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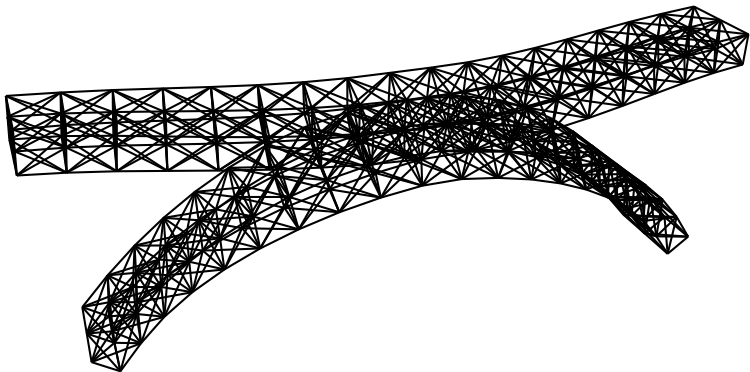
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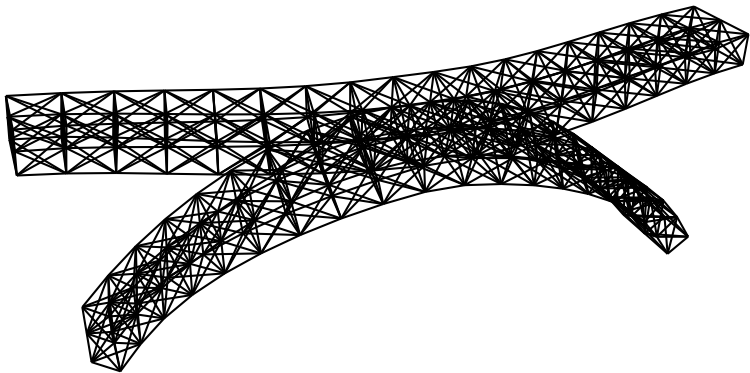
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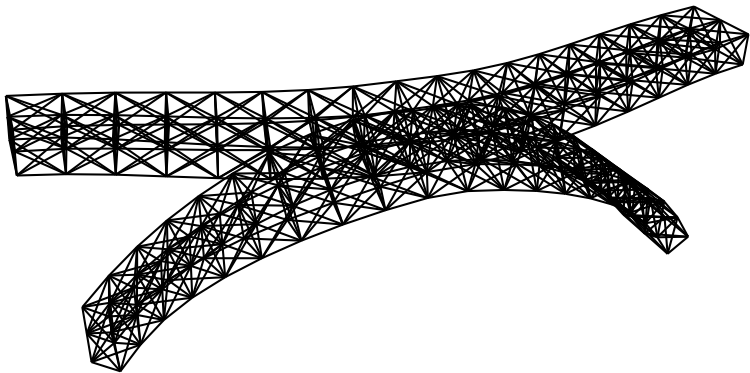
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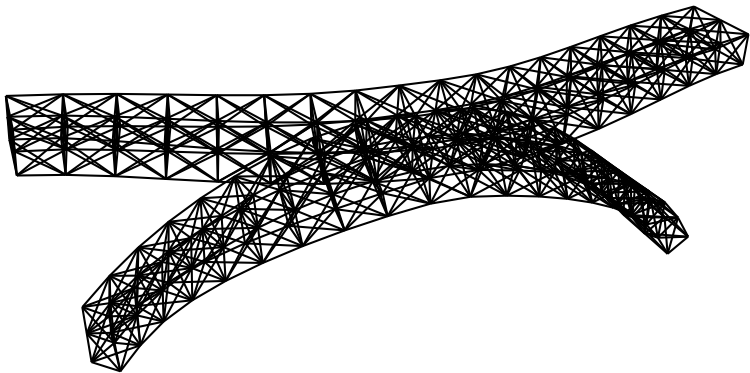
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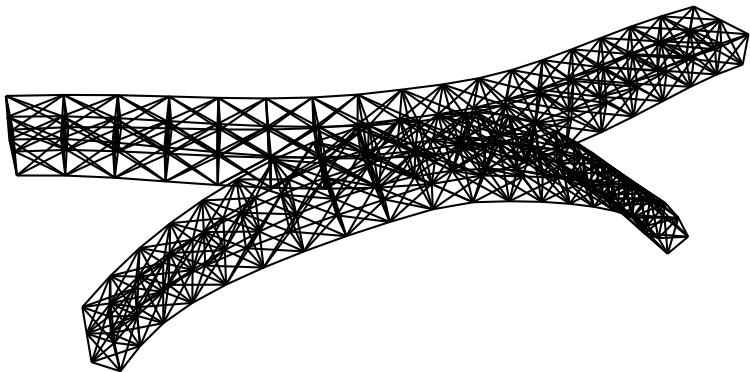
The bridge



The bridge



The bridge



References

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- [2] J. Bonet and R.D. Wood. *Nonlinear Continuum Mechanics for Finite Element Analysis*. Cambridge University Press, 2008.
- [3] M.E. Gurtin, E. Fried, and L. Anand. *The Mechanics and Thermodynamics of Continua*. Cambridge University Press, 2010.
- [4] A. Quarteroni, R. Sacco, and F. Saleri. *Numerical Mathematics*. Springer, 1991.