Proof of Individual Agent Opinion Convergence in a Weakly Connected Influence Graph Using Classic Update Function

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NOTE: I need help with the definition of path, I really do not know how to define it the best way. I also do not know to make Theorem 3 better. But the rest of the things you pointed out are fixed (if I am not mistaken).

Definition 1. The *classic update-function*, is defined as:

$$Bel_p^{t+1}(a_i|a_j) = Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \tag{1}$$

Definition 2. While the *overall classic update*, is defined as:

$$Bel_p^{t+1}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j) \tag{2}$$

Definition 3. We say a influence graph In over agents A is *weakly connected* if for all a_i, $a_j \in A$, there exist a_{k_1},a_{k_2},...,$a_{k_l} \subseteq A$ such that In(a_i,a_{k_1}) > 0, In(a_{k_l},a_j) > 0, and for $m = 1$,...,$l-1$, In(a_{k_m},$a_{k_{m+1}}$) > 0.

Definition 4. max^t and min^t are the maximum and minimum belief values in a given instant t, respectively.

To prove our conjecture, let's do some simplifications:

$$\begin{aligned}
Bel_p^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j). \\
&= \frac{1}{|A|} \sum_{a_j \in A} \left(Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))\right) \\
&= Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \qquad (3)
\end{aligned}$$

Since we have a finite number of beliefs and $\forall a_i \in A : \ Bel_p^t(a_i) \in [0, 1]$, there are always min^t and a max^t. We shall also note that, by the Squeeze Theorem, individual agent opinion converges to the same value if and only if $lim_{t\to\infty}\, min^t = lim_{t\to\infty}\, max^t$.

Thus, since we want to prove that it always converges, if $min^t = max^t$ we have nothing to prove, so assume $min^t \neq max^t$.

Lemma 1. *In a weakly connected graph and under classic belief update:*

$$\forall t \text{ and } \forall a_i \in A : min^t \leq Bel_p^{t+1}(a_i) \leq max^t$$

Proof. By the equation 3:

$$Bel_p^{t+1}(a_i) = Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))$$

Trying to maximize the right-hand side, we can substitute $Bel_p^t(a_j)$ by max^t, this turns our equation into an inequality, since $\forall a_j \in A$, $Bel_p^t(a_j) \leq max^t$, by the definition of max^t. That makes the terms inside the summation either equal or smaller than 0, thus:

$$\begin{aligned}
Bel_p^{t+1}(a_i) &\leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(max^t - Bel_p^t(a_i)) \\
&\leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} (max^t - Bel_p^t(a_i)) && \text{(since } In(a_j,a_i) \leq 1 \text{ and} \\
& && max^t - Bel_p^t(a_i) \geq 0) \\
&= Bel_p^t(a_i) + \frac{|A|}{|A|}(max^t - Bel_p^t(a_i)) \\
&= Bel_p^t(a_i) + max^t - Bel_p^t(a_i) \\
Bel_p^{t+1}(a_i) &\leq max^t
\end{aligned} \tag{4}$$

Since a_i was arbitrary, the Lemma is true for all agents. The same reasoning can be used to show the equivalent property for min^t □

Corollary 1. *In a weakly connected influence graph and a classic update function,* $max^{t+1} \leq max^t$ *and* $min^{t+1} \geq min^t$ *for all* t.

Proof. The result of Lemma 1 tells us that all beliefs in the time $t + 1$ are either smaller than max^t or equal to max^t, thus, since max^{t+1} must be one of those elements, $max^{t+1} \leq max^t$. And the same reasoning can be used for min^t. □

Corollary 2. $\lim_{t\to\infty} max^t = U$ *and* $\lim_{t\to\infty} min^t = L$ *for some* U, $L \in [0, 1]$.

Proof. Since both max^t and min^t are bounded between 0 and 1 by the definition of belief; and Lemma 1 showed us that they are monotonic, according to the Monotonic Convergence Theorem, the limits exist. □

Now that we have those properties, our proof will follow by showing that an agent a_i that holds some belief $Bel_p^t(a_i)$ will influence every other agent by the time $t+|A|-1$. To see this, we must open the definition of belief throughout time. But before we do this, let's jump into some small definitions and corollaries that will help us on the way.

Definition 5. Let's call the sequence $P(a_i \to a_j) = (a_i, a_k, ..., a_{k+l}, a_j)$ a *path* from a_i to a_j, if:

- All elements on the sequence are different.
- The first element in the sequence is a_i.
- The last element in the sequence is a_j.
- If a_n is the n'th element in the sequence, if it has a successor a_{n+1}, $In(a_n, a_{n+1}) > 0$.

Note that many paths from a_i to a_j can exist, although our notation isn't enough to differentiate them. But in subsequent steps we will only need one of those paths, so the notation shouldn't be a problem.

Definition 6. Let's denote by $|P(a_i \to a_j)|$ the size of a simple path from a_i to a_j, which we define as the number of elements in the sequence $P(a_i \to a_j) - 1$.

Corollary 3. $\forall P(a_i \to a_j)$, $|P(a_i \to a_j)| \leq |A| - 1$.

Proof. This follows directly from the definition of path. Since it doesn't have repeated elements and we have $|A|$ agents, the simple path can't have more than $|A|$ elements, since the size of a simple path is defined as the number of elements minus one, the maximum size is $|A| - 1$. □

Lemma 2. *$\forall x, \forall t$ and $\forall a_i$, if $Bel_p^t(a_i) \leq x$:*

$$Bel_p^{t+1}(a_i) \leq x + \frac{1}{|A|} \sum_{a_j \in A} (In(a_j, a_i) \left(Bel_p^t(a_j) - x\right)$$

Proof.

$$\begin{aligned}
Bel_p^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} \left(Bel_p^t(a_i) + In(a_j, a_i) \left(Bel_p^t(a_j) - Bel_p^t(a_i)\right)\right) \\
&= \frac{1}{|A|} \sum_{a_j \in A} \left(Bel_p^t(a_i)(1 - In(a_j, a_i)) + In(a_j, a_i) Bel_p^t(a_j)\right) \\
&\leq \frac{1}{|A|} \sum_{a_j \in A} \left(x.(1 - In(a_j, a_i)) + In(a_j, a_i) Bel_p^t(a_j)\right) \\
&= x + \frac{1}{|A|} \sum_{a_j \in A} \left(In(a_j, a_i) \left(Bel_p^t(a_j) - x\right)\right)
\end{aligned}$$

□

Lemma 3. *$\forall a_i, a_k \in A$ and $\forall n \geq 1$ and $\forall t$:*

$$Bel_p^{t+n}(a_i) \leq max^t + \frac{1}{|A|} \left(In(a_k, a_i)(Bel_p^{t+n-1}(a_k) - max^t)\right) \quad (5)$$

Proof. By the Definitions 1 and 2:

$$Bel_p^{t+n}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} Bel_p^{t+n}(a_i | a_j)$$

$$Bel_p^{t+n}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} \left(Bel_p^{t+n-1}(a_i) + In(a_j, a_i)(Bel_p^{t+n-1}(a_j) - Bel_p^{t+n-1}(a_i)) \right)$$

Since $Bel_p^{t+n}(a_i) \leq max^{t+n} \leq max^{t+n-1}$ according to Corollary 1, we can use Lemma 2:

$$\begin{aligned} Bel_p^{t+n}(a_i) &\leq \frac{1}{|A|} \sum_{a_j \in A} \left(max^{t+n-1} + In(a_j, a_i)(Bel_p^{t+n-1}(a_j) - max^{t+n-1}) \right) \\ &= max^{t+n-1} + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^{t+n-1}(a_j) - max^{t+n-1}) \end{aligned}$$

To make our Lemma useful in future manipulations, we will take an arbitrary element a_k out of the summation:

$$\begin{aligned} Bel_p^{t+n}(a_i) \leq max^{t+n-1} + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_k\}} \left(In(a_j, a_i)(Bel_p^{t+n-1}(a_j) - max^{t+n-1}) \right) \\ + \frac{1}{|A|} (In(a_k, a_i)(Bel_p^{t+n-1}(a_k) - max^{t+n-1})) \end{aligned}$$

Since max^{t+n-1} is the greatest belief possible in that time step, the summation can be at most 0, thus:

$$Bel_p^{t+n}(a_i) \leq max^{t+n-1} + \frac{1}{|A|} In(a_*^t, a_i) \left(Bel_p^{t+n-1}(a_k) - max^{t+n-1} \right)$$

Since max doesn't increase throughout time, $max^t \leq max^{t+n-1}$, thus:

$$Bel_p^{t+n}(a_i) \leq max^t + \frac{1}{|A|} In(a_k, a_i) \left(Bel_p^{t+n-1}(a_k) - max^t \right)$$

□

Definition 7. Let's denote by In_{min} the smallest influence that's different from 0 in the influence graph.

Using the same notation we used in Corollary 2, let's call $\lim_{t \to \infty} max^t = U$ and $\lim_{t \to \infty} min^t = L$.
Now that we have all of these tools, let's jump to Lemma 1 which will be a tool in the most important part of the proof. Calling a_*^t one agent who holds the belief min^t in the time t:

Theorem 1. $\forall t$ *and* $\forall a_i \in A$:

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t - \delta^t, \text{ with } \delta^t = \left(\frac{In_{min}}{|A|} \right)^{|P(a_*^t \to a_i)|} .(U - L).$$

Proof. By equation 3:

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) = Bel_p^{t+|P(a_*^t \to a_i)|-1}(a_i) + \frac{1}{|A|}\sum_{a_j \in A} Bel_p^{t+|P(a_*^t \to a_i)|-1}(a_i|a_j)$$

What we will do now is separate, at each step, one element of the summation and apply Lemma 3 to modify our inequality. But we will be careful when choosing the elements we separate from the summation. We will separate from the summation the elements in $P(a_*^t \to a_i)$, starting from the end of the simple path until we get to a_*^t. To simplify our notation, let's index the elements in the simple path from a_*^t to a_i, starting from the end of the simple path (since we are backtracking it will make more sense) by calling a_n the n^{th} element from the end to the beginning of the sequence (excluding a_i itself). Thus, by Lemma 3:

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t + \frac{1}{|A|} In(a_1, a_i)(Bel_p^{t+|P(a_*^t \to a_i)|-1}(a_1) - max^t)$$

Note, now, that if $|P(a_*^t, a_i)| = 1$, we could prove our result. Instead of showing it I will expand this two more times, show the general version and then prove the Lemma for all cases. Using Lemma 3 again:

$$\begin{aligned}
&Bel_p^{t+|P(a_*^t \to a_i)|}(a_i)\\
&\leq max^t + \frac{1}{|A|} In(a_1, a_i)(Bel_p^{t+|P(a_*^t \to a_i)|-1}(a_1) - max^t)\\
&\leq max^t + \frac{1}{|A|} In(a_1, a_i)\left(\left(max^t + \frac{1}{|A|} In(a_2, a_1)(Bel_p^{t+|P(a_*^t \to a_i)|-2}(a_2) - max^t)\right) - max^t\right)\\
&= max^t + \frac{1}{|A|} In(a_1, a_i)\left(\frac{1}{|A|} In(a_2, a_1)(Bel_p^{t+|P(a_*^t \to a_i)|-2}(a_2) - max^t)\right)\\
&= max^t + \frac{1}{|A|^2} In(a_2, a_1) In(a_1, a_i)(Bel_p^{t+|P(a_*^t \to a_i)|-2}(a_2) - max^t)\\
&\leq max^t + \frac{1}{|A|^2} In(a_2, a_1) In(a_1, a_i) \times\\
&\quad \left(\left(max^t + \frac{1}{|A|} In(a_3, a_2)\left(Bel_p^{t+|P(a_*^t \to a_i)|-3}(a_3) - max^t\right)\right) - max^t\right)\\
&= max^t + \frac{1}{|A|^2} In(a_2, a_1) In(a_1, a_i)\left(\frac{1}{|A|} In(a_3, a_2)\left(Bel_p^{t+|P(a_*^t \to a_i)|-3}(a_3)\right) - max^t\right)\\
&= max^t + \frac{1}{|A|^3} In(a_3, a_2) In(a_2, a_1) In(a_1, a_i)\left(Bel_p^{t+|P(a_*^t \to a_i)|-3}(a_3) - max^t\right)
\end{aligned}$$

We can see a patter forming since the equation above has the same form of the one before it, and this pattern will continue throughout time. Now, denoting P_{In} the product of the influences in the simple path ($P_{In} = In(a_*^t, a_{|P(a_*^t, a_i)|}) \times ... \times In(a_1, a_i)$), we can write the generalized version of the inequality above as:

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t + \frac{P_{In}}{|A|^{|P(a_*^t \to a_i)|}}.(Bel_p^t(a_*^t) - max^t)$$

$$= max^t + \frac{P_{In}}{|A|^{|P(a_*^t \to a_i)|}}.(min^t - max^t)$$

This inequality comes from the fact that the simple path ends after $|P(a_*^t \to a_i)|$ steps with a_*^t as the start of the path, and, by definition, the belief of a_*^t in the time t is min^t.

Since the rightmost term in the equation is either equal to or smaller than 0, to make the inequality hold for all a_i's, we shall substitute P_{In} by the smallest value possible. According to Definition 7, In_{min} is the smallest positive influence in the graph. By the definition of simple path (5) all influences are positive, thus the smallest possible value of P_{In} is $In_{min}^{|P(a_*^t \to a_i)|}$. Thus:

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t + \left(\frac{In_{min}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(min^t - max^t)$$

According to Corollary 2, the maximum value of min^t is L and the minimum value of max^t is U, those are the values we should plug to maintain the inequality:

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t + \left(\frac{In_{min}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(L - U)$$
$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t - \left(\frac{In_{min}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U - L)$$
$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t - \delta^t$$

□

Lemma 4.

$$\sum_{a_j \in A} In(a_j, a_i)\left(Bel_p^t(a_j) - Bel_p^t(a_i)\right) = \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i)\left(Bel_p^t(a_j) - Bel_p^t(a_i)\right)$$

Proof.

$$\sum_{a_j \in A} In(a_j, a_i)\left(Bel_p^t(a_j) - Bel_p^t(a_i)\right)$$
$$= \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i)\left(Bel_p^t(a_j) - Bel_p^t(a_i)\right) + In(a_i, a_i)(Bel_p^t(a_i) - Bel_p^t(a_i))$$
$$= \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i)\left(Bel_p^t(a_j) - Bel_p^t(a_i)\right)$$

□

Lemma 5. *If* $Bel_p^{t+n}(a_i) \leq max^t - \gamma$, $\gamma \geq 0$ *and* $n \geq 0$, *then* $Bel_p^{t+n+1}(a_i) \leq max^t - \frac{\gamma}{|A|}$.

Proof.

$$Bel_p^{t+n+1}(a_i) = Bel_p^{t+n}(a_i) + \frac{1}{|A|}\sum_{a_j \in A} In(a_j, a_i)\left(Bel_p^{t+n}(a_j) - Bel_p^{t+n}(a_i)\right)$$

$$= Bel_p^{t+n}(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i) \left(Bel_p^{t+n}(a_j) - Bel_p^{t+n}(a_i)\right) \quad \text{(Lemma 4)}$$

$$\leq max^t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i) \left(Bel_p^{t+n}(a_j) - max^t + \gamma\right) \quad \text{(Lemma 2)}$$

$$\leq max^t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i) \left(max^t - max^t + \gamma\right)$$

$$= max^t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i) \left(\gamma\right)$$

$$\leq max^t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} (\gamma)$$

$$= max^t - \gamma + \frac{(|A| - 1)(\gamma)}{|A|}$$

$$= max^t + \frac{(\gamma)((-|A|) + (|A| - 1))}{|A|}$$

$$= max^t - \frac{\gamma}{|A|}$$

□

Theorem 2. *$\forall a_i \in A: \ max^t - Bel_p^{t+|A|-1}(a_i) \geq \epsilon, \ with \ \epsilon = \left(\frac{In_{min}}{|A|}\right)^{|A|-1}.(U - L).$*

Proof. Let's keep the notation of the previous Lemma and call a_*^t the agent that holds the belief min^t in the time t.
First we should note that, if $|P(a_*^t \to a_i)| = |A| - 1$, our theorem is true by Theorem 1 and we nothing to prove.
Else if $|P(a_*^t \to a_i)| \neq |A| - 1$, then $|P(a_*^t \to a_i)| < |A| - 1$ according to Corollary 3.
According to Theorem 1:

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t - \left(\frac{In_{min}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U - L)$$

To keep things simple let's keep the notation from Theorem 1 and call:

$$\delta^t = \left(\frac{In_{min}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U - L)$$

.
Now it is easy to see that we can apply Lemma 5 successively:

$$Bel_p^{t+|P(a_*^t \to a_i)|+1}(a_i) \leq max^t - \frac{\delta^t}{|A|}$$

$$\Downarrow$$

$$Bel_p^{t+|P(a_*^t \to a_i)|+2}(a_i) \leq max^t - \frac{\delta^t}{|A|^2}$$

$$\Downarrow$$

$$Bel_p^{t+|P(a_*^t \to a_i)|+3}(a_i) \leq max^t - \frac{\delta^t}{|A|^3}$$

If we do it $|A| - |P(a_*^t \to a_i)| - 1$ times we get:

$$\begin{aligned}
Bel_p^{t+|P(a_*^t \to a_i)|+|A|-|P(a_*^t \to a_i)|-1}(a_i) &\leq max^t - \frac{\delta^t}{|A|^{|A|-|P(a_*^t \to a_i)|-1}} \\
Bel_p^{t+|A|-1}(a_i) &\leq max^t - \frac{\delta^t}{|A|^{|A|-|P(a_*^t \to a_i)|-1}} \\
Bel_p^{t+|A|-1}(a_i) &\leq max^t - \frac{\left(\frac{In_{min}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U-L)}{|A|^{|A|-|P(a_*^t \to a_i)|-1}} \\
Bel_p^{t+|A|-1}(a_i) &\leq max^t - \frac{In_{min}^{|P(a_*^t \to a_i)|}.(U-L)}{|A|^{|A|-1}} \\
Bel_p^{t+|A|-1}(a_i) &\leq max^t - \left(\frac{In_{min}}{|A|}\right)^{|A|-1}.(U-L) \\
Bel_p^{t+|A|-1}(a_i) - max^t &\leq - \left(\frac{In_{min}}{|A|}\right)^{|A|-1}.(U-L) \\
max^t - Bel_p^{t+|A|-1}(a_i) &\geq \left(\frac{In_{min}}{|A|}\right)^{|A|-1}.(U-L) \\
max^t - Bel_p^{t+|A|-1}(a_i) &\geq \epsilon
\end{aligned}$$

□

Corollary 4. $max^t - max^{t+|A|-1} \geq \epsilon$

Proof. Since $max^{t+|A|-1}$ must be one of the beliefs in the time $t+|A|-1$ and, according to Theorem 2 all of them are smaller than max^t by a factor of at least ϵ, $max^{t+|A|-1}$ must also be smaller than max^t by a factor of at least ϵ. □

Theorem 3. $\lim_{t\to\infty} max^t = U = \lim_{t\to\infty} min^t = L$

Proof. Suppose, by contradiction, that $U \neq L$. Plugging this values into the ϵ formula show us that $\epsilon \neq 0$. Since, according to Theorem 2, $max^{t+|A|-1}$ is smaller than max^t by a factor of ϵ, we can finally reach to a contradiction and end our proof.

To see this contradiction, let's assume we did $v = (|A| - 1)\left(\lceil \frac{1}{\epsilon} \rceil + 1\right)$ time steps after $t = 0$. Since max diminishes by at least ϵ at each $|A| - 1$ steps:

$$max^0 \geq max^v + \epsilon \left(\left\lceil \frac{1}{\epsilon} \right\rceil + 1\right)$$

$$max^0 - \epsilon \left(\left\lceil \frac{1}{\epsilon} \right\rceil + 1\right) \geq max^v$$

But $\epsilon.\left(\lceil \frac{1}{\epsilon} \rceil + 1\right) > 1$, thus $\max^0 < \epsilon.\left(\lceil \frac{1}{\epsilon} \rceil + 1\right)$. And this would imply that $max^v < 0$, which contradicts the definition of belief!

Since assuming that $U \neq L$ led us to a contradiction we can conclude that $U = L$. This result implies that all agents belief converge to the same value, as we wanted to prove. □