Proof of Belief Convergence and Other Discoveries

Weakly Connected - Classic Update

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min^t and max^t definition

Definition

Let's call min^t and max^t the minimum and maximum of the beliefs in time t over all agents, respectively. Thus:

$$min^t = \min_{a_i \in A} Bel_p^t(a_i) \text{ and } max^t = \max_{a_i \in A} Bel_p^t(a_i)$$

Useful lemmas

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Lemma

$$\lim_{t\to\infty} min^t = L \ and\ \lim_{t\to\infty} max^t = U \ \ for\ some\ L,\ U \in [0,1].$$

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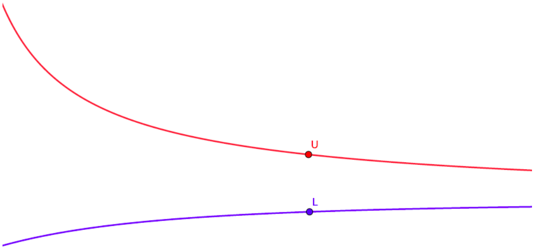

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Let's denote by $P(a_i \to a_j)$ *a path from* a_i *to* a_j *and let's call* $|P(a_i \to a_j)|$ *the size of this path.*

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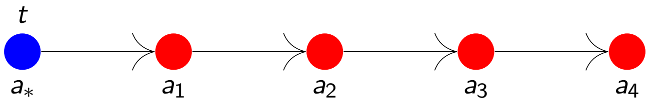
Theorem

$\forall t$ *and* $\forall a_i \in A$:

$$Bel_p^{t+|P(a_*^t \rightarrow a_i)|}(a_i) \leq max^t - \delta^t, \text{ with } \delta^t = \left(\frac{In_{min}}{|A|}\right)^{|P(a_*^t \rightarrow a_i)|}.(U - L).$$

What the theorem means?

- The theorem states that the influence exerted by a_*^t guarantees that, by the time $t + |P(a_*^t, a_i)|$, the belief of any agent a_i is smaller than max^t by a factor of at least δ^t.



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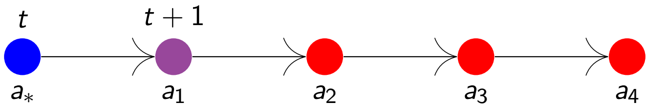

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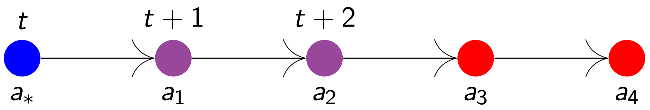

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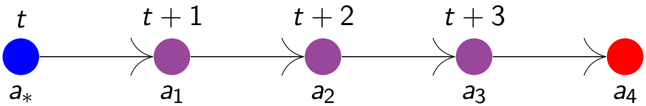

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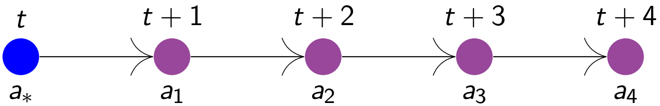

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- But it does not convey us much information, because each one of them is influenced in a different time.
- To solve this we can use two important pieces of information to acquire an idea about the agents belief in the same time step.

Information 1

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$$\forall a_i, a_j \in A, \ |P(a_i \to a_j)| \leq |A| - 1.$$

- Thus, any agent a_i will be influenced by a_*^t in $|A| - 1$ steps maximum.

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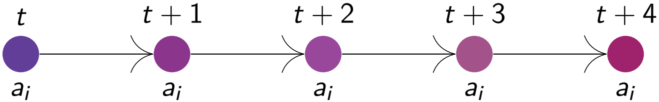

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- Intuition:
  - Every agent a_i was influenced by a_*^t in some time step between t and $t + |A| - 1$.
  - Since every agent influences itself throughout time, we can find an *constant* ϵ which is a common factor of influence of a_*^t over every agent in the time $t + |A| - 1$.

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  - The maximum an influence must travel to reach an agent is $|A|-1$ which is the exponent in the formula.

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  - Even when min_t is closer to max_t (which happens when $min_t = L$ and $max_t = U$), a_*^t still influences the agent that holds the belief max_t.
  - This is the "worst case scenario", thus this ϵ that holds in every case.

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Corollary

$$max^t - \epsilon \geq max^{t+|A|-1}.$$

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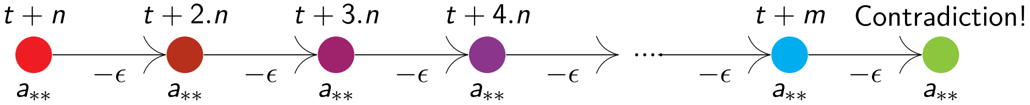

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Proof.

For every a_i: $\lim_{t\to\infty} min^t \leq \lim_{t\to\infty} Bel_p^t(a_i) \leq \lim_{t\to\infty} max^t$.

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Proof.

For every a_i: $\lim_{t\to\infty} min^t \leq \lim_{t\to\infty} Bel_p^t(a_i) \leq \lim_{t\to\infty} max^t$.

We can then use the Squeeze Theorem to show that for every a_i: $\lim_{t\to\infty} Bel_p^t(a_i) = U$. □

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- When $0 \neq f_{confbias}$ the proof is basically the same but with a new constant, f_{cbmin}.
- This constant has the same role as In_{min} in the proof showed above.

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  - Case 2: Exits some agent a_k that has belief $B \neq 0$ and $B \neq 1$.
    - We can use same technique used with a_* in the proof above.
    - Tracing the influence of a_k to every agent guarantees that no longer exists $f_{confbias} = 0$, and then we fall on the general case.

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- The idea is, for short, to use the proof showed above in weakly connected subgraphs and thus guaranteeing conversion of each subgraph.

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- Experiments showed that, under the backfire-effect update function, *min* and *max* are not monotonic.
- Since this property is crucial for the proof showed above, I don't think that this is the way to prove convergence in this case.

Other discoveries

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- In a clique, under the classic update function, the speed of convergence is directly proportional to the influence. (proved)
- In some graphs, the initial belief of some agents does not affect their own belief in the limit. (found via experiments).
- In the graph "unrelenting influencers" the belief in the limit *seems* to be equal to the average of the beliefs of the influencers weighted by it's influence (found via experiments).