Proof of Belief Convergence

Scratch of Generalization for Weakly Connected Graphs

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Influencers definition

Definition

Let's call I_i the influencers of the agent a_i, which is defined as:

$$\{a_j \in A \mid \exists\ P(a_j|a_i)\ \}$$

An useful lemma

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Proof.

Let's suppose that $\exists P(a_j|a_i)$.
Now suppose $a_k \in I_j$. By definition this means that $\exists P(a_k|a_j)$. But since $\exists P(a_j|a_i)$, there will also exist $P(a_k|a_i)$. Thus, by definition of influencers $a_k \in I_i$. Since a_k was arbitrary, $I_j \subseteq I_i$.
Proving the other side of the implication is trivial given the definition of influencers. $\square$

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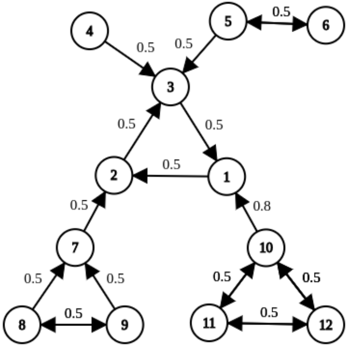
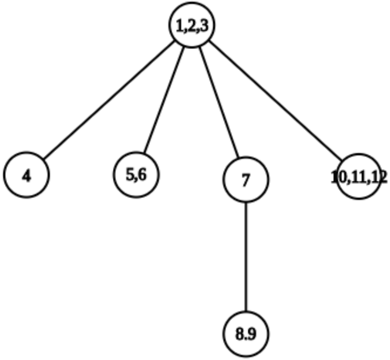


Partial order representation

Drawing a partial order representation of the influencers of the previous graph we end up with:



Lemma

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Proof.

Suppose $a_j, a_k \in I_i$. Thus $\exists P(a_k|a_i)$ and $\exists P(a_j|a_i)$.
Since I_i is minimal, there must exist $P(a_i|a_j)$ and $P(a_i|a_k)$. This is true because, according to Lemma 1, $I_j \subseteq I_i$ and $I_k \subseteq I_i$. But if $\nexists P(a_i|a_j)$ or $\nexists P(a_i|a_k)$ then either $I_j \subset \; N_i$ or $I_k \subset I_i$. Which contradicts our hypothesis. Since $\exists P(a_k|a_i) \wedge \exists P(a_i|a_j)$, $\exists P(a_k|a_j)$. And the same goes the other way around, since both a_j and a_k where arbitrary, it is true for all, which means there is a path between any two agents in the influencers. Does it is strongly connected. □

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- Proving the statement above we could prove convergence for the whole graph, since it can be proved that the SCC graph is a DAG and, thus, it has a topological sort (Cormen, 3rd edition in portuguese, pg 449). Using the topological sort we can prove convergence without any problems.

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- Also, the simulation I made in the graph showed above actually resulted in the fact that changing the beliefs of agents who aren't from minimal influencers do not change the beliefs in the limit.
- What I believe now is that the elements who are not minimal converge to the weighted average of all paths from elements in minimal influencers multiplied by its beliefs.