Proof of Belief Convergence

Weakly Connected - Classic Update

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min^t and max^t definition

Definition

Let's call min^t and max^t the minimum and maximum of the beliefs in time t over all agents, respectively. Thus:

$$min^t = \min_{a_i \in A} Bel_p^t(a_i) \text{ and } max^t = \max_{a_i \in A} Bel_p^t(a_i)$$

Useful lemmas

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Lemma

$$\lim_{t\to\infty} min^t = L \ \ and\ \lim_{t\to\infty} max^t = U\ \ for\ some\ L,\ U \in [0, 1].$$

Proof main idea

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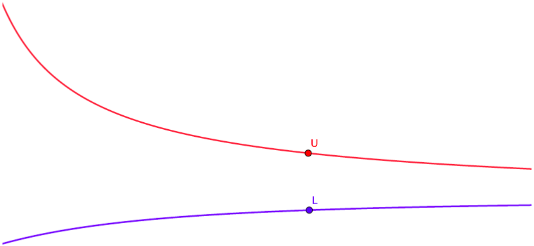

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Let's denote by $P(a_i \to a_j)$ *a path from* a_i *to* a_j *and let's call* $|P(a_i \to a_j)|$ *the size of this path.*

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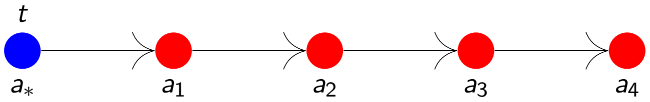
Theorem

$\forall t$ *and* $\forall a_i \in A$:

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t - \delta^t, \text{ with } \delta^t = \left(\frac{In_{min}}{|A|}\right)^{|P(a_*^t \to a_i)|} .(U - L).$$

But what the theorem means?

- It means that the influence exerted by a_*^t guarantees that, by the time $t + |P(a_*^t, a_i)|$, the belief of any agent a_i is smaller than max^t by a factor of at least δ^t.



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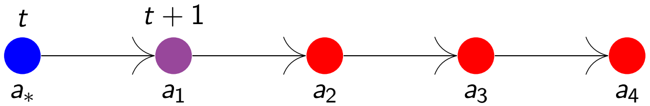

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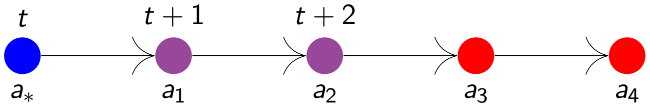

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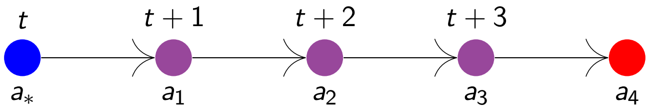

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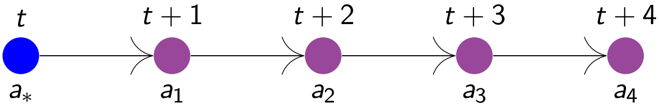

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- But we can use two important pieces of information to acquire an idea about the agents in the same time step.

Information 1

Lemma

$$\forall a_i, a_j \in A, \ |P(a_i \rightarrow a_j)| \leq |A| - 1.$$

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$$\forall a_i, a_j \in A,\ |P(a_i \to a_j)| \leq |A| - 1.$$

- Thus, $\forall a_i$, a_i will be influenced by a_*^t in $|A|-1$ steps maximum.

Information 2

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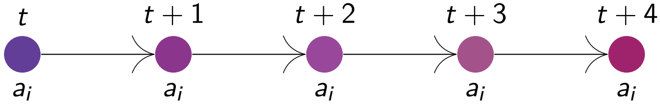

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Intuition behind the theorem:

- a_*^t (the agent who holds the belief min^t) influenced every agent by δ^t in some time step between t and $t + |A| - 1$.
- Since every agent influences itself throughout time, we can find an *constant* ϵ which is a common factor of influence of a_*^t over every agent in the time $t + |A| - 1$.

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- $\frac{1}{|A|}$ is a part of ϵ because, the bigger the size of the society, the less an agent alone influences other (or even the agent influences itself).
- The part inside the parenthesis is elevated to the $|A|-1^{th}$ power because it is the maximum amount the influence of a_*^t must travel to reach a node.

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- Now the most important: the influence is proportional to the difference of U and L, which are the limits of max and min.
- The intuition behind it is that, even when min_t is closer to max_t (which happens when $min_t = L$ and $max_t = U$), a^t_* still influences the agent that holds the belief max_t, since this is the "worst case scenario", this is a bound ϵ that we holds in every case.

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$$max^t - \epsilon \geq max^{t+|A|-1}.$$

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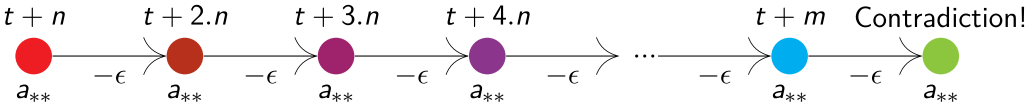

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Proof.

Since all beliefs must be between *min* and *max*, if their limits to infinity are equal, all limits of the beliefs to infinity are equal, as we wanted to prove. □

Generalizing: Confirmation bias

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- In the general case (when $0 < f_{confbias}$) the proof is basically the same but with a new constant, which is $f_{confbiasmin}$ and it has the same role as In_{min} from the proof showed above.

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- Case 2: Exits some agent a_k that has belief B, and $B \neq 0$ and $B \neq 1$.
- In case 2 I applied the same technique used with a_* in the proof for classic update, tracing the influence of a_k to every agent, guarantees that no longer exists $f_{confirmation-bias} = 0$, and then we fall on the general case.

Generalizing: All graphs

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- The idea is, for short, use the proof showed above in weakly connected subgraphs and, this way, guaranteeing conversion of each subgraph.

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- Experiments showed that, under the backfire-effect update function, *min* and *max* are not monotonic, which is crucial for the proof showed above, thus I don't think that this is the way to prove convergence in this case.

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- In a clique, under the classic update function, the speed of convergence is directly proportional to the influence. (proved)
- In some graphs, the initial belief of some agents does not affect their own belief in the limit. (found via experiments).
- In the graph "unrelenting influencers" the belief in the limit seems to be equal to the average of the beliefs of the influencers weighted by it's influence (found via experiments).