

Proof of Individual Agent Opinion Convergence in a Strongly Connected Influence Graph Using Classic Update Function

Bernardo T. Amorim
bernardoamorim@dcc.ufmg.br

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In the classic update function, $Bel_p^{t+1}(a_i|a_j)$ can be written in the following form:

Definition 1 $Bel_p^{t+1}(a_i|a_j) = Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))$.

And the classic update function, $Bel_p^{t+1}(a_i)$ is written as:

Definition 2 $Bel_p^{t+1}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j)$.

And let's define a strongly connected graph as:

Definition 3 *A strongly connected influence graph social network in which every agent exerts influence on every other agent: $In(a_i, a_j) > 0$, for every i, j .*

Definition 4 max_t and min_t are the maximum and minimum belief values in a given instant t , respectively.

To prove our conjecture, let's do some simplifications:

$$\begin{aligned} Bel_p^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j) \\ &= \frac{1}{|A|} \sum_{a_j \in A} (Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))) \\ &= Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \end{aligned}$$

Since belief values are finite, by the well-ordering principle we always have a min_t and a max_t . It is easy to see, by the squeeze theorem, that individual agent opinion converges to the same value if and only if $\lim_{t \rightarrow \infty} min_t = \lim_{t \rightarrow \infty} max_t$.

Thus, since we want to prove that it always converges, if $min_t = max_t$ we have nothing to prove, so assume $min_t \neq max_t$.

Lemma 1 *In a strongly connected graph and under classic belief update, if $max_t \neq min_t$:*

$$\forall a_j \in A : Bel_p^{t+1}(a_j) < max_t \tag{1}$$

and:

$$\forall a_j \in A : Bel_p^{t+1}(a_j) > min_t \tag{2}$$

Proof of Lemma 1

To show this we are gonna consider two cases:

Case 1 - $Bel_p^t(a_i) = max_t$:

$$\begin{aligned} Bel_p^{t+1}(a_i) &= Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \\ &= max_t + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - max_t) \end{aligned}$$

Since max_t is, by definition, the biggest element of the set of all beliefs in the instant t , then $Bel_p^t(a_j) - max_t \leq 0$ for every a_j . This implies that:

$$In(a_j, a_i)(Bel_p^t(a_j) - max_t) \leq 0$$

Since $In(a_j, a_i) > 0$.

Given that we assumed that $min_t \neq max_t$, there exists at least one a_j , such that $Bel_p^t(a_j) \neq max_t$, thus, since all influence are positive:

$$\frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - max_t) < 0$$

Thus $Bel_p^{t+1}(a_i) = Bel_p^t(a_i)$ plus a negative number, which implies that

$$\begin{aligned} Bel_p^{t+1}(a_i) &< Bel_p^t(a_i) \\ Bel_p^{t+1}(a_i) &< max_t \end{aligned}$$

Case 2 - $Bel_p^t(a_i) < max_t$:

We will prove this case by contradiction, suppose:

$$\begin{aligned} max_t &\leq Bel_p^{t+1}(a_i) \\ max_t &\leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \end{aligned}$$

We can take a_i itself out of the summation:

$$\begin{aligned} max_t &\leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) + \frac{1}{|A|}(Bel_p^t(a_i) - Bel_p^t(a_i)) \\ &\leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \end{aligned}$$

Looking at the summation, given that $\forall a_j, a_i \in A$:

$$0 < In(a_j, a_i) \leq 1$$

and:

$$Bel_p^t(a_j) - Bel_p^t(a_i) \leq max_t - Bel_p^t(a_i)$$

We can say that:

$$\frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \leq \frac{|A| - 1}{|A|} (max_t - Bel_p^t(a_i))$$

Thus:

$$\begin{aligned} max_t &\leq Bel_p^{t+1}(a_i) \\ max_t &\leq \frac{|A| - 1}{|A|} (max_t - Bel_p^t(a_i)) + Bel_p^t(a_i) \\ max_t &\leq \frac{(|A| - 1) max_t}{|A|} + \frac{Bel_p^t(a_i)}{|A|} \\ max_t - \frac{(|A| - 1) max_t}{|A|} &\leq \frac{Bel_p^t(a_i)}{|A|} \\ \frac{max_t}{|A|} &\leq \frac{Bel_p^t(a_i)}{|A|} \\ max_t &\leq Bel_p^t(a_i) \end{aligned}$$

Which is a contradiction, since we assumed $Bel_p^t(a_i) > max_t$. Thus:

$$max_t > Bel_p^{t+1}(a_i)$$

Since a_i was arbitrary and the cases covered are exhaustive, the lemma is true for all agents. The same reasoning can be used to show the equivalent property for min_t

□

Corollary 1 *In a strongly connected influence graph and a classic update function if $\min_t \neq \max_t$, then $\max_{t+1} < \max_t$ and $\min_{t+1} > \min_t$.*

The result of lemma 1 tells us that all beliefs in the time $t + 1$ are smaller than \max_t , thus, since \max_{t+1} must be one of those elements, $\max_{t+1} < \max_t$. And the same reasoning can be used for \min_t .

Corollary 2 *$\lim_{t \rightarrow \infty} \max_t = L$ and $\lim_{t \rightarrow \infty} \min_t = M$ for some $L, M \in [0, 1]$.*

Since both \max_t and \min_t are bounded between 0 and 1 by the definition of belief; and lemma 1 showed us that they are monotonic, according to the monotonic convergence theorem, the limits exist.

Definition 5 *Let's denote by In_{\min} the smallest influence in the influence graph. Keep in mind that $In_{\min} > 0$ since we are working with a strongly connected influence graph.*

Using the same notation we used in corollary 2, let's call $\lim_{t \rightarrow \infty} \max_t = L$ and $\lim_{t \rightarrow \infty} \min_t = M$.

Lemma 2 $\forall t$ and $\forall a_i \in A$: $Bel_p^{t+1}(a_i) - \max_t \leq \epsilon$, with $\epsilon = \frac{In_{\min}(L-M)}{|A|}$.

To prove this lemma, first we will try to find the biggest $Bel_p^{t+1}(a_i)$ possible. Now let's start with the formula of belief:

$$Bel_p^{t+1}(a_i) = Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))$$

To achieve our goal, in each step we will choose the values in the right side of the equation in a way that maximizes it. Trying to do so will lead us to ϵ .

The first thing we will do is separate from the summation the element a_k , which we define as the agent who holds the belief \min_t in that arbitrary time step.

$$\begin{aligned} Bel_p^{t+1}(a_i) &= Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_k\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) + \frac{In(a_k, a_i)(Bel_p^t(a_k) - Bel_p^t(a_i))}{|A|} \\ &= Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_k\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) + \frac{In(a_k, a_i)(\min_t - Bel_p^t(a_i))}{|A|} \end{aligned}$$

Now trying to maximize the rightmost term in the inequality, we shall see that, by the definition of \min_t : $\min_t - Bel_p^t(a_i) \leq 0$. If $\min_t - Bel_p^t(a_i)$ values 0, the influence that multiplies it doesn't make any difference, but if it is different of 0 we want the influence to be as small as possible, which is In_{\min} .

$$Bel_p^{t+1}(a_i) \leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_k\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) + \frac{In_{\min}(\min_t - Bel_p^t(a_i))}{|A|}$$

Now it's time to choose the value of the $Bel_p^t(a_j)$ for all a_j 's that maximizes the right side. Since this part is always positive, we shall pick the maximum value possible, which is \max_t .

$$Bel_p^{t+1}(a_i) \leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_k\}} In(a_j, a_i)(\max_t - Bel_p^t(a_i)) + \frac{In_{\min}(\min_t - Bel_p^t(a_i))}{|A|}$$

Now looking at the terms inside the summation, since $\max_t - Bel_p^t(a_i) \geq 0$, the influence that maximizes it is the biggest one possible, which is 1, thus: $\forall a_i, a_j \in A \setminus \{a_k\} : In(a_j, a_i) = 1$.

$$\begin{aligned} Bel_p^{t+1}(a_i) &\leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_k\}} (\max_t - Bel_p^t(a_i)) + \frac{In_{\min}(\min_t - Bel_p^t(a_i))}{|A|} \\ &\leq Bel_p^t(a_i) + \frac{(|A| - 1)(\max_t - Bel_p^t(a_i))}{|A|} + \frac{In_{\min}(\min_t - Bel_p^t(a_i))}{|A|} \\ &\leq Bel_p^t(a_i) + \frac{(|A| - 1)(\max_t - Bel_p^t(a_i)) + In_{\min}(\min_t - Bel_p^t(a_i))}{|A|} \\ &\leq \frac{|A| Bel_p^t(a_i) + (|A| - 1)(\max_t - Bel_p^t(a_i)) + In_{\min}(\min_t - Bel_p^t(a_i))}{|A|} \\ &\leq \frac{(|A| - 1)\max_t - Bel_p^t(a_i) + In_{\min}(\min_t - Bel_p^t(a_i))}{|A|} \\ &\leq \frac{(|A| - 1)\max_t + Bel_p^t(a_i)(1 - In_{\min}) + In_{\min}\min_t}{|A|} \end{aligned}$$

These manipulations made it clear which value of $Bel_p^t(a_i)$ we should choose to achieve our goal, and it is $Bel_p^t(a_i) = max_t$.

$$\begin{aligned}
Bel_p^{t+1}(a_i) &\leq \frac{(|A| - 1) max_t + max_t (1 - In_{min}) + In_{min} min_t}{|A|} \\
&\leq \frac{|A| max_t - max_t In_{min} + In_{min} min_t}{|A|} \\
&\leq \frac{|A| max_t + In_{min}(min_t - max_t)}{|A|} \\
&\leq max_t + \frac{In_{min}(min_t - max_t)}{|A|}
\end{aligned}$$

Now we shall remember that, since max_t is decreasing and min_t is increasing, our choice to make the right side as big as possible is to plug it's limits, which gives us:

$$Bel_p^{t+1}(a_i) \leq max_t + \frac{In_{min}(L - M)}{|A|}$$

Since we tried to build the greater $Bel_p^{t+1}(a_i)$ we could, and reached this inequality, we can say that the greater $Bel_p^{t+1}(a_i)$ possible is equal to $max_t + \frac{In_{min}(min_t - max_t)}{|A|}$.

With this, we must be able to show that, even with the greater $Bel_p^{t+1}(a_i)$ possible, the difference between it and max_t is still ϵ since this property is valid for the greater $Bel_p^{t+1}(a_i)$ we can build, the difference will be either equal or bigger for a_j 's that hold smaller beliefs, thus, for all.

Let's call Bel_{max}^{t+1} the maximum belief we could build.

$$\begin{aligned}
Bel_{max}^{t+1} &= max_t + \frac{In_{min}(L - M)}{|A|} \\
Bel_{max}^{t+1} - max_t &= \frac{In_{min}(L - M)}{|A|}
\end{aligned}$$

Thus: $\forall t$ and $\forall a_i \in A$: $Bel_p^{t+1}(a_i) - max_t \leq \epsilon$, with $\epsilon = \frac{In_{min}(L - M)}{|A|}$

□

Corollary 3 $max_{t+1} + \epsilon \leq max_t$

Since max_{t+1} must be one of the beliefs in the time $t + 1$ and, according to Lemma 2, all of them are smaller than max_t by at least ϵ , max_{t+1} must also be smaller than max_t by a factor of at least ϵ .

Theorem 1 $\lim_{t \rightarrow \infty} max_t = L = \lim_{t \rightarrow \infty} min_t = M$

Suppose, by contradiction, that $L \neq M$. Plugging this values into the ϵ formula show us that $\epsilon \neq 0$. Since, according to lemma 2, max_{t+1} is smaller than max_t by a factor of ϵ . With all of this we can finally reach to a contradiction and end our proof.

To see this contradiction, let's assume we did $\lceil \frac{1}{\epsilon} \rceil + 1 = v$ timesteps after some time t . Since max diminishes by at least ϵ at each step:

$$\begin{aligned}
max_t &\geq max_v + \epsilon \left(\lceil \frac{1}{\epsilon} \rceil + 1 \right) \\
max_t &\geq max_v + \epsilon v \\
max_t - \epsilon v &\geq max_v
\end{aligned}$$

But $\epsilon v > 1$, and this would imply that $max_v < 0$, which is a contradiction!

Since assuming that $L \neq M$ leads us to a contradiction we can conclude that $L = M$. This result implies that all agents belief converge to the same value, as we wanted to prove.

□