

# Proof of Individual Agent Opinion Convergence in a Weakly Connected Influence Graph Using Classic Update Function

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**Definition 1.** The *classic update-function*, is defined as:

$$B^{t+1}(a_i|a_j) = B^t(a_i) + I(a_j, a_i)(B^t(a_j) - B^t(a_i)) \quad (1)$$

**Definition 2.** While the *overall classic update*, is defined as:

$$B^{t+1}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} B^{t+1}(a_i|a_j) \quad (2)$$

**Definition 3.** We say a influence graph  $\text{In}$  over agents  $A$  is *weakly connected* if for all  $a_i, a_j \in A$ , there exist  $a_{k_1}, a_{k_2}, \dots, a_{k_l} \subseteq A$  such that  $\text{In}(a_i, a_{k_1}) > 0$ ,  $\text{In}(a_{k_l}, a_j) > 0$ , and for  $m = 1, \dots, l - 1$ ,  $\text{In}(a_{k_m}, a_{k_{m+1}}) > 0$ .

**Definition 4.**  $\max^t$  and  $\min^t$  are the maximum and minimum belief values in a given instant  $t$ , respectively. Thus:

$$\min^t = \min_{a_i \in A} B^t(a_i) \text{ and } \max^t = \max_{a_i \in A} B^t(a_i).$$

To prove our conjecture, let's do some simplifications:

$$\begin{aligned} B^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} B^{t+1}(a_i|a_j). \\ &= \frac{1}{|A|} \sum_{a_j \in A} (B^t(a_i) + I(a_j, a_i)(B^t(a_j) - B^t(a_i))) \\ &= B^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} I(a_j, a_i)(B^t(a_j) - B^t(a_i)) \end{aligned} \quad (3)$$

Since we have a finite number of beliefs and  $\forall a_i \in A : B^t(a_i) \in [0, 1]$ , there are always  $\min^t$  and a  $\max^t$ . We shall also note that, by the Squeeze Theorem, individual agent opinion converges to the same value if and only if  $\lim_{t \rightarrow \infty} \min^t = \lim_{t \rightarrow \infty} \max^t$ .

Thus, since we want to prove that it always converges, if  $\min^t = \max^t$  we have nothing to prove, so assume  $\min^t \neq \max^t$ .

**Lemma 1.** *In a weakly connected graph and under classic belief update:*

$$\forall t \text{ and } \forall a_i \in A : \min^t \leq B^{t+1}(a_i) \leq \max^t$$

*Proof.* By the equation 3:

$$B^{t+1}(a_i) = B^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} I(a_j, a_i)(B^t(a_j) - B^t(a_i))$$

Trying to maximize the right-hand side, we can substitute  $B^t(a_j)$  by  $\max^t$ , this turns our equation into an inequality, since  $\forall a_j \in A, B^t(a_j) \leq \max^t$ , by the definition of  $\max^t$ . That makes the terms inside the summation either equal or smaller than 0, thus:

$$\begin{aligned} B^{t+1}(a_i) &\leq B^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} I(a_j, a_i)(\max^t - B^t(a_i)) \\ &\leq B^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} (\max^t - B^t(a_i)) && (\text{since } I(a_j, a_i) \leq 1 \text{ and } \max^t - B^t(a_i) \geq 0) \\ &= B^t(a_i) + \frac{|A|}{|A|}(\max^t - B^t(a_i)) \\ &= B^t(a_i) + \max^t - B^t(a_i) \\ &= \max^t \end{aligned} \tag{4}$$

Since  $a_i$  was arbitrary, the Lemma is true for all agents. The same reasoning can be used to show the equivalent property for  $\min^t$   $\square$

**Corollary 1.** *In a weakly connected influence graph and a classic update function,  $\max^{t+1} \leq \max^t$  and  $\min^{t+1} \geq \min^t$  for all  $t$ .*

*Proof.* The result of Lemma 1 tells us that all beliefs in the time  $t + 1$  are either smaller than  $\max^t$  or equal to  $\max^t$ , thus, since  $\max^{t+1}$  must be one of those elements,  $\max^{t+1} \leq \max^t$ . And the same reasoning can be used for  $\min^t$ .  $\square$

**Corollary 2.**  $\lim_{t \rightarrow \infty} \max^t = U$  and  $\lim_{t \rightarrow \infty} \min^t = L$  for some  $U, L \in [0, 1]$ .

*Proof.* Since both  $\max^t$  and  $\min^t$  are bounded between 0 and 1 by the definition of belief; and Lemma 1 showed us that they are monotonic, according to the Monotonic Convergence Theorem, the limits exist.  $\square$

Now that we have those properties, our proof will follow by showing that an agent  $a_i$  that holds some belief  $B^t(a_i)$  will influence every other agent by the time  $t + |A| - 1$ . To see this, we must open the definition of belief throughout time. But before we do this, let's jump into some small definitions and corollaries that will help us on the way.

**Definition 5.** Let's call the sequence  $P(a_i \rightarrow a_j) = (a_i, a_k, \dots, a_{k+l}, a_j)$  a *simple path* from  $a_i$  to  $a_j$ , if:

- All elements on the sequence are different.
- The first element in the sequence is  $a_i$ .
- The last element in the sequence is  $a_j$ .
- If  $a_n$  is the  $n$ 'th element in the sequence, if it has a successor  $a_{n+1}$ ,  $In(a_n, a_{n+1}) > 0$ .

Note that many simple paths from  $a_i$  to  $a_j$  can exist, although our notation isn't enough to differentiate them. But in subsequent steps we will only need one of those simple paths, so the notation shouldn't be a problem.

**Definition 6.** Let's denote by  $|P(a_i \rightarrow a_j)|$  the size of a simple path from  $a_i$  to  $a_j$ , which we define as the number of elements in the sequence  $P(a_i \rightarrow a_j) - 1$ .

**Corollary 3.**  $\forall P(a_i \rightarrow a_j), |P(a_i \rightarrow a_j)| \leq |A| - 1$ .

*Proof.* This follows directly from the definition of a simple path. Since it doesn't have repeated elements and we have  $|A|$  agents, the simple path can't have more than  $|A|$  elements, since the size of a simple path is defined as the number of elements minus one, the maximum size is  $|A| - 1$ .  $\square$

**Lemma 2.**  $\forall x, \forall t$  and  $\forall a_i$ , if  $B^t(a_i) \leq x$ :

$$B^{t+1}(a_i) \leq x + \frac{1}{|A|} \sum_{a_j \in A} (I(a_j, a_i) (B^t(a_j) - x))$$

*Proof.*

$$\begin{aligned} B^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} B^{t+1}(a_i | a_j) \\ &= \frac{1}{|A|} \sum_{a_j \in A} (B^t(a_i) + I(a_j, a_i) (B^t(a_j) - B^t(a_i))) \\ &= \frac{1}{|A|} \sum_{a_j \in A} (B^t(a_i)(1 - I(a_j, a_i)) + I(a_j, a_i)B^t(a_j)) \\ &\leq \frac{1}{|A|} \sum_{a_j \in A} (x \cdot (1 - I(a_j, a_i)) + I(a_j, a_i)B^t(a_j)) \\ &= x + \frac{1}{|A|} \sum_{a_j \in A} (I(a_j, a_i) (B^t(a_j) - x)) \end{aligned}$$

$\square$

**Lemma 3.**  $\forall a_i, a_k \in A$  and  $\forall n \geq 1$  and  $\forall t$ :

$$B^{t+n}(a_i) \leq \max^t + \frac{1}{|A|} (I(a_k, a_i)(B^{t+n-1}(a_k) - \max^t)) \quad (5)$$

*Proof.* By the Definitions 1 and 2:

$$B^{t+n}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} B^{t+n}(a_i|a_j)$$

$$B^{t+n}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} (B^{t+n-1}(a_i) + I(a_j, a_i)(B^{t+n-1}(a_j) - B^{t+n-1}(a_i)))$$

Since  $B^{t+n}(a_i) \leq \max^{t+n} \leq \max^{t+n-1}$  according to Corollary 1, we can use Lemma 2:

$$B^{t+n}(a_i) \leq \frac{1}{|A|} \sum_{a_j \in A} (\max^{t+n-1} + I(a_j, a_i)(B^{t+n-1}(a_j) - \max^{t+n-1}))$$

$$= \max^{t+n-1} + \frac{1}{|A|} \sum_{a_j \in A} I(a_j, a_i)(B^{t+n-1}(a_j) - \max^{t+n-1})$$

To make our Lemma useful in future manipulations, we will take an arbitrary element  $a_k$  out of the summation:

$$B^{t+n}(a_i) \leq \max^{t+n-1} + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_k\}} (I(a_j, a_i)(B^{t+n-1}(a_j) - \max^{t+n-1}))$$

$$+ \frac{1}{|A|} (I(a_k, a_i)(B^{t+n-1}(a_k) - \max^{t+n-1}))$$

Since  $\max^{t+n-1}$  is the greatest belief possible in that time step, the summation can be at most 0, thus:

$$B^{t+n}(a_i) \leq \max^{t+n-1} + \frac{1}{|A|} I(a_k, a_i) (B^{t+n-1}(a_k) - \max^{t+n-1})$$

Since  $\max$  doesn't increase throughout time,  $\max^{t+n-1} \leq \max^t$ , thus:

$$B^{t+n}(a_i) \leq \max^t + \frac{1}{|A|} I(a_k, a_i) (B^{t+n-1}(a_k) - \max^t)$$

□

**Definition 7.** Let's denote by  $I_{min}$  the smallest influence that's different from 0 in the influence graph.

Using the same notation we used in Corollary 2, let's call  $\lim_{t \rightarrow \infty} \max^t = U$  and  $\lim_{t \rightarrow \infty} \min^t = L$ .

Now that we have all of these tools, let's jump to Theorem 1 which will be a tool in the most important part of the proof. Calling  $a_*^t$  one agent who holds the belief  $\min^t$  in the time  $t$ :

**Theorem 1.**  $\forall t$  and  $\forall a_i \in A$  :

$$B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) \leq \max^t - \delta^t, \text{ with } \delta^t = \left(\frac{I_{\min}}{|A|}\right)^{|P(a_*^t \rightarrow a_i)|} \cdot (U - L).$$

*Proof.* By equation 3:

$$B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) = Bel_p^{t+|P(a_*^t \rightarrow a_i)|-1}(a_i) + \frac{1}{|A|} \sum_{a_j \in A} B^{t+|P(a_*^t \rightarrow a_i)|-1}(a_i|a_j)$$

What we will do now is separate, at each step, one element of the summation and apply Lemma 3 to modify our inequality. But we will be careful when choosing the elements we separate from the summation. We will separate from the summation the elements in  $P(a_*^t \rightarrow a_i)$ , starting from the end of the simple path until we get to  $a_*^t$ . To simplify our notation, let's index the elements in the simple path from  $a_*^t$  to  $a_i$ , starting from the end of the simple path (since we are backtracking it will make more sense) by calling  $a_n$  the  $n^{th}$  element from the end to the beginning of the sequence (excluding  $a_i$  itself). Thus, by Lemma 3:

$$B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) \leq \max^t + \frac{1}{|A|} In(a_1, a_i)(B^{t+|P(a_*^t \rightarrow a_i)|-1}(a_1) - \max^t)$$

Note, that if  $|P(a_*^t, a_i)| = 1$ , we could prove our result. Instead of showing it I will expand this two more times, show the general version and then prove the Theorem for all cases. Using Lemma 3 again:

$$\begin{aligned} & B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) \\ & \leq \max^t + \frac{1}{|A|} In(a_1, a_i)(B^{t+|P(a_*^t \rightarrow a_i)|-1}(a_1) - \max^t) \\ & \leq \max^t + \frac{1}{|A|} In(a_1, a_i) \left( \left( \max^t + \frac{1}{|A|} In(a_2, a_1)(B^{t+|P(a_*^t \rightarrow a_i)|-2}(a_2) - \max^t) \right) - \max^t \right) \\ & = \max^t + \frac{1}{|A|} In(a_1, a_i) \left( \frac{1}{|A|} In(a_2, a_1)(B^{t+|P(a_*^t \rightarrow a_i)|-2}(a_2) - \max^t) \right) \\ & = \max^t + \frac{1}{|A|^2} In(a_2, a_1) In(a_1, a_i)(B^{t+|P(a_*^t \rightarrow a_i)|-2}(a_2) - \max^t) \\ & \leq \max^t + \frac{1}{|A|^2} In(a_2, a_1) In(a_1, a_i) \times \\ & \quad \left( \left( \max^t + \frac{1}{|A|} In(a_3, a_2) \left( B^{t+|P(a_*^t \rightarrow a_i)|-3}(a_3) - \max^t \right) \right) - \max^t \right) \\ & = \max^t + \frac{1}{|A|^2} In(a_2, a_1) In(a_1, a_i) \left( \frac{1}{|A|} In(a_3, a_2) \left( B^{t+|P(a_*^t \rightarrow a_i)|-3}(a_3) \right) - \max^t \right) \\ & = \max^t + \frac{1}{|A|^3} In(a_3, a_2) In(a_2, a_1) In(a_1, a_i) \left( B^{t+|P(a_*^t \rightarrow a_i)|-3}(a_3) - \max^t \right) \end{aligned}$$

We can see a pattern forming since the equation above has the same form of the one before it, and this pattern will continue throughout time. Now, denoting  $P_{In}$  the product of the influences in the simple path ( $P_{In} = I(a_*^t, a_{|P(a_*^t, a_i)|}) \times \dots \times I(a_1, a_i)$ ), we can write the generalized version of the inequality above as:

$$B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) \leq \max^t + \frac{P_{In}}{|A|^{|P(a_*^t \rightarrow a_i)|}} \cdot (Bel_p^t(a_*^t) - \max^t)$$

$$= max^t + \frac{P_{In}}{|A|^{|P(a_*^t \rightarrow a_i)|}} \cdot (min^t - max^t)$$

This inequality comes from the fact that the simple path ends after  $|P(a_*^t \rightarrow a_i)|$  steps with  $a_*^t$  as the start of the path, and, by definition, the belief of  $a_*^t$  in the time  $t$  is  $min^t$ .

Since the rightmost term in the equation is either equal to or smaller than 0, to make the inequality hold for all  $a_i$ 's, we shall substitute  $P_{In}$  by the smallest value possible. According to Definition 7,  $I_{min}$  is the smallest positive influence in the graph. By the definition of simple path (5) all influences are positive, thus the smallest possible value of  $P_{In}$  is  $I_{min}^{|P(a_*^t \rightarrow a_i)|}$ . Thus:

$$B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) \leq max^t + \left( \frac{I_{min}}{|A|} \right)^{|P(a_*^t \rightarrow a_i)|} \cdot (min^t - max^t)$$

According to Corollary 2, the maximum value of  $min^t$  is  $L$  and the minimum value of  $max^t$  is  $U$ , those are the values we should plug to maintain the inequality:

$$\begin{aligned} B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) &\leq max^t + \left( \frac{I_{min}}{|A|} \right)^{|P(a_*^t \rightarrow a_i)|} \cdot (L - U) \\ B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) &\leq max^t - \left( \frac{I_{min}}{|A|} \right)^{|P(a_*^t \rightarrow a_i)|} \cdot (U - L) \\ B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) &\leq max^t - \delta^t \end{aligned}$$

□

**Lemma 4.**

$$\sum_{a_j \in A} I(a_j, a_i) (B^t(a_j) - B^t(a_i)) = \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i) (B^t(a_j) - B^t(a_i))$$

*Proof.*

$$\begin{aligned} &\sum_{a_j \in A} I(a_j, a_i) (B^t(a_j) - B^t(a_i)) \\ &= \sum_{a_j \in A \setminus \{a_i\}} (I(a_j, a_i) (B^t(a_j) - B^t(a_i))) + In(a_i, a_i) (B^t(a_i) - B^t(a_i)) \\ &= \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i) (B^t(a_j) - B^t(a_i)) \end{aligned}$$

□

**Lemma 5.** If  $B^{t+n}(a_i) \leq max^t - \gamma$ ,  $\gamma \geq 0$  and  $n \geq 0$ , then  $B^{t+n+1}(a_i) \leq max^t - \frac{\gamma}{|A|}$ .

*Proof.*

$$B^{t+n+1}(a_i) = B^{t+n}(a_i) + \frac{1}{|A|} \sum_{a_j \in A} I(a_j, a_i) (B^{t+n}(a_j) - B^{t+n}(a_i))$$

$$\begin{aligned}
&= B^{t+n}(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i) (B^{t+n}(a_j) - B^{t+n}(a_i)) \quad (\text{Lemma 4}) \\
&\leq \max^t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i) (B^{t+n}(a_j) - \max^t + \gamma) \quad (\text{Lemma 2}) \\
&\leq \max^t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i) (\max^t - \max^t + \gamma) \\
&= \max^t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} I(a_j, a_i) (\gamma) \\
&\leq \max^t - \gamma + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_i\}} (\gamma) \\
&= \max^t - \gamma + \frac{(|A| - 1)(\gamma)}{|A|} \\
&= \max^t + \frac{(\gamma)((-|A|) + (|A| - 1))}{|A|} \\
&= \max^t - \frac{\gamma}{|A|}
\end{aligned}$$

□

**Theorem 2.**  $\forall a_i \in A : B^{t+|A|-1}(a_i) \leq \max^t - \epsilon$ , with  $\epsilon = \left(\frac{I_{min}}{|A|}\right)^{|A|-1} \cdot (U - L)$ .

*Proof.* Let's keep the notation of the previous Theorem and call  $a_*^t$  the agent that holds the belief  $\min^t$  in the time  $t$ .

First we should note that, if  $|P(a_*^t \rightarrow a_i)| = |A| - 1$ , our theorem is true by Theorem 1 and we nothing to prove.

Else if  $|P(a_*^t \rightarrow a_i)| \neq |A| - 1$ , then  $|P(a_*^t \rightarrow a_i)| < |A| - 1$  according to Corollary 3. According to Theorem 1:

$$B^{t+|P(a_*^t \rightarrow a_i)|}(a_i) \leq \max^t - \left(\frac{I_{min}}{|A|}\right)^{|P(a_*^t \rightarrow a_i)|} \cdot (U - L)$$

To keep things simple let's keep the notation from Theorem 1 and call:

$$\delta^t = \left(\frac{I_{min}}{|A|}\right)^{|P(a_*^t \rightarrow a_i)|} \cdot (U - L)$$

Now it is easy to see that we can apply Lemma 5 successively:

$$\begin{aligned}
B^{t+|P(a_*^t \rightarrow a_i)|+1}(a_i) &\leq \max^t - \frac{\delta^t}{|A|} \\
&\Downarrow \\
B^{t+|P(a_*^t \rightarrow a_i)|+2}(a_i) &\leq \max^t - \frac{\delta^t}{|A|^2} \\
&\Downarrow
\end{aligned}$$

$$B^{t+|P(a_*^t \rightarrow a_i)|+3}(a_i) \leq max^t - \frac{\delta^t}{|A|^3}$$

If we do it  $|A| - |P(a_*^t \rightarrow a_i)| - 1$  times we get:

$$\begin{aligned} B^{t+|P(a_*^t \rightarrow a_i)|+|A|-|P(a_*^t \rightarrow a_i)|-1}(a_i) &\leq max^t - \frac{\delta^t}{|A||A|-|P(a_*^t \rightarrow a_i)|-1} \\ B^{t+|A|-1}(a_i) &\leq max^t - \frac{\delta^t}{|A||A|-|P(a_*^t \rightarrow a_i)|-1} \\ B^{t+|A|-1}(a_i) &\leq max^t - \frac{\left(\frac{I_{min}}{|A|}\right)^{|P(a_*^t \rightarrow a_i)|} \cdot (U - L)}{|A||A|-|P(a_*^t \rightarrow a_i)|-1} \\ B^{t+|A|-1}(a_i) &\leq max^t - \frac{I_{min}^{|P(a_*^t \rightarrow a_i)|} \cdot (U - L)}{|A||A|-1} \\ B^{t+|A|-1}(a_i) &\leq max^t - \left(\frac{I_{min}}{|A|}\right)^{|A|-1} \cdot (U - L) \\ B^{t+|A|-1}(a_i) &\leq max^t - \epsilon \end{aligned}$$

□

**Corollary 4.**  $max^{t+|A|-1} \leq max^t - \epsilon$

*Proof.* Since  $max^{t+|A|-1}$  must be one of the beliefs in the time  $t+|A|-1$  and, according to Theorem 2 all of them are smaller than  $max^t$  by a factor of at least  $\epsilon$ ,  $max^{t+|A|-1}$  must also be smaller than  $max^t$  by a factor of at least  $\epsilon$ . □

**Theorem 3.**  $\lim_{t \rightarrow \infty} max^t = U = \lim_{t \rightarrow \infty} min^t = L$

*Proof.* Suppose, by contradiction, that  $U \neq L$ . Plugging this values into the  $\epsilon$  formula show us that  $\epsilon > 0$ . Since, according to Theorem 2,  $max^{t+|A|-1}$  is smaller than  $max^t$  by a factor of  $\epsilon$ , we can finally reach to a contradiction and end our proof.

To see this contradiction, let's assume we did  $v = (|A| - 1) \left(\left\lceil \frac{1}{\epsilon} \right\rceil + 1\right)$  time steps after  $t = 0$ . Since  $max$  diminishes by at least  $\epsilon$  at each  $|A| - 1$  steps:

$$\begin{aligned} max^0 &\geq max^v + \epsilon \left( \left\lceil \frac{1}{\epsilon} \right\rceil + 1 \right) \\ max^0 - \epsilon \left( \left\lceil \frac{1}{\epsilon} \right\rceil + 1 \right) &\geq max^v \end{aligned}$$

But  $\epsilon \cdot \left(\left\lceil \frac{1}{\epsilon} \right\rceil + 1\right) > 1$ , thus  $max^0 < \epsilon \cdot \left(\left\lceil \frac{1}{\epsilon} \right\rceil + 1\right)$ . And this would imply that  $max^v < 0$ , which contradicts the definition of belief!

Since assuming that  $U \neq L$  led us to a contradiction we can conclude that  $U = L$ . This result implies that all agents belief converge to the same value, as we wanted to prove. □