Proof of Individual Belief Convergence Constant Clique Graph Using Classic Update

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Definition 1. $Bel_p^{t+1}(a_i|a_j) = Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^{t+1}(a_j) - Bel_p^t(a_i))$.

Definition 2. $Bel_p^{t+1}(a_i) = \frac{1}{|A|}\sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j)$.

Definition 3. A constant clique influence graph $In^{constant-clique}$ representing an idealized totally connected social network in which every agent exerts considerable influence on every other agent: $In(a_i,a_j) = c$, $(0 < c \leq 1)$, for every i, j.

Since in a constant clique the influence is constant (we called this constant c), we can write:

$$Bel_p(a_{t+1}|a_i)j = Bel_p^t(a_i) + c\ (Bel_p^t(a_j) - Bel_p^t(a_i)).$$

Thus:

$$Bel_p^{t+1}(a_i) = \frac{1}{|A|}\sum_{a_j \in A}\left(Bel_p^t(a_i) + c\left(Bel_p^t(a_j) - Bel_p^t(a_i)\right)\right).$$

We can then separate the summation and write it as:

$$Bel_p^{t+1}(a_i) = \frac{1}{|A|}\left(\sum_{a_j \in A} Bel_p^t(a_i) + \sum_{a_j \in A} c\left(Bel_p^t(a_j) - Bel_p^t(a_i)\right)\right).$$

In the first summation, a_i is independent of a_j. Since there are $|A|$ a_j's:

$$\begin{aligned}
Bel_p^{t+1}(a_i) &= \frac{1}{|A|}\ |A| \times Bel_p^t(a_i) + \frac{1}{|A|}\ \sum_{a_j \in A} c\ (Bel_p^t(a_j) - Bel_p^t(a_i)). \\
&= Bel_p^t(a_i) + \frac{1}{|A|}\ \sum_{a_j \in A} c\ (Bel_p^t(a_j) - Bel_p^t(a_i)). \\
&= Bel_p^t(a_i) + \frac{c}{|A|}\ \sum_{a_j \in A}(Bel_p^t(a_j) - Bel_p^t(a_i)).
\end{aligned}$$

Separating the summation again:

$$Bel_p^{t+1}(a_i) = Bel_p^t(a_i) + \frac{c}{|A|} \left(\sum_{a_j \in A} Bel_p^t(a_j) - \sum_{a_j \in A} Bel_p^t(a_i) \right).$$

From this we can see in the second summation that the terms are independent of the indices, thus:

$$\begin{aligned}
Bel_p^{t+1}(a_i) &= Bel_p^t(a_i) + \frac{c}{|A|} \left(\sum_{a_j \in A} Bel_p^t(a_j) - |A| \times Bel_p^t(a_i) \right) \\
&= Bel_p^t(a_i) + \frac{c}{|A|} \left(\sum_{a_j \in A} Bel_p^t(a_j) \right) - \frac{c}{|A|} \left(|A| \times Bel_p^t(a_i) \right) \\
&= Bel_p^t(a_i) - c \times Bel_p^t(a_i) + \frac{c}{|A|} \left(\sum_{a_j \in A} Bel_p^t(a_j) \right) \\
&= (1 - c) \times Bel_p^t(a_i) + \frac{c}{|A|} \left(\sum_{a_j \in A} Bel_p^t(a_j) \right)
\end{aligned}$$

Now that we simplified $Bel_p^{t+1}(a_i)$, we will show that the summation of all beliefs keeps constant throughout all t's:

$$\sum_{a_i \in A} Bel_p^{t+1}(a_i) = \sum_{a_i \in A} \left((1 - c) \times Bel_p^t(a_i) + \frac{c}{|A|} \left(\sum_{a_j \in A} Bel_p^t(a_j) \right) \right)$$

It is clear that the summation of all beliefs keeps constant in a time stamp t, thus let's call this summation S_t:

Lemma 1. *Given a constant clique influence graph with S_t as the sum of the beliefs in the "t'th moment.*

$$S_{t+1} = S_t$$

Proof.

$$\begin{aligned}
\sum_{a_i \in A} Bel_p^{t+1}(a_i) &= \sum_{a_i \in A} \left((1 - c) \times Bel_p^t(a_i) + \frac{c}{|A|} S_t \right) \\
&= \sum_{a_i \in A} \left((1 - c) \times Bel_p^t(a_i) \right) + \sum_{a_i \in A} \left(\frac{c}{|A|} S_t \right) \\
&= (1 - c) \times \sum_{a_i \in A} Bel_p^t(a_i) + \sum_{a_i \in A} \left(\frac{c}{|A|} S_t \right)
\end{aligned}$$

By definition of S_t, the we can write:

$$\sum_{a_i \in A} Bel_p^{t+1}(a_i) = (1-c)\ S_t + \sum_{a_i \in A} \left(\frac{c}{|A|}\ S_t \right)$$

Since the second summation is independent of a_i:

$$\begin{aligned}\sum_{a_i \in A} Bel_p^{t+1}(a_i) &= (1-c)\ S_t + |A| \frac{c}{|A|}\ S_t \\ &= (1-c)\ S_t + c\ S_t \\ \sum_{a_i \in A} Bel_p^{t+1}(a_i) &= S_t\end{aligned}$$

Now we showed that, for every t, $S_{t+1} = S_t$, we can used induction to show our hypothesis. □

Theorem 1. *In a constant clique with constant influence c, $(0 < c \leq 1)$, all agents believes converge to the same value:* $\frac{S_t}{|A|}$

Proof. As we showed before $Bel_p^{t+1}(a_i)$ can be written as:

$$Bel_p^{t+1}(a_i) = (1-c) \times Bel_p^t(a_i) + \frac{c}{|A|}\ S_t$$

To simplify the notation, we will call $k = 1 - c$ and $C = \frac{c}{|A|} \times S_t$ and $x_t = Bel_p^t(a_i)$, we can do this *w.l.o.g.* because we proved that S_t is constant throughout the time, thus both k and C are constant:

$$Bel_p^{t+1}(a_i) = k \times Bel_p^t(a_i) + C$$

If we expand this, we can see a pattern:

$$\begin{aligned}Bel_p^{t+2}(a_i) &= k\ x_{t+1} + C \\ &= k\ (k\ x_t + C) + C \\ &= k^2\ x_t + k\ C + C \\ &\Downarrow \\ Bel_p^{t+3}(a_i) &= k\ x_{t+2} + C \\ &= k\ (k^2\ x_t + k\ C + C) + C \\ &= k^3\ x_t + k^2\ C + k\ C + C\end{aligned}$$

Generically we can write $Bel_p^{t+n}(a_i)$ as:

$$\begin{aligned}Bel_p^{t+n}(a_i) &= k^n\ x_t + \sum_{i=0}^{i=n-1} (k^i\ C) \\ &= k^n\ x_t + C \times \sum_{i=0}^{i=n-1} k^i\end{aligned}$$

Plugging $t = 0$ we get a formula for the value of $Bel_p^n(a_i)$ through time:

$$
\begin{aligned}
Bel_p^n(a_i) &= k^n \; x_0 + \sum_{i=0}^{i=n-1} (k^i \; C) \\
&= k^n \; x_0 + C \times \sum_{i=0}^{i=n-1} k^i
\end{aligned}
$$

As $n \to \infty$, k^n clearly goes to 0, since $k = 1 - c$ and $0 < c \leq 1$. Thus:

$$
\begin{aligned}
\lim_{n\to\infty} Bel_p^n(a_i) &= \lim_{n\to\infty} \left(C \times \sum_{i=0}^{i=n-1} k^i \right) \\
&= C \times \lim_{n\to\infty} \sum_{i=0}^{i=n-1} k^i
\end{aligned}
$$

This summation is a very known series, the geometric series, since $0 \leq k < 1$, and we know this result equals:

$$
\lim_{n\to\infty} Bel_p^n(a_i) = C \; \frac{1}{1-k}
$$

Which by the definition of C and k equals:

$$
\begin{aligned}
\lim_{n\to\infty} Bel_p^n(a_i) &= \left(\frac{c}{|A|} \; S \right) \; \left(\frac{1}{c} \right) \\
&= \frac{S}{|A|}
\end{aligned}
$$

Since a_i is arbitrary, all a_i's converge for the same value, as we wanted to prove. $\square$