Proof of Individual Belief Convergence in a Weakly Connected Influence Graph Using Confirmation Bias Update

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Definition 1. The *confirmation-bias update-function*, is defined as:

$$Bel_p^{t+1}(a_i|a_j) = Bel_p^t(a_i) + f_{cb}.In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \tag{1}$$

While f_{cb} is defined as $1 - |Bel_p^t(a_j) - Bel_p^t(a_i)|$.

Definition 2. While the *overall confirmation-bias update*, is defined as:

$$Bel_p^{t+1}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j) \tag{2}$$

Definition 3. We say a influence graph In over agents A is *weakly connected* if for all a_i, $a_j \in A$, there exist a_{k_1},a_{k_2},...,$a_{k_l} \subseteq A$ such that In(a_i,a_{k_1}) > 0, In(a_{k_l},a_j) > 0, and for $m = 1$,...,$l-1$, In(a_{k_m},$a_{k_{m+1}}$) > 0.

Definition 4. max^t and min^t are the maximum and minimum belief values in a given instant t, respectively. Thus:

$$min^t = \underset{a_i \in A}{min}\ Bel_p^t(a_i) \text{ and } max^t = \underset{a_i \in A}{max}\ Bel_p^t(a_i).$$

To prove our conjecture, let's do some simplifications:

$$\begin{aligned}
Bel_p^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j). \\
&= \frac{1}{|A|} \sum_{a_j \in A} \left(Bel_p^t(a_i) + f_{cb}.In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))\right) \\
&= Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} f_{cb}.In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \qquad (3)
\end{aligned}$$

Since we have a finite number of beliefs and $\forall a_i \in A :\ Bel_p^t(a_i) \in [0, 1]$, there are always min^t and a max^t. We shall also note that, by the Squeeze Theorem, individual agent opinion converges to the same value if and only if $lim_{t\to\infty}\ min^t = lim_{t\to\infty}\ max^t$.

Thus, since we want to prove that it always converges, if $min^t = max^t$ we have nothing to prove, so assume $min^t \neq max^t$. *We will also assume from now on that no agent has belief* 0 *or* 1, which will guarantee us that $f_{cb} > 0$. We will address the case in which there are beliefs equal to 0 or 1 later, because we will use a similar reasoning used in a part of this proof, thus it will make more sense.

Lemma 1. *In a weakly connected graph and under confirmation-bias belief update:*

$$\forall t \text{ and } \forall a_i \in A : min^t \leq Bel_p^{t+1}(a_i) \leq max^t$$

Proof. By the equation 3:

$$Bel_p^{t+1}(a_i) = Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} f_{cb}.In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))$$

Trying to maximize the right-hand side, we can substitute $Bel_p^t(a_j)$ by max^t, this turns our equation into an inequality, since $\forall a_j \in A$, $Bel_p^t(a_j) \leq max^t$, by the definition of max^t. That makes the terms inside the summation either equal or smaller than 0, thus:

$$\begin{aligned}
Bel_p^{t+1}(a_i) &\leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} f_{cb}.In(a_j, a_i)(max^t - Bel_p^t(a_i)) \\
&\leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} f_{cb}.(max^t - Bel_p^t(a_i)) && \text{(since } In(a_j,a_i) \leq 1 \text{ and} \\
& && max^t - Bel_p^t(a_i) \geq 0) \\
&\leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} (max^t - Bel_p^t(a_i)) && \text{(since } f_{cb} \leq 1 \text{ and} \\
& && max^t - Bel_p^t(a_i) \geq 0) \\
&= Bel_p^t(a_i) + \frac{|A|}{|A|}(max^t - Bel_p^t(a_i)) \\
&= Bel_p^t(a_i) + max^t - Bel_p^t(a_i) \\
Bel_p^{t+1}(a_i) &\leq max^t
\end{aligned} \tag{4}$$

Since a_i was arbitrary, the Lemma is true for all agents. The same reasoning can be used to show the equivalent property for min^t □

Corollary 1. *In a weakly connected influence graph and a confirmation-bias update function, $max^{t+1} \leq max^t$ and $min^{t+1} \geq min^t$ for all t.*

Proof. The result of Lemma 1 tells us that all beliefs in the time $t+1$ are either smaller than max^t or equal to max^t, thus, since max^{t+1} must be one of those elements, $max^{t+1} \leq max^t$. And the same reasoning can be used for min^t. □

Corollary 2. $\lim_{t \to \infty} max^t = U$ *and* $\lim_{t \to \infty} min^t = L$ *for some* U, $L \in [0, 1]$.

Proof. Since both max^t and min^t are bounded between 0 and 1 by the definition of belief; and Lemma 1 showed us that they are monotonic, according to the Monotonic Convergence Theorem, the limits exist. □

Now that we have those properties, our proof will follow by showing that an agent a_i that holds some belief $Bel_p^t(a_i)$ will influence every other agent by the time $t+|A|-1$. To see this, we must open the definition of belief throughout time. But before we do this, let's jump into some small definitions and corollaries that will help us on the way.

Definition 5. Let's call the sequence $P(a_i \to a_j) = (a_i, a_k, ..., a_{k+l}, a_j)$ a *path* from a_i to a_j, if:

- All elements on the sequence are different.
- The first element in the sequence is a_i.
- The last element in the sequence is a_j.
- If a_n is the n'th element in the sequence, if it has a successor a_{n+1}, $In(a_n, a_{n+1}) > 0$.

Note that many paths from a_i to a_j can exist, although our notation isn't enough to differentiate them. But in subsequent steps we will only need one of those paths, so the notation shouldn't be a problem.

Definition 6. Let's denote by $|P(a_i \to a_j)|$ the size of a simple path from a_i to a_j, which we define as the number of elements in the sequence $P(a_i \to a_j) - 1$.

Corollary 3. $\forall P(a_i \to a_j)$, $|P(a_i \to a_j)| \leq |A| - 1$.

Proof. This follows directly from the definition of path. Since it doesn't have repeated elements and we have $|A|$ agents, the simple path can't have more than $|A|$ elements, since the size of a simple path is defined as the number of elements minus one, the maximum size is $|A| - 1$. □

Lemma 2. *$\forall x, \forall t$ and $\forall a_i$, if $Bel_p^t(a_i) \leq x$:*

$$Bel_p^{t+1}(a_i) \leq x + \frac{1}{|A|} \sum_{a_j \in A} f_{cb}.In(a_j, a_i) \left(Bel_p^t(a_j) - x\right)$$

Proof.

$$\begin{aligned}
Bel_p^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} \left(Bel_p^t(a_i) + f_{cb}.In(a_j, a_i) \left(Bel_p^t(a_j) - Bel_p^t(a_i)\right)\right) \\
&= \frac{1}{|A|} \sum_{a_j \in A} \left(Bel_p^t(a_i)(1 - f_{cb}.In(a_j, a_i)) + f_{cb}.In(a_j, a_i) Bel_p^t(a_j)\right) \\
&\leq \frac{1}{|A|} \sum_{a_j \in A} \left(x.(1 - f_{cb}.In(a_j, a_i)) + f_{cb}.In(a_j, a_i) Bel_p^t(a_j)\right) \\
&= x + \frac{1}{|A|} \sum_{a_j \in A} f_{cb}.In(a_j, a_i) \left(Bel_p^t(a_j) - x\right)
\end{aligned}$$

□

Lemma 3. *$\forall a_i, a_k \in A$ and $\forall n \geq 1$ and $\forall t$:*

$$Bel_p^{t+n}(a_i) \leq max^t + \frac{1}{|A|} f_{cb}.In(a_k, a_i)(Bel_p^{t+n-1}(a_k) - max^t) \tag{5}$$

Proof. By the Definitions 1 and 2:

$$Bel_p^{t+n}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} Bel_p^{t+n}(a_i|a_j)$$
$$Bel_p^{t+n}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} \left(Bel_p^{t+n-1}(a_i) + f_{cb}.In(a_j, a_i)(Bel_p^{t+n-1}(a_j) - Bel_p^{t+n-1}(a_i))\right)$$

Since $Bel_p^{t+n}(a_i) \leq max^{t+n} \leq max^{t+n-1}$ according to Corollary 1, we can use Lemma 2:

$$\begin{aligned}Bel_p^{t+n}(a_i) &\leq \frac{1}{|A|} \sum_{a_j \in A} \left(max^{t+n-1} + f_{cb}.In(a_j, a_i)(Bel_p^{t+n-1}(a_j) - max^{t+n-1})\right) \\ &= max^{t+n-1} + \frac{1}{|A|} \sum_{a_j \in A} f_{cb}.In(a_j, a_i)(Bel_p^{t+n-1}(a_j) - max^{t+n-1})\end{aligned}$$

To make our Lemma useful in future manipulations, we will take an arbitrary element a_k out of the summation:

$$\begin{aligned}Bel_p^{t+n}(a_i) \leq max^{t+n-1} &+ \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_k\}} \left(f_{cb}.In(a_j, a_i)(Bel_p^{t+n-1}(a_j) - max^{t+n-1})\right) \\ &+ \frac{1}{|A|}(f_{cb}.In(a_k, a_i)(Bel_p^{t+n-1}(a_k) - max^{t+n-1}))\end{aligned}$$

Since max^{t+n-1} is the greatest belief possible in that time step, the summation can be at most 0, thus:

$$Bel_p^{t+n}(a_i) \leq max^{t+n-1} + \frac{1}{|A|} f_{cb}.In(a_k, a_i) \left(Bel_p^{t+n-1}(a_k) - max^{t+n-1}\right)$$

Since max doesn't increase throughout time, $max^{t+n-1} \leq max^t$, thus:

$$Bel_p^{t+n}(a_i) \leq max^t + \frac{1}{|A|} f_{cb}.In(a_k, a_i) \left(Bel_p^{t+n-1}(a_k) - max^t\right)$$

□

Definition 7. Let's denote by In_{min} the smallest influence that's different from 0 in the influence graph.

Definition 8. Let's denote by f_{cbmin} the smallest f_{cb} in our influence graph. Note that, this f_{cb} is greater than 0 because of our assumption that no agents have belief 0 or 1. Note, also, that the minimum f_{cb} occurs between max^0 and min^0, does it does not diminishes throughout time, according to 1.

Using the same notation we used in Corollary 2, let's call $\lim_{t\to\infty} max^t = U$ and $\lim_{t\to\infty} min^t = L$.
Now that we have all of these tools, let's jump to Lemma 1 which will be a tool in the most important part of the proof. Calling a_*^t one agent who holds the belief min^t in the time t:

Theorem 1. *$\forall t$ and $\forall a_i \in A$:*

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t - \delta^t, \text{ with } \delta^t = \left(\frac{In_{min}.f_{cbmin}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U - L).$$

Proof. By equation 3:

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) = Bel_p^{t+|P(a_*^t \to a_i)|-1}(a_i) + \frac{1}{|A|}\sum_{a_j \in A} Bel_p^{t+|P(a_*^t \to a_i)|-1}(a_i|a_j)$$

What we will do now is separate, at each step, one element of the summation and apply Lemma 3 to modify our inequality. But we will be careful when choosing the elements we separate from the summation. We will separate from the summation the elements in $P(a_*^t \to a_i)$, starting from the end of the simple path until we get to a_*^t. To simplify our notation, let's index the elements in the simple path from a_*^t to a_i, starting from the end of the simple path (since we are backtracking it will make more sense) by calling a_n the n^{th} element from the end to the beginning of the sequence (excluding a_i itself). Thus, by Lemma 3:

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t + \frac{1}{|A|} f_{cb}.In(a_1, a_i)(Bel_p^{t+|P(a_*^t \to a_i)-1|}(a_1) - max^t)$$

Note, now, that if $|P(a_*^t, a_i)| = 1$, we could prove our result. Instead of showing it I will expand this two more times, show the general version and then prove the Lemma for all cases. Note also that f_{cb} notation does not differentiate between the agents it came from or the time, but this should not be a problem, because we will use f_{cbmin} in the future. The notation is already big enough so I will keep using f_{cb}. Using Lemma 3 again:

$$\begin{aligned}
&Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \\
&\leq max^t + \frac{1}{|A|} f_{cb}.In(a_1, a_i)(Bel_p^{t+|P(a_*^t \to a_i)-1|}(a_1) - max^t) \\
&\leq max^t + \frac{1}{|A|} f_{cb}.In(a_1, a_i)\left(\left(max^t + \frac{1}{|A|} f_{cb}.In(a_2, a_1)(Bel_p^{t+|P(a_*^t \to a_i)-2|}(a_2) - max^t)\right) - max^t\right) \\
&= max^t + \frac{1}{|A|} f_{cb}.In(a_1, a_i)\left(\frac{1}{|A|} f_{cb}.In(a_2, a_1)(Bel_p^{t+|P(a_*^t \to a_i)-2|}(a_2) - max^t)\right) \\
&= max^t + \frac{1}{|A|^2} f_{cb}.f_{cb}.In(a_2, a_1)In(a_1, a_i)(Bel_p^{t+|P(a_*^t \to a_i)-2|}(a_2) - max^t) \\
&\leq max^t + \frac{1}{|A|^2} f_{cb}.f_{cb}.In(a_2, a_1)In(a_1, a_i) \times \\
&\quad \left(\left(max^t + \frac{1}{|A|} f_{cb}.In(a_3, a_2)\left(Bel_p^{t+|P(a_*^t \to a_i)|-3}(a_3) - max^t\right)\right) - max^t\right)
\end{aligned}$$

$$= max^t + \frac{1}{|A|^2} f_{cb}.f_{cb}.In(a_2,a_1)In(a_1,a_i)\left(\frac{1}{|A|} f_{cb}.In(a_3,a_2)\left(Bel_p^{t+|P(a_*^t \to a_i)|-3}(a_3)\right) - max^t\right)$$

$$= max^t + \frac{1}{|A|^3} f_{cb}.f_{cb}.f_{cb}.In(a_3,a_2)In(a_2,a_1)In(a_1,a_i)\left(Bel_p^{t+|P(a_*^t \to a_i)|-3}(a_3) - max^t\right)$$

We can see a patter forming since the equation above has the same form of the one before it, and this pattern will continue throughout time. Now, denoting P_{In} the product of the influences in the simple path ($P_{In} = In(a_*^t, a_{|P(a_*^t, a_i)|}) \times ... \times In(a_1, a_i)$), we can write the generalized version of the inequality above as:

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \le max^t + \frac{P_{In}}{|A|^{|P(a_*^t \to a_i)|}}(Bel_p^t(a_*^t) - max^t).(f_{cb})(\text{appears } |P(a_*^t \to a_i)| \text{ times})$$

$$= max^t + \frac{P_{In}}{|A|^{|P(a_*^t \to a_i)|}}.(min^t - max^t)(f_{cb}...f_{cb})$$

This inequality comes from the fact that the simple path ends after $|P(a_*^t \to a_i)|$ steps with a_*^t as the start of the path, and, by definition, the belief of a_*^t in the time t is min^t.

Since the rightmost term in the equation is either equal to or smaller than 0, to make the inequality hold for all a_i's, we shall substitute P_{In} by the smallest value possible. According to Definition 7, In_{min} is the smallest positive influence in the graph. By the definition of simple path (5) all influences are positive, thus the smallest possible value of P_{In} is $In_{min}^{|P(a_*^t \to a_i)|}$. Thus:

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \le max^t + \left(\frac{In_{min}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(min^t - max^t)(f_{cb}...f_{cb})$$

Using the same reasoning we must replace all f_{cb} by the smallest value they can assume, which is, by definition f_{cbmin}.

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \le max^t + \left(\frac{In_{min}.f_{cbmin}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(min^t - max^t)$$

According to Corollary 2, the maximum value of min^t is L and the minimum value of max^t is U, those are the values we should plug to maintain the inequality:

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \le max^t + \left(\frac{In_{min}.f_{cbmin}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(L - U)$$

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \le max^t - \left(\frac{In_{min}.f_{cbmin}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U - L)$$

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \le max^t - \delta^t$$

□

Lemma 4.

$$\sum_{a_j \in A} f_{cb}.In(a_j,a_i)\left(Bel_p^t(a_j) - Bel_p^t(a_i)\right) = \sum_{a_j \in A\setminus\{a_i\}} f_{cb}.In(a_j,a_i)\left(Bel_p^t(a_j) - Bel_p^t(a_i)\right)$$

Proof.

$$\begin{aligned}
&\sum_{a_j \in A} f_{cb}.In(a_j,a_i)\left(Bel_p^t(a_j) - Bel_p^t(a_i)\right) \\
&= \sum_{a_j \in A\setminus\{a_i\}} f_{cb}.In(a_j,a_i)\left(Bel_p^t(a_j) - Bel_p^t(a_i)\right) + f_{cb}.In(a_i,a_i)(Bel_p^t(a_i) - Bel_p^t(a_i)) \\
&= \sum_{a_j \in A\setminus\{a_i\}} f_{cb}.In(a_j,a_i)\left(Bel_p^t(a_j) - Bel_p^t(a_i)\right)
\end{aligned}$$

□

Lemma 5. *If* $Bel_p^{t+n}(a_i) \leq max^t - \gamma$, $\gamma \geq 0$ *and* $n \geq 0$, *then* $Bel_p^{t+n+1}(a_i) \leq max^t - \frac{\gamma}{|A|}$.

Proof.

$$\begin{aligned}
Bel_p^{t+n+1}(a_i) &= Bel_p^{t+n}(a_i) + \frac{1}{|A|}\sum_{a_j \in A} f_{cb}.In(a_j,a_i)\left(Bel_p^{t+n}(a_j) - Bel_p^{t+n}(a_i)\right) \\
&= Bel_p^{t+n}(a_i) + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_i\}} f_{cb}.In(a_j,a_i)\left(Bel_p^{t+n}(a_j) - Bel_p^{t+n}(a_i)\right) \quad \text{(Lemma 4)} \\
&\leq max^t - \gamma + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_i\}} f_{cb}.In(a_j,a_i)\left(Bel_p^{t+n}(a_j) - max^t + \gamma\right) \quad \text{(Lemma 2)} \\
&\leq max^t - \gamma + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_i\}} f_{cb}.In(a_j,a_i)\left(max^t - max^t + \gamma\right) \\
&= max^t - \gamma + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_i\}} f_{cb}.In(a_j,a_i)\,(\gamma) \\
&\leq max^t - \gamma + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_i\}} (\gamma) \\
&= max^t - \gamma + \frac{(|A|-1)(\gamma)}{|A|} \\
&= max^t + \frac{(\gamma)((-|A|) + (|A|-1))}{|A|} \\
&= max^t - \frac{\gamma}{|A|}
\end{aligned}$$

□

Theorem 2. $\forall a_i \in A:\ Bel_p^{t+|A|-1}(a_i) \leq max^t - \epsilon$, *with* $\epsilon = \left(\frac{In_{min}\cdot f_{cbmin}}{|A|}\right)^{|A|-1}.(U-L)$.

Proof. Let's keep the notation of the previous Lemma and call a_*^t the agent that holds the belief min^t in the time t.

First we should note that, if $|P(a_*^t \to a_i)| = |A| - 1$, our theorem is true by Theorem 1 and we nothing to prove.
Else if $|P(a_*^t \to a_i)| \neq |A| - 1$, then $|P(a_*^t \to a_i)| < |A| - 1$ according to Corollary 3.
According to Theorem 1:

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t - \left(\frac{In_{min}.f_{cbmin}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U - L)$$

To keep things simple let's keep the notation from Theorem 1 and call:

$$\delta^t = \left(\frac{In_{min}.f_{cbmin}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U - L)$$

.
Now it is easy to see that we can apply Lemma 5 successively:

$$Bel_p^{t+|P(a_*^t \to a_i)|+1}(a_i) \leq max^t - \frac{\delta^t}{|A|}$$

$$\Downarrow$$

$$Bel_p^{t+|P(a_*^t \to a_i)|+2}(a_i) \leq max^t - \frac{\delta^t}{|A|^2}$$

$$\Downarrow$$

$$Bel_p^{t+|P(a_*^t \to a_i)|+3}(a_i) \leq max^t - \frac{\delta^t}{|A|^3}$$

If we do it $|A| - |P(a_*^t \to a_i)| - 1$ times we get:

$$\begin{aligned}
Bel_p^{t+|P(a_*^t \to a_i)|+|A|-|P(a_*^t \to a_i)|-1}(a_i) &\leq max^t - \frac{\delta^t}{|A|^{|A|-|P(a_*^t \to a_i)|-1}} \\
Bel_p^{t+|A|-1}(a_i) &\leq max^t - \frac{\delta^t}{|A|^{|A|-|P(a_*^t \to a_i)|-1}} \\
Bel_p^{t+|A|-1}(a_i) &\leq max^t - \frac{\left(\frac{In_{min}.f_{cbmin}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U - L)}{|A|^{|A|-|P(a_*^t \to a_i)|-1}} \\
Bel_p^{t+|A|-1}(a_i) &\leq max^t - \frac{(In_{min}.f_{cbmin})^{|P(a_*^t \to a_i)|}.(U - L)}{|A|^{|A|-1}} \\
Bel_p^{t+|A|-1}(a_i) &\leq max^t - \left(\frac{In_{min}.f_{cbmin}}{|A|}\right)^{|A|-1}.(U - L) \\
Bel_p^{t+|A|-1}(a_i) - max^t &\leq -\left(\frac{In_{min}.f_{cbmin}}{|A|}\right)^{|A|-1}.(U - L) \\
max^t - Bel_p^{t+|A|-1}(a_i) &\geq \left(\frac{In_{min}.f_{cbmin}}{|A|}\right)^{|A|-1}.(U - L) \\
max^t - Bel_p^{t+|A|-1}(a_i) &\geq \epsilon \\
max^t - \epsilon &\geq Bel_p^{t+|A|-1}(a_i)
\end{aligned}$$

□

Corollary 4. $max^{t+|A|-1} \leq max^t - \epsilon$

Proof. Since $max^{t+|A|-1}$ must be one of the beliefs in the time $t+|A|-1$ and, according to Theorem 2 all of them are smaller than max^t by a factor of at least ϵ, $max^{t+|A|-1}$ must also be smaller than max^t by a factor of at least ϵ. □

Theorem 3. $\lim_{t\to\infty} max^t = U = \lim_{t\to\infty} min^t = L$

Proof. Suppose, by contradiction, that $U \neq L$. Plugging this values into the ϵ formula show us that $\epsilon \neq 0$. Since, according to Theorem 2, $max^{t+|A|-1}$ is smaller than max^t by a factor of ϵ, we can finally reach to a contradiction and end our proof.

To see this contradiction, let's assume we did $v = (|A|-1)\left(\lceil\frac{1}{\epsilon}\rceil + 1\right)$ time steps after $t = 0$. Since max diminishes by at least ϵ at each $|A|-1$ steps:

$$max^0 \geq max^v + \epsilon \left(\left\lceil\frac{1}{\epsilon}\right\rceil + 1\right)$$

$$max^0 - \epsilon \left(\left\lceil\frac{1}{\epsilon}\right\rceil + 1\right) \geq max^v$$

But $\epsilon.\left(\lceil\frac{1}{\epsilon}\rceil + 1\right) > 1$, thus $\max^0 < \epsilon.\left(\lceil\frac{1}{\epsilon}\rceil + 1\right)$. And this would imply that $max^v < 0$, which contradicts the definition of belief!

Since assuming that $U \neq L$ led us to a contradiction we can conclude that $U = L$. This result implies that all agents belief converge to the same value, as we wanted to prove. □

Everything showed above was based on assumption that $f_{cb} > 0$, but this is not always true. f_{cb} can equal 0 when we have agents with belief 0 and 1 in the same graph (note that those beliefs are always maximum and minimum, thus they can't exist in subsequent steps if they did not exist in the beginning according to 1). Now we will address those cases:

- Case 1: $\forall a_i \in A,\ Bel_p^0(a_i) = 0$ or $Bel_p^0(a_i) = 1$.
  In this case our graph converges trivially (but necessarily to the same value), because every agent is not influenced by an agent that has a different belief, thus this graph is constant throughout time (and it is the only case in which not all beliefs converge to the same value).

- Case 2: $\exists a_{**} \in A,\ Bel_p^0(a_{**}) \neq 0$ and $Bel_p^0(a_{**}) \neq 1$.
  What we will show now is that from this case we can reach the general case, in which $f_{cb} > 0$. The idea is similar to the one used in Theorem 1, we will use a_{**} to influence every agent in our influence graph. Since a_{**} does not have belief equal to 0 or 1 it can influence every agent (because $f_{cb} \neq 0$), and it will influence every agent out of the extremes. After we do this every agent will have a belief different from the extremes and we will fall on the general case, which we have already proved.

Lemma 6. *$\forall a_i \in A, \forall t$:*

$$\text{If } 0 < Bel_p^t(a_i) < 1, \text{ then } 0 < Bel_p^{t+1}(a_i) < 1.$$

Proof. By equation 3 and Lemma 4:

$$\begin{aligned}
Bel_p^{t+1}(a_i) &= Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j\in A} f_{cb}.In(a_j,a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \\
&= Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j\in A\setminus a_i} f_{cb}.In(a_j,a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \\
&\leq Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j\in A\setminus a_i} f_{cb}.In(a_j,a_i)(1 - Bel_p^t(a_i)) \\
&\leq Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j\in A\setminus a_i} f_{cb}.(1 - Bel_p^t(a_i)) \\
&\leq Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j\in A\setminus a_i} (1 - Bel_p^t(a_i)) \\
&= Bel_p^t(a_i) + \frac{(|A|-1).(1 - Bel_p^t(a_i))}{|A|} \\
&= \frac{|A|.Bel_p^t(a_i) + (|A|-1).(1 - Bel_p^t(a_i))}{|A|} \\
&= \frac{Bel_p^t(a_i)(|A| - (|A|-1)) + (|A|-1)}{|A|} \\
&= \frac{Bel_p^t(a_i) + (|A|-1)}{|A|} \\
&= 1 + \frac{Bel_p^t(a_i) - 1}{|A|}
\end{aligned} \tag{6}$$

But since $Bel_p^t(a_i) < 1$, $\frac{Bel_p^t(a_i)-1}{|A|} < 0$, thus $Bel_p^{t+1}(a_i) < 1$ as we wanted to show. The same can be done to show that $0 < Bel_p^t(a_i)$. □

Now it gets easy to show that we will fall on the general case, in $t = 1$ a_{**} influences all vertices a_j in which $|P(a_{**} \to a_j)| = 1$. This makes that $\forall t > 0$, $0 < Bel_p^t(a_j) < 1$, according to Lemma 6.
After this we can use the a_j's from the next step to influence the more agents out of the extremes. It isn't hard to see that, doing this guarantees that, after $|A| - 1$ steps every belief isn't extreme and, thus, we fall on the general case.