Proof of Individual Belief Convergence in a Strongly Connected Influence Graph Using Confirmation Bias Update

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Definition 1. The *confirmation-bias factor* is the defined as:

$$f^t_{cb}(a_i, a_j) = 1 - |B^t(a_j) - B^t(a_i)|$$

Definition 2. The *confirmation-bias update-function* is defined as:

$$B^{t+1}(a_i|a_j) = B^t(a_i) + f^t_{cb}(a_i, a_j).I(a_j, a_i)(B^t(a_j) - B^t(a_i)) \tag{1}$$

Definition 3. While the *overall confirmation-bias update*, is defined as:

$$B^{t+1}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} B^{t+1}(a_i|a_j) \tag{2}$$

Definition 4. We say a influence graph In over agents A is *strongly connected* if for all a_i, $a_j \in A$, there exist a_{k_1},a_{k_2},...,$a_{k_l} \subseteq A$ such that $I(a_i, a_{k_1}) > 0$, $I(a_{k_l}, a_j) > 0$, and for $m = 1$,...,$l - 1$, $I(a_{k_m}, a_{k_{m+1}}) > 0$.

Definition 5. max^t and min^t are the maximum and minimum belief values in a given instant t, respectively. Thus:

$$min^t = \underset{a_i \in A}{min}\ B^t(a_i) \text{ and } max^t = \underset{a_i \in A}{max}\ B^t(a_i).$$

To prove our conjecture, let's do some simplifications:

$$\begin{aligned}
B^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} B^{t+1}(a_i|a_j). \\
&= \frac{1}{|A|} \sum_{a_j \in A} \big(B^t(a_i) + f^t_{cb}(a_i, a_j).I(a_j, a_i)(B^t(a_j) - B^t(a_i))\big) \\
&= B^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} f^t_{cb}(a_i, a_j).I(a_j, a_i)(B^t(a_j) - B^t(a_i)) \qquad (3)
\end{aligned}$$

Since we have a finite number of agents and $\forall a_i \in A:\ B^t(a_i) \in [0, 1]$, there are always min^t and a max^t. We shall also note that, by the Squeeze Theorem, individual agent opinion converges to the same value if and only if $lim_{t\to\infty}\ min^t = lim_{t\to\infty}\ max^t$.

Since we want to prove that polarization always converges, if $min^t = max^t$ we have nothing to prove, so assume from now on $min^t \neq max^t$. *We will also assume from now on that no agent has belief* 0 *or* 1, which will guarantee us that $\forall t$ and $\forall a_i, a_j \in A$, $f^t_{cb}(a_i, a_j) > 0$. The case in which there are beliefs equal to 0 or 1 will be addressed later.

Lemma 1. *Under the confirmation-bias belief update:*

$$\forall t \ \text{and}\ \forall a_i \in A : min^t \leq B^{t+1}(a_i) \leq max^t$$

Proof. By the equation 3:

$$B^{t+1}(a_i) = B^t(a_i) + \frac{1}{|A|}\sum_{a_j\in A} f^t_{cb}(a_i, a_j)I(a_j, a_i)(B^t(a_j) - B^t(a_i))$$

Substituting $B^t(a_j)$ by max^t turns our equation into an inequality, since $\forall a_j \in A$, $B^t(a_j) \leq max^t$ and also makes the terms inside the summation either equal to or greater than 0. Thus:

$$\begin{aligned}
B^{t+1}(a_i) &\leq B^t(a_i) + \frac{1}{|A|}\sum_{a_j\in A} f^t_{cb}(a_i, a_j).I(a_j, a_i)(max^t - B^t(a_i)) && \text{(since } I(a_j, a_i) \geq 0 \text{ and } f^t_{cb}(a_i, a_j) \geq 0)\\
&\leq B^t(a_i) + \frac{1}{|A|}\sum_{a_j\in A} f^t_{cb}(a_i, a_j).(max^t - B^t(a_i)) && \text{(since } I(a_j, a_i) \leq 1 \text{ and } max^t - B^t(a_i) \geq 0)\\
&\leq B^t(a_i) + \frac{1}{|A|}\sum_{a_j\in A} (max^t - B^t(a_i)) && \text{(since } f^t_{cb}(a_i, a_j) \leq 1 \text{ and } max^t - B^t(a_i) \geq 0)\\
&= B^t(a_i) + \frac{|A|}{|A|}(max^t - B^t(a_i))\\
&= B^t(a_i) + max^t - B^t(a_i)\\
B^{t+1}(a_i) &\leq max^t
\end{aligned} \tag{4}$$

Since a_i was arbitrary, the Lemma is true for all agents. The same reasoning can be used to show the equivalent property for min^t □

Corollary 1. *In a strongly connected influence graph under the confirmation-bias update function:*

$$max^{t+1} \leq max^t \ \text{and}\ min^{t+1} \geq min^t \ \text{for all}\ t.$$

Proof. Lemma 1 tells us that all beliefs in the time $t+1$ are either smaller or equal to max^t. Since max^{t+1} must be one of those beliefs, $max^{t+1} \leq max^t$. The same reasoning can be used for min^t. □

Corollary 2. $\lim_{t\to\infty} max^t = U$ *and* $\lim_{t\to\infty} min^t = L$ *for some* $U,\ L \in [0,1]$.

Proof. Both max^t and min^t are bounded between 0 and 1 and Lemma 1 showed us that they are monotonic. According to the Monotonic Convergence Theorem, this guarantees that the limits exist. □

The proof will follow by showing that an agent a_i that holds some belief $B^t(a_i)$ influences every other agent by the time $t + |A| - 1$. Before we do this, let's jump into some small definitions and corollaries that will help us on the way.

Definition 6. A *simple path* $P(a_i \to a_j)$ from agent a_i to agent a_j is a sequence $(a_i, a_k, ..., a_{k+l}, a_j)$ of agents such that:

- All elements on the sequence are different.
- The first element in the sequence is a_i.
- The last element in the sequence is a_j.
- If a_n is the n'th element in the sequence, if it has a successor a_{n+1}, $I(a_n, a_{n+1}) > 0$.

Many simple paths from a_i to a_j can exist, although our notation isn't enough to differentiate them. But in subsequent steps we will only need one of those simple paths, so the notation shouldn't be a problem.

Definition 7. Denote by $|P(a_i \to a_j)|$ the *length* of a simple path from a_i to a_j, which we define as the number of elements in the sequence $P(a_i \to a_j)$ minus 1.

Corollary 3. $\forall P(a_i \to a_j),\ |P(a_i \to a_j)| \leq |A| - 1$.

Proof. A simple path doesn't have repeated elements and we have $|A|$ agents, thus simple path can't have more than $|A|$ elements. According to Definition 7, the length of a simple path is defined as the number of elements minus one, thus maximum length is $|A| - 1$. □

Lemma 2. $\forall x, \forall t$ *and* $\forall a_i$, *if* $B^t(a_i) \leq x$*:*

$$B^{t+1}(a_i) \leq x + \frac{1}{|A|} \sum_{a_j \in A} f^t_{cb}(a_i, a_j).I(a_j, a_i) \left(B^t(a_j) - x\right)$$

Proof.

$$\begin{aligned}
B^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} \left(B^t(a_i) + f^t_{cb}(a_i, a_j).I(a_j, a_i) \left(B^t(a_j) - B^t(a_i)\right)\right) \\
&= \frac{1}{|A|} \sum_{a_j \in A} \left(B^t(a_i)(1 - f^t_{cb}(a_i, a_j).I(a_j, a_i)) + f^t_{cb}(a_i, a_j).I(a_j, a_i) B^t(a_j)\right) \\
&\leq \frac{1}{|A|} \sum_{a_j \in A} \left(x.(1 - f^t_{cb}(a_i, a_j).I(a_j, a_i)) + f^t_{cb}(a_i, a_j).I(a_j, a_i) B^t(a_j)\right) \\
&= x + \frac{1}{|A|} \sum_{a_j \in A} f^t_{cb}(a_i, a_j).I(a_j, a_i) \left(B^t(a_j) - x\right)
\end{aligned}$$

□

Lemma 3. *$\forall a_i, a_k \in A$ and $\forall n \geq 1$ and $\forall t$:*

$$B^{t+n}(a_i) \leq max^t + \frac{1}{|A|} f_{cb}^{t+n-1}(a_i, a_j).I(a_k, a_i)(B^{t+n-1}(a_k) - max^t) \tag{5}$$

Proof. By the Definitions 2 and 3:

$$\begin{aligned} B^{t+n}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} B^{t+n}(a_i|a_j) \\ &= \frac{1}{|A|} \sum_{a_j \in A} \left(B^{t+n-1}(a_i) + f_{cb}^{t+n-1}(a_i, a_j).I(a_j, a_i)(B^{t+n-1}(a_j) - B^{t+n-1}(a_i)) \right) \end{aligned}$$

According to Corollary 1: $B^{t+n}(a_i) \leq max^{t+n} \leq max^{t+n-1}$. Thus we can use Lemma 2:

$$\begin{aligned} B^{t+n}(a_i) &\leq \frac{1}{|A|} \sum_{a_j \in A} \left(max^{t+n-1} + f_{cb}^{t+n-1}(a_i, a_j).I(a_j, a_i)(B^{t+n-1}(a_j) - max^{t+n-1}) \right) \\ &= max^{t+n-1} + \frac{1}{|A|} \sum_{a_j \in A} f_{cb}^{t+n-1}(a_i, a_j).I(a_j, a_i)(B^{t+n-1}(a_j) - max^{t+n-1}) \end{aligned}$$

To make our Lemma useful in future manipulations, we will take an arbitrary element a_k out of the summation :

$$\begin{aligned} B^{t+n}(a_i) \leq max^{t+n-1} &+ \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_k\}} \left(f_{cb}^{t+n-1}(a_i, a_j).I(a_j, a_i)(B^{t+n-1}(a_j) - max^{t+n-1}) \right) \\ &+ \frac{1}{|A|} f_{cb}^{t+n-1}(a_i, a_j).I(a_k, a_i)(B^{t+n-1}(a_k) - max^{t+n-1})) \end{aligned}$$

Since max^{t+n-1} is the greatest belief possible in that time step, the summation can be at most 0, thus:

$$B^{t+n}(a_i) \leq max^{t+n-1} + \frac{1}{|A|} f_{cb}^{t+n-1}(a_i, a_j).I(a_k, a_i) \left(B^{t+n-1}(a_k) - max^{t+n-1} \right)$$

Since max doesn't increase throughout time, $max^{t+n-1} \leq max^t$. Thus:

$$B^{t+n}(a_i) \leq max^t + \frac{1}{|A|} f_{cb}^{t+n-1}(a_i, a_j).I(a_k, a_i) \left(B^{t+n-1}(a_k) - max^t \right)$$

□

Definition 8. Denote by I_{min} the smallest positive influence in the influence graph.

Definition 9. Let's denote by f_{cbmin} the smallest f_{cb} in our society throughout time. Note that, this f_{cb} is greater than 0 because of our assumption that no agents have belief 0 or 1. Note, also, that the minimum f_{cb} occurs between max^0 and min^0, does it is constant and does not diminishes throughout time, according to 1.

Using the same notation we used in Corollary 2, let's call $\lim_{t\to\infty} max^t = U$ and $\lim_{t\to\infty} min^t = L$. Denoting by a_*^t one agent who holds the belief min^t in the time t:

Theorem 1. *$\forall t$ and $\forall a_i \in A$:*

$$B^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t - \delta^t, \text{ with } \delta^t = \left(\frac{I_{min}.f_{cbmin}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U-L).$$

Proof. By equation 3:

$$B^{t+|P(a_*^t \to a_i)|}(a_i) = Bel_p^{t+|P(a_*^t \to a_i)|-1}(a_i) + \frac{1}{|A|}\sum_{a_j \in A} B^{t+|P(a_*^t \to a_i)|-1}(a_i|a_j)$$

We will now separate, at each step, a carefully chosen element of the summation and apply Lemma 3 to modify our inequality. The chosen elements will be the ones in $P(a_*^t \to a_i)$, starting from the end of the simple path until we get to a_*^t.

To simplify the notation, let's index the elements in the simple path from a_*^t to a_i, starting from the end of the simple path (since we are backtracking) by calling a_n the n^{th} element from the end to the beginning of the sequence (excluding a_i itself).

By Lemma 3:

$$B^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t + \frac{1}{|A|} f_{cb}^{t+|P(a_*^t \to a_i)-1|}(a_i, a_1).I(a_1, a_i)(B^{t+|P(a_*^t \to a_i)-1|}(a_1) - max^t)$$

If $|P(a_*^t, a_i)| = 1$, we could prove our result. Instead of showing it I will expand this two more times to show the general formula.

Using Lemma 3:

$$\begin{aligned}
&B^{t+|P(a_*^t \to a_i)|}(a_i) \\
&\leq max^t + \frac{1}{|A|} f_{cb}^{t+|P(a_*^t \to a_i)-1|}(a_i, a_1).I(a_1, a_i)(B^{t+|P(a_*^t \to a_i)-1|}(a_1) - max^t) \\
&\leq max^t + \frac{1}{|A|} f_{cb}^{t+|P(a_*^t \to a_i)-1|}(a_i, a_1).I(a_1, a_i)\times \\
&\left(\left(max^t + \frac{1}{|A|} f_{cb}^{t+|P(a_*^t \to a_i)-2|}(a_1, a_2).I(a_2, a_1)(B^{t+|P(a_*^t \to a_i)-2|}(a_2) - max^t)\right) - max^t\right) \\
&= max^t + \frac{1}{|A|} f_{cb}^{t+|P(a_*^t \to a_i)-1|}(a_i, a_1).I(a_1, a_i)\times \\
&\left(\frac{1}{|A|} f_{cb}^{t+|P(a_*^t \to a_i)-2|}(a_1, a_2).I(a_2, a_1)(B^{t+|P(a_*^t \to a_i)-2|}(a_2) - max^t)\right) \\
&= max^t + \frac{1}{|A|^2} f_{cb}^{t+|P(a_*^t \to a_i)-1|}(a_i, a_1).f_{cb}^{t+|P(a_*^t \to a_i)-2|}(a_1, a_2).I(a_2, a_1)I(a_1, a_i)\times \\
&(B^{t+|P(a_*^t \to a_i)-2|}(a_2) - max^t) \\
&\leq max^t + \frac{1}{|A|^2} f_{cb}^{t+|P(a_*^t \to a_i)-1|}(a_i, a_1).f_{cb}^{t+|P(a_*^t \to a_i)-2|}(a_1, a_2).I(a_2, a_1)I(a_1, a_i)\ \times
\end{aligned}$$

$$\left(\left(max^t + \frac{1}{|A|} f_{cb}^{t+|P(a_*^t \to a_i)-3|}(a_2, a_3).I(a_3, a_2)\left(B^{t+|P(a_*^t \to a_i)|-3}(a_3) - max^t\right)\right) - max^t\right)$$
$$= max^t + \frac{1}{|A|^2} f_{cb}^{t+|P(a_*^t \to a_i)-1|}(a_i, a_1).f_{cb}^{t+|P(a_*^t \to a_i)-2|}(a_1, a_2).I(a_2, a_1)I(a_1, a_i)\times$$
$$\left(\frac{1}{|A|} f_{cb}^{t+|P(a_*^t \to a_i)-3|}(a_2, a_3).I(a_3, a_2)\left(B^{t+|P(a_*^t \to a_i)|-3}(a_3)\right) - max^t\right)$$
$$= max^t + \frac{1}{|A|^3} f_{cb}^{t+|P(a_*^t \to a_i)-1|}(a_i, a_1).f_{cb}^{t+|P(a_*^t \to a_i)-2|}(a_1, a_2).f_{cb}^{t+|P(a_*^t \to a_i)-3|}(a_2, a_3)\times$$
$$I(a_3, a_2)I(a_2, a_1)I(a_1, a_i)\left(B^{t+|P(a_*^t \to a_i)|-3}(a_3) - max^t\right)$$

We can see a pattern forming and this pattern will continue throughout time. Denoting P_{In} the product of the influences in the simple path ($P_{In} = I(a_*^t, a_{|P(a_*^t, a_i)|}) \times ... \times I(a_1, a_i)$), and denoting by F_{cb} the product of the f_{cb}'s we can write the general version of the inequality above as:

$$\begin{aligned} B^{t+|P(a_*^t \to a_i)|}(a_i) &\leq max^t + \frac{P_{In}.F_{cb}}{|A|^{|P(a_*^t \to a_i)|}}(Bel_p^t(a_*^t) - max^t) \\ &= max^t + \frac{P_{In}.F_{cb}}{|A|^{|P(a_*^t \to a_i)|}}.(min^t - max^t) \end{aligned} \tag{6}$$

The rightmost term in the equation is either equal to or smaller than 0 thus, to make the inequality hold for all a_i's, we shall substitute P_{In} by the smallest value possible. By the Definition 8, I_{min} is the smallest positive influence in the graph and according to Definition 6 the influences in a simple path are positive. Thus:

$$B^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t + \left(\frac{I_{min}}{|A|}\right)^{|P(a_*^t \to a_i)|}.F_{cb}.(min^t - max^t)$$

Using the same reasoning we must replace all f_{cb} by the smallest value they can assume, which is f_{cbmin}:

$$B^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t + \left(\frac{I_{min}.f_{cbmin}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(min^t - max^t)$$

According to Corollary 2, the maximum value of min^t is L and the minimum value of max^t is U, thus:

$$\begin{aligned} B^{t+|P(a_*^t \to a_i)|}(a_i) &\leq max^t + \left(\frac{I_{min}.f_{cbmin}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(L - U) \\ &\leq max^t - \left(\frac{I_{min}.f_{cbmin}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U - L) \\ &\leq max^t - \delta^t \end{aligned}$$

□

Lemma 4.

$$\sum_{a_j \in A} f_{cb}^t(a_i,a_j).I(a_j,a_i)\left(B^t(a_j)-B^t(a_i)\right) = \sum_{a_j \in A\setminus\{a_i\}} f_{cb}^t(a_i,a_j).I(a_j,a_i)\left(B^t(a_j)-B^t(a_i)\right)$$

Proof.

$$\begin{aligned}
&\sum_{a_j \in A} f_{cb}^t(a_i,a_j).I(a_j,a_i)\left(B^t(a_j)-B^t(a_i)\right)\\
&= \sum_{a_j \in A\setminus\{a_i\}} f_{cb}^t(a_i,a_j).I(a_j,a_i)\left(B^t(a_j)-B^t(a_i)\right) + f_{cb}^t(a_i,a_i).I(a_i,a_i)(B^t(a_i)-B^t(a_i))\\
&= \sum_{a_j \in A\setminus\{a_i\}} f_{cb}^t(a_i,a_j).I(a_j,a_i)\left(B^t(a_j)-B^t(a_i)\right)
\end{aligned}$$

□

Lemma 5. *If $B^{t+n}(a_i) \leq max^t - \gamma$, $\gamma \geq 0$ and $n \geq 0$, then $B^{t+n+1}(a_i) \leq max^t - \frac{\gamma}{|A|}$.*

Proof.

$$\begin{aligned}
B^{t+n+1}(a_i) &= B^{t+n}(a_i) + \frac{1}{|A|}\sum_{a_j \in A} f_{cb}^{t+n}(a_i,a_j).I(a_j,a_i)\left(B^{t+n}(a_j)-B^{t+n}(a_i)\right)\\
&= B^{t+n}(a_i) + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_i\}} f_{cb}^{t+n}(a_i,a_j).I(a_j,a_i)\left(B^{t+n}(a_j)-B^{t+n}(a_i)\right) && \text{(Lemma 4)}\\
&\leq max^t - \gamma + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_i\}} f_{cb}^{t+n}(a_i,a_j).I(a_j,a_i)\left(B^{t+n}(a_j)-max^t+\gamma\right) && \text{(Lemma 2)}\\
&\leq max^t - \gamma + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_i\}} f_{cb}^{t+n}(a_i,a_j).I(a_j,a_i)\left(max^t-max^t+\gamma\right)\\
&= max^t - \gamma + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_i\}} f_{cb}^{t+n}(a_i,a_j).I(a_j,a_i)\left(\gamma\right)\\
&\leq max^t - \gamma + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_i\}} (\gamma)\\
&= max^t - \gamma + \frac{(|A|-1)(\gamma)}{|A|}\\
&= max^t + \frac{(\gamma)((-|A|)+(|A|-1))}{|A|}\\
&= max^t - \frac{\gamma}{|A|}
\end{aligned}$$

□

Theorem 2. $\forall a_i \in A: \; B^{t+|A|-1}(a_i) \leq max^t - \epsilon$, *with* $\epsilon = \left(\frac{I_{min}.f_{cbmin}}{|A|}\right)^{|A|-1}.(U-L)$.

Proof. Keeping the notation of Theorem 1, let's call a_*^t one agent that holds the belief min^t in the time t.

Note that, if $|P(a_*^t \to a_i)| = |A|-1$, our theorem is true by Theorem 1 and we nothing to prove.

Else if $|P(a_*^t \to a_i)| \neq |A|-1$, then $|P(a_*^t \to a_i)| < |A|-1$ according to Corollary 3.

According to Theorem 1:

$$B^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t - \left(\frac{I_{min}.f_{cbmin}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U-L)$$

To keep things simple let's keep the notation from Theorem 1 and call:

$$\delta^t = \left(\frac{I_{min}.f_{cbmin}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U-L)$$

.
Now it is easy to see that we can apply Lemma 5 successively:

$$B^{t+|P(a_*^t \to a_i)|+1}(a_i) \leq max^t - \frac{\delta^t}{|A|}$$
$$\Downarrow$$
$$B^{t+|P(a_*^t \to a_i)|+2}(a_i) \leq max^t - \frac{\delta^t}{|A|^2}$$
$$\Downarrow$$
$$B^{t+|P(a_*^t \to a_i)|+3}(a_i) \leq max^t - \frac{\delta^t}{|A|^3}$$

If we do it $|A| - |P(a_*^t \to a_i)| - 1$ times we get:

$$B^{t+|P(a_*^t \to a_i)|+|A|-|P(a_*^t \to a_i)|-1}(a_i) \leq max^t - \frac{\delta^t}{|A|^{|A|-|P(a_*^t \to a_i)|-1}}$$
$$\Downarrow$$
$$\begin{aligned}
B^{t+|A|-1}(a_i) &\leq max^t - \frac{\delta^t}{|A|^{|A|-|P(a_*^t \to a_i)|-1}} \\
&\leq max^t - \frac{\left(\frac{I_{min}.f_{cbmin}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U-L)}{|A|^{|A|-|P(a_*^t \to a_i)|-1}} \\
&\leq max^t - \frac{(I_{min}.f_{cbmin})^{|P(a_*^t \to a_i)|}.(U-L)}{|A|^{|A|-1}} \\
&\leq max^t - \left(\frac{I_{min}.f_{cbmin}}{|A|}\right)^{|A|-1}.(U-L) \\
&\leq max^t - \epsilon
\end{aligned}$$

□

Corollary 4. $max^{t+|A|-1} \leq max^t - \epsilon$

Proof. Since $max^{t+|A|-1}$ is one of the beliefs in the time $t + |A| - 1$ and, according to Theorem 2 all of them are smaller than max^t by a factor of at least ϵ, $max^{t+|A|-1}$ must also be smaller than max^t by a factor of at least ϵ. $\square$

Theorem 3. $\lim_{t\to\infty} max^t = U = \lim_{t\to\infty} min^t = L$

Proof. Suppose, by contradiction, that $U \neq L$. Plugging this values into the ϵ formula show us that $\epsilon > 0$.

Let's assume we did $v = (|A| - 1)\left(\lceil \frac{1}{\epsilon} \rceil + 1\right)$ time steps after $t = 0$. Since max diminishes by at least ϵ at each $|A| - 1$ steps:

$$max^0 \geq max^v + \epsilon \left(\left\lceil \frac{1}{\epsilon} \right\rceil + 1\right)$$

Since $\epsilon. \left(\lceil \frac{1}{\epsilon} \rceil + 1\right) > 1$ and $0 \leq max^v \leq 1$, this would imply that $max^0 \geq 1$ contradicting the definition of belief!

Since assuming that $U \neq L$ led us to a contradiction: $U = L$. $\square$

Theorem 4. $\forall a_i, a_j \in A, \lim\limits_{t\to\infty} B^t(a_i) = \lim\limits_{t\to\infty} B^t(a_j)$

Proof. Since $L \leq \lim\limits_{t\to\infty} B^t(a_i) \leq U$ and $L = U$: $L = B^t(a_i) = U$. And the same can be showed for $B^t(a_j)$. $\square$

Everything showed above was based on assumption that $f_{cb} > 0$, but this false when we have agents with belief 0 and 1 in the same graph. To address this case we must note that the beliefs 0 and 1 are always maximum and minimum thus, according to Corollary 1 at the time t no agent has belief 0 or belief 1, there will never be an agent with those beliefs in subsequent steps, thus we only have to look at the initial state of our society. We will divide the possible situations in two:

- Case 1: $\forall a_i \in A: \ B^0(a_i) = 0$ or $B^0(a_i) = 1$.
  In this case our graph converges trivially (but necessarily to the same value), because every agent is not influenced by an agent that has a different belief, thus this graph is constant throughout time.

- Case 2: $\exists a_{**} \in A, \ B^0(a_{**}) \neq 0$ and $B^0(a_{**}) \neq 1$.
  From this situation we can reach the general case, in which $f_{cb} > 0$. The idea to prove this is similar to the one used in Theorem 1. Using a_{**} to influence every agent we can guarantee that no agent will have belief 0 or 1:

Lemma 6. *$\forall a_i \in A, \forall t$:*

$$\textit{If } 0 < B^t(a_i) < 1, \textit{ then } 0 < B^{t+1}(a_i) < 1.$$

Proof. By Equation 3 and Lemma 4:

$$B^{t+1}(a_i) = B^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} f^t_{cb}(a_i, a_j).I(a_j, a_i)(B^t(a_j) - B^t(a_i))$$

$$
\begin{aligned}
&= B^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus a_i} f^t_{cb}(a_i, a_j).I(a_j, a_i)(B^t(a_j) - B^t(a_i)) \\
&\leq B^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus a_i} f^t_{cb}(a_i, a_j).I(a_j, a_i)(1 - B^t(a_i)) && \text{(since } f^t_{cb}(a_i, a_j).I(a_j, a_i) \geq 0) \\
&\leq B^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus a_i} f^t_{cb}(a_i, a_j).(1 - B^t(a_i)) && \text{(since } 1 - B^t(a_i) \geq 0 \\
& && \text{and } I(a_j, a_i) \leq 1) \\
&\leq B^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus a_i} (1 - B^t(a_i)) && \text{(since } 1 - B^t(a_i) \geq 0 \\
& && \text{and } f^t_{cb}(a_i, a_j) \leq 1) \\
&= B^t(a_i) + \frac{(|A| - 1).(1 - B^t(a_i))}{|A|} \\
&= \frac{|A|.B^t(a_i) + (|A| - 1).(1 - B^t(a_i))}{|A|} \\
&= \frac{B^t(a_i)(|A| - (|A| - 1)) + (|A| - 1)}{|A|} \\
&= \frac{B^t(a_i) + (|A| - 1)}{|A|} \\
&= 1 + \frac{B^t(a_i) - 1}{|A|}
\end{aligned}
$$

Since $B^t(a_i) < 1$, $\frac{B^t(a_i)-1}{|A|} < 0$. Thus $B^{t+1}(a_i) < 1$ as we wanted to prove. The same can be done to show that $0 < B^t(a_i)$. □

Now it gets easy to see that we will fall on the general case:

At time $t = 1$ a_{**} influences all agents a_j in which $|P(a_{**} \to a_j)| = 1$ this makes so that $\forall t > 0$, $0 < B^t(a_j) < 1$, according to Lemma 6.

We can now use those a_j's from previous step to influence the more agents out of the extremes. It isn't hard to see that, doing this repeatedly guarantees that, after $|A| - 1$ steps every belief is different from 0 and 1. We then fall on the general case, which have already proved convergence for.