

Proof of Individual Agent Opinion Convergence in a Weakly Connected Influence Graph Using Classic Update Function

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Definition 1. The *classic update-function*, is defined as:

$$B^{t+1}(a_i|a_j) = B^t(a_i) + I(a_j, a_i)(B^t(a_j) - B^t(a_i)) \quad (1)$$

Definition 2. While the *overall classic update*, is defined as:

$$B^{t+1}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} B^{t+1}(a_i|a_j) \quad (2)$$

Definition 3. We say a influence graph In over agents A is *weakly connected* if for all $a_i, a_j \in A$, there exist $a_{k_1}, a_{k_2}, \dots, a_{k_l} \subseteq A$ such that $\text{In}(a_i, a_{k_1}) > 0$, $\text{In}(a_{k_l}, a_j) > 0$, and for $m = 1, \dots, l - 1$, $\text{In}(a_{k_m}, a_{k_{m+1}}) > 0$.

Definition 4. \max^t and \min^t are the maximum and minimum belief values in a given instant t , respectively. Thus:

$$\min^t = \min_{a_i \in A} B^t(a_i) \text{ and } \max^t = \max_{a_i \in A} B^t(a_i).$$

To prove our conjecture, let's do some simplifications:

$$\begin{aligned} B^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} B^{t+1}(a_i|a_j). \\ &= \frac{1}{|A|} \sum_{a_j \in A} (B^t(a_i) + I(a_j, a_i)(B^t(a_j) - B^t(a_i))) \\ &= B^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} I(a_j, a_i)(B^t(a_j) - B^t(a_i)) \end{aligned} \quad (3)$$

Since we have a finite number of beliefs and $\forall a_i \in A : B^t(a_i) \in [0, 1]$, there are always \min^t and a \max^t . We shall also note that, by the Squeeze Theorem, individual agent opinion converges to the same value if and only if $\lim_{t \rightarrow \infty} \min^t = \lim_{t \rightarrow \infty} \max^t$.

Since we want to prove that it always converges, if $\min^t = \max^t$ we have nothing to prove, so assume from now on that $\min^t \neq \max^t$.

Lemma 1. *In a weakly connected graph and under classic belief update:*

$$\forall t \text{ and } \forall a_i \in A : \min^t \leq B^{t+1}(a_i) \leq \max^t$$

Proof. By the equation 3:

$$B^{t+1}(a_i) = B^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} I(a_j, a_i)(B^t(a_j) - B^t(a_i))$$

Substituting $B^t(a_j)$ by \max^t turns our equation into an inequality, since $\forall a_j \in A$, $B^t(a_j) \leq \max^t$ and also makes the terms inside the summation either equal to or smaller than 0. Thus:

$$\begin{aligned} B^{t+1}(a_i) &\leq B^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} I(a_j, a_i)(\max^t - B^t(a_i)) \\ &\leq B^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} (\max^t - B^t(a_i)) && (\text{since } In(a_j, a_i) \leq 1 \text{ and} \\ &&& \max^t - B^t(a_i) \geq 0) \\ &= B^t(a_i) + \frac{|A|}{|A|}(\max^t - B^t(a_i)) \\ &= B^t(a_i) + \max^t - B^t(a_i) \\ &= \max^t \end{aligned} \tag{4}$$

Since a_i was arbitrary, the Lemma is true for all agents. The same reasoning can be used to show the equivalent property for \min^t \square

Corollary 1. *In a weakly connected influence graph under the confirmation-bias update function:*

$$\max^{t+1} \leq \max^t \text{ and } \min^{t+1} \geq \min^t \text{ for all } t.$$

Proof. Lemma 1 tells us that all beliefs in the time $t + 1$ are either smaller or equal to \max^t . Since \max^{t+1} must be one of those beliefs, $\max^{t+1} \leq \max^t$. The same reasoning can be used for \min^t . \square

Corollary 2. $\lim_{t \rightarrow \infty} \max^t = U$ and $\lim_{t \rightarrow \infty} \min^t = L$ for some $U, L \in [0, 1]$.

Proof. Both \max^t and \min^t are bounded between 0 and 1 and Lemma 1 showed us that they are monotonic. According to the Monotonic Convergence Theorem, this guarantees that the limits exist. \square

The proof will follow by showing that an agent a_i that holds some belief $B^t(a_i)$ influences every other agent by the time $t + |A| - 1$. Before we do this, let's jump into some small definitions and corollaries that will help us on the way.

Definition 5. Let's call the sequence $P(a_i \rightarrow a_j) = (a_i, a_k, \dots, a_{k+l}, a_j)$ a *simple path* from a_i to a_j , if:

- All elements on the sequence are different.
- The first element in the sequence is a_i .
- The last element in the sequence is a_j .
- If a_n is the n 'th element in the sequence, if it has a successor a_{n+1} , $In(a_n, a_{n+1}) > 0$.

Many simple paths from a_i to a_j can exist, although our notation isn't enough to differentiate them. But in subsequent steps we will only need one of those simple paths, so the notation shouldn't be a problem.

Definition 6. Denote by $|P(a_i \rightarrow a_j)|$ the *size* of a simple path from a_i to a_j , which we define as the number of elements in the sequence $P(a_i \rightarrow a_j) - 1$.

Corollary 3. $\forall P(a_i \rightarrow a_j), |P(a_i \rightarrow a_j)| \leq |A| - 1$.

Proof. A simple path doesn't have repeated elements and we have $|A|$ agents, thus simple path can't have more than $|A|$ elements. According to Definition 6, the size of a simple path is defined as the number of elements minus one, thus maximum size is $|A| - 1$. \square

Lemma 2. $\forall x, \forall t$ and $\forall a_i$, if $B^t(a_i) \leq x$:

$$B^{t+1}(a_i) \leq x + \frac{1}{|A|} \sum_{a_j \in A} (I(a_j, a_i) (B^t(a_j) - x))$$

Proof.

$$\begin{aligned} B^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} B^{t+1}(a_i | a_j) \\ &= \frac{1}{|A|} \sum_{a_j \in A} (B^t(a_i) + I(a_j, a_i) (B^t(a_j) - B^t(a_i))) \\ &= \frac{1}{|A|} \sum_{a_j \in A} (B^t(a_i)(1 - I(a_j, a_i)) + I(a_j, a_i)B^t(a_j)) \\ &\leq \frac{1}{|A|} \sum_{a_j \in A} (x \cdot (1 - I(a_j, a_i)) + I(a_j, a_i)B^t(a_j)) \\ &= x + \frac{1}{|A|} \sum_{a_j \in A} (I(a_j, a_i) (B^t(a_j) - x)) \end{aligned}$$

\square

Lemma 3. $\forall a_i, a_k \in A$ and $\forall n \geq 1$ and $\forall t$:

$$B^{t+n}(a_i) \leq \max^t + \frac{1}{|A|} (I(a_k, a_i)(B^{t+n-1}(a_k) - \max^t)) \quad (5)$$

Proof. By the Definitions 1 and 2:

$$B^{t+n}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} B^{t+n}(a_i|a_j)$$

$$B^{t+n}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} (B^{t+n-1}(a_i) + I(a_j, a_i)(B^{t+n-1}(a_j) - B^{t+n-1}(a_i)))$$

According to Corollary 1: $B^{t+n}(a_i) \leq \max^{t+n} \leq \max^{t+n-1}$. Thus we can use Lemma 2:

$$B^{t+n}(a_i) \leq \frac{1}{|A|} \sum_{a_j \in A} (\max^{t+n-1} + I(a_j, a_i)(B^{t+n-1}(a_j) - \max^{t+n-1}))$$

$$= \max^{t+n-1} + \frac{1}{|A|} \sum_{a_j \in A} I(a_j, a_i)(B^{t+n-1}(a_j) - \max^{t+n-1})$$

To make our Lemma useful in future manipulations, we will take an arbitrary element a_k out of the summation:

$$B^{t+n}(a_i) \leq \max^{t+n-1} + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_k\}} (I(a_j, a_i)(B^{t+n-1}(a_j) - \max^{t+n-1}))$$

$$+ \frac{1}{|A|} (I(a_k, a_i)(B^{t+n-1}(a_k) - \max^{t+n-1}))$$

Since \max^{t+n-1} is the greatest belief possible in that time step, the summation can be at most 0, thus:

$$B^{t+n}(a_i) \leq \max^{t+n-1} + \frac{1}{|A|} I(a_k, a_i) (B^{t+n-1}(a_k) - \max^{t+n-1})$$

Since \max doesn't increase throughout time, $\max^{t+n-1} \leq \max^t$. Thus:

$$B^{t+n}(a_i) \leq \max^t + \frac{1}{|A|} I(a_k, a_i) (B^{t+n-1}(a_k) - \max^t)$$

□

Definition 7. Denote by I_{min} the smallest positive influence in the influence graph.

Using the same notation we used in Corollary 2, let's call $\lim_{t \rightarrow \infty} \max^t = U$ and $\lim_{t \rightarrow \infty} \min^t = L$. Denoting by a_*^t one agent who holds the belief \min^t in the time t :

Theorem 1. $\forall t$ and $\forall a_i \in A$:

$$B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) \leq \max^t - \delta^t, \text{ with } \delta^t = \left(\frac{I_{min}}{|A|} \right)^{|P(a_*^t \rightarrow a_i)|} \cdot (U - L).$$

Proof. By equation 3:

$$B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) = Bel_p^{t+|P(a_*^t \rightarrow a_i)|-1}(a_i) + \frac{1}{|A|} \sum_{a_j \in A} B^{t+|P(a_*^t \rightarrow a_i)|-1}(a_i|a_j)$$

We will now separate, at each step, a carefully chosen element of the summation and apply Lemma 3 to modify our inequality. The chosen elements will be the ones in $P(a_*^t \rightarrow a_i)$, starting from the end of the simple path until we get to a_*^t .

To simplify the notation, let's index the elements in the simple path from a_*^t to a_i , starting from the end of the simple path (since we are backtracking) by calling a_n the n^{th} element from the end to the beginning of the sequence (excluding a_i itself).

By Lemma 3:

$$B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) \leq max^t + \frac{1}{|A|} In(a_1, a_i)(B^{t+|P(a_*^t \rightarrow a_i)|-1}(a_1) - max^t)$$

If $|P(a_*^t, a_i)| = 1$, we could prove our result. Instead of showing it I will expand this two more times to show the general formula.

Using Lemma 3:

$$\begin{aligned} & B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) \\ & \leq max^t + \frac{1}{|A|} In(a_1, a_i)(B^{t+|P(a_*^t \rightarrow a_i)|-1}(a_1) - max^t) \\ & \leq max^t + \frac{1}{|A|} In(a_1, a_i) \left(\left(max^t + \frac{1}{|A|} In(a_2, a_1)(B^{t+|P(a_*^t \rightarrow a_i)|-2}(a_2) - max^t) \right) - max^t \right) \\ & = max^t + \frac{1}{|A|} In(a_1, a_i) \left(\frac{1}{|A|} In(a_2, a_1)(B^{t+|P(a_*^t \rightarrow a_i)|-2}(a_2) - max^t) \right) \\ & = max^t + \frac{1}{|A|^2} In(a_2, a_1) In(a_1, a_i)(B^{t+|P(a_*^t \rightarrow a_i)|-2}(a_2) - max^t) \\ & \leq max^t + \frac{1}{|A|^2} In(a_2, a_1) In(a_1, a_i) \times \\ & \quad \left(\left(max^t + \frac{1}{|A|} In(a_3, a_2) \left(B^{t+|P(a_*^t \rightarrow a_i)|-3}(a_3) - max^t \right) \right) - max^t \right) \\ & = max^t + \frac{1}{|A|^2} In(a_2, a_1) In(a_1, a_i) \left(\frac{1}{|A|} In(a_3, a_2) \left(B^{t+|P(a_*^t \rightarrow a_i)|-3}(a_3) \right) - max^t \right) \\ & = max^t + \frac{1}{|A|^3} In(a_3, a_2) In(a_2, a_1) In(a_1, a_i) \left(B^{t+|P(a_*^t \rightarrow a_i)|-3}(a_3) - max^t \right) \end{aligned}$$

We can see a pattern forming and this pattern will continue throughout time. Denoting P_{In} the product of the influences in the simple path ($P_{In} = I(a_*^t, a_{|P(a_*^t, a_i)|}) \times \dots \times I(a_1, a_i)$), we can write the general version of the inequality above as:

$$B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) \leq max^t + \frac{P_{In}}{|A|^{|P(a_*^t \rightarrow a_i)|}} \cdot (Bel_p^t(a_*^t) - max^t)$$

$$= \max^t + \frac{P_{In}}{|A|^{|P(a_*^t \rightarrow a_i)|}} \cdot (\min^t - \max^t)$$

The rightmost term in the equation is either equal to or smaller than 0 thus, to make the inequality hold for all a_i 's, we shall substitute P_{In} by the smallest value possible. By the Definition 7, I_{min} is the smallest positive influence in the graph and according to Definition 5 the influences in a simple path are positive. Thus:

$$B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) \leq \max^t + \left(\frac{I_{min}}{|A|} \right)^{|P(a_*^t \rightarrow a_i)|} \cdot (\min^t - \max^t)$$

According to Corollary 2, the maximum value of \min^t is L and the minimum value of \max^t is U , thus:

$$\begin{aligned} B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) &\leq \max^t + \left(\frac{I_{min}}{|A|} \right)^{|P(a_*^t \rightarrow a_i)|} \cdot (L - U) \\ &\leq \max^t - \left(\frac{I_{min}}{|A|} \right)^{|P(a_*^t \rightarrow a_i)|} \cdot (U - L) \\ &\leq \max^t - \delta^t \end{aligned}$$

□

Lemma 4.

$$\sum_{a_j \in A} I(a_j, a_i) (B^t(a_j) - B^t(a_i)) = \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i) (B^t(a_j) - B^t(a_i))$$

Proof.

$$\begin{aligned} &\sum_{a_j \in A} I(a_j, a_i) (B^t(a_j) - B^t(a_i)) \\ &= \sum_{a_j \in A \setminus \{a_i\}} (I(a_j, a_i) (B^t(a_j) - B^t(a_i))) + I(a_i, a_i) (B^t(a_i) - B^t(a_i)) \\ &= \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i) (B^t(a_j) - B^t(a_i)) \end{aligned}$$

□

Lemma 5. If $B^{t+n}(a_i) \leq \max^t - \gamma$, $\gamma \geq 0$ and $n \geq 0$, then $B^{t+n+1}(a_i) \leq \max^t - \frac{\gamma}{|A|}$.

Proof.

$$\begin{aligned} B^{t+n+1}(a_i) &= B^{t+n}(a_i) + \frac{1}{|A|} \sum_{a_j \in A} I(a_j, a_i) (B^{t+n}(a_j) - B^{t+n}(a_i)) \\ &= B^{t+n}(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i) (B^{t+n}(a_j) - B^{t+n}(a_i)) \quad (\text{Lemma 4}) \end{aligned}$$

$$\begin{aligned}
&\leq \max^t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i) (B^{t+n}(a_j) - \max^t + \gamma) \quad (\text{Lemma 2}) \\
&\leq \max^t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i) (\max^t - \max^t + \gamma) \\
&= \max^t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i) (\gamma) \\
&\leq \max^t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} (\gamma) \\
&= \max^t - \gamma + \frac{(|A| - 1)(\gamma)}{|A|} \\
&= \max^t + \frac{(\gamma)((-|A|) + (|A| - 1))}{|A|} \\
&= \max^t - \frac{\gamma}{|A|}
\end{aligned}$$

□

Theorem 2. $\forall a_i \in A : B^{t+|A|-1}(a_i) \leq \max^t - \epsilon$, with $\epsilon = \left(\frac{I_{min}}{|A|}\right)^{|A|-1} \cdot (U - L)$.

Proof. Keeping the notation of Theorem 1, let's call a_*^t one agent that holds the belief min^t in the time t .

Note that, if $|P(a_*^t \rightarrow a_i)| = |A| - 1$, our theorem is true by Theorem 1 and we nothing to prove.

Else if $|P(a_*^t \rightarrow a_i)| \neq |A| - 1$, then $|P(a_*^t \rightarrow a_i)| < |A| - 1$ according to Corollary 3.

According to Theorem 1:

$$B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) \leq \max^t - \left(\frac{I_{min}}{|A|}\right)^{|P(a_*^t \rightarrow a_i)|} \cdot (U - L)$$

To keep things simple let's keep the notation from Theorem 1 and call:

$$\delta^t = \left(\frac{I_{min}}{|A|}\right)^{|P(a_*^t \rightarrow a_i)|} \cdot (U - L)$$

Now it is easy to see that we can apply Lemma 5 successively:

$$\begin{aligned}
B^{t+|P(a_*^t \rightarrow a_i)|+1}(a_i) &\leq \max^t - \frac{\delta^t}{|A|} \\
&\Downarrow \\
B^{t+|P(a_*^t \rightarrow a_i)|+2}(a_i) &\leq \max^t - \frac{\delta^t}{|A|^2} \\
&\Downarrow
\end{aligned}$$

$$B^{t+|P(a_*^t \rightarrow a_i)|+3}(a_i) \leq max^t - \frac{\delta^t}{|A|^3}$$

If we do it $|A| - |P(a_*^t \rightarrow a_i)| - 1$ times we get:

$$\begin{aligned} B^{t+|P(a_*^t \rightarrow a_i)|+|A|-|P(a_*^t \rightarrow a_i)|-1}(a_i) &\leq max^t - \frac{\delta^t}{|A|^{|A|-|P(a_*^t \rightarrow a_i)|-1}} \\ &\Downarrow \\ B^{t+|A|-1}(a_i) &\leq max^t - \frac{\delta^t}{|A|^{|A|-|P(a_*^t \rightarrow a_i)|-1}} \\ &\leq max^t - \frac{\left(\frac{I_{min}}{|A|}\right)^{|P(a_*^t \rightarrow a_i)|} \cdot (U - L)}{|A|^{|A|-|P(a_*^t \rightarrow a_i)|-1}} \\ &\leq max^t - \frac{I_{min}^{|P(a_*^t \rightarrow a_i)|} \cdot (U - L)}{|A|^{|A|-1}} \\ &\leq max^t - \left(\frac{I_{min}}{|A|}\right)^{|A|-1} \cdot (U - L) \\ &\leq max^t - \epsilon \end{aligned}$$

□

Corollary 4. $max^{t+|A|-1} \leq max^t - \epsilon$

Proof. Since $max^{t+|A|-1}$ is one of the beliefs in the time $t + |A| - 1$ and, according to Theorem 2 all of them are smaller than max^t by a factor of at least ϵ , $max^{t+|A|-1}$ must also be smaller than max^t by a factor of at least ϵ . □

Theorem 3. $\lim_{t \rightarrow \infty} max^t = U = \lim_{t \rightarrow \infty} min^t = L$

Proof. Suppose, by contradiction, that $U \neq L$. Plugging this values into the ϵ formula show us that $\epsilon > 0$.

Let's assume we did $v = (|A| - 1) \left(\left\lceil \frac{1}{\epsilon} \right\rceil + 1\right)$ time steps after $t = 0$. Since max diminishes by at least ϵ at each $|A| - 1$ steps:

$$max^0 \geq max^v + \epsilon \left(\left\lceil \frac{1}{\epsilon} \right\rceil + 1 \right)$$

Since $\epsilon \cdot \left(\left\lceil \frac{1}{\epsilon} \right\rceil + 1\right) > 1$ and $0 \leq max^v \leq 1$, this would imply that $max^0 \geq 1$ contradicting the definition of belief!

Since assuming that $U \neq L$ led us to a contradiction: $U = L$. □

Theorem 4. $\forall a_i, a_j \in A, \lim_{t \rightarrow \infty} B^t(a_i) = \lim_{t \rightarrow \infty} B^t(a_j)$

Proof. Since $L \leq \lim_{t \rightarrow \infty} B^t(a_i) \leq U$ and $L = U$: $L = B^t(a_i) = U$. And the same can be showed for $B^t(a_j)$. □