

Proof of Belief Convergence

Weakly Connected - Classic Update

Bernardo Amorim

Universidade Federal de Minas Gerais

May 12, 2020



\min^t and \max^t definition

Definition

Let's call \min^t and \max^t the minimum and maximum of the beliefs in the time t , respectively:

$$\min^t = \min_{a_i \in A} Bel_p^t(a_i) \text{ and } \max^t = \max_{a_i \in A} Bel_p^t(a_i)$$

Lemma

$$\forall t, \min^{t+1} \geq \min^t \text{ and } \max^{t+1} \leq \max^t$$

Lemma

$$\forall t, \min^{t+1} \geq \min^t \text{ and } \max^{t+1} \leq \max^t$$

- This means that both *min* and *max* are monotonic, and according to the Monotonic Convergence Theorem, this implies that they converge (since beliefs are bounded between 0 and 1):

Lemma

$$\forall t, \min^{t+1} \geq \min^t \text{ and } \max^{t+1} \leq \max^t$$

- This means that both \min and \max are monotonic, and according to the Monotonic Convergence Theorem, this implies that they converge (since beliefs are bounded between 0 and 1):

Lemma

$$\lim_{t \rightarrow \infty} \min^t = L \text{ and } \lim_{t \rightarrow \infty} \max^t = U \text{ for some } L, U \in [0, 1].$$

Proof main idea

- Since we now that *min* and *max* converge, our proof relies on the fact that all beliefs converge to the same value if and only if $L = U$.

Proof main idea

- Since we now that *min* and *max* converge, our proof relies on the fact that all beliefs converge to the same value if and only if $L = U$.
- Thus we now must show that $L = U$, otherwise a situation like this could occur:

Theorem 1

Definition

Let's denote by $P(a_i \rightarrow a_j)$ a path from a_i to a_j and let's call $|P(a_i \rightarrow a_j)|$ the size of this path.

Theorem 1

Definition

Let's denote by $P(a_i \rightarrow a_j)$ a path from a_i to a_j and let's call $|P(a_i \rightarrow a_j)|$ the size of this path.

Definition

Let's call a_^t an element that holds the belief \min^t in the time t .*

Theorem 1

Definition

Let's denote by $P(a_i \rightarrow a_j)$ a path from a_i to a_j and let's call $|P(a_i \rightarrow a_j)|$ the size of this path.

Definition

Let's call a_*^t an element that holds the belief \min^t in the time t .

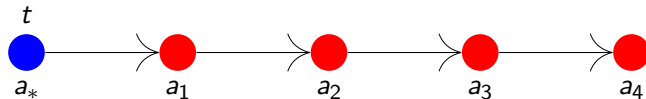
Theorem

$\forall t$ and $\forall a_i \in A$:

$$\text{Bel}_p^{t+|P(a_*^t \rightarrow a_i)|}(a_i) \leq \max^t - \delta^t, \text{ with } \delta^t = \left(\frac{\ln_{\min}}{|A|}\right)^{|P(a_*^t \rightarrow a_i)|} \cdot (U - L).$$

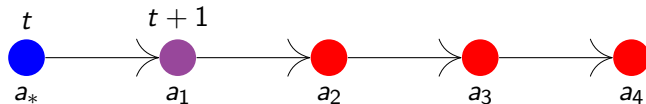
But what the theorem means?

- It means that, by the time $t + |P(a_*^t, a_i)|$, a_i will be influenced by a_*^t by a factor of δ^t .



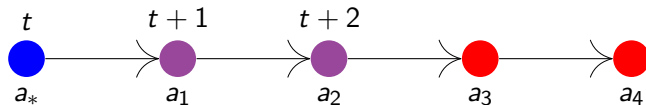
But what the theorem means?

- It means that, by the time $t + |P(a_*^t, a_i)|$, a_i will be influenced by a_*^t by a factor of δ^t .



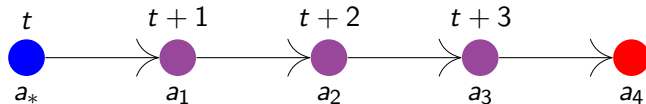
But what the theorem means?

- It means that, by the time $t + |P(a_*^t, a_i)|$, a_i will be influenced by a_*^t by a factor of δ^t .



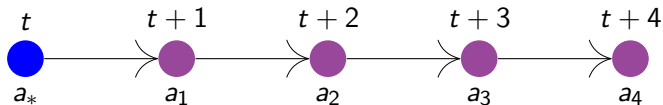
But what the theorem means?

- It means that, by the time $t + |P(a_*^t, a_i)|$, a_i will be influenced by a_*^t by a factor of δ^t .



But what the theorem means?

- It means that, by the time $t + |P(a_*^t, a_i)|$, a_i will be influenced by a_*^t by a factor of δ^t .



One problem

- Although we know that all agents are influenced by a factor of δ^t , it does not convey us much information, because each one of them is influenced in a different time.

One problem

- Although we know that all agents are influenced by a factor of δ^t , it does not convey us much information, because each one of them is influenced in a different time.
- But we can use two important informations to acquire an idea about the agents in the same time step.

Lemma

$$\forall a_i, a_j \in A, |P(a_i \rightarrow a_j)| \leq |A| - 1.$$

Lemma

$\forall a_i, a_j \in A, |P(a_i \rightarrow a_j)| \leq |A| - 1.$

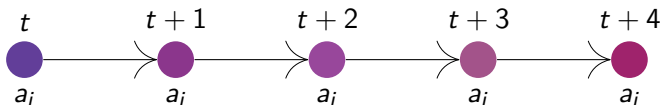
- Thus, $\forall a_i, a_j$ will be influenced by a_*^t in $|A| - 1$ steps maximum.

- An agent is influenced by what it believed in the past.

- An agent is influenced by what it believed in the past.
- Thus, although every agent receives the influence of a_*^t in a different time step, all of them are still influenced by it at time $t + |A| - 1$.

Information 2

- An agent is influenced by what it believed in the past.
- Thus, although every agent receives the influence of a_*^t in a different time step, all of them are still influenced by it at time $t + |A| - 1$.



Theorem 2

Definition

$$ln_{min} = \min_{a_i, a_j \in A} ln(a_i, a_j)$$

Theorem 2

Definition

$$ln_{min} = \min_{a_i, a_j \in A} ln(a_i, a_j)$$

Theorem

$$\forall a_i \in A: \max^t - Bel_p^{t+|A|-1}(a_i) \geq \epsilon, \text{ with } \epsilon = \left(\frac{ln_{min}}{|A|}\right)^{|A|-1} \cdot (U - L).$$

Theorem 2

Definition

$$ln_{min} = \min_{a_i, a_j \in A} ln(a_i, a_j)$$

Theorem

$$\forall a_i \in A: \max^t - Bel_p^{t+|A|-1}(a_i) \geq \epsilon, \text{ with } \epsilon = \left(\frac{ln_{min}}{|A|}\right)^{|A|-1} \cdot (U - L).$$

Intuition behind the theorem:

Theorem 2

Definition

$$ln_{min} = \min_{a_i, a_j \in A} ln(a_i, a_j)$$

Theorem

$$\forall a_i \in A: \max^t - Bel_p^{t+|A|-1}(a_i) \geq \epsilon, \text{ with } \epsilon = \left(\frac{ln_{min}}{|A|}\right)^{|A|-1} \cdot (U - L).$$

Intuition behind the theorem:

- a_*^t (the agent who holds the belief min^t) influenced every agent by δ^t in some time step between t and $t + |A| - 1$.

Theorem 2

Definition

$$ln_{min} = \min_{a_i, a_j \in A} ln(a_i, a_j)$$

Theorem

$$\forall a_i \in A: max^t - Bel_p^{t+|A|-1}(a_i) \geq \epsilon, \text{ with } \epsilon = \left(\frac{ln_{min}}{|A|}\right)^{|A|-1} \cdot (U - L).$$

Intuition behind the theorem:

- a_*^t (the agent who holds the belief min^t) influenced every agent by δ^t in some time step between t and $t + |A| - 1$.
- Since every agent influences itself throughout time, we can find an ϵ which is a common factor of influence of a_*^t over every agent in the time $t + |A| - 1$.

What is ϵ ?

$$\epsilon = \left(\frac{\ln_{min}}{|A|} \right)^{|A|-1} . (U - L)$$

What is ϵ ?

$$\epsilon = \left(\frac{\ln_{min}}{|A|} \right)^{|A|-1} . (U - L)$$

Intuition:

What is ϵ ?

$$\epsilon = \left(\frac{ln_{min}}{|A|} \right)^{|A|-1} \cdot (U - L)$$

Intuition:

- ln_{min} is a part of ϵ because, in the worst case, the a_*^t influences by the smallest amount possible, which is ln_{min} .

What is ϵ ?

$$\epsilon = \left(\frac{In_{min}}{|A|} \right)^{|A|-1} \cdot (U - L)$$

Intuition:

- In_{min} is a part of ϵ because, in the worst case, the a_*^t influences by the smallest amount possible, which is In_{min} .
- $\frac{1}{|A|}$ is a part of ϵ because, the bigger the size of the society, the less an agent alone influences other (or even the agent influences itself).

What is ϵ ?

$$\epsilon = \left(\frac{In_{min}}{|A|} \right)^{|A|-1} \cdot (U - L)$$

Intuition:

- In_{min} is a part of ϵ because, in the worst case, the a_*^t influences by the smallest amount possible, which is In_{min} .
- $\frac{1}{|A|}$ is a part of ϵ because, the bigger the size of the society, the less an agent alone influences other (or even the agent influences itself).
- The part inside the parenthesis is elevated to the $|A|-1^{th}$ power because it is the maximum amount the influence of a_*^t must travel to reach a node.

What is ϵ ?

$$\epsilon = \left(\frac{\ln_{min}}{|A|} \right)^{|A|-1} \cdot (U - L)$$

- Now the most important: the influence is proportional to the difference of U and L , which are the limits of *max* and *min*.

What is ϵ ?

$$\epsilon = \left(\frac{\ln_{min}}{|A|} \right)^{|A|-1} \cdot (U - L)$$

- Now the most important: the influence is proportional to the difference of U and L , which are the limits of max and min .
- The intuition behind it is that, even when min_t is closer to max_t (which happens when $min_t = L$ and $max_t = U$), a_*^t still influences the agent that holds the belief max_t , since this is the “worst case scenario”, this is a bound ϵ that we holds in every case.

Corollary 4

- Now we know that every is influenced, at every $|A|-1$ time steps, by a factor of ϵ .

Corollary 4

- Now we know that every is influenced, at every $|A|-1$ time steps, by a factor of ϵ .
- Since $\max_{t+|A|-1}$ must be one of these beliefs, we gain a corollary:

Corollary 4

- Now we know that every is influenced, at every $|A|-1$ time steps, by a factor of ϵ .
- Since $\max_{t+|A|-1}$ must be one of these beliefs, we gain a corollary:

Corollary

$$\max^t - \max^{t+|A|-1} \geq \epsilon$$

Theorem 3

- Now an observation is sufficient to end our proof.

Theorem 3

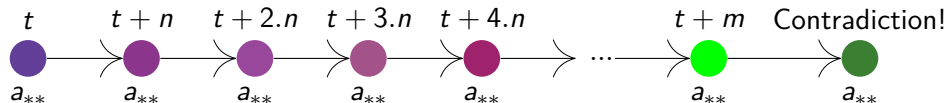
- Now an observation is sufficient to end our proof.
- If we assume that $L \neq U$, ϵ is positive, but this leads us to a contradiction (because we can reduce max until it is smaller than 0).

Theorem 3

- Now an observation is sufficient to end our proof.
- If we assume that $L \neq U$, ϵ is positive, but this leads us to a contradiction (because we can reduce max until it is smaller than 0).
- To see it, let's denote by a_{**}^t the agent who holds the belief max_t in the time t and denote call $n = |A| - 1$, and $m = \left(\left\lceil \frac{1}{\epsilon} \right\rceil + 1\right)$:

Theorem 3

- Now an observation is sufficient to end our proof.
- If we assume that $L \neq U$, ϵ is positive, but this leads us to a contradiction (because we can reduce max until it is smaller than 0).
- To see it, let's denote by a_{**}^t the agent who holds the belief max_t in the time t and denote call $n = |A| - 1$, and $m = \left(\left\lceil \frac{1}{\epsilon} \right\rceil + 1\right)$:



Theorem 3

- Thus, if we assume $L \neq U$, max gets smaller than 0, which contradicts the definition of belief, thus $L = U$:

Theorem 3

- Thus, if we assume $L \neq U$, max gets smaller than 0, which contradicts the definition of belief, thus $L = U$:

Theorem

$$\lim_{t \rightarrow \infty} max^t = U = \lim_{t \rightarrow \infty} min^t = L$$

Theorem 3

- Thus, if we assume $L \neq U$, max gets smaller than 0, which contradicts the definition of belief, thus $L = U$:

Theorem

$$\lim_{t \rightarrow \infty} max^t = U = \lim_{t \rightarrow \infty} min^t = L$$

- Since all beliefs must be between min and max , if their limits to infinity are equal, all limits of the beliefs to infinity are equal, as we wanted to prove.

Next steps: Confirmation bias

- I think that changing this proof to also hold for confirmation bias shouldn't be too hard.

Next steps: Confirmation bias

- I think that changing this proof to also hold for confirmation bias shouldn't be too hard.
- I believe that we may just need to add another constant, which would be $f_{confirmation-bias-min}$, which would be the smallest $f_{confirmation-bias}$ on that society (it is not hard to see that the minimum $f_{confirmation-bias}$ throughout time is monotonic, thus the smallest one exists in $t = 0$).

Next steps: Confirmation bias

There exists corner we must address with this approach, though:

Next steps: Confirmation bias

There exists corner we must address with this approach, though:

$f_{\text{confirmation-bias-min}}$ might equal 0. When this happens we have two cases:

Next steps: Confirmation bias

There exists corner we must address with this approach, though:

$f_{confirmation-bias-min}$ might equal 0. When this happens we have two cases:

- Every agent has belief either 0 or 1 in time $t = 0$, in this case beliefs converge trivially, but not to the same value.

Next steps: Confirmation bias

There exists corner we must address with this approach, though:

$f_{\text{confirmation-bias-min}}$ might equal 0. When this happens we have two cases:

- Every agent has belief either 0 or 1 in time $t = 0$, in this case beliefs converge trivially, but not to the same value.
- Exits some agent a_k that has belief B , and $B \neq 0$ and $B \neq 1$.

Next steps: Confirmation bias

There exists corner we must address with this approach, though:

$f_{\text{confirmation-bias-min}}$ might equal 0. When this happens we have two cases:

- Every agent has belief either 0 or 1 in time $t = 0$, in this case beliefs converge trivially, but not to the same value.
- Exists some agent a_k that has belief B , and $B \neq 0$ and $B \neq 1$.
- In this case my idea is that we should use the same approach we used with a_* in the proof for classic update, tracing the influence of a_k to every agent, this would guarantee that no longer exists $f_{\text{confirmation-bias}} = 0$, and then we would fall on the general case.

Next steps: Backfire-Effect

- Unfortunately, I think that none of what was used above can also be used for the backfire-effect.

Next steps: Backfire-Effect

- Unfortunately, I think that none of what was used above can also be used for the backfire-effect.
- Experiments showed that, under the backfire-effect update function, *min* and *max* are not monotonic, which is crucial for the proof showed above, thus I don't think that this is the way to prove convergence in this case.