Proof of Individual Agent Opinion Convergence in a Strongly Connected Influence Graph Using Classic Update Function

Bernardo T. Amorim
bernardoamorim@dcc.ufmg.br

March 2020

In the classic update function, $Bel_p^{t+1}(a_i|a_j)$ can be written in the following form:

Definition 1 $Bel_p^{t+1}(a_i|a_j) = Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))$.

And the classic update function, $Bel_p^{t+1}(a_i)$ is written as:

Definition 2 $Bel_p^{t+1}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} Bel_p^t(a_i|a_j)$.

And let's define a strongly connected graph as:

Definition 3 *A strongly connected influence graph social network in which every agent exerts influence on every other agent:* $In(a_i,a_j) > 0$, *for every* i, j.

Definition 4 max_t *and* min_t *are the maximum and minimum belief values in a given instant* t*, respectively.*

To prove our conjecture, let's do some simplifications:

$$
\begin{aligned}
Bel_p^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} Bel_p^t(a_i|a_j). \\
&= \frac{1}{|A|} \sum_{a_j \in A} \left(Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))\right) \\
&= Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))
\end{aligned}
$$

Since belief values are finite, by the well-ordering principle we always have a min_t and a max_t. It is easy to see, by the squeeze theorem, that individual agent opinion converges to the same value if and only if $lim_{t\to\infty}\ min_t = lim_{t\to\infty}\ max_t$.

Thus, since we want to prove that it always converges, if $min_t = max_t$ we have nothing to prove, so assume $min_t \neq max_t$.

Lemma 1 *In a strongly connected influence graph and a classic update function if $min_t \neq max_t$, then $max_{t+1} < max_t$ and $min_{t+1} > min_t$.*

Proof of Lemma 1

Suppose $Bel_p^t(a_i) = max_t$. Thus:

$$\begin{aligned} Bel_p^{t+1}(a_i) &= Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \\ &= max_t + \frac{1}{|A|}\sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - max_t) \end{aligned}$$

Since max_t is, by definition, the biggest element of the set of all beliefs in the instant t, then $Bel_p^t(a_j) - max_t \leq 0$ for every a_j. This implies that:

$$In(a_j, a_i)(Bel_p^t(a_j) - max_t) \leq 0$$

Since $In(a_j, a_i) > 0$.

Given that we assumed that $min_t \neq max_t$, there exits at least one a_j, such that $Bel_p^t(a_j) \neq max_t$, thus, since all influence are positive:

$$\frac{1}{|A|}\sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - max_t) < 0$$

Thus $Bel_p^{t+1}(a_i) = Bel_p^t(a_i)$ plus a negative number, which implies that

$$Bel_p^{t+1}(a_i) < Bel_p^t(a_i)$$

Since a_i was arbitrary this is true for all agents that had $Bel_p^t(a_i) = max_t$, thus:

$$max_{t+1} < max_t$$

$\square$

And exactly the same reasoning can be used to show that under this conditions $min_{t+1} > min_t$, completing our proof.

Now that we showed that min and max are monotonic, since they are bounded between 0 and 1 by the definition of belief, we gain a corollary by the monotone convergence theorem:

Corollary 1 *$lim_{t\to\infty} max_t = L$ and $lim_{t\to\infty} min_t = M$ for some L, $M \in \mathbb{R}$.*

With this, we can finally get to our theorem:

Theorem 1 *$lim_{t\to\infty} max_t = lim_{t\to\infty} Bel_p^t(a_j)$ for every a_j*

Let's define $lim_{t\to\infty} max_t = L$. Since we showed max_i converges, for some a_i:

$$lim_{t\to\infty} Bel_p^t(a_i) = lim_{t\to\infty} Bel_p^{t+1}(a_i) = lim_{t\to\infty} max_t = L$$

Which implies that:

$$\begin{aligned} L &= lim_{t\to\infty}\ Bel_p^t(a_i) + \frac{1}{|A|}\sum_{a_j\in A} In(a_j,a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \\ &= lim_{t\to\infty}\ max_t + \frac{1}{|A|}\sum_{a_j\in A} In(a_j,a_i)(Bel_p^t(a_j) - max_t) \\ &= L + lim_{t\to\infty}\ \frac{1}{|A|}\sum_{a_j\in A} In(a_j,a_i)(Bel_p^t(a_j) - max_t) \end{aligned}$$

Which shows that:

$$lim_{t\to\infty}\ \frac{1}{|A|}\sum_{a_j\in A} In(a_j,a_i)(Bel_p^t(a_j) - max_t) = 0$$

Since $Bel_p^t(a_j) \leq max_t$ by definition, this means that, for this to be equal to 0, $Bel_p^t(a_j) = max_t$ for every a_j, thus all beliefs converge, as we wanted to show.

$\square$