Proof of Individual Agent Opinion Convergence in a Strongly Connected Influence Graph Using Classic Update Function

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In the classic update function, $Bel_p^{t+1}(a_i|a_j)$ can be written in the following form:

Definition 1 $Bel_p^{t+1}(a_i|a_j) = Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))$.

And the classic update function, $Bel_p^{t+1}(a_i)$ is written as:

Definition 2 $Bel_p^{t+1}(a_i) = \frac{1}{|A|}\sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j)$.

And let's define a strongly connected graph as:

Definition 3 *A strongly connected influence graph social network in which every agent exerts influence on every other agent: $In(a_i,a_j) > 0$, for every i, j.*

Definition 4 *max_t and min_t are the maximum and minimum belief values in a given instant t, respectively.*

To prove our conjecture, let's do some simplifications:

$$\begin{aligned}
Bel_p^{t+1}(a_i) &= \frac{1}{|A|}\sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j). \\
&= \frac{1}{|A|}\sum_{a_j \in A} \big(Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))\big) \\
&= Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))
\end{aligned}$$

Since belief values are finite, by the well-ordering principle we always have a min_t and a max_t. It is easy to see, by the squeeze theorem, that individual agent opinion converges to the same value if and only if $lim_{t\to\infty}\ min_t = lim_{t\to\infty}\ max_t$.

Thus, since we want to prove that it always converges, if $min_t = max_t$ we have nothing to prove, so assume $min_t \neq max_t$.

Lemma 1 *In a strongly connected graph and under classic belief update, if $max_t \neq min_t$:*

$$\forall a_j \in A : Bel_p^{t+1}(a_j) < max_t \tag{1}$$

and:

$$\forall a_j \in A : Bel_p^{t+1}(a_j) > min_t \tag{2}$$

Proof of Lemma 1

To show this we are gonna consider two cases:

Case 1 - $Bel_p^{t+1}(a_i) = max_t$:

$$\begin{aligned}
Bel_p^{t+1}(a_i) &= Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \\
&= max_t + \frac{1}{|A|}\sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - max_t)
\end{aligned}$$

Since max_t is, by definition, the biggest element of the set of all beliefs in the instant t, then $Bel_p^t(a_j) - max_t \leq 0$ for every a_j. This implies that:

$$In(a_j, a_i)(Bel_p^t(a_j) - max_t) \leq 0$$

Since $In(a_j, a_i) > 0$.

Given that we assumed that $min_t \neq max_t$, there exits at least one a_j, such that $Bel_p^t(a_j) \neq max_t$, thus, since all influence are positive:

$$\frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - max_t) < 0$$

Thus $Bel_p^{t+1}(a_i) = Bel_p^t(a_i)$ plus a negative number, which implies that

$$\begin{aligned} Bel_p^{t+1}(a_i) &< Bel_p^t(a_i) \\ Bel_p^{t+1}(a_i) &< max_t \end{aligned}$$

Case 2 - $Bel_p^{t+1}(a_i) < max_t$:
We will prove this case by contradiction, suppose:

$$\begin{aligned} max_t &\leq Bel_p^{t+1}(a_i) \\ max_t &\leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \end{aligned}$$

We can take a_i itself out of the summation:

$$\begin{aligned} max_t &\leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) + \frac{1}{|A|}(Bel_p^t(a_i) - Bel_p^t(a_i)) \\ &\leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \end{aligned}$$

Looking at the summation in the right side of the equation, given that $\forall a_j, a_i \in A$:

$$0 < In(a_j, a_i) \leq 1$$

and:

$$Bel_p^t(a_j) - Bel_p^t(a_i) \leq max_t - Bel_p^t(a_i)$$

We can say that:

$$\frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \leq \frac{|A| - 1}{|A|} \left(max_t - Bel_p^t(a_i)\right)$$

Thus:

$$\begin{aligned} max_t &\leq Bel_p^{t+1}(a_i) \\ max_t &\leq \frac{|A| - 1}{|A|} \left(max_t - Bel_p^t(a_i)\right) + Bel_p^t(a_i) \\ max_t &\leq \frac{(|A| - 1)\ max_t}{|A|} + \frac{Bel_p^t(a_i)}{|A|} \\ max_t - \frac{(|A| - 1)\ max_t}{|A|} &\leq \frac{Bel_p^t(a_i)}{|A|} \\ \frac{max_t}{|A|} &\leq \frac{Bel_p^t(a_i)}{|A|} \\ max_t &\leq Bel_p^t(a_i) \end{aligned}$$

Which is a contradiction, since we assumed $Bel_p^t(a_i) > max_t$. Thus:

$$max_t > Bel_p^{t+1}(a_i)$$

Since a_i was arbitrary and the cases covered are exhaustive, the lemma is true for all agents. The same reasoning can be used to show the equivalent property for min_t

□

Corollary 1 *In a strongly connected influence graph and a classic update function if* $min_t \neq max_t$*, then* $max_{t+1} < max_t$ *and* $min_{t+1} > min_t$*.*

The result of lemma 1 tells us that all beliefs in the time $t+1$ are smaller than max_t, thus, since max_{t+1} must be one of those elements, $max_{t+1} < max_t$. And the same reasoning can be used for min_t.

Corollary 2 $lim_{t\to\infty} max_t = L$ *and* $lim_{t\to\infty} min_t = M$ *for some* $L,\, M \in [0,1]$*.*

Since both max_t and min_t are bounded between 0 and 1 by the definition of belief; and lemma 1 showed us that they are monotonic, according to the monotonic convergence theorem, the limits exist.

Definition 5 *Let's denote by* In_{min} *the smallest influence in the influence graph. Keep in mind that* $In_{min} > 0$ *since we are working with a strongly connected influence graph.*

Using the same notation we used in corollary 2, let's call $lim_{t\to\infty} max_t = L$ and $lim_{t\to\infty} min_t = M$.

Lemma 2 $\forall t$ *and* $\forall a_i \in A$*:* $Bel_p^{t+1}(a_i) - max_t \leq \epsilon$*, with* $\epsilon = \frac{In_{min}(L-M)}{|A|}$*.*

To prove this lemma we will show that, even in the "worst case", the property above holds. Our goal then is to guarantee that it is in fact the worst case and that it implies our lemma. Now, let's start with the formula of belief:

$$Bel_p^{t+1}(a_i) = Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))$$

Since to prove our statement we must show that an inequality holds true, we will start with the above equation and try to maximize the right side as much as we can, this will lead us to ϵ.

The first thing we will do is separate from the summation the element a_k, which we define as the element that holds the belief min_t in that arbitrary time step.

$$\begin{aligned} Bel_p^{t+1}(a_i) &= Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_k\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) + \frac{In(a_k, a_i)(Bel_p^t(a_k) - Bel_p^t(a_i))}{|A|} \\ &= Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_k\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) + \frac{In(a_k, a_i)(min_t - Bel_p^t(a_i))}{|A|} \end{aligned}$$

By the definition of min_t, $min_t - Bel_p^t(a_i) \leq 0$. If it values 0, the influence that multiplies it doesn't make any difference, but if it is different of 0 we want the influence to be as small as possible, which is In_{min}.

$$Bel_p^{t+1}(a_i) \leq Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_k\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) + \frac{In_{min}(min_t - Bel_p^t(a_i))}{|A|}$$

Now it's time to choose the "worst" value of $Bel_p^t(a_j)$, and this value is max_t.

$$Bel_p^{t+1}(a_i) \leq Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_k\}} In(a_j, a_i)(max_t - Bel_p^t(a_i)) + \frac{In_{min}(min_t - Bel_p^t(a_i))}{|A|}$$

Since $max_t - Bel_p^t(a_i) \geq 0$, the "worst case" for the influence is $\forall a_i, a_j \in A \setminus \{a_k\} :\ In(a_j, a_i) = 1$.

$$\begin{aligned} Bel_p^{t+1}(a_i) &\leq Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_k\}} (max_t - Bel_p^t(a_i)) + \frac{In_{min}(min_t - Bel_p^t(a_i))}{|A|} \\ &\leq Bel_p^t(a_i) + \frac{|A|-1}{|A|}(max_t - Bel_p^t(a_i)) + \frac{In_{min}(min_t - Bel_p^t(a_i))}{|A|} \\ &\leq Bel_p^t(a_i) + \frac{(|A|-1)\,(max_t - Bel_p^t(a_i)) + In_{min}(min_t - Bel_p^t(a_i))}{|A|} \\ &\leq \frac{|A|\ Bel_p^t(a_i) + (|A|-1)\,(max_t - Bel_p^t(a_i)) + In_{min}(min_t - Bel_p^t(a_i))}{|A|} \\ &\leq \frac{(|A|-1)\,max_t - Bel_p^t(a_i) + In_{min}(min_t - Bel_p^t(a_i))}{|A|} \\ &\leq \frac{(|A|-1)\,max_t + Bel_p^t(a_i)\,(1 - In_{min}) + In_{min} min_t}{|A|} \end{aligned}$$

These manipulations made it clear which choice of $Bel_p^t(a_i)$ is the "worst", and it is $Bel_p^t(a_i) = max_t$.

$$
\begin{aligned}
Bel_p^{t+1}(a_i) &\leq \frac{(|A|-1)\,max_t + max_t\,(1-In_{min}) + In_{min}min_t}{|A|} \\
&\leq \frac{|A|\ max_t - max_t\ In_{min} + In_{min}\ min_t}{|A|} \\
&\leq \frac{|A|\ max_t + In_{min}(min_t - max_t)}{|A|} \\
&\leq max_t + \frac{In_{min}(min_t - max_t)}{|A|}
\end{aligned}
$$

Now we shall remember that, since max_t is decreasing and min_t is increasing, our choice to make the right side as big as possible is to plug it's limits, which gives us:

$$Bel_p^{t+1}(a_i) \leq max_t + \frac{In_{min}(L-M)}{|A|}$$

Since we tried to build the greater $Bel_p^{t+1}(a_i)$ we could, and reached this inequality, we can say that the greater $Bel_p^{t+1}(a_i) = max_t + \frac{In_{min}(min_t - max_t)}{|A|}$.

With this, we must be able to show that, even with the greater $Bel_p^{t+1}(a_i)$ possible, the difference between it and max_t is still ϵ since this property is valid for the greater $Bel_p^{t+1}(a_i)$ we can build, the different will be either equal or bigger for smaller ones, thus, for all.

Let's call Bel_{max}^{t+1} the maximum belief we could build.

$$
\begin{aligned}
Bel_{max}^{t+1} &= max_t + \frac{In_{min}(L-M)}{|A|} \\
Bel_{max}^{t+1} - max_t &= \frac{In_{min}(L-M)}{|A|}
\end{aligned}
$$

Thus: $\forall t$ and $\forall a_i \in A$: $Bel_p^{t+1}(a_i) - max_t \leq \epsilon$, with $\epsilon = \frac{In_{min}(L-M)}{|A|}$

□

Corollary 3 $max_{t+1} + \epsilon \leq max_t$

Since max_{t+1} must be one of the beliefs in the time $t+1$ and, according to Lemma 2, all of them are smaller than max_t by at least ϵ, max_{t+1} must also be smaller than max_t by a factor of at least ϵ.

Theorem 1 $lim_{t\to\infty} max_t = L = lim_{t\to\infty} min_t = M$

Suppose, by contradiction, that $L \neq M$. This means that $\epsilon \neq 0$. Since, according to lemma 2, max_{t+1} is smaller than max_t by a factor of ϵ. If ϵ is constant we reach to a contradiction because: when t tend to infinity it would get smaller than L. But L is the limit of max_t which is decreasing. Since assuming that $L \neq M$ leads us to a contradiction, $L = M$. Which implies that all agents belief converge to the same value, as we wanted to prove.

□