

# Proof of Individual Agent Opinion Convergence in a Strongly Connected Influence Graph Using Classic Update Function

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In the classic update function,  $Bel_p^{t+1}(a_i|a_j)$  can be written in the following form:

**Definition 1**  $Bel_p^{t+1}(a_i|a_j) = Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))$ .

And the classic update function,  $Bel_p^{t+1}(a_i)$  is written as:

**Definition 2**  $Bel_p^{t+1}(a_i) = \frac{1}{|A|} \sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j)$ .

And let's define a strongly connected graph as:

**Definition 3** *A strongly connected influence graph social network in which every agent exerts influence on every other agent:  $In(a_i, a_j) > 0$ , for every  $i, j$ .*

**Definition 4**  $max_t$  and  $min_t$  are the maximum and minimum belief values in a given instant  $t$ , respectively.

To prove our conjecture, let's do some simplifications:

$$\begin{aligned} Bel_p^{t+1}(a_i) &= \frac{1}{|A|} \sum_{a_j \in A} Bel_p^{t+1}(a_i|a_j) \\ &= \frac{1}{|A|} \sum_{a_j \in A} (Bel_p^t(a_i) + In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))) \\ &= Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) \end{aligned}$$

Since belief values are finite, by the well-ordering principle we always have a  $min_t$  and a  $max_t$ . It is easy to see, by the squeeze theorem, that individual agent opinion converges to the same value if and only if  $\lim_{t \rightarrow \infty} min_t = \lim_{t \rightarrow \infty} max_t$ .

Thus, since we want to prove that it always converges, if  $min_t = max_t$  we have nothing to prove, so assume  $min_t \neq max_t$ .

**Lemma 1** *In a strongly connected graph and under classic belief update, if  $max_t \neq min_t$ :*

$$\forall a_i \in A : Bel_p^{t+1}(a_i) < max_t \tag{1}$$

and:

$$\forall a_i \in A : Bel_p^{t+1}(a_i) > min_t \tag{2}$$

**Proof of Lemma 1**

By definition:

$$Bel_p^{t+1}(a_i) = Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))$$

Now since  $max_t \neq min_t$ , there is at least one value between  $a_j$ 's, such that  $Bel_p^t(a_j) < max_t$ , thus replacing all  $Bel_p^t(a_j)$  by  $max_t$ , we make the right side strictly greater than the left one:

$$\begin{aligned}
Bel_p^{t+1}(a_i) &< Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(max_t - Bel_p^t(a_i)) \\
&< Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} (max_t - Bel_p^t(a_i)) \\
&< Bel_p^t(a_i) + \frac{|A|}{|A|} (max_t - Bel_p^t(a_i)) \\
&< Bel_p^t(a_i) + max_t - Bel_p^t(a_i) \\
Bel_p^{t+1}(a_i) &< max_t
\end{aligned}$$

Since  $a_i$  was arbitrary, the lemma is true for all agents. The same reasoning can be used to show the equivalent property for  $min_t$

□

**Corollary 1** *In a strongly connected influence graph and a classic update function if  $min_t \neq max_t$ , then  $max_{t+1} < max_t$  and  $min_{t+1} > min_t$ .*

**Proof of Corollary 1**

The result of lemma 1 tells us that all beliefs in the time  $t + 1$  are smaller than  $max_t$ , thus, since  $max_{t+1}$  must be one of those elements,  $max_{t+1} < max_t$ . And the same reasoning can be used for  $min_t$ .

**Corollary 2**  *$\lim_{t \rightarrow \infty} max_t = L$  and  $\lim_{t \rightarrow \infty} min_t = M$  for some  $L, M \in [0, 1]$ .*

**Proof of Corollary 2**

Since both  $max_t$  and  $min_t$  are bounded between 0 and 1 by the definition of belief; and lemma 1 showed us that they are monotonic, according to the monotonic convergence theorem, the limits exist.

**Definition 5** *Let's denote by  $In_{min}$  the smallest influence in the influence graph. Keep in mind that  $In_{min} > 0$  since we are working with a strongly connected influence graph.*

Using the same notation we used in corollary 2, let's call  $\lim_{t \rightarrow \infty} max_t = L$  and  $\lim_{t \rightarrow \infty} min_t = M$ .

**Lemma 2**  $\forall t$  and  $\forall a_i \in A$ :  $max_t - Bel_p^{t+1}(a_i) \geq \epsilon$ , with  $\epsilon = \frac{In_{min}(L-M)}{|A|}$ .

**Proof of Lemma 2**

To prove this lemma, first we will try to find the biggest  $Bel_p^{t+1}(a_i)$  possible. Now let's start with the formula of belief:

$$Bel_p^{t+1}(a_i) = Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i))$$

To achieve our goal, in each step we will choose the values in the right side of the equation in a way that maximizes it. Trying to do so will guarantee us that the inequality holds for every  $Bel_p^{t+1}(a_i)$  and will lead us to  $\epsilon$ .

The first thing we will do is separate from the summation the element  $a_k$ , which we define as the agent who holds the belief  $min_t$  in that arbitrary time step.

$$\begin{aligned}
Bel_p^{t+1}(a_i) &= Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_k\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) + \frac{In(a_k, a_i)(Bel_p^t(a_k) - Bel_p^t(a_i))}{|A|} \\
&= Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_k\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) + \frac{In(a_k, a_i)(min_t - Bel_p^t(a_i))}{|A|}
\end{aligned}$$

Now trying to maximize the rightmost term in the inequality, we shall see that, by the definition of  $min_t$ :  $min_t - Bel_p^t(a_i) \leq 0$ . If  $min_t - Bel_p^t(a_i)$  values 0, the influence that multiplies it doesn't make any difference, but if it is different of 0 we want the influence to be as small as possible, which is  $In_{min}$ .

$$Bel_p^{t+1}(a_i) \leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_k\}} In(a_j, a_i)(Bel_p^t(a_j) - Bel_p^t(a_i)) + \frac{In_{min}(min_t - Bel_p^t(a_i))}{|A|}$$

Now it's time to choose the value of  $Bel_p^t(a_j)$  for all  $a_j$ 's that maximizes the right side. Since this part is always positive, we shall pick the maximum value possible, which is  $max_t$ .

$$Bel_p^{t+1}(a_i) \leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_k\}} In(a_j, a_i)(max_t - Bel_p^t(a_i)) + \frac{In_{min}(min_t - Bel_p^t(a_i))}{|A|}$$

Now looking at the terms inside the summation, since  $max_t - Bel_p^t(a_i) \geq 0$ , the influence that maximizes it is the biggest one possible, which is 1, thus:  $\forall a_i, a_j \in A \setminus \{a_k\} : In(a_j, a_i) = 1$ .

$$\begin{aligned} Bel_p^{t+1}(a_i) &\leq Bel_p^t(a_i) + \frac{1}{|A|} \sum_{a_j \in A \setminus \{a_k\}} (max_t - Bel_p^t(a_i)) + \frac{In_{min}(min_t - Bel_p^t(a_i))}{|A|} \\ &\leq Bel_p^t(a_i) + \frac{(|A| - 1)(max_t - Bel_p^t(a_i))}{|A|} + \frac{In_{min}(min_t - Bel_p^t(a_i))}{|A|} \\ &\leq Bel_p^t(a_i) + \frac{(|A| - 1)(max_t - Bel_p^t(a_i)) + In_{min}(min_t - Bel_p^t(a_i))}{|A|} \\ &\leq \frac{|A| Bel_p^t(a_i) + (|A| - 1)(max_t - Bel_p^t(a_i)) + In_{min}(min_t - Bel_p^t(a_i))}{|A|} \\ &\leq \frac{(|A| - 1)max_t - Bel_p^t(a_i) + In_{min}(min_t - Bel_p^t(a_i))}{|A|} \\ &\leq \frac{(|A| - 1)max_t + Bel_p^t(a_i)(1 - In_{min}) + In_{min}min_t}{|A|} \end{aligned}$$

These manipulations made it clear which value of  $Bel_p^t(a_i)$  we should choose to achieve our goal, and it is  $Bel_p^t(a_i) = max_t$ .

$$\begin{aligned} Bel_p^{t+1}(a_i) &\leq \frac{(|A| - 1)max_t + max_t(1 - In_{min}) + In_{min}min_t}{|A|} \\ &\leq \frac{|A| max_t - In_{min} max_t + In_{min} min_t}{|A|} \\ &\leq \frac{|A| max_t + In_{min}(min_t - max_t)}{|A|} \\ &\leq max_t + \frac{In_{min}(min_t - max_t)}{|A|} \end{aligned}$$

Now we shall remember that, since  $max_t$  is decreasing and  $min_t$  is increasing, our choice to make the right side as big as possible is to plug it's limits, which gives us:

$$\begin{aligned} Bel_p^{t+1}(a_i) &\leq max_t + \frac{In_{min}(M - L)}{|A|} \\ Bel_p^{t+1}(a_i) - max_t &\leq \frac{In_{min}(M - L)}{|A|} \\ max_t - Bel_p^{t+1}(a_i) &\geq \frac{In_{min}(L - M)}{|A|} \\ max_t - Bel_p^{t+1}(a_i) &\geq \epsilon \end{aligned}$$

□

**Corollary 3**  $max_{t+1} + \epsilon \leq max_t$

**Proof of Corollary 3**

Since  $max_{t+1}$  must be one of the beliefs in the time  $t + 1$  and, according to Lemma 2, all of them are smaller than  $max_t$  by at least  $\epsilon$ ,  $max_{t+1}$  must also be smaller than  $max_t$  by a factor of at least  $\epsilon$ .

**Theorem 1**  $\lim_{t \rightarrow \infty} max_t = L = \lim_{t \rightarrow \infty} min_t = M$

**Proof of Theorem 1**

Suppose, by contradiction, that  $L \neq M$ . Plugging this values into the  $\epsilon$  formula show us that  $\epsilon \neq 0$ . Since, according to lemma 2,  $max_{t+1}$  is smaller than  $max_t$  by a factor of  $\epsilon$ . With all of this we can finally reach to a contradiction and end our proof.

To see this contradiction, let's assume we did  $v = \lceil \frac{1}{\epsilon} \rceil + 1$  timesteps after some  $t = 0$ . Since  $max$  diminishes by at least  $\epsilon$  at each step:

$$\begin{aligned} max_0 &\geq max_v + \epsilon \left( \left\lceil \frac{1}{\epsilon} \right\rceil + 1 \right) \\ max_0 &\geq max_v + \epsilon.v \\ max_0 - \epsilon.v &\geq max_v \end{aligned}$$

But  $\epsilon.v > 1$ , thus  $\max_0 < \epsilon.v$ . And this would imply that  $\max_v < 0$ , which contradicts the definition of belief!

Since assuming that  $L \neq M$  leads us to a contradiction we can conclude that  $L = M$ . This result implies that all agents belief converge to the same value, as we wanted to prove.

□