Proof of Individual Agent Opinion Convergence in a Strongly Connected Influence Graph Using Classic Update Function

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Definition 1. The *classic update-function*, is defined as:

$$B^{t+1}(a_i|a_j) = B^t(a_i) + I(a_j, a_i)(B^t(a_j) - B^t(a_i)) \tag{1}$$

Definition 2. While the *overall classic update*, is defined as:

$$B^{t+1}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} B^{t+1}(a_i|a_j) \tag{2}$$

Definition 3. We say a influence graph In over agents A is *strongly connected* if for all a_i, $a_j \in A$, there exist a_{k_1},a_{k_2},...,$a_{k_l} \subseteq A$ such that In(a_i,a_{k_1}) > 0, In(a_{k_l},a_j) > 0, and for $m = 1$,...,$l-1$, In(a_{k_m},$a_{k_{m+1}}$) > 0.

Definition 4. max^t and min^t are the maximum and minimum belief values in a given instant t, respectively. Thus:

$$min^t = \underset{a_i \in A}{min}\ B^t(a_i) \text{ and } max^t = \underset{a_i \in A}{max}\ B^t(a_i).$$

To prove our conjecture, let's do some simplifications:

$$\begin{aligned}
B^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} B^{t+1}(a_i|a_j). \\
&= \frac{1}{|A|} \sum_{a_j \in A} \big(B^t(a_i) + I(a_j, a_i)(B^t(a_j) - B^t(a_i))\big) \\
&= B^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} I(a_j, a_i)(B^t(a_j) - B^t(a_i)) \qquad (3)
\end{aligned}$$

Since we have a finite number of beliefs and $\forall a_i \in A :\ B^t(a_i) \in [0, 1]$, there are always min^t and a max^t. We shall also note that, by the Squeeze Theorem, individual agent opinion converges to the same value if and only if $lim_{t\to\infty}\ min^t = lim_{t\to\infty}\ max^t$.

Since we want to prove that it always converges, if $min^t = max^t$ we have nothing to prove, so assume from now on that $min^t \neq max^t$.

Lemma 1. *Under the classic belief update:*

$$\forall t \text{ and } \forall a_i \in A : min^t \le B^{t+1}(a_i) \le max^t$$

Proof. By the equation 3:

$$B^{t+1}(a_i) = B^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A} I(a_j, a_i)(B^t(a_j) - B^t(a_i))$$

Substituting $B^t(a_j)$ by max^t turns our equation into an inequality, since $\forall a_j \in A$, $B^t(a_j) \le max^t$ and also makes the terms inside the summation either equal to or greater than 0. Thus:

$$\begin{aligned}
B^{t+1}(a_i) &\le B^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A} I(a_j, a_i)(max^t - B^t(a_i)) \\
&\le B^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A} (max^t - B^t(a_i)) && \text{(since } In(a_j,a_i) \le 1 \text{ and} \\
& && max^t - B^t(a_i) \ge 0) \\
&= B^t(a_i) + \frac{|A|}{|A|}(max^t - B^t(a_i)) \\
&= B^t(a_i) + max^t - B^t(a_i) \\
&= max^t
\end{aligned} \tag{4}$$

Since a_i was arbitrary, the Lemma is true for all agents. The same reasoning can be used to show the equivalent property for min^t □

Corollary 1. *In a strongly connected influence graph under the classic update function:*

$$max^{t+1} \le max^t \text{ and } min^{t+1} \ge min^t \text{ for all } t.$$

Proof. Lemma 1 tells us that all beliefs in the time $t+1$ are either smaller or equal to max^t. Since max^{t+1} must be one of those beliefs, $max^{t+1} \le max^t$. The same reasoning can be used for min^t. □

Corollary 2. $\lim_{t\to\infty} max^t = U$ *and* $\lim_{t\to\infty} min^t = L$ *for some* U, $L \in [0, 1]$.

Proof. Both max^t and min^t are bounded between 0 and 1 and Lemma 1 showed us that they are monotonic. According to the Monotonic Convergence Theorem, this guarantees that the limits exist. □

The proof will follow by showing that an agent a_i that holds some belief $B^t(a_i)$ influences every other agent by the time $t + |A| - 1$. Before we do this, let's jump into some small definitions and corollaries that will help us on the way.

Definition 5. Let's call the sequence $P(a_i \to a_j) = (a_i, a_k, ..., a_{k+l}, a_j)$ a *simple path* from a_i to a_j, if:

- All elements on the sequence are different.

- The first element in the sequence is a_i.
- The last element in the sequence is a_j.
- If a_n is the n'th element in the sequence, if it has a successor a_{n+1}, $In(a_n, a_{n+1}) > 0$.

Many simple paths from a_i to a_j can exist, although our notation isn't enough to differentiate them. But in subsequent steps we will only need one of those simple paths, so the notation shouldn't be a problem.

Definition 6. Denote by $|P(a_i \to a_j)|$ the *size* of a simple path from a_i to a_j, which we define as the number of elements in the sequence $P(a_i \to a_j) - 1$.

Corollary 3. *$\forall P(a_i \to a_j)$, $|P(a_i \to a_j)| \leq |A| - 1$.*

Proof. A simple path doesn't have repeated elements and we have $|A|$ agents, thus simple path can't have more than $|A|$ elements. According to Definition 6, the size of a simple path is defined as the number of elements minus one, thus maximum size is $|A| - 1$. □

Lemma 2. *$\forall x, \forall t$ and $\forall a_i$, if $B^t(a_i) \leq x$:*

$$B^{t+1}(a_i) \leq x + \frac{1}{|A|} \sum_{a_j \in A} (I(a_j, a_i) \left(B^t(a_j) - x\right)$$

Proof.

$$\begin{aligned}
B^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} B^{t+1}(a_i|a_j) \\
&= \frac{1}{|A|} \sum_{a_j \in A} \left(B^t(a_i) + I(a_j, a_i) \left(B^t(a_j) - B^t(a_i)\right)\right) \\
&= \frac{1}{|A|} \sum_{a_j \in A} \left(B^t(a_i)(1 - I(a_j, a_i)) + I(a_j, a_i) B^t(a_j)\right) \\
&\leq \frac{1}{|A|} \sum_{a_j \in A} \left(x.(1 - I(a_j, a_i)) + I(a_j, a_i) B^t(a_j)\right) \\
&= x + \frac{1}{|A|} \sum_{a_j \in A} \left(I(a_j, a_i) \left(B^t(a_j) - x\right)\right)
\end{aligned}$$

□

Lemma 3. *$\forall a_i, a_k \in A$ and $\forall n \geq 1$ and $\forall t$:*

$$B^{t+n}(a_i) \leq max^t + \frac{1}{|A|} \left(I(a_k, a_i)(B^{t+n-1}(a_k) - max^t)\right) \tag{5}$$

Proof. By the Definitions 1 and 2:

$$B^{t+n}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} B^{t+n}(a_i|a_j)$$

$$B^{t+n}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} \left(B^{t+n-1}(a_i) + I(a_j, a_i)(B^{t+n-1}(a_j) - B^{t+n-1}(a_i)) \right)$$

According to Corollary 1: $B^{t+n}(a_i) \leq max^{t+n} \leq max^{t+n-1}$. Thus we can use Lemma 2:

$$\begin{aligned} B^{t+n}(a_i) &\leq \frac{1}{|A|} \sum_{a_j \in A} \left(max^{t+n-1} + I(a_j, a_i)(B^{t+n-1}(a_j) - max^{t+n-1}) \right) \\ &= max^{t+n-1} + \frac{1}{|A|} \sum_{a_j \in A} I(a_j, a_i)(B^{t+n-1}(a_j) - max^{t+n-1}) \end{aligned}$$

To make our Lemma useful in future manipulations, we will take an arbitrary element a_k out of the summation:

$$\begin{aligned} B^{t+n}(a_i) \leq max^{t+n-1} &+ \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_k\}} \left(I(a_j, a_i)(B^{t+n-1}(a_j) - max^{t+n-1}) \right) \\ &+ \frac{1}{|A|} (I(a_k, a_i)(B^{t+n-1}(a_k) - max^{t+n-1})) \end{aligned}$$

Since max^{t+n-1} is the greatest belief possible in that time step, the summation can be at most 0, thus:

$$B^{t+n}(a_i) \leq max^{t+n-1} + \frac{1}{|A|} In(a_k, a_i) \left(B^{t+n-1}(a_k) - max^{t+n-1} \right)$$

Since max doesn't increase throughout time, $max^{t+n-1} \leq max^t$. Thus:

$$B^{t+n}(a_i) \leq max^t + \frac{1}{|A|} I(a_k, a_i) \left(B^{t+n-1}(a_k) - max^t \right)$$

□

Definition 7. Denote by I_{min} the smallest positive influence in the influence graph.

Using the same notation we used in Corollary 2, let's call $\lim_{t \to \infty} max^t = U$ and $\lim_{t \to \infty} min^t = L$. Denoting by a_*^t one agent who holds the belief min^t in the time t:

Theorem 1. $\forall t$ *and* $\forall a_i \in A$:

$$B^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t - \delta^t, \text{ with } \delta^t = \left(\frac{I_{min}}{|A|} \right)^{|P(a_*^t \to a_i)|} .(U - L).$$

Proof. By equation 3:

$$B^{t+|P(a_*^t \to a_i)|}(a_i) = Bel_p^{t+|P(a_*^t \to a_i)|-1}(a_i) + \frac{1}{|A|} \sum_{a_j \in A} B^{t+|P(a_*^t \to a_i)|-1}(a_i|a_j)$$

We will now separate, at each step, a carefully chosen element of the summation and apply Lemma 3 to modify our inequality. The chosen elements will be the ones in $P(a_*^t \rightarrow a_i)$, starting from the end of the simple path until we get to a_*^t.

To simplify the notation, let's index the elements in the simple path from a_*^t to a_i, starting from the end of the simple path (since we are backtracking) by calling a_n the n^{th} element from the end to the beginning of the sequence (excluding a_i itself).

By Lemma 3:

$$B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) \leq max^t + \frac{1}{|A|} In(a_1, a_i)(B^{t+|P(a_*^t \rightarrow a_i)-1|}(a_1) - max^t)$$

If $|P(a_*^t, a_i)| = 1$, we could prove our result. Instead of showing it I will expand this two more times to show the general formula.

Using Lemma 3:

$$\begin{aligned}
&B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) \\
&\leq max^t + \frac{1}{|A|} In(a_1, a_i)(B^{t+|P(a_*^t \rightarrow a_i)-1|}(a_1) - max^t) \\
&\leq max^t + \frac{1}{|A|} In(a_1, a_i) \left(\left(max^t + \frac{1}{|A|} In(a_2, a_1)(B^{t+|P(a_*^t \rightarrow a_i)-2|}(a_2) - max^t) \right) - max^t \right) \\
&= max^t + \frac{1}{|A|} In(a_1, a_i) \left(\frac{1}{|A|} In(a_2, a_1)(B^{t+|P(a_*^t \rightarrow a_i)-2|}(a_2) - max^t) \right) \\
&= max^t + \frac{1}{|A|^2} In(a_2, a_1) In(a_1, a_i)(B^{t+|P(a_*^t \rightarrow a_i)-2|}(a_2) - max^t) \\
&\leq max^t + \frac{1}{|A|^2} In(a_2, a_1) In(a_1, a_i) \times \\
&\quad \left(\left(max^t + \frac{1}{|A|} In(a_3, a_2) \left(B^{t+|P(a_*^t \rightarrow a_i)|-3}(a_3) - max^t \right) \right) - max^t \right) \\
&= max^t + \frac{1}{|A|^2} In(a_2, a_1) In(a_1, a_i) \left(\frac{1}{|A|} In(a_3, a_2) \left(B^{t+|P(a_*^t \rightarrow a_i)|-3}(a_3) \right) - max^t \right) \\
&= max^t + \frac{1}{|A|^3} In(a_3, a_2) In(a_2, a_1) In(a_1, a_i) \left(B^{t+|P(a_*^t \rightarrow a_i)|-3}(a_3) - max^t \right)
\end{aligned}$$

We can see a pattern forming and this pattern will continue throughout time. Denoting P_{In} the product of the influences in the simple path ($P_{In} = I(a_*^t, a_{|P(a_*^t, a_i)|}) \times ... \times I(a_1, a_i)$), we can write the general version of the inequality above as:

$$\begin{aligned}
B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) &\leq max^t + \frac{P_{In}}{|A|^{|P(a_*^t \rightarrow a_i)|}} . (Bel_p^t(a_*^t) - max^t) \\
&= max^t + \frac{P_{In}}{|A|^{|P(a_*^t \rightarrow a_i)|}} . (min^t - max^t)
\end{aligned}$$

The rightmost term in the equation is either equal to or smaller than 0 thus, to make the inequality hold for all a_i's, we shall substitute P_{In} by the smallest value possible.

By the Definition 7, I_{min} is the smallest positive influence in the graph and according to Definition 5 the influences in a simple path are positive. Thus:

$$B^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t + \left(\frac{I_{min}}{|A|}\right)^{|P(a_*^t \to a_i)|} .(min^t - max^t)$$

According to Corollary 2, the maximum value of min^t is L and the minimum value of max^t is U, thus:

$$\begin{aligned} B^{t+|P(a_*^t \to a_i)|}(a_i) &\leq max^t + \left(\frac{I_{min}}{|A|}\right)^{|P(a_*^t \to a_i)|} .(L - U) \\ &\leq max^t - \left(\frac{I_{min}}{|A|}\right)^{|P(a_*^t \to a_i)|} .(U - L) \\ &\leq max^t - \delta^t \end{aligned}$$

□

Lemma 4.

$$\sum_{a_j \in A} I(a_j, a_i) \left(B^t(a_j) - B^t(a_i)\right) = \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i) \left(B^t(a_j) - B^t(a_i)\right)$$

Proof.

$$\begin{aligned} &\sum_{a_j \in A} I(a_j, a_i) \left(B^t(a_j) - B^t(a_i)\right) \\ &= \sum_{a_j \in A \setminus \{a_i\}} \left(I(a_j, a_i) \left(B^t(a_j) - B^t(a_i)\right)\right) + In(a_i, a_i)(B^t(a_i) - B^t(a_i)) \\ &= \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i) \left(B^t(a_j) - B^t(a_i)\right) \end{aligned}$$

□

Lemma 5. *If $B^{t+n}(a_i) \leq max^t - \gamma$, $\gamma \geq 0$ and $n \geq 0$, then $B^{t+n+1}(a_i) \leq max^t - \frac{\gamma}{|A|}$.*

Proof.

$$\begin{aligned} B^{t+n+1}(a_i) &= B^{t+n}(a_i) + \frac{1}{|A|} \sum_{a_j \in A} I(a_j, a_i) \left(B^{t+n}(a_j) - B^{t+n}(a_i)\right) \\ &= B^{t+n}(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i) \left(B^{t+n}(a_j) - B^{t+n}(a_i)\right) && \text{(Lemma 4)} \\ &\leq max^t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i) \left(B^{t+n}(a_j) - max^t + \gamma\right) && \text{(Lemma 2)} \\ &\leq max^t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i) \left(max^t - max^t + \gamma\right) \end{aligned}$$

$$
\begin{aligned}
&= max^t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i)\,(\gamma) \\
&\leq max^t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} (\gamma) \\
&= max^t - \gamma + \frac{(|A| - 1)(\gamma)}{|A|} \\
&= max^t + \frac{(\gamma)((-|A|) + (|A| - 1))}{|A|} \\
&= max^t - \frac{\gamma}{|A|}
\end{aligned}
$$

□

Theorem 2. *$\forall a_i \in A :\ B^{t+|A|-1}(a_i) \leq max^t - \epsilon$, with $\epsilon = \left(\frac{I_{min}}{|A|}\right)^{|A|-1}.(U - L)$.*

Proof. Keeping the notation of Theorem 1, let's call a_*^t one agent that holds the belief min^t in the time t.

Note that, if $|P(a_*^t \to a_i)| = |A| - 1$, our theorem is true by Theorem 1 and we nothing to prove.

Else if $|P(a_*^t \to a_i)| \neq |A| - 1$, then $|P(a_*^t \to a_i)| < |A| - 1$ according to Corollary 3.

According to Theorem 1:

$$B^{t+|P(a_*^t \to a_i)|}(a_i) \leq max^t - \left(\frac{I_{min}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U - L)$$

To keep things simple let's keep the notation from Theorem 1 and call:

$$\delta^t = \left(\frac{I_{min}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U - L)$$

.
Now it is easy to see that we can apply Lemma 5 successively:

$$
\begin{gathered}
B^{t+|P(a_*^t \to a_i)|+1}(a_i) \leq max^t - \frac{\delta^t}{|A|} \\
\Downarrow \\
B^{t+|P(a_*^t \to a_i)|+2}(a_i) \leq max^t - \frac{\delta^t}{|A|^2} \\
\Downarrow \\
B^{t+|P(a_*^t \to a_i)|+3}(a_i) \leq max^t - \frac{\delta^t}{|A|^3}
\end{gathered}
$$

If we do it $|A| - |P(a_*^t \to a_i)| - 1$ times we get:

$$B^{t+|P(a_*^t \to a_i)|+|A|-|P(a_*^t \to a_i)|-1}(a_i) \leq max^t - \frac{\delta^t}{|A|^{|A|-|P(a_*^t \to a_i)|-1}}$$

$$\Downarrow$$

$$\begin{aligned}
B^{t+|A|-1}(a_i) &\leq max^t - \frac{\delta^t}{|A|^{|A|-|P(a_*^t \to a_i)|-1}} \\
&\leq max^t - \frac{\left(\frac{I_{min}}{|A|}\right)^{|P(a_*^t \to a_i)|}.(U-L)}{|A|^{|A|-|P(a_*^t \to a_i)|-1}} \\
&\leq max^t - \frac{I_{min}^{|P(a_*^t \to a_i)|}.(U-L)}{|A|^{|A|-1}} \\
&\leq max^t - \left(\frac{I_{min}}{|A|}\right)^{|A|-1}.(U-L) \\
&\leq max^t - \epsilon
\end{aligned}$$

□

Corollary 4. $max^{t+|A|-1} \leq max^t - \epsilon$

Proof. Since $max^{t+|A|-1}$ is one of the beliefs in the time $t + |A| - 1$ and, according to Theorem 2 all of them are smaller than max^t by a factor of at least ϵ, $max^{t+|A|-1}$ must also be smaller than max^t by a factor of at least ϵ. □

Theorem 3. $\lim_{t\to\infty} max^t = U = \lim_{t\to\infty} min^t = L$

Proof. Suppose, by contradiction, that $U \neq L$. Plugging this values into the ϵ formula show us that $\epsilon > 0$.

Let's assume we did $v = (|A| - 1)\left(\lceil \frac{1}{\epsilon} \rceil + 1\right)$ time steps after $t = 0$. Since max diminishes by at least ϵ at each $|A| - 1$ steps:

$$max^0 \geq max^v + \epsilon \left(\left\lceil \frac{1}{\epsilon} \right\rceil + 1\right)$$

Since $\epsilon.\left(\lceil \frac{1}{\epsilon} \rceil + 1\right) > 1$ and $0 \leq max^v \leq 1$, this would imply that $max^0 \geq 1$ contradicting the definition of belief!

Since assuming that $U \neq L$ led us to a contradiction: $U = L$. □

Theorem 4. $\forall a_i, a_j \in A, \lim\limits_{t\to\infty} B^t(a_i) = \lim\limits_{t\to\infty} B^t(a_j)$

Proof. Since $L \leq \lim\limits_{t\to\infty} B^t(a_i) \leq U$ and $L = U$: $L = B^t(a_i) = U$. And the same can be showed for $B^t(a_j)$. □