Proof of Individual Agent Opinion Convergence in a Weakly Connected Influence Graph Using Classic Update Function

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Definition 1. The *classic update-function*, is defined as:

$$Bel_p^{t+1}(a_i|a_j) = Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)). \tag{1}$$

Definition 2. While the *overall classic update*, is defined as:

$$Bel_p^{t+1}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j). \tag{2}$$

Definition 3. We say a influence graph In over agents A is *weakly connected* if for all a_i, $a_j \in A$, there exist a_{k_1},a_{k_2},...,$a_{k_l} \subseteq A$ such that In(a_i,a_{k_1}) > 0, In(a_{k_l},a_j) > 0, and for $m = 1$,...,$l-1$, In(a_{k_m},$a_{k_{m+1}}$) > 0.

Definition 4. max_t and min_t are the maximum and minimum belief values in a given instant t, respectively.

To prove our conjecture, let's do some simplifications:

$$\begin{aligned}
Bel_p^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j). \\
&= \frac{1}{|A|} \sum_{a_j \in A} \big(Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))\big) \\
&= Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))
\end{aligned}$$

Since we have a finite number of beliefs and $\forall a_i \in A : \; Bel_p^t(a_i) \in [0,1]$, there are always min_t and a max_t. We shall also note that, by the squeeze theorem, individual agent opinion converges to the same value if and only if $lim_{t\to\infty}\, min_t = lim_{t\to\infty}\, max_t$.

Thus, since we want to prove that it always converges, if $min_t = max_t$ we have nothing to prove, so assume $min_t \neq max_t$.

Lemma 1. *In a weakly connected graph and under classic belief update, if $max_t \neq min_t$:*

$$\forall a_i \in A : Bel_p^{t+1}(a_i) \leq max_t \tag{3}$$

and:

$$\forall a_i \in A : Bel_p^{t+1}(a_i) \geq min_t \tag{4}$$

Proof. By definition:

$$Bel_p^{t+1}(a_i) = Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))$$

Trying to maximize the right side, we can substitute $Bel_p^t(a_j)$ by max_t, this turns our equation into an inequity, since $\forall a_j \in A$, $Bel_p^t(a_j) \leq max_t$, by the definition of max_t. That makes the terms inside the summation either equal or smaller than 0, thus:

$$\begin{aligned}
Bel_p^{t+1}(a_i) &\leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(max_t - Bel_p^t(a_i)) \\
&= Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} (max_t - Bel_p^t(a_i)) && \text{(since } In(a_j,a_i) \leq 1\text{)} \\
&= Bel_p^t(a_i) + \frac{|A|}{|A|}(max_t - Bel_p^t(a_i)) \\
&= Bel_p^t(a_i) + max_t - Bel_p^t(a_i) \\
Bel_p^{t+1}(a_i) &\leq max_t
\end{aligned}$$

Since a_i was arbitrary, the lemma is true for all agents. The same reasoning can be used to show the equivalent property for min_t □

Corollary 1. *In a weakly connected influence graph and a classic update function, if $min_t \neq max_t$, then $max_{t+1} \leq max_t$ and $min_{t+1} \geq min_t$.*

Proof. The result of Lemma 1 tells us that all beliefs in the time $t + 1$ are either smaller than max_t or equal to max_t, thus, since max_{t+1} must be one of those elements, $max_{t+1} \leq max_t$. And the same reasoning can be used for min_t. □

Corollary 2. $\lim_{t\to\infty} max_t = U$ *and* $\lim_{t\to\infty} min_t = L$ *for some* U, $L \in [0, 1]$.

Proof. Since both max_t and min_t are bounded between 0 and 1 by the definition of belief; and Lemma 1 showed us that they are monotonic, according to the monotonic convergence theorem, the limits exist. □

Now that we have this properties, our proof will follow by showing that an agent a_i that holds some belief $Bel_p^t(a_i)$ will influence every other agent by the time $t + |A| - 1$. To see this, we must open the definition of belief throughout time. But before we do this, let's jump into some small definitions and corollaries that will help us on the way.

Definition 5. Let's call the sequence $P(a_i|a_j) = (a_i, a_k, ..., a_{k+l})$ a *path* from a_i to a_j, if:

- All elements on the sequence are different.
- The first element in the sequence is a_i.
- If a_n is the n'th element in the sequence, if it has a successor a_{n+1}, $In(a_n, a_{n+1}) > 0$.
- If a_n is the last element in the sequence $In(a_n, a_j) > 0$.

Definition 6. Let's denote by $|P(a_i|a_j)|$ the size of a path from a_i to a_j, which we define as the number of elements in the sequence $P(a_i|a_j)$.

Corollary 3. *In a weakly connected influence graph, there is always a path from a_i to a_j.*

Proof. This follows almost instantly by the definition of a weakly connected influence graph. The only thing we must address is that, in the definition of weakly connected, the sequence of influence that it guarantees that exist may have repeated elements, but if it does, we might be able to convince ourselves that it cycles to the same point, thus taking these cycles out we have a path. □

Corollary 4. $|P(a_j|a_j)| \leq |A| - 1$.

Proof. This follows directly from the definition of path, since it doesn't have repeated elements, and we have $|A|$ agents, the path can't have more than $|A| - 1$ elements (remembering that the last element doesn't appear in the sequence). □

Corollary 5. $\forall a_i, a_k \in A$ *and* $\forall a \geq 1$*:*

$$Bel_p^{t+a}(a_i) \leq max_t + \frac{1}{|A|}\left(In(a_k, a_i)(Bel_p^{t+a-1}(a_k) - max_t)\right)$$

Proof. By the definition of belief:

$$Bel_p^{t+a}(a_i) = Bel_p^{t+a-1}(a_i) + \frac{1}{|A|}\sum_{a_j \in A} In(a_j, a_i)(Bel_p^{t+a-1}(a_j) - Bel_p^{t+a-1}(a_i))$$

The belief in the next time step is maximum when the element itself is the maximum, thus:

$$Bel_p^{t+a}(a_i) \leq max_{t+a-1} + \frac{1}{|A|}\sum_{a_j \in A} In(a_j, a_i)(Bel_p^{t+a-1}(a_j) - max_{t+a-1})$$

To make our corollary useful in future manipulations, we will take an arbitrary element a_k out of the summation:

$$\begin{aligned} &Bel_p^{t+a}(a_i) \leq \\ &max_{t+a-1} + \frac{1}{|A|}\sum_{a_j \in A\setminus\{a_k\}} In(a_j, a_i)(Bel_p^{t+a-1}(a_j) - max_{t+a-1}) + Bel_p^{t+a-1}(a_i|a_k) \end{aligned}$$

Since max_{t+a-1} is the greatest belief possible in that time step, the summation can be at most 0, thus:

$$\begin{aligned} Bel_p^{t+a}(a_i) &\leq max_{t+a-1} + Bel_p^{t+a-1}(a_i|a_k) \\ &= max_{t+a-1} + In(a_k, a_i)\left(Bel_p^{t+a-1}(a_k) - Bel_p^{t+a-1}(a_i)\right) \\ &= max_{t+a-1} + In(a_k, a_i)\left(Bel_p^{t+a-1}(a_k) - max_{t+a-1}\right) \end{aligned}$$

Since max decreases throughout time:

$$Bel_p^{t+a}(a_i) \leq max_t + In(a_k, a_i)\left(Bel_p^{t+a-1}(a_k) - max_t\right)$$

□

Definition 7. Let's denote by $In_{path-min}$ smallest product of the influences of subsequent agents among all paths in the graph. In other words, given all paths in a graph, if we multiply the influences when we go from the start of the path until it's end we have some product as a result (which is positive), $In_{path-min}$ is the smallest value among those products.

Using the same notation we used in Corollary 2, let's call $\lim_{t\to\infty} max_t = U$ and $\lim_{t\to\infty} min_t = L$.

Now that we have all of these tools, let's jump into what is probably the most important part in the proof:

Lemma 2. *$\forall a_i \in A:\ max_t - Bel_p^{t+|A|-1}(a_i) \geq \epsilon$, with $\epsilon = \frac{In_{path-min}\cdot(U-L)}{|A|^{|A|-1}}$.*

Ps.: After reading this proof again I found one mistake in this lemma. I claim to proof that this property holds independently of the size of the path, but I proved only for the case in which $|P(a_k|a_i)| = |A|-1$. I believe that expanding the belief definition until $Bel_p^{t+|P(a_k|a_i)|-1}$ appear may be enough to show this property, but those are just ideas until we find a valid proof to it. Anyway, there it goes the proof with the fallacies listed above. Sorry for the inconvenience.

Proof. By definition:

$$Bel_p^{t+|A|-1}(a_i) = Bel_p^{t+|A|-2}(a_i) + \frac{1}{|A|}\sum_{a_j\in A} Bel_p^{t+|A|-2}(a_i|a_j)$$

What we will do now is separate, at each step, one element of the summation and apply corollary 5 to modify our inequity. But we will be smart when choosing the elements we separate from the summation. Denoting by a_k the agent that holds the belief min_t in the time t, we will separate from the summation the elements in $P(a_k|a_i)$, starting from the end of the path until we get to a_k.

To simplify our notation, let's index the elements in the path from a_k to a_i, starting from the end of the path (since we are backtracking it will make more sense) by calling a_n the n'th element from the end to the beginning of the sequence.

Thus, by corollary 5:

$$Bel_p^{t+|A|-1}(a_i) \leq max_t + \frac{1}{|A|}In(a_1, a_i)(Bel_p^{t+|A|-2}(a_1) - max_t)$$

Note, now, that if $|P(a_k, a_i)| = 1$, our hypothesis would have been proved, Because a_1 would be a_k itself, which can be at maximum L, according to corollary 2. In this case some manipulations would lead us to our goal. Since we must prove the general case, lets backtrack one more step to see the pattern forming:

Using corollary 5 again:

$$\begin{aligned}
&Bel_p^{t+|A|-1}(a_i) \\
&\leq max_t + \frac{1}{|A|} In(a_1, a_i)(Bel_p^{t+|A|-2}(a_1) - max_t) \\
&\leq max_t + \frac{1}{|A|} In(a_1, a_i) \left(\left(max_t + \frac{1}{|A|} In(a_2, a_1)(Bel_p^{t+|A|-3}(a_2) - max_t) \right) - max_t \right) \\
&= max_t + \frac{1}{|A|} In(a_1, a_i) \left(\frac{1}{|A|} In(a_2, a_1)(Bel_p^{t+|A|-3}(a_2) - max_t) \right) \\
&= max_t + \frac{1}{|A|^2} In(a_2, a_1) In(a_1, a_i)(Bel_p^{t+|A|-3}(a_2) - max_t)
\end{aligned}$$

The equation above has the same form of the one before it, and this pattern will continue throughout time. Now, denoting as P_{In} the product of the influences in the path, we can write the generalized version of the inequality above as:

$$Bel_p^{t+|A|-1}(a_i) \leq max_t + \frac{P_{In}(min_t - max_t)}{|A|^{|P(a_k|a_i)|}}$$

This formulas comes from the fact that the path ends after $|P(a_k|a_i)|$ steps with a_k as the end of the path, and, by definition, the belief of a_k is min_t.

Since the rightmost term in the equation is either equal to or smaller than 0, to make the inequality hold for all a_i's, we shall substitute P_{In} by the smallest value possible, which is, by definition, $In_{path-min}$. Thus:

$$Bel_p^{t+|A|-1}(a_i) \leq max_t + \frac{In_{path-min}(min_t - max_t)}{|A|^{|P(a_k|a_i)|}}$$

Using the same argument that the rightmost term is either negative or smaller that 0, we shall substitute denominator by the biggest one possible to guarantee that the inequality holds for all a_i's, and this value is the size of the biggest path, which is, according to 4, $|A| - 1$. Thus:

$$Bel_p^{t+|A|-1}(a_i) \leq max_t + \frac{In_{path-min}(min_t - max_t)}{|A|^{|A|-1}}$$

Since the maximum value of min_t is L and the minimum value of max_t is U, those are the values we should plug to maintain the inequity:

$$Bel_p^{t+|A|-1}(a_i) \leq max_t + \frac{In_{path-min}(L-U)}{|A|^{|A|-1}}$$
$$Bel_p^{t+|A|-1}(a_i) - max_t \leq \frac{In_{path-min}(L-U)}{|A|^{|A|-1}}$$
$$max_t - Bel_p^{t+|A|-1}(a_i) \geq \frac{In_{path-min}(U-L)}{|A|^{|A|-1}}$$
$$max_t - Bel_p^{t+|A|-1}(a_i) \geq \epsilon$$

□

Corollary 6. $max_t - max_{t+|A|-1} \geq \epsilon$

Proof. Since $max_{t+|A|-1}$ must be one of the beliefs in the time $t+|A|-1$ and, according to lemma 2 all of them are smaller than max_t by a factor of at least ϵ, $max_{t+|A|-1}$ must also be smaller than max_t by a factor of at least ϵ. □

Theorem 1. $\lim_{t\to\infty} max_t = U = \lim_{t\to\infty} min_t = L$

Proof. Suppose, by contradiction, that $U \neq L$. Plugging this values into the ϵ formula show us that $\epsilon \neq 0$. Since, according to Lemma 2, $max_{t+|A|-1}$ is smaller than max_t by a factor of ϵ, we can finally reach to a contradiction and end our proof.

To see this contradiction, let's assume we did $v = (|A|-1)\left(\left\lceil\frac{1}{\epsilon}\right\rceil + 1\right)$ time steps after $t = 0$. Since max diminishes by at least ϵ at each $|A|-1$ steps:

$$max_0 \geq max_v + \epsilon \left(\left\lceil\frac{1}{\epsilon}\right\rceil + 1\right)$$
$$max_0 - \epsilon \left(\left\lceil\frac{1}{\epsilon}\right\rceil + 1\right) \geq max_v$$

But $\epsilon.\left(\left\lceil\frac{1}{\epsilon}\right\rceil + 1\right) > 1$, thus $\max_0 < \epsilon.\left(\left\lceil\frac{1}{\epsilon}\right\rceil + 1\right)$. And this would imply that $max_v < 0$, which contradicts the definition of belief!

Since assuming that $U \neq L$ led us to a contradiction we can conclude that $U = L$. This result implies that all agents belief converge to the same value, as we wanted to prove. □