Proof of Belief Convergence

Weakly Connected - Classic Update

Bernardo Amorim

Universidade Federal de Minas Gerais

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UFMG

min^t and max^t definition

Definition

Let's call min^t and max^t the minimum and maximum of the beliefs in the time t, respectively:

$$min^t = \min_{a_i \in A} Bel_p^t(a_i) \text{ and } max^t = \max_{a_i \in A} Bel_p^t(a_i)$$

Useful lemmas

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Lemma

$$\lim_{t \to \infty} min^{t} = L\ \ and\ \lim_{t \to \infty} max^{t} = U\ \ for\ some\ L,\ U \in [0, 1].$$

Proof main idea

- Since we now that *min* and *max* converge, our proof relies on the fact that all beliefs converge to the same value if and only if $L = U$.

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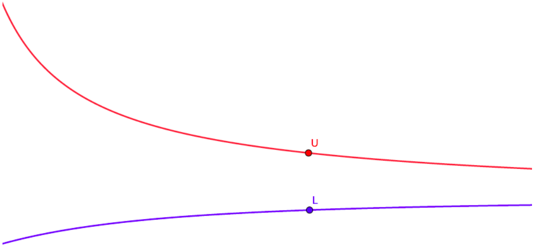

Theorem 1

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Let's denote by $P(a_i \to a_j)$ *a path from* a_i *to* a_j *and let's call* $|P(a_i \to a_j)|$ *the size of this path.*

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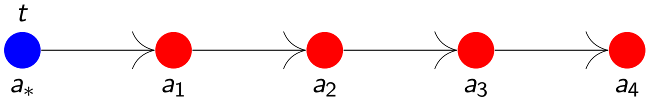
Theorem

$\forall t$ *and* $\forall a_i \in A$:

$$Bel_p^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t - \delta^t, \text{ with } \delta^t = \left(\frac{In_{min}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U - L).$$

But what the theorem means?

- It means that, by the time $t + |P(a_*^t, a_i)|$, a_i will be influenced by a_*^t by a factor of δ^t.



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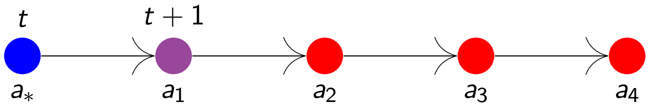

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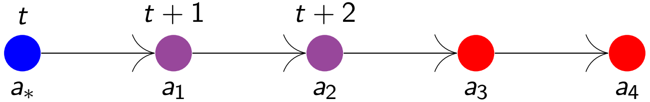

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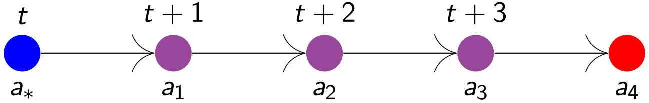

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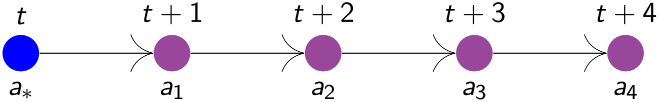

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- But we can use two important informations to acquire an idea about the agents in the same time step.

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- Thus, $\forall a_i$, a_i will be influenced by a_*^t in $|A| - 1$ steps maximum.

Information 2

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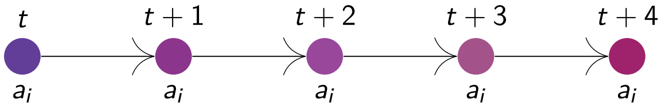

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$$\forall a_i \in A: \;\; max^t - Bel_p^{t+|A|-1}(a_i) \geq \epsilon, \; with \; \epsilon = \left(\frac{In_{min}}{|A|}\right)^{|A|-1}.(U - L).$$

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Intuition behind the theorem:

- a_*^t (the agent who holds the belief min^t) influenced every agent by δ^t in some time step between t and $t + |A| - 1$.
- Since every agent influences itself throughout time, we can find an ϵ which is a common factor of influence of a_*^t over every agent in the time $t + |A| - 1$.

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- $\frac{1}{|A|}$ is a part of ϵ because, the bigger the size of the society, the less an agent alone influences other (or even the agent influences itself).
- The part inside the parenthesis is elevated to the $|A|-1^{th}$ power because it is the maximum amount the influence of a_*^t must travel to reach a node.

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- Now the most important: the influence is proportional to the difference of U and L, which are the limits of *max* and *min*.
- The intuition behind it is that, even when min_t is closer to max_t (which happens when $min_t = L$ and $max_t = U$), a_*^t still influences the agent that holds the belief max_t, since this is the "worst case scenario", this is a bound ϵ that we holds in every case.

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Corollary

$max^t - max^{t+|A|-1} \geq \epsilon$

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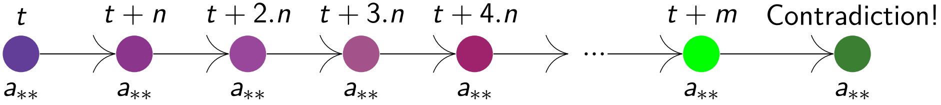

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$\lim_{t \to \infty} max^t = U = \lim_{t \to \infty} min^t = L$

- Since all beliefs must be between min and max, if their limits to infinity are equal, all limits of the beliefs to infinity are equal, as we wanted to prove.