Proof of Individual Agent Opinion Convergence in a Clique Using Classic Update Function

Bernardo Teixeira de Amorim Neto
bernardoamorim@dcc.ufmg.br

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1 Introduction

Given the original article's unproven conjectures about the polarization model, this article proves a smaller conjecture: the convergence of individual agent belief in cliques with constant influence using the classic update measure.

2 Definitions

In the classic update function, $Bel_p^{t+1}(a_i|a_j)$ can be written in the following form:

Definition 1 $Bel_p^{t+1}(a_i|a_j) = Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^{t+1}(a_j) - Bel_p^t(a_i))$.

And the classic update function, $Bel_p^{t+1}(a_i)$ is written as:

Definition 2 $Bel_p^{t+1}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j)$.

And let's define a clique as:

Definition 3 *A clique influence graph In^{clique} representing an idealized totally connected social network in which every agent exerts considerable influence on every other agent: $In(a_i,a_j) = c$, $(0 < c \leq 1)$, for every i, j.*

3 Proofs

3.1 Some simplifications

Since in a clique the influence is constant (we called this constant c), we can write:

$$Bel_p^{t+1}(a_i|a_j) = Bel_p^t(a_i) + c\ (Bel_p^t(a_j) - Bel_p^t(a_i)).$$

Thus:

$$Bel_p^{t+1}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} \left(Bel_p^t(a_i) + c\left(Bel_p^t(a_j) - Bel_p^t(a_i)\right)\right).$$

We can then separate the summation and write it as:

$$Bel_p^{t+1}(a_i) = \frac{1}{|A|} \left(\sum_{a_j \in A} Bel_p^t(a_i) + \sum_{a_j \in A} c\left(Bel_p^t(a_j) - Bel_p^t(a_i)\right)\right).$$

In the first summation, a_i is independent of a_j. Since there are $|A|$ a_j's:

$$\begin{aligned}
Bel_p^{t+1}(a_i) &= \frac{1}{|A|}\ |A| \times Bel_p^t(a_i) + \frac{1}{|A|}\ \sum_{a_j \in A} c\ (Bel_p^t(a_j) - Bel_p^t(a_i)). \\
&= Bel_p^t(a_i) + \frac{1}{|A|}\ \sum_{a_j \in A} c\ (Bel_p^t(a_j) - Bel_p^t(a_i)). \\
&= Bel_p^t(a_i) + \frac{c}{|A|}\ \sum_{a_j \in A} (Bel_p^t(a_j) - Bel_p^t(a_i)).
\end{aligned}$$

Separating the summation again:

$$Bel_p^{t+1}(a_i) = Bel_p^t(a_i) + \frac{c}{|A|}\ \left(\sum_{a_j \in A} Bel_p^t(a_j) - \sum_{a_j \in A} Bel_p^t(a_i)\right).$$

From this we can see in the second summation that the terms are independent of the indices, thus:

$$\begin{aligned}
Bel_p^{t+1}(a_i) &= Bel_p^t(a_i) + \frac{c}{|A|}\ \left(\sum_{a_j \in A} Bel_p^t(a_j) - |A| \times Bel_p^t(a_i)\right) \\
&= Bel_p^t(a_i) + \frac{c}{|A|}\ \left(\sum_{a_j \in A} Bel_p^t(a_j)\right) - \frac{c}{|A|}\ \left(|A| \times Bel_p^t(a_i)\right) \\
&= Bel_p^t(a_i) - c \times Bel_p^t(a_i) + \frac{c}{|A|}\ \left(\sum_{a_j \in A} Bel_p^t(a_j)\right) \\
&= (1-c) \times Bel_p^t(a_i) + \frac{c}{|A|}\ \left(\sum_{a_j \in A} Bel_p^t(a_j)\right)
\end{aligned}$$

Now that we simplified $Bel_p^{t+1}(a_i)$, we will show that the summation of all beliefs keeps constant throughout all t's:

$$\sum_{a_i \in A} Bel_p^{t+1}(a_i) = \sum_{a_i \in A} \left((1-c) \times Bel_p^t(a_i) + \frac{c}{|A|} \left(\sum_{a_j \in A} Bel_p^t(a_j) \right) \right)$$

It is clear that the summation of all beliefs keeps constant in a time stamp t, thus let's call this summation S_t:

Lemma 1 *Given a clique influence graph with S_t as the sum of the beliefs in the "t'th moment.*

$$S_{t+1} = S_t$$

3.2 Proof of Lemma 1

$$\begin{aligned}
\sum_{a_i \in A} Bel_p^{t+1}(a_i) &= \sum_{a_i \in A} \left((1-c) \times Bel_p^t(a_i) + \frac{c}{|A|} \ S_t \right) \\
&= \sum_{a_i \in A} \left((1-c) \times Bel_p^t(a_i) \right) + \sum_{a_i \in A} \left(\frac{c}{|A|} \ S_t \right) \\
&= (1-c) \times \sum_{a_i \in A} Bel_p^t(a_i) + \sum_{a_i \in A} \left(\frac{c}{|A|} \ S_t \right)
\end{aligned}$$

By definition of S_t, the we can write:

$$\sum_{a_i \in A} Bel_p^{t+1}(a_i) = (1-c) \ S_t + \sum_{a_i \in A} \left(\frac{c}{|A|} \ S_t \right)$$

Since the second summation is independent of a_i:

$$\begin{aligned}
\sum_{a_i \in A} Bel_p^{t+1}(a_i) &= (1-c) \ S_t + |A| \frac{c}{|A|} \ S_t \\
&= (1-c) \ S_t + c \ S_t \\
\sum_{a_i \in A} Bel_p^{t+1}(a_i) &= S_t
\end{aligned}$$

□

Now we showed that, for every t, $S_{t+1} = S_t$, by induction it shows that all S_t have the same value.

Theorem 1 *In a clique with constant influence c, $(0 < c \leq 1)$, all agents believes converge to the same value: $\frac{S_t}{|A|}$*

3.3 Proof of theorem 1

As we showed before $Bel_p^{t+1}(a_i)$ can be written as:

$$Bel_p^{t+1}(a_i) = (1-c) \times Bel_p^t(a_i) + \frac{c}{|A|}\ S_t$$

To simplify the notation, we will call $k = 1 - c$ and $C = \frac{c}{|A|} \times S_t$ and $x_t = Bel_p^t(a_i)$, we can do this *w.l.o.g.* because we proved that S_t is constant throughout the time, thus both k and C are constant:

$$Bel_p^{t+1}(a_i) = k \times Bel_p^t(a_i) + C$$

If we expand this, we can see a pattern:

$$\begin{aligned} Bel_p^{t+2}(a_i) &= k\ x_{t+1} + C \\ &= k\ (k\ x_t + C) + C \\ &= k^2\ x_t + k\ C + C \end{aligned}$$

$$\begin{aligned} Bel_p^{t+3}(a_i) &= k\ x_{t+2} + C \\ &= k\ (k^2\ x_t + k\ C + C) + C \\ &= k^3\ x_t + k^2\ C + k\ C + C \end{aligned}$$

Generically we can write $Bel_p^{t+n}(a_i)$ as:

$$\begin{aligned} Bel_p^{t+n}(a_i) &= k^n\ x_t + \sum_{i=0}^{i=n-1} (k^i\ C) \\ &= k^n\ x_t + C \times \sum_{i=0}^{i=n-1} k^i \end{aligned}$$

Plugging $t = 0$ we get a formula for the value of $Bel_p^n(a_i)$ through time:

$$\begin{aligned} Bel_p^{n}(a_i) &= k^n\ x_0 + \sum_{i=0}^{i=n-1} (k^i\ C) \\ &= k^n\ x_0 + C \times \sum_{i=0}^{i=n-1} k^i \end{aligned}$$

As $n \to \infty$, k^n clearly goes to 0, since $k = 1 - c$ and $0 < c \leq 1$. Thus:

$$\begin{aligned} \lim_{n\to\infty} Bel_p^n(a_i) &= \lim_{n\to\infty} \left(C \times \sum_{i=0}^{i=n-1} k^i \right) \\ &= C \times \lim_{n\to\infty} \sum_{i=0}^{i=n-1} k^i \end{aligned}$$

This summation is a very known series, the geometric series, since $0 \leq k < 1$, and we know this result equals:

$$\lim_{n\to\infty} Bel_p^n(a_i) = C \; \frac{1}{1-k}$$

Which by the definition of C and k equals:

$$\begin{aligned}\lim_{n\to\infty} Bel_p^n(a_i) &= \left(\frac{c}{|A|} \; S\right) \; (\frac{1}{c}) \\ &= \frac{S}{|A|}\end{aligned}$$

□

Since a_i is arbitrary, all a_i's converge for the same value, as we wanted to prove.

4 Conclusion

With this we complete out proofs, and they make a lot of sense intuitively. In a totally connected society and a classic update, the opinions converge to their averages.