Proof of Individual Agent Opinion Convergence in a Weakly Connected Influence Graph Using Classic Update Function

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Definition 1. The *classic update-function*, is defined as:

$$Bel_p^{t+1}(a_i|a_j) = Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \tag{1}$$

Definition 2. While the *overall classic update*, is defined as:

$$Bel_p^{t+1}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j) \tag{2}$$

Definition 3. We say a influence graph In over agents A is *weakly connected* if for all a_i, $a_j \in A$, there exist a_{k_1},a_{k_2},...,$a_{k_l} \subseteq A$ such that In(a_i,a_{k_1}) > 0, In(a_{k_l},a_j) > 0, and for $m = 1$,...,$l-1$, In(a_{k_m},$a_{k_{m+1}}$) > 0.

Definition 4. max_t and min_t are the maximum and minimum belief values in a given instant t, respectively.

To prove our conjecture, let's do some simplifications:

$$\begin{aligned}
Bel_p^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j). \\
&= \frac{1}{|A|} \sum_{a_j \in A} \big(Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))\big) \\
&= Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))
\end{aligned} \tag{3}$$

Since we have a finite number of beliefs and $\forall a_i \in A : \ Bel_p^t(a_i) \in [0, 1]$, there are always min_t and a max_t. We shall also note that, by the Squeeze Theorem, individual agent opinion converges to the same value if and only if $lim_{t\to\infty} min_t = lim_{t\to\infty} max_t$.

Thus, since we want to prove that it always converges, if $min_t = max_t$ we have nothing to prove, so assume $min_t \neq max_t$.

Lemma 1. *In a weakly connected graph and under classic belief update, if $max_t \neq min_t$:*

$$\forall a_i \in A : min_t \leq Bel_p^{t+1}(a_i) \leq max_t$$

Proof. By the equation 3:

$$Bel_p^{t+1}(a_i) = Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))$$

Trying to maximize the right side, we can substitute $Bel_p^t(a_j)$ by max_t, this turns our equation into an inequity, since $\forall a_j \in A$, $Bel_p^t(a_j) \leq max_t$, by the definition of max_t. That makes the terms inside the summation either equal or smaller than 0, thus:

$$\begin{aligned} Bel_p^{t+1}(a_i) &\leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(max_t - Bel_p^t(a_i)) \\ &= Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} (max_t - Bel_p^t(a_i)) && \text{(since } In(a_j,a_i) \leq 1) \\ &= Bel_p^t(a_i) + \frac{|A|}{|A|}(max_t - Bel_p^t(a_i)) \\ &= Bel_p^t(a_i) + max_t - Bel_p^t(a_i) \\ Bel_p^{t+1}(a_i) &\leq max_t \end{aligned} \tag{4}$$

Since a_i was arbitrary, the Lemma is true for all agents. The same reasoning can be used to show the equivalent property for min_t □

Corollary 1. *In a weakly connected influence graph and a classic update function, if $min_t \neq max_t$, then $max_{t+1} \leq max_t$ and $min_{t+1} \geq min_t$.*

Proof. The result of Lemma 1 tells us that all beliefs in the time $t+1$ are either smaller than max_t or equal to max_t, thus, since max_{t+1} must be one of those elements, $max_{t+1} \leq max_t$. And the same reasoning can be used for min_t. □

Corollary 2. $\lim_{t\to\infty} max_t = U$ *and* $\lim_{t\to\infty} min_t = L$ *for some U, $L \in [0, 1]$.*

Proof. Since both max_t and min_t are bounded between 0 and 1 by the definition of belief; and Lemma 1 showed us that they are monotonic, according to the Monotonic Convergence Theorem, the limits exist. □

Now that we have those properties, our proof will follow by showing that an agent a_i that holds some belief $Bel_p^t(a_i)$ will influence every other agent by the time $t+|A|-1$. To see this, we must open the definition of belief throughout time. But before we do this, let's jump into some small definitions and corollaries that will help us on the way.

Definition 5. Let's call the sequence $P(a_i \to a_j) = (a_i, a_k, ..., a_{k+l}, a_j)$ a *path* from a_i to a_j, if:

- All elements on the sequence are different.

- The first element in the sequence is a_i.
- The last element in the sequence is a_j.
- If a_n is the n'th element in the sequence, if it has a successor a_{n+1}, $In(a_n, a_{n+1}) > 0$.

Note that many paths from a_i to a_j can exist, although our notation isn't enough to differentiate them. But in subsequent steps we will only need one of those paths, so the notation shouldn't be a problem.

Definition 6. Let's denote by $|P(a_i \to a_j)|$ the size of a path from a_i to a_j, which we define as the number of elements in the sequence $P(a_i \to a_j) - 1$.

Corollary 3. $|P(a_i \to a_j)| \leq |A| - 1$.

Proof. This follows directly from the definition of path. Since it doesn't have repeated elements and we have $|A|$ agents, the path can't have more than $|A|$ elements, since the size of a path is defined as the number of elements minus one, the maximum size is $|A| - 1$. □

Lemma 2. *$\forall a_i, a_k \in A$ and $\forall n \geq 1$:*

$$Bel_p^{t+n}(a_i) \leq max_t + \frac{1}{|A|}\left(In(a_k, a_i)(Bel_p^{t+n-1}(a_k) - max_t)\right) \tag{5}$$

Proof. By the Definitions 1 and 2:

$$Bel_p^{t+n}(a_i) = \frac{1}{|A|}\sum_{a_j \in A} Bel_p^{t+n}(a_i|a_j)$$

$$Bel_p^{t+n}(a_i) = \frac{1}{|A|}\sum_{a_j \in A}\left(Bel_p^{t+n-1}(a_i) + In(a_j, a_i)(Bel_p^{t+n-1}(a_j) - Bel_p^{t+n-1}(a_i))\right)$$

Now note that the belief in the next time step is maximized when the belief of a_i itself is the maximum one possible. Given this property, we can replace $Bel_p^{t+n-1}(a_i)$ by max_{t+n-1} and thus turn our equation into an inequity:

$$\begin{aligned}Bel_p^{t+n}(a_i) &\leq \frac{1}{|A|}\sum_{a_j \in A}\left(max_{t+n-1} + In(a_j, a_i)(Bel_p^{t+n-1}(a_j) - max_{t+n-1})\right)\\ &= max_{t+n-1} + \frac{1}{|A|}\sum_{a_j \in A} In(a_j, a_i)(Bel_p^{t+n-1}(a_j) - max_{t+n-1})\end{aligned}$$

To make our Lemma useful in future manipulations, we will take an arbitrary element a_k out of the summation:

$$\begin{aligned}Bel_p^{t+n}(a_i) \leq max_{t+n-1} &+ \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_k\}}\left(In(a_j, a_i)(Bel_p^{t+n-1}(a_j) - max_{t+n-1})\right)\\ &+ \frac{1}{|A|}(In(a_k, a_i)(Bel_p^{t+n-1}(a_k) - max_{t+n-1}))\end{aligned}$$

Since max_{t+n-1} is the greatest belief possible in that time step, the summation can be at most 0, thus:

$$Bel_p^{t+n}(a_i) \leq max_{t+n-1} + \frac{1}{|A|}\left(In(a_k, a_i)\left(Bel_p^{t+n-1}(a_k) - max_{t+n-1}\right)\right)$$

Since max doesn't increase throughout time, $max_t \leq max_{t+n-1}$, thus:

$$Bel_p^{t+n}(a_i) \leq max_t + \frac{1}{|A|}\left(In(a_k, a_i)\left(Bel_p^{t+n-1}(a_k) - max_t\right)\right)$$

□

Definition 7. Let's denote by In_{min} the smallest influence that's different from 0 in the influence graph.

Using the same notation we used in Corollary 2, let's call $\lim_{t\to\infty} max_t = U$ and $\lim_{t\to\infty} min_t = L$.
Now that we have all of these tools, let's jump to Lemma 1 which will be a tool in the most important part of the proof. Calling a_k the agent who holds the belief min_t in the time t:

Theorem 1. *$\forall a_i \in A: \; Bel_p^{t+|P(a_k\to a_i)|}(a_i) \leq max_t - \delta$, with $\delta = \left(\frac{In_{min}}{|A|}\right)^{|P(a_k\to a_i)|} .(U-L)$.*

Proof. By equation 3:

$$Bel_p^{t+|P(a_k\to a_i)|}(a_i) = Bel_p^{t+|P(a_k\to a_i)|-1}(a_i) + \frac{1}{|A|}\sum_{a_j\in A} Bel_p^{t+|P(a_k\to a_i)|-1}(a_i|a_j)$$

What we will do now is separate, at each step, one element of the summation and apply Lemma 2 to modify our inequity. But we will be careful when choosing the elements we separate from the summation. We will separate from the summation the elements in $P(a_k \to a_i)$, starting from the end of the path until we get to a_k. To simplify our notation, let's index the elements in the path from a_k to a_i, starting from the end of the path (since we are backtracking it will make more sense) by calling a_n the n^{th} element from the end to the beginning of the sequence. Thus, by Lemma 2:

$$Bel_p^{t+|P(a_k\to a_i)|}(a_i) \leq max_t + \frac{1}{|A|} In(a_1, a_i)(Bel_p^{t+|P(a_k\to a_i)|-1}(a_1) - max_t)$$

Note, now, that if $|P(a_k, a_i)| = 1$, we could prove our result. Instead of showing it I will expand this two more times, show the general version and then prove the Lemma for all cases. Using Lemma 2 again:

$$\begin{aligned}
&Bel_p^{t+|P(a_k\to a_i)|}(a_i)\\
&\leq max_t + \frac{1}{|A|} In(a_1, a_i)(Bel_p^{t+|P(a_k\to a_i)|-1}(a_1) - max_t)\\
&\leq max_t + \frac{1}{|A|} In(a_1, a_i)\left(\left(max_t + \frac{1}{|A|} In(a_2, a_1)(Bel_p^{t+|P(a_k\to a_i)|-2}(a_2) - max_t)\right) - max_t\right)
\end{aligned}$$

$$
\begin{aligned}
&= max_t + \frac{1}{|A|} In(a_1, a_i) \left(\frac{1}{|A|} In(a_2, a_1)(Bel_p^{t+|P(a_k \to a_i)|-2}(a_2) - max_t) \right) \\
&= max_t + \frac{1}{|A|^2} In(a_2, a_1) In(a_1, a_i)(Bel_p^{t+|P(a_k \to a_i)|-2}(a_2) - max_t) \\
&\leq max_t + \frac{1}{|A|^2} In(a_2, a_1) In(a_1, a_i) \times \\
&\quad \left(\left(max_t + \frac{1}{|A|} In(a_3, a_2) \left(Bel_p^{t+|P(a_k \to a_i)|-3}(a_3) - max_t \right) \right) - max_t \right) \\
&= max_t + \frac{1}{|A|^2} In(a_2, a_1) In(a_1, a_i) \left(\frac{1}{|A|} In(a_3, a_2) \left(Bel_p^{t+|P(a_k \to a_i)|-3}(a_3) \right) - max_t \right) \\
&= max_t + \frac{1}{|A|^3} In(a_3, a_2) In(a_2, a_1) In(a_1, a_i) \left(Bel_p^{t+|P(a_k \to a_i)|-3}(a_3) - max_t \right)
\end{aligned}
$$

We can see a patter forming since the equation above has the same form of the one before it, and this pattern will continue throughout time. Now, denoting P_{In} the product of the influences in the path ($P_{In} = In(a_k, a_{|P(a_k,a_i)|}) \times ... \times In(a_1, a_i)$), we can write the generalized version of the inequity above as:

$$
\begin{aligned}
Bel_p^{t+|P(a_k \to a_i)|}(a_i) &\leq max_t + \frac{P_{In}}{|A|^{|P(a_k \to a_i)|}}.(Bel_p^t(a_k) - max_t) \\
&= max_t + \frac{P_{In}}{|A|^{|P(a_k \to a_i)|}}.(min_t - max_t)
\end{aligned}
$$

This inequity comes from the fact that the path ends after $|P(a_k \to a_i)|$ steps with a_k as the start of the path, and, by definition, the belief of a_k in the time t is min_t.

Since the rightmost term in the equation is either equal to or smaller than 0, to make the inequality hold for all a_i's, we shall substitute P_{In} by the smallest value possible. According to Definition 7, In_{min} is the smallest positive influence in the graph. By the definition of path (5) all influences are positive, thus the smallest possible value of P_{In} is $In_{min}^{|P(a_k \to a_i)|}$. Thus:

$$
Bel_p^{t+|P(a_k \to a_i)|}(a_i) \leq max_t + \left(\frac{In_{min}}{|A|} \right)^{|P(a_k \to a_i)|}.(min_t - max_t)
$$

According to Corollary 2, the maximum value of min_t is L and the minimum value of max_t is U, those are the values we should plug to maintain the inequity:

$$
\begin{aligned}
Bel_p^{t+|P(a_k \to a_i)|}(a_i) &\leq max_t + \left(\frac{In_{min}}{|A|} \right)^{|P(a_k \to a_i)|}.(L - U) \\
Bel_p^{t+|P(a_k \to a_i)|}(a_i) &\leq max_t - \left(\frac{In_{min}}{|A|} \right)^{|P(a_k \to a_i)|}.(U - L) \\
Bel_p^{t+|P(a_k \to a_i)|}(a_i) &\leq max_t - \delta
\end{aligned}
$$

$\square$

Lemma 3.

$$\sum_{a_j \in A} In(a_j, a_i) \left(Bel_p^t(a_j) - Bel_p^t(a_i)\right) = \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i) \left(Bel_p^t(a_j) - Bel_p^t(a_i)\right)$$

Proof.

$$\begin{aligned}
&\sum_{a_j \in A} In(a_j, a_i) \left(Bel_p^t(a_j) - Bel_p^t(a_i)\right) \\
&= \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i) \left(Bel_p^t(a_j) - Bel_p^t(a_i)\right) + In(a_i, a_i)(Bel_p^t(a_i) - Bel_p^t(a_i)) \\
&= \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i) \left(Bel_p^t(a_j) - Bel_p^t(a_i)\right)
\end{aligned}$$

□

Lemma 4. *If* $Bel_p^t(a_i) \leq x$*:*

$$Bel_p^{t+1}(a_i) \leq x + \frac{1}{|A|} \sum_{a_j \in A} (In(a_j, a_i) \left(Bel_p^t(a_j) - x\right)$$

Proof.

$$\begin{aligned}
Bel_p^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} \left(Bel_p^t(a_i) + In(a_j, a_i) \left(Bel_p^t(a_j) - Bel_p^t(a_i)\right)\right) \\
&= \frac{1}{|A|} \sum_{a_j \in A} \left(Bel_p^t(a_i)(1 - In(a_j, a_i)) + In(a_j, a_i) Bel_p^t(a_j)\right) \\
&\leq \frac{1}{|A|} \sum_{a_j \in A} \left(x.(1 - In(a_j, a_i)) + In(a_j, a_i) Bel_p^t(a_j)\right) \\
&= x + \frac{1}{|A|} \sum_{a_j \in A} \left(In(a_j, a_i) \left(Bel_p^t(a_j) - x\right)\right)
\end{aligned}$$

□

Lemma 5. *If* $Bel_p^{t+x}(a_i) \leq max_t - \gamma$*,* $\gamma \geq 0$ *and* $x \geq 0$*, then* $Bel_p^{t+x+1}(a_i) \leq max_t - \frac{\gamma}{|A|}$*.*

Proof.

$$\begin{aligned}
Bel_p^{t+x+1}(a_i) &= Bel_p^{t+x}(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i) \left(Bel_p^{t+x}(a_j) - Bel_p^{t+x}(a_i)\right) \\
&= Bel_p^{t+x}(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i) \left(Bel_p^{t+x}(a_j) - Bel_p^{t+x}(a_i)\right) \quad \text{(Lemma 3)} \\
&\leq max_t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i) \left(Bel_p^{t+x}(a_j) - max_t + \gamma\right) \quad \text{(Lemma 4)} \\
&\leq max_t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i) \left(max_t - max_t + \gamma\right)
\end{aligned}$$

$$
\begin{aligned}
&= max_t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i)\,(\gamma) \\
&\leq max_t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} (\gamma) \\
&= max_t - \gamma + \frac{(|A| - 1)(\gamma)}{|A|} \\
&= max_t + \frac{(\gamma)((-|A|) + (|A| - 1))}{|A|} \\
Bel_p^{t+x+1}(a_i) &\leq max_t - \frac{\gamma}{|A|}
\end{aligned}
$$

□

Theorem 2. *$\forall a_i \in A: \; max_t - Bel_p^{t+|A|-1}(a_i) \geq \epsilon,$ with $\epsilon = \left(\frac{In_{min}}{|A|}\right)^{|A|-1}.(U - L).$*

Proof. Let's keep the notation of the previous Lemma and call a_k the agent that holds the belief min_t in the time t.
First we should note that, if $|P(a_k \to a_i)| = |A| - 1$, our theorem is true by Theorem 1 and we nothing to prove.
Else if $|P(a_k \to a_i)| \neq |A| - 1$, then $|P(a_k \to a_i)| < |A| - 1$ according to Corollary 3.
According to Theorem 1:

$$
Bel_p^{t+|P(a_k \to a_i)|}(a_i) \leq max_t - \left(\frac{In_{min}}{|A|}\right)^{|P(a_k \to a_i)|}.(U - L)
$$

To keep things simple let's keep the notation from Theorem 1 and call:

$$
\delta = \left(\frac{In_{min}}{|A|}\right)^{|P(a_k \to a_i)|}.(U - L)
$$

.
Now it is easier to see that we can apply Lemma 5 successively:

$$
\begin{aligned}
Bel_p^{t+|P(a_k \to a_i)|+1}(a_i) &\leq max_t - \frac{\delta}{|A|} \\
Bel_p^{t+|P(a_k \to a_i)|+2}(a_i) &\leq max_t - \frac{\delta}{|A|^2} \\
Bel_p^{t+|P(a_k \to a_i)|+3}(a_i) &\leq max_t - \frac{\delta}{|A|^3}
\end{aligned}
$$

If we do it $|A| - |P(a_k \to a_i)| - 1$ times we get:

$$
\begin{aligned}
Bel_p^{t+|P(a_k \to a_i)|+|A|-|P(a_k \to a_i)|-1}(a_i) &\leq max_t - \frac{\delta}{|A|^{|A|-|P(a_k \to a_i)|-1}} \\
Bel_p^{t+|A|-1}(a_i) &\leq max_t - \frac{\delta}{|A|^{|A|-|P(a_k \to a_i)|-1}} \\
Bel_p^{t+|A|-1}(a_i) &\leq max_t - \frac{\left(\frac{In_{min}}{|A|}\right)^{|P(a_k \to a_i)|}.(U - L)}{|A|^{|A|-|P(a_k \to a_i)|-1}}
\end{aligned}
$$

$$
\begin{aligned}
Bel_p^{t+|A|-1}(a_i) &\leq max_t - \frac{In_{min}^{|P(a_k \to a_i)|}.(U-L)}{|A|^{|A|-1}} \\
Bel_p^{t+|A|-1}(a_i) &\leq max_t - \left(\frac{In_{min}}{|A|}\right)^{|A|-1}.(U-L) \\
Bel_p^{t+|A|-1}(a_i) - max_t &\leq - \left(\frac{In_{min}}{|A|}\right)^{|A|-1}.(U-L) \\
max_t - Bel_p^{t+|A|-1}(a_i) &\geq \left(\frac{In_{min}}{|A|}\right)^{|A|-1}.(U-L) \\
max_t - Bel_p^{t+|A|-1}(a_i) &\geq \epsilon
\end{aligned}
$$

□

Corollary 4. $max_t - max_{t+|A|-1} \geq \epsilon$

Proof. Since $max_{t+|A|-1}$ must be one of the beliefs in the time $t+|A|-1$ and, according to Theorem 2 all of them are smaller than max_t by a factor of at least ϵ, $max_{t+|A|-1}$ must also be smaller than max_t by a factor of at least ϵ. □

Theorem 3. $\lim_{t\to\infty} max_t = U = \lim_{t\to\infty} min_t = L$

Proof. Suppose, by contradiction, that $U \neq L$. Plugging this values into the ϵ formula show us that $\epsilon \neq 0$. Since, according to Theorem 2, $max_{t+|A|-1}$ is smaller than max_t by a factor of ϵ, we can finally reach to a contradiction and end our proof.

To see this contradiction, let's assume we did $v = (|A|-1)\left(\left\lceil\frac{1}{\epsilon}\right\rceil + 1\right)$ time steps after $t = 0$. Since max diminishes by at least ϵ at each $|A|-1$ steps:

$$
\begin{aligned}
max_0 &\geq max_v + \epsilon\ \left(\left\lceil\frac{1}{\epsilon}\right\rceil + 1\right) \\
max_0 - \epsilon\ \left(\left\lceil\frac{1}{\epsilon}\right\rceil + 1\right) &\geq max_v
\end{aligned}
$$

But $\epsilon.\left(\left\lceil\frac{1}{\epsilon}\right\rceil + 1\right) > 1$, thus $\max_0 < \epsilon.\left(\left\lceil\frac{1}{\epsilon}\right\rceil + 1\right)$. And this would imply that $max_v < 0$, which contradicts the definition of belief!

Since assuming that $U \neq L$ led us to a contradiction we can conclude that $U = L$. This result implies that all agents belief converge to the same value, as we wanted to prove. □