Subjective Logic

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April 7, 2020

Obs: This notes are basically a clone of José's with some extra teaks and notes of my own.

1 Introduction

Question 1.1. What is $p(y \parallel x)$ and $p(x \mathbin{\widetilde{\parallel}} y)$.

2 Elements of Subjective Opinions

2.1 Motivation for the Opinion Representation

For decision makers it can make a big difference whether probabilities are confident or uncertain. Decision makers should instead request additional evidence so the analysts can produce more confident conclusion probabilities about hypotheses of interest.

2.2 Flexibility of Representation

There can be multiple equivalent formal representations of subjective opinions.

2.3 Domains and Hyperdomains

Definition 2.1. *(Hyperdomain)* Let $\mathbb{X}$ be a domain, and let $\mathcal{P}(\mathbb{X})$ denote the powerset of $\mathbb{X}$. The powerset contains all subsets of $\mathbb{X}$, including the empty set $\{\varnothing\}$, and the domain $\mathbb{X}$ itself. The *hyperdomain* denoted $\mathcal{R}(\mathbb{X})$ is the reduced powerset of $\mathbb{X}$, i.e. the powerset excluding the empty-set $\{\varnothing\}$ and the domain value $\{\mathbb{X}\}$. The hyperdomain is expressed as

$$\text{Hyperdomain: } \mathcal{R}(\mathbb{X}) = \mathcal{P} \setminus \{\{\mathbb{X}\}, \{\varnothing\}\} \tag{2.1}$$

Question 2.1. I don't know if this is important, but I don't understand exactly how indexing works by the way that is explained in the book.

Definition 2.2. *(Composite set)* Let $\mathbb{X}$ be a domain of cardinality k, where $\mathcal{R}(\mathbb{X})$ is its hyperdomain of cardinality κ. Every proper subset $x \subset \mathbb{X}$ of cardinality $|x| \geq 2$ is a *composite value*. The set of composite values is the *composite set*, denoted $\mathcal{C}(\mathbb{X})$ and defined as:

$$\text{Composite set: } \mathcal{C}(\mathbb{X}) = \{x \subset \mathbb{X} \text{ where } \; |x| \geq 2\} \tag{2.2}$$

2.4 Random Variables and Hypervariables

Definition 2.3. *(Hypervariable)* Let $\mathbb{X}$ be a domain with coresponding hyperdomain $\mathcal{R}(\mathbb{X})$ A variable X takes its value from $\mathcal{R}(\mathbb{X})$ is a hypervariable.

Note 2.1. The events analyzed must be mutually exclusive.

2.5 Belief Mass Distribution and Uncertainty Mass

Definition 2.4. *(Belief Mass Distribution)* Let $\mathbb{X}$ be a domain with corresponding hyperdomain $\mathcal{R}(\mathbb{X})$, and let X be a variable over those domains. A belief mass distribution denote $\mathbf{b}_X$ assigns belief mass to possible values of the variable X. In the case of a random variable $X \in \mathbb{X}$, the belief mass distribution applies to domain $\mathbb{X}$, and in the case of a hypervariable $X \in \mathcal{R}(\mathbb{X})$ the belief mass distribution applies to hyperdomain $\mathcal{R}(\mathbb{X})$. This is formally defined as follows.

$$\begin{array}{l}\text{Multinomial belief mass distribution: } \mathbf{b}_X : \mathbb{X} \to [0,\ 1], \\ \text{with the additivity requirement: } u_X + \sum_{x \in \mathbb{X}} \mathbf{b}_X(x) = 1.\end{array} \tag{2.3}$$

$$\begin{array}{l}\text{Hypernominal belief mass distribution: } \mathbf{b}_X : \mathcal{R}(\mathbb{X}) \to [0,\ 1], \\ \text{with the additivity requirement: } u_X + \sum_{x \in \mathcal{R}(\mathbb{X})} \mathbf{b}_X(x) = 1.\end{array} \tag{2.4}$$

The sub-additivity of belief mass distributions is complemented by *uncertainty mass* denoted u_X.

2.6 Base Rate Distributions

Definition 2.5. *(Base Rate Distribution)* Let $\mathbb{X}$ be a domain, and let X be a random variable in $\mathbb{X}$. The base rate distribution $\mathbf{a}_X$ assigns base rate probability to possible values of $X \in \mathbb{X}$, and is an additive probability distribution, formally expressed as:

$$\begin{array}{l}\text{Base rate distribution: } \mathbf{a}_X : \mathbb{X} \to [0,\ 1], \\ \text{with the additivity requirement: } \sum_{x \in \mathbb{X}} \mathbf{a}_X(x) = 1.\end{array} \tag{2.5}$$

Definition 2.6. *(Base Rate Distribution over Values in a Hyperdomain)* Let $\mathbb{X}$ be a domain with corresponding hyperdomain $\mathcal{R}(\mathbb{X})$, and let X be a variable over those domains. Assume the base rate distribution $\mathbf{a}_X$ over the domain $\mathbb{X}$ according to Definition 2.5. The base rate $\mathbf{a}_X$ for a composite value $x \in \mathcal{R}(\mathbb{X})$ can be computed as follows:

$$\text{Base rate over composite values: } \mathbf{a}_X(x_i) = \sum_{\substack{x_j \in \mathbb{X} \\ x_j \subseteq x_i}} \mathbf{a}_X(x_j), \ \forall x_i \in \mathcal{R}(\mathbb{X}). \tag{2.6}$$

Definition 2.7. *(Relative Base Rate)* Assume a domain $\mathbb{X}$ of cardinality k, and the corresponding hyperdomain $\mathcal{R}(\mathbb{X})$. Let X be a hypervariable over $\mathcal{R}(\mathbb{X})$. Assume that a base rate distribution $\mathbf{a}_X$ is defined over $\mathbb{X}$ according to Definition 2.6. Then the base rate of a value x relative to a value v_i is expressed as the relative base rate $\mathbf{a}_X(x|x_i)$ defined below.

$$\mathbf{a}_X(x|x_i) = \frac{\mathbf{a}_X(x \cap x_i)}{\mathbf{a}_X(x_i)}, \ \forall x, x_i \in \mathcal{R}(\mathbb{X}), \text{ where } \mathbf{a}_X(x_i) \neq 0. \tag{2.7}$$

In the case when $\mathbf{a}_X(x_i) = 0$, then $\mathbf{a}_X(x|x_i) = 0$. Alternatively it can simply be assumed that $a_X(x_i) > 0$, for every $x_i \in \mathbb{X}$, meaning that everything we include in the domain has a non-zero base rate of occurrence in general.

2.7 Probability Distributions

Definition 2.8. *(Probability Distribution)* Let $\mathbb{X}$ be a domain with corresponding hyperdomain $\mathcal{R}(\mathbb{X})$, and let X denote a variable in $\mathbb{X}$ or in $\mathcal{R}(\mathbb{X})$. The standard probability distribution $\mathbf{p}_X$ assigns probabilities to possible values of $X \in \mathbb{X}$. The hyper-probability distribution $\mathbf{p}^{\mathrm{H}}_X$ assigns probabilities to possible values of $X \in \mathcal{R}(\mathbb{X})$. These distributions are formally defined below:

$$\begin{aligned} &\text{Probability distribution: } \mathbf{p}_X : \mathbb{X} \to [0,\ 1], \\ &\text{with the additivity requirement: } \textstyle\sum_{x \in \mathbb{X}} \mathbf{p}_X(x) = 1. \end{aligned} \tag{2.8}$$

$$\begin{aligned} &\text{Hyper-probability distribution: } \mathbf{p}^{\mathrm{H}}_X : \mathcal{R}(\mathbb{X}) \to [0,\ 1], \\ &\text{with the additivity requirement: } \textstyle\sum_{x \in \mathcal{R}(\mathbb{X})} \mathbf{p}^{\mathrm{H}}_X(x) = 1. \end{aligned} \tag{2.9}$$

3 Opinion Representations

3.1 Opinion Classes

The opinion itself is a composite function $\omega^A_X = (\mathbf{b}_X, u_X, \mathbf{a}_X)$, consisting of the belief mass distribution $\mathbf{b}_X$, the uncertainty mass u_X, and the base rate distribution $\mathbf{a}_X$.

Classes:

- *Binomial*: Domain $\mathbb{X}$ and variable X are binary.

- *Multinomial*: Domain larger than binary and the variable is a random variable $X \in \mathbb{X}$.
- *Hypernomial*: Domain larger than binary and the variable is a hypervariable $X \in \mathcal{R}(\mathbb{X})$.

Levels of confidence of a opinion:

- *Vacuous*: $u_X = 1$.
- *Uncertain*: $0 < u_X < 1$.
- *Dogmatic*: $u_X = 0$.
- *Absolute*: One single value is TRUE by assigning belief mass 1 to that value.

3.2 Aleatory and Epistemic Opinions

- *Aleatory Uncertainty*, which is the same as statistical uncertainty, express that we do not know the outcome each time we run the same experiment, we only know the long-term relative frequency of outcomes. E.g.: Flip a coin.
- *Epistemic Uncertainty*, aka systematic uncertainty, express that we could in principle know the outcome of a specific or future or past event, but that we do not have enough evidence to know it exactly. E.g.: Assassination of President Kennedy.

Question 3.1. First-order and second-order opinions?

Question 3.2. Projected probability?

High aleatory/epistemic uncertainty is consistent with both high and low uncertainty mass.

- **An aleatory Opinion** applies to a variable governed by a frequentist process, and that represents the (uncertain) likelihood of values of the variable in any unknown past or future instance of the process. An aleatory opinion can naturaly have an arbitrary uncertainty mass.
- **An epistemic Opinion** applies to a variable that is assumed to be non-frequentist, and that represents the (uncertain) likelihood of values of the variable in a specific unknown past or future instance.

3.3 Binomial Opinions

3.3.1 Binomial Opinion Representation

Definition 3.1. *Binomial Opinion* Let $\mathbb{X} = \{x, \overline{x}\}$ be a binary domain with binomial random variable $X \in \mathbb{X}$. A binomial opinion about the truth/presence of value x is the ordered quadruplet $\omega_x = (b_x, d_x, u_x, a_x)$, where the additivity requirement

$$b_x + d_x + u_x = 1 \tag{3.1}$$

is satisfied, and where the respective parameters are defined as

- b_x: *belief mass* in support of x being TRUE (i.e. $X = x$),
- d_x: *disbelief mass* in support of x being FALSE (i.e. $X = \overline{x}$)
- u_x: *uncertainty mass* representing the vacuity of evidence,
- a_x: *base rate*, i.e. prior probability of x without any evidence.

The projected probability of a binomial opinion about value x is defined by the following equation.

$$\mathrm{P}(x) = b_x + a_x u_x. \tag{3.2}$$

The variance of binomial options is expressed as

$$\mathrm{Var}(x) = \frac{\mathrm{P}(x)(1 - \mathrm{P}(x))u_x}{W + u_x}, \tag{3.3}$$

where W denotes non-informative prior weight, which must be set to $W = 2$ as explained in Section 3.5.2. Binomial opinion variance is derived from the variance of the Beta PDF.

3.3.2 The Beta Binomial Model

Definition 3.2. *(Beta Probability Density Function)* Assume a binaru domain $\mathbb{X} = \{x, \overline{x}\}$ and a random variable $X \in \mathbb{X}$. Let p denote the continuous probability function $p : X \to [0, 1]$ where $p(x) + p(\overline{x}) = 1$. For compactness of notation e define $p_x \equiv p\,(x)$ and $p_{\overline{x}} \equiv p\,(\overline{x})$.

The parameter α represents evidence/observations of $X = x$, and the parameter β represents evidence/observations of $X = \overline{x}$. With p_x as variable, the Beta probability density function $\mathrm{Beta}(p_x, \alpha, \beta)$ is the function expressed as

$$\mathrm{Beta}(p_x, \alpha, \beta) \ : \ [0, 1] \to \mathbb{R}_{\leq 0}, \text{ where} \tag{3.4}$$

$$\mathrm{Beta}(p_x, \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}(p_x)^{\alpha - 1}(1 - p_x)^{\beta - 1}, \ \alpha > 0, \ \beta > 0, \tag{3.5}$$

with the restrictions that $p_(x) \neq 0$ if $\alpha < 1$, and $p(x) \neq 1$ if $\beta \leq 1$.

Note 3.1. The part of the equation that evolves the gamma PDF exists only to ensure that the integral of the PDF equals 1.

Assume that x represents a frequentist event. Let r_x (or r_s) denote the number of observations of x (or $\overline{x}$). With the evidence observations, the base rate a_x and the non-informative prior weight W, the α and β parameters can be expressed as:

$$\begin{cases} \alpha = r_x + a_x W, \\ \beta = s_x + (1 - a_x). \end{cases} \tag{3.6}$$

The evidence notation of the Beta PDF is denoted by $\mathrm{Beta}^{\mathrm{e}}(p_x, r_x, s_x, a_x)$.

Note 3.2. $W = 2$ because, with $r_x = 0$ and $s_x = 0$, and $a_x = \frac{1}{2}$, the Beta PDF $(p_x^{\alpha-1}(1-p_x)^{\beta-1})$ becomes a constant, which is equivalent to the uniform PDF [0,1]. This makes sense intuitively, if we don't have any evidence and our base rates are the same for both events, any event is perceived as equally likely.

Expected probability:

$$\mathrm{E}(x) = \frac{r_x + a_x W}{r_x + s_x + W} \tag{3.7}$$

Variance:

$$\mathrm{Var}(x) = \frac{\mathrm{P}(x)(1 - \mathrm{P}(x))u_x}{W + u_x} \tag{3.8}$$

3.3.3 Mapping Between a Binomial Opinion and a Beta PDF

Definition 3.3. *(Mapping: Binomial Opinion $\leftrightarrow$) Beta PDF)* Let $\omega_x = (b_x, d_x, u_x, a_x)$ be a binomial opinion, and let $p(x)$ be a probability distribution, both over the same binomial random variable X. Let $\mathrm{Beta}^{\mathrm{e}}(p_x, r_x, s_x, a_x)$ a Beta PDF over the probability variable p_x defined as a function of r_x, s_x and a_x according. The opinion ω_x and the Beta PDF $\mathrm{Beta}^{\mathrm{e}}(p_x, r_x, s_x, a_x)$ are equivalent through the following mapping:

$$\begin{cases} b_x = \dfrac{r_x}{W + r_x + s_x}, \\ d_x = \dfrac{s_x}{W + r_x + s_x}, \\ u_x = \dfrac{W}{W + r_x + s_x} \end{cases} \Leftrightarrow \begin{cases} \begin{cases} r_x = \dfrac{b_x W}{u_x}, \\ s_x = \dfrac{d_x W}{u_x}, \\ 1 = b_x + d_x + u_x \end{cases} & \text{if } u \neq 0 \\ \begin{cases} r_x = b_x \cdot \infty, \\ s_x = d_x \cdot \infty, \\ 1 = b_x + d_x. \end{cases} & \text{if } u \neq 0 \end{cases} \tag{3.9}$$

The equivalence between binomial opinions and Beta PDFs is very powerful, because subjective-logic operators (SL operators) can then be applied to Beta PDFs, and statistics operations for Beta PDFs can be applied to opinions. In addiction, it makes it possible to determine binomial opinions from statistical observations.

3.4 Multinomial Opinions

3.4.1 The Multinomial Opinion Representation

Definition 3.4. *(Multinomial Opinion)* Let $\mathbb{X}$ be a domain larger than binary, i.e. so that $k = |X| > 2$. Let X be a random variable in $\mathbb{X}$. A multinomial opinion over the random variable X is the ordered triplet $\omega_X = (\mathbf{b}_X, u_X, \mathbf{a}_X)$ where

- $\mathbf{b}_X$ is a belief mass distribution over X,
- u_X is the uncertainty mass which represents the vacuity of evidence,
- $\mathbf{a}_X$ is a base rate distribution over $\mathbb{X}$,

and the multinomial additivity requirement of Eq.(2.3) is satisfied.

A multinomial opinion contains $(2k+1)$ parameters. However, given the belief and uncertainty mass additivity of Eq.(2.3), and the base rate additivity of Eq.(2.5), multinomial opinions only have $(2k-1)$ degrees of freedom.

Question 3.3. What is degrees of freedom?

The projected probability distribution of multinomial opinions is defined by:

$$\mathbf{P}_X(x) = \mathbf{b}_X(x) + \mathbf{a}_X(x)u_X, \ \forall x \in \mathbb{X}. \tag{3.10}$$

The variance of multinomial opinions is expressed as

$$\mathrm{Var}_X = \frac{\mathbf{P}_X(x)(1 - \mathbf{P}_X(x)u_X)}{W + u_X}, \tag{3.11}$$

where W denotes non-informative prior weight, which must be set to $W = 2$.

3.4.2 The Dirichlet Multinomial Model

Definition 3.5. *(Dirichlet Probability Density Function)* Let $\mathbb{X}$ be a domain consisting of k mutually disjoint values. Let α_X represent the strength vector over the values of $\mathbb{X}$, and let $\mathbf{p}_X$ denote the probability distribution over $\mathbb{X}$. With $\mathbf{p}_X$ as a k-dimensional variable, the Dirichlet PDF denoted $\mathrm{Dir}(\mathbf{p}_X, \alpha_X)$ is expressed as:

$$\mathrm{Dir}(\mathbf{p}_X, \alpha_X) = \frac{\Gamma\left(\sum\limits_{x\in\mathbb{X}} \alpha_X(x)\right)}{\prod\limits_{x\in\mathbb{X}} \Gamma(\alpha_X(x))} = \prod_{x\in\mathbb{X}} \mathbf{p}_X(x)^{(\alpha_X(x)-1)}, \text{ where } \alpha_X(x) \geq 0, \tag{3.12}$$

with the restrictions that $\mathbf{p}_X(x) \neq 0$ if $\alpha_X(x) < 1$.

The evidence representation of the Dirichlet PDF is denoted by $\mathrm{Dir}^{\mathrm{e}}_X(\mathbf{p}_X, \mathbf{r}_X, \mathbf{a}_X)$, where the total strength $\alpha_X(x)$ for each value $x \in \curvearrowleft$ can be expressed as

$$\alpha_X(x) = \mathbf{r}_X(x) + \mathbf{a}_X(x)W, \text{ where } \mathbf{r}_X(x) \geq 0 \ \forall x \in \mathbb{X}. \tag{3.13}$$

The evidence-Dirichlet PDF is expressed in terms of the evidence vector $\mathbf{r}_X$, where $\mathbf{r}_X(x)$ is the evidence for outcome $x \in \mathbb{X}$. In addition, the base rate distribution $\mathbf{a}_X$ and the non-informative prior weight W are parameters in the expression for the evidence-Dirichlet PDF.

The expected distribution over $\mathbb{X}$ can be written as

$$\mathbf{E}_X(x) = \frac{\mathbf{r}_X(x) + \mathbf{a}_X(x)W}{W + \sum\limits_{x_j \in \mathbb{X}} \mathbf{r}_X(x_j)} \ \forall x \in \mathbb{X}. \tag{3.14}$$

The variance of the Dirichlet is defined by

$$\mathrm{Var}_X(x) = \frac{\mathbf{P}_X(x)(1 - \mathbf{P}_X(x))}{W + u_X}. \tag{3.15}$$

3.4.3 Visualising Dirichlet Probability Density Functions

Dirichlet PDFs over ternary domains are the largest that can be practically visualized.

The Figure 3.4 from the book shows graphical representations with non-informative prior Dirichlet PDF, and posterior Dirichlet PDF. At the second case, $\mathbf{r}_X$ is not constant on 0.

3.4.4 Coarsening Example: From Ternary to Binary

Question 3.4. What does coarsening mean?

This subsubsection shows a ternary domain $\{x_1, x_2, x_3\}$ and it reduces to binary domain making $\overline{x}_1 = \{x_2, x_3\}$. This way makes it possible to visualize prior and posteriori Beta PDFs for $p(x_1)$. Here I realize that didn't understand at section 3.3.2 that a Beta (and Dirithlet here) is a function from probability to probability. The PDF I saw most so far was from Normal Distribution and was a function from a random variable to probability. The take-way here (actually from subsubsection 3.3.2) is that the expected probability (3.7 and 3.14) is the posteriori probability, i.e. with the new evidences.

3.4.5 Mapping Between Multinomial Opinion and Dirichlet PDF