Proof of Individual Agent Opinion Convergence in a Strongly Connected Influence Graph Using Classic Update Function

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In the classic update function, $Bel_p^{t+1}(a_i|a_j)$ can be written in the following form:

Definition 1 $Bel_p^{t+1}(a_i|a_j) = Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))$.

And the classic update function, $Bel_p^{t+1}(a_i)$ is written as:

Definition 2 $Bel_p^{t+1}(a_i) = \frac{1}{|A|}\sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j)$.

And let's define a strongly connected graph as:

Definition 3 *A strongly connected influence graph social network in which every agent exerts influence on every other agent: $In(a_i,a_j) > 0$, for every i, j.*

Definition 4 *max_t and min_t are the maximum and minimum belief values in a given instant t, respectively.*

To prove our conjecture, let's do some simplifications:

$$
\begin{aligned}
Bel_p^{t+1}(a_i) &= \frac{1}{|A|}\sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j). \\
&= \frac{1}{|A|}\sum_{a_j \in A} \left(Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))\right) \\
&= Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))
\end{aligned}
$$

Since belief values are finite, by the well-ordering principle we always have a min_t and a max_t. It is easy to see, by the squeeze theorem, that individual agent opinion converges to the same value if and only if $lim_{t\to\infty}\ min_t = lim_{t\to\infty}\ max_t$.

Thus, since we want to prove that it always converges, if $min_t = max_t$ we have nothing to prove, so assume $min_t \neq max_t$.

Lemma 1 *In a strongly connected graph and under classic belief update, if $max_t \neq min_t$:*

$$\forall a_i \in A : Bel_p^{t+1}(a_i) < max_t \tag{1}$$

and:

$$\forall a_i \in A : Bel_p^{t+1}(a_i) > min_t \tag{2}$$

Proof of Lemma 1

By definition:

$$Bel_p^{t+1}(a_i) = Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))$$

Now since $max_t \neq min_t$, there is at least one value between a_j's, such that $Bel_p^t(a_j) < max_t$, thus replacing all $Bel_p^t(a_j)$ by max_t, we make the right side strictly greater than the left one:

$$\begin{aligned}
Bel_p^{t+1}(a_i) &< Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A} In(a_j, a_i)(max_t - Bel_p^t(a_i)) \\
&< Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A}(max_t - Bel_p^t(a_i)) \\
&< Bel_p^t(a_i) + \frac{|A|}{|A|}(max_t - Bel_p^t(a_i)) \\
&< Bel_p^t(a_i) + max_t - Bel_p^t(a_i) \\
Bel_p^{t+1}(a_i) &< max_t
\end{aligned}$$

Since a_i was arbitrary, the lemma is true for all agents. The same reasoning can be used to show the equivalent property for min_t

□

Corollary 1 *In a strongly connected influence graph and a classic update function if $min_t \neq max_t$, then $max_{t+1} < max_t$ and $min_{t+1} > min_t$.*

The result of lemma 1 tells us that all beliefs in the time $t+1$ are smaller than max_t, thus, since max_{t+1} must be one of those elements, $max_{t+1} < max_t$. And the same reasoning can be used for min_t.

Corollary 2 *$lim_{t\to\infty} max_t = L$ and $lim_{t\to\infty} min_t = M$ for some L, $M \in [0, 1]$.*

Since both max_t and min_t are bounded between 0 and 1 by the definition of belief; and lemma 1 showed us that they are monotonic, according to the monotonic convergence theorem, the limits exist.

Definition 5 *Let's denote by In_{min} the smallest influence in the influence graph. Keep in mind that $In_{min} > 0$ since we are working with a strongly connected influence graph.*

Using the same notation we used in corollary 2, let's call $lim_{t\to\infty} max_t = L$ and $lim_{t\to\infty} min_t = M$.

Lemma 2 *$\forall t$ and $\forall a_i \in A$: $max_t - Bel_p^{t+1}(a_i) \geq \epsilon$, with $\epsilon = \frac{In_{min}(L-M)}{|A|}$.*

To prove this lemma, first we will try to find the biggest $Bel_p^{t+1}(a_i)$ possible. Now let's start with the formula of belief:

$$Bel_p^{t+1}(a_i) = Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))$$

To achieve our goal, in each step we will choose the values in the right side of the equation in a way that maximizes it. Trying to do so will guarantee us that the inequality holds for every $Bel_p^{t+1}(a_i)$ and will lead us to ϵ.

The first thing we will do is separate from the summation the element a_k, which we define as the agent who holds the belief min_t in that arbitrary time step.

$$\begin{aligned}
Bel_p^{t+1}(a_i) &= Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_k\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) + \frac{In(a_k, a_i)(Bel_p^t(a_k) - Bel_p^t(a_i))}{|A|} \\
&= Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_k\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) + \frac{In(a_k, a_i)(min_t - Bel_p^t(a_i))}{|A|}
\end{aligned}$$

Now trying to maximize the rightmost term in the inequality, we shall see that, by the definition of min_t: $min_t - Bel_p^t(a_i) \leq 0$. If $min_t - Bel_p^t(a_i)$ values 0, the influence that multiplies it doesn't make any difference, but if it is different of 0 we want the influence to be as small as possible, which is In_{min}.

$$Bel_p^{t+1}(a_i) \leq Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_k\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) + \frac{In_{min}(min_t - Bel_p^t(a_i))}{|A|}$$

Now it's time to choose the value of $Bel_p^t(a_j)$ for all a_j's that maximizes the right side. Since this part is always positive, we shall pick the maximum value possible, which is max_t.

$$Bel_p^{t+1}(a_i) \leq Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_k\}} In(a_j, a_i)(max_t - Bel_p^t(a_i)) + \frac{In_{min}(min_t - Bel_p^t(a_i))}{|A|}$$

Now looking at the terms inside the summation, since $max_t - Bel_p^t(a_i) \geq 0$, the influence that maximizes it is the biggest one possible, which is 1, thus: $\forall a_i, a_j \in A \setminus \{a_k\}: \ In(a_j, a_i) = 1$.

$$
\begin{aligned}
Bel_p^{t+1}(a_i) &\leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_k\}} (max_t - Bel_p^t(a_i)) + \frac{In_{min}(min_t - Bel_p^t(a_i))}{|A|} \\
&\leq Bel_p^t(a_i) + \frac{(|A|-1)(max_t - Bel_p^t(a_i))}{|A|} + \frac{In_{min}(min_t - Bel_p^t(a_i))}{|A|} \\
&\leq Bel_p^t(a_i) + \frac{(|A|-1)\,(max_t - Bel_p^t(a_i)) + In_{min}(min_t - Bel_p^t(a_i))}{|A|} \\
&\leq \frac{|A|\ Bel_p^t(a_i) + (|A|-1)\,(max_t - Bel_p^t(a_i)) + In_{min}(min_t - Bel_p^t(a_i))}{|A|} \\
&\leq \frac{(|A|-1)\,max_t - Bel_p^t(a_i) + In_{min}(min_t - Bel_p^t(a_i))}{|A|} \\
&\leq \frac{(|A|-1)\,max_t + Bel_p^t(a_i)\,(1 - In_{min}) + In_{min}min_t}{|A|}
\end{aligned}
$$

These manipulations made it clear which value of $Bel_p^t(a_i)$ we should choose to achieve our goal, and it is $Bel_p^t(a_i) = max_t$.

$$
\begin{aligned}
Bel_p^{t+1}(a_i) &\leq \frac{(|A|-1)\,max_t + max_t\,(1 - In_{min}) + In_{min}min_t}{|A|} \\
&\leq \frac{|A|\ max_t - In_{min}\ max_t + In_{min}\ min_t}{|A|} \\
&\leq \frac{|A|\ max_t + In_{min}(min_t - max_t)}{|A|} \\
&\leq max_t + \frac{In_{min}(min_t - max_t)}{|A|}
\end{aligned}
$$

Now we shall remember that, since max_t is decreasing and min_t is increasing, our choice to make the right side as big as possible is to plug it's limits, which gives us:

$$
\begin{aligned}
Bel_p^{t+1}(a_i) &\leq max_t + \frac{In_{min}(M - L)}{|A|} \\
Bel_p^{t+1}(a_i) - max_t &\leq \frac{In_{min}(M - L)}{|A|} \\
max_t - Bel_p^{t+1}(a_i) &\geq \frac{In_{min}(L - M)}{|A|} \\
max_t - Bel_p^{t+1}(a_i) &\geq \epsilon
\end{aligned}
$$

□

Corollary 3 $max_{t+1} + \epsilon \leq max_t$

Since max_{t+1} must be one of the beliefs in the time $t+1$ and, according to Lemma 2, all of them are smaller than max_t by at least ϵ, max_{t+1} must also be smaller than max_t by a factor of at least ϵ.

Theorem 1 $lim_{t \to \infty} max_t = L = lim_{t \to \infty} min_t = M$

Suppose, by contradiction, that $L \neq M$. Plugging this values into the ϵ formula show us that $\epsilon \neq 0$. Since, according to lemma 2, max_{t+1} is smaller than max_t by a factor of ϵ. With all of this we can finally reach to a contradiction and end our proof.

To see this contradiction, let's assume we did $v = \lceil \frac{1}{\epsilon} \rceil + 1$ timesteps after some $t = 0$. Since max diminishes by at least ϵ at each step:

$$
\begin{aligned}
max_0 &\geq max_v + \epsilon \left(\left\lceil \frac{1}{\epsilon} \right\rceil + 1 \right) \\
max_0 &\geq max_v + \epsilon.v \\
max_0 - \epsilon.v &\geq max_v
\end{aligned}
$$

But $\epsilon.v > 1$, thus $\max_0 < \epsilon.v$. And this would imply that $max_v < 0$, which contradicts the definition of belief!

Since assuming that $L \neq M$ leads us to a contradiction we can conclude that $L = M$. This result implies that all agents belief converge to the same value, as we wanted to prove.

□