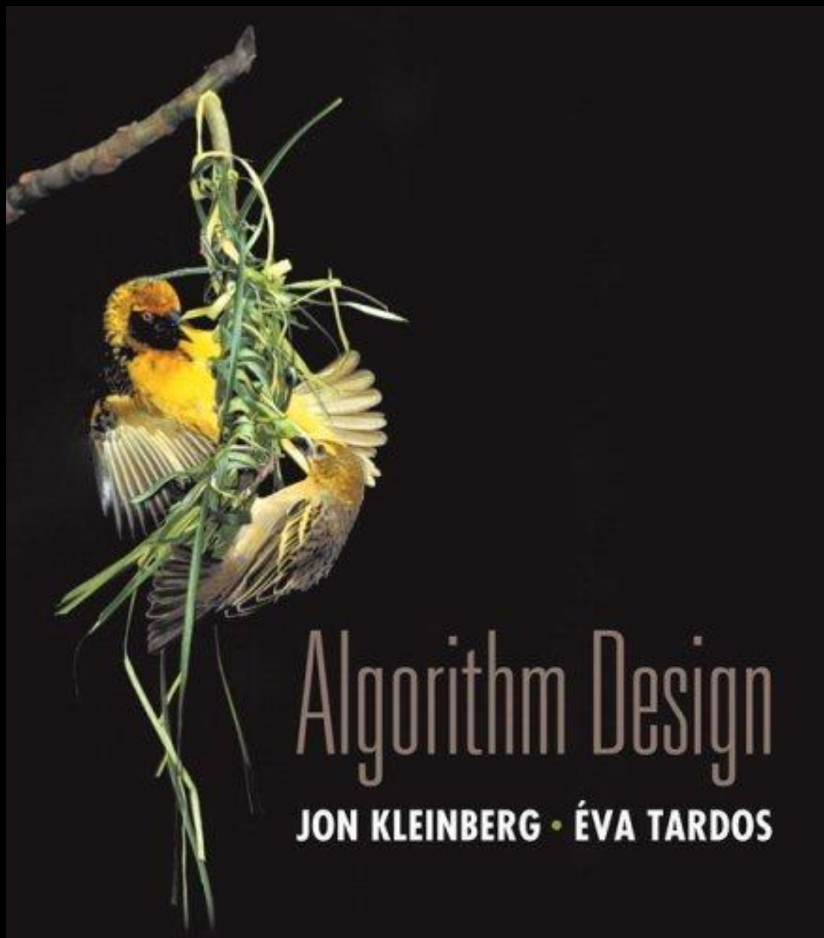


Chapter 6

Dynamic Programming

Algorithmic Paradigms

Greed. Build up a solution incrementally, myopically optimizing some local criterion.

Divide-and-conquer. Break up a problem into two sub-problems, solve each sub-problem independently, and combine solution to sub-problems to form solution to original problem.

Dynamic programming. Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems.

Dynamic Programming History

Bellman. Pioneered the systematic study of dynamic programming in the 1950s.

Etymology.

- Dynamic programming = planning over time.
- Secretary of Defense was hostile to mathematical research.
- Bellman sought an impressive name to avoid confrontation.
  - "it's impossible to use dynamic in a pejorative sense"
  - "something not even a Congressman could object to"

Reference: Bellman, R. E. *Eye of the Hurricane, An Autobiography.*

Dynamic Programming Applications

Areas.

- Bioinformatics.
- Control theory.
- Information theory.
- Operations research.
- Computer science: theory, graphics, AI, systems, ...

Some famous dynamic programming algorithms.

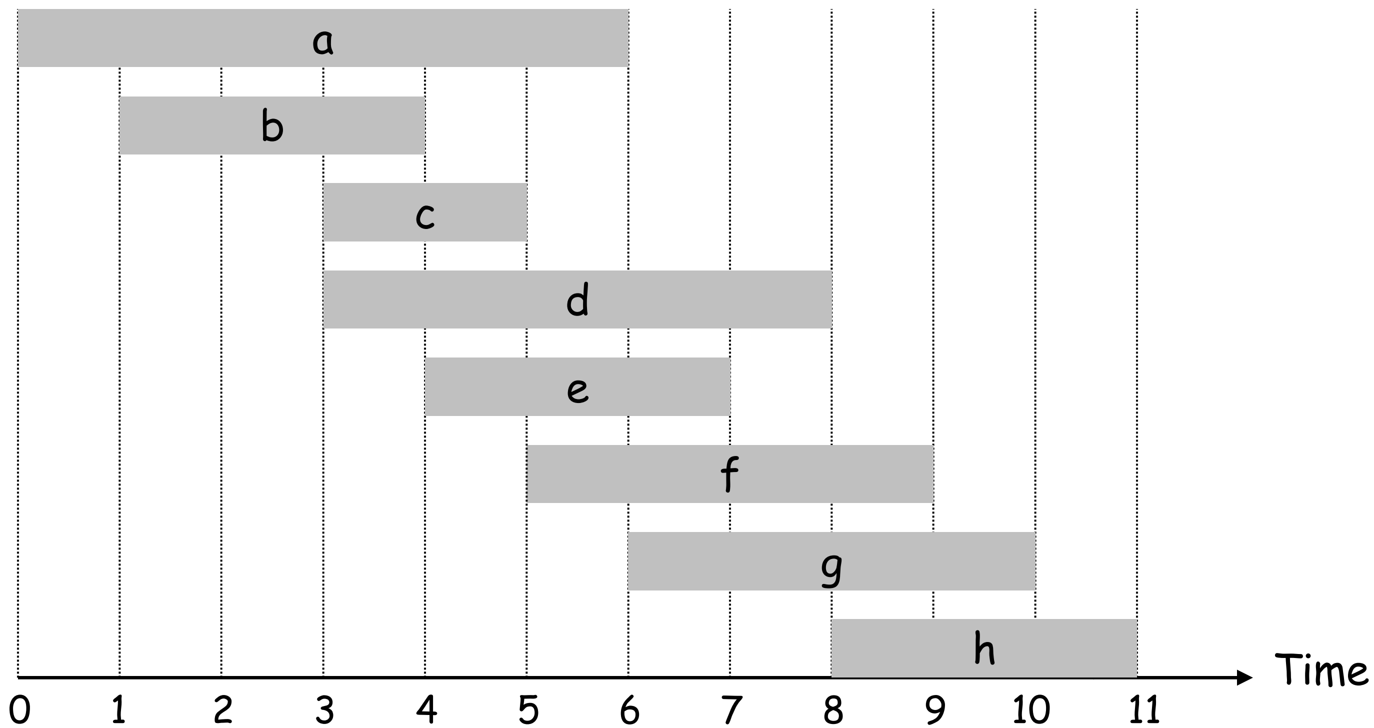
- Viterbi for hidden Markov models.
- Unix diff for comparing two files.
- Smith-Waterman for sequence alignment.
- Bellman-Ford for shortest path routing in networks.
- Cocke-Kasami-Younger for parsing context free grammars.

6.1 Weighted Interval Scheduling

Weighted Interval Scheduling

Weighted interval scheduling problem.

- Job j starts at s_j, finishes at f_j, and has weight or value v_j .
- Two jobs compatible if they don't overlap.
- Goal: find maximum weight subset of mutually compatible jobs.

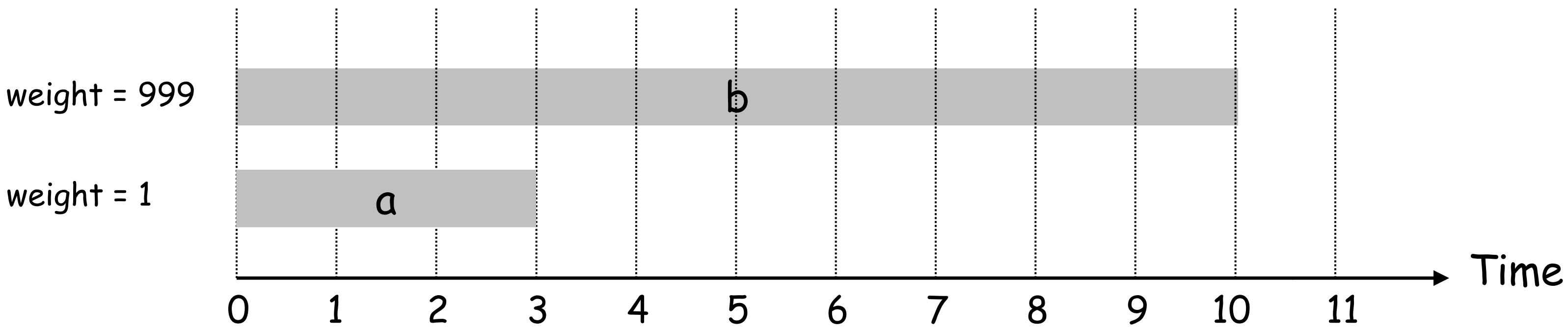


Unweighted Interval Scheduling Review

Recall. Greedy algorithm works if all weights are 1.

- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

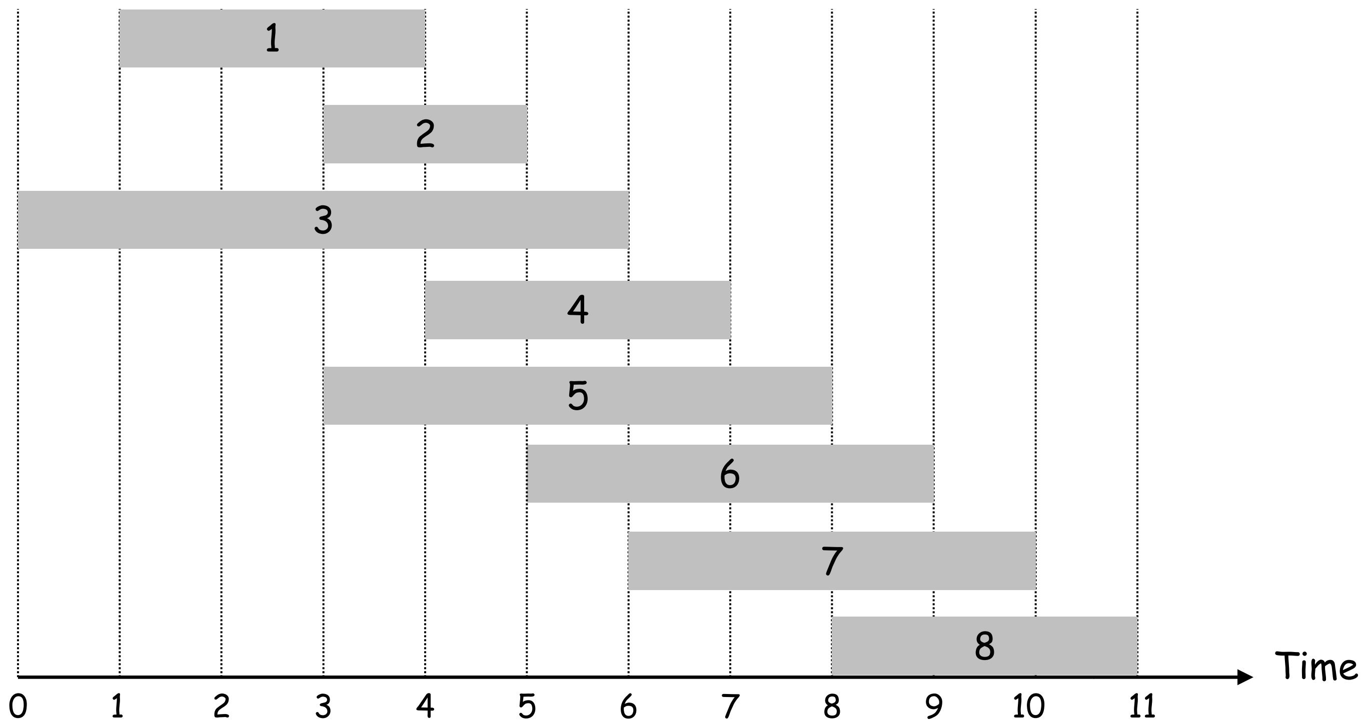
Observation. Greedy algorithm can fail spectacularly if arbitrary weights are allowed.



Weighted Interval Scheduling

Notation. Label jobs by finishing time: $f_1 \leq f_2 \leq \ldots \leq f_n$.
Def. $p(j)$ = largest index $i < j$ such that job i is compatible with j.

Ex: $p(8) = 5$, $p(7) = 3$, $p(2) = 0$.



Dynamic Programming: Binary Choice

Notation. OPT(j) = value of optimal solution to the problem consisting of job requests 1, 2, ..., j.

- Case 1: OPT selects job j.
  - can't use incompatible jobs { p(j) + 1, p(j) + 2, ..., j - 1 }
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., p(j)

optimal substructure

- Case 2: OPT does not select job j.
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., j-1

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max \{ v_j + OPT(p(j)),\ OPT(j-1) \} & \text{otherwise} \end{cases}$$

Weighted Interval Scheduling: Brute Force

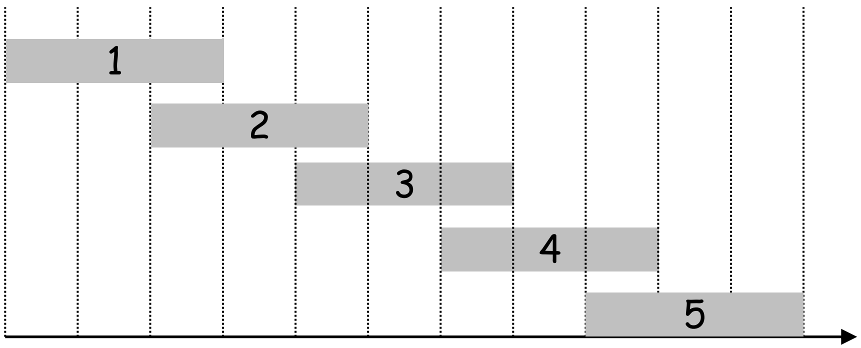
Brute force algorithm.

```
Input: n, s1,…,sn , f1,…,fn , v1,…,vn

Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.

Compute p(1), p(2), …, p(n)

Compute-Opt(j) {
   if (j = 0)
      return 0
   else
      return max(vj + Compute-Opt(p(j)), Compute-Opt(j-1))
}
```

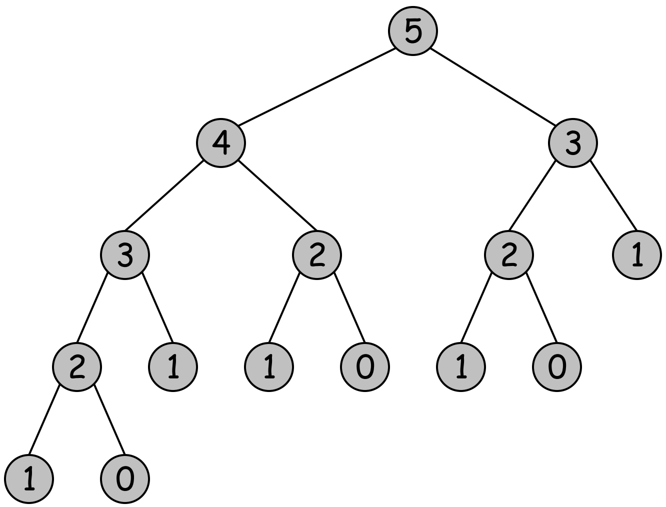
Weighted Interval Scheduling: Brute Force

Observation. Recursive algorithm fails spectacularly because of redundant sub-problems $\Rightarrow$ exponential algorithms.

Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence.



p(1) = 0, p(j) = j-2



Weighted Interval Scheduling: Memoization

Memoization. Store results of each sub-problem in a cache; lookup as needed.

```
Input: n, s1,…,sn , f1,…,fn , v1,…,vn

Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.
Compute p(1), p(2), …, p(n)

M[0] = 0
for j = 1 to n      ← global array
   M[j] = empty

M-Compute-Opt(j) {
   if (M[j] is empty)
      M[j] = max(wj + M-Compute-Opt(p(j)), M-Compute-Opt(j-1))
   return M[j]
}
```

Automated Memoization

Automated memoization. Many functional programming languages (e.g., Lisp) have built-in support for memoization.

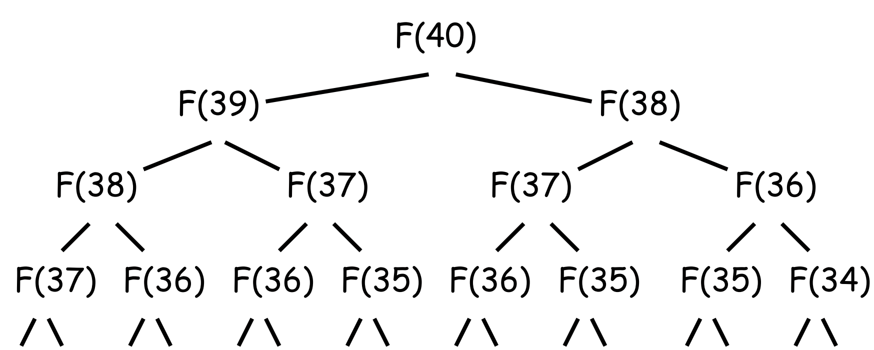
Q. Why not in imperative languages (e.g., Java)?

```
(defun F (n)
  (if
    (<= n 1)
    n
    (+ (F (- n 1)) (F (- n 2)))))
```

Lisp (efficient)

```
static int F(int n) {
   if (n <= 1) return n;
   else return F(n-1) + F(n-2);
}
```

Java (exponential)



Weighted Interval Scheduling: Finding a Solution

Q. Dynamic programming algorithms computes optimal value. What if we want the solution itself?

A. Do some post-processing.

```
Run M-Compute-Opt(n)
Run Find-Solution(n)

Find-Solution(j) {
   if (j = 0)
      output nothing
   else if (vj + M[p(j)] > M[j-1])
      print j
      Find-Solution(p(j))
   else
      Find-Solution(j-1)
}
```

- # of recursive calls $\leq n \Rightarrow O(n)$.

Weighted Interval Scheduling: Bottom-Up

Bottom-up dynamic programming. Unwind recursion.

```
Input: n, s_1,…,s_n , f_1,…,f_n , v_1,…,v_n

Sort jobs by finish times so that f_1 ≤ f_2 ≤ ... ≤ f_n.

Compute p(1), p(2), …, p(n)

Iterative-Compute-Opt {
   M[0] = 0
   for j = 1 to n
      M[j] = max(v_j + M[p(j)], M[j-1])
}
```

Maior subsequência crescente

Maior subsequência crescente

Subsequência crescente

- Seja A=(a(1),a(2),….,a(n)) uma sequência de números reais distintos.
- Encontrar a maior subsequência crescente A

- Exemplo A= (2, 3, 14, 5, 9, 8, 4)
- A maior subsequência crescente de A é 2,3,5,9

Maior subsequência crescente

- Seja L(i) : tamanho da maior subsequência crescente que termina em a(i)

- Exemplo A= (2, 3, 14, 5, 9, 8, 4)
- L(1)=1, L(2)=2, L(3)=3, L(4)=3, L(5)=4,L(6)=4,L(7)=3

- O tamanho da maior subsequência crescente é

$$\max \{ L(1), L(2), \ldots, L(n) \}$$

- Temos a seguinte equação para L

$$L(j) = \max\_i \quad \{ 1+L(i) \mid i < j \;\; e \;\; a(j) > a(i) \}$$

Maior subsequência crescente

```
Input: n, a1,…,an ,


M[1] = 1
for j = 2 to n
   M[j] = empty

M-Opt(j) {
   if (M[j] is empty)
        M[j]=1
        for i=1 to j-1
          if a(i)< a(j)
               M[j] = max( M[j], 1+ M-Opt(i) )
          end if
         end for
   end if
  return M[j]
}
```

Maior subsequência crescente

- Análise

- Complexidade = soma dos custos de todas as chamadas

- Vamos analisar o custo devido M-Opt(i)
  - M-OPT(i) é chamdo no máximo (n-1) vezes
  - O custo da primeira vez que M-OPT(i) é executado é O(i-1)
  - O custo de cada uma das demais chamada é O(1)
  - Portanto, o custo total devido a M-OPT(i) é O(n)

  - Somando para todo i temos um custo total de $O(n^2)$

Maior subsequência crescente

- Exercícios

  - Obter a maior subsequência crescente dado que o vetor M já está preenchido
  - Criar uma versão iterativa do algoritmo

  * Provar que toda sequencia tem uma subsequência crescente de tamanho n^(0.5) ou uma subsequência decrescente de tamanho n^(0.5)

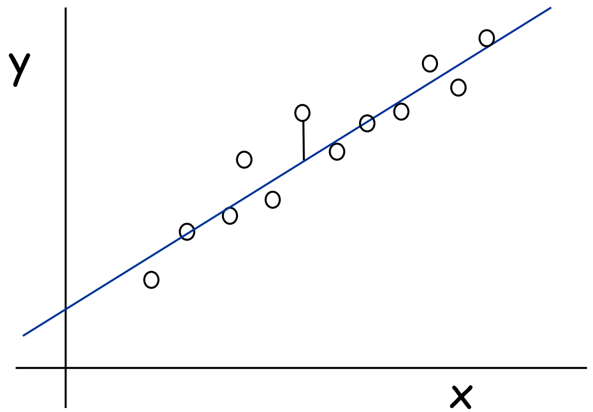
6.3 Segmented Least Squares

Segmented Least Squares

Least squares.

- Foundational problem in statistic and numerical analysis.
- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.
- Find a line $y = ax + b$ that minimizes the sum of the squared error:

$$SSE = \sum_{i=1}^{n} (y_i - a x_i - b)^2$$



Solution. Calculus $\Rightarrow$ min error is achieved when

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2}, \quad b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

Segmented Least Squares

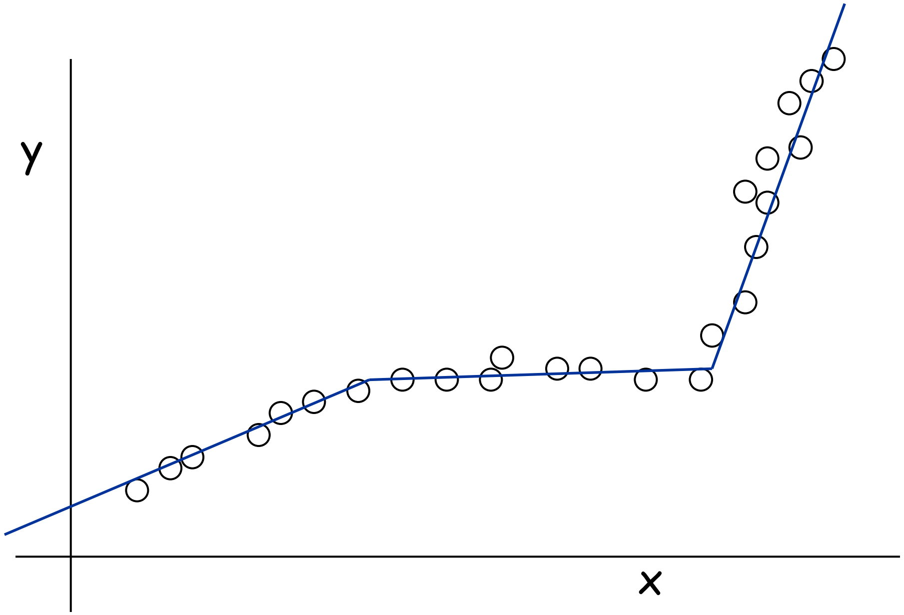
Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with
- $x_1 < x_2 < \ldots < x_n$, find a sequence of lines that minimizes $f(x)$.

Q. What's a reasonable choice for $f(x)$ to balance accuracy and parsimony?

goodness of fit

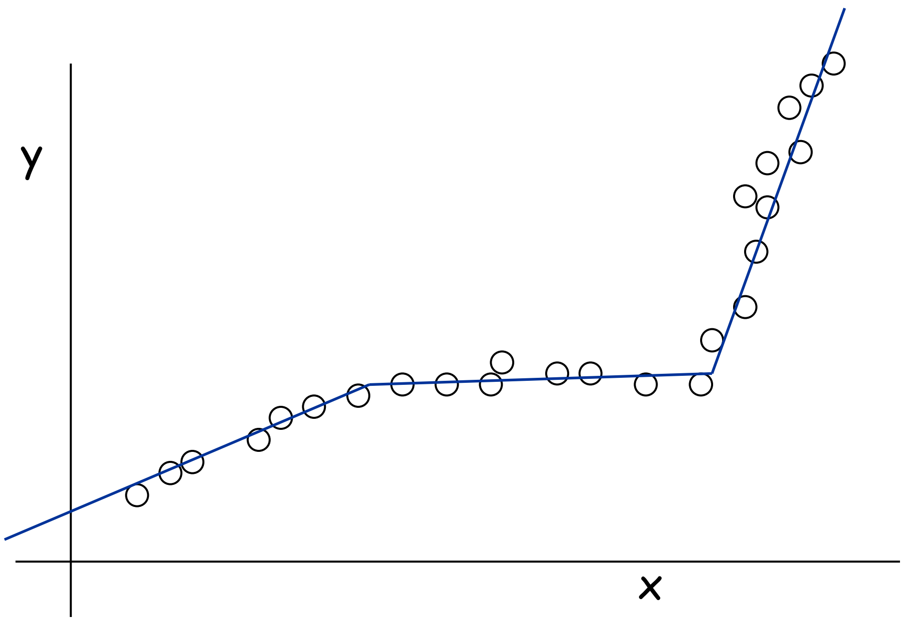
number of lines



Segmented Least Squares

Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with
- $x_1 < x_2 < \ldots < x_n$, find a sequence of lines that minimizes:
  - the sum of the sums of the squared errors E in each segment
  - the number of lines L
- Tradeoff function: $E + c\,L$, for some constant $c > 0$.



Dynamic Programming: Multiway Choice

Notation.

- $OPT(j)$ = minimum cost for points $p_1, p_{i+1}, \ldots, p_j$.
- $e(i, j)$ = minimum sum of squares for points $p_i, p_{i+1}, \ldots, p_j$.

To compute OPT(j):

- Last segment uses points $p_i, p_{i+1}, \ldots, p_j$ for some i.
- Cost = $e(i, j) + c + OPT(i-1)$.

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \min\limits_{1 \le i \le j} \{ \; e(i,j) + c + OPT(i-1) \} & \text{otherwise} \end{cases}$$

Segmented Least Squares: Algorithm

```
INPUT: n, p_1,…,p_N , c

Segmented-Least-Squares() {
   M[0] = 0
   for j = 1 to n
      for i = 1 to j
         compute the least square error e_ij for
         the segment p_i,…, p_j

   for j = 1 to n
      M[j] = min_{1 ≤ i ≤ j} (e_ij + c + M[i-1])

   return M[n]
}
```

Running time. $O(n^3)$. can be improved to $O(n^2)$ by pre-computing various statistics

- Bottleneck = computing $e(i, j)$ for $O(n^2)$ pairs, $O(n)$ per pair using previous formula.

6.4 Knapsack Problem

Knapsack Problem

Knapsack problem.

- Given n objects and a "knapsack."
- Item i weighs $w_i > 0$ kilograms and has value $v_i > 0$.
- Knapsack has capacity of W kilograms.
- Goal: fill knapsack so as to maximize total value.

Ex: { 3, 4 } has value 40.

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Greedy: repeatedly add item with maximum ratio v_i / w_i.

Ex: { 5, 2, 1 } achieves only value = 35 $\Rightarrow$ greedy not optimal.

Dynamic Programming: False Start

Def. OPT(i) = max profit subset of items 1, …, i.

- Case 1: OPT does not select item i.
  - OPT selects best of { 1, 2, …, i-1 }
- Case 2: OPT selects item i.
  - accepting item i does not immediately imply that we will have to reject other items
  - without knowing what other items were selected before i, we don't even know if we have enough room for i

Conclusion. Need more sub-problems!

Dynamic Programming: Adding a New Variable

Def. OPT(i, w) = max profit subset of items 1, …, i with weight limit w.

- Case 1: OPT does not select item i.
  - OPT selects best of { 1, 2, …, i-1 } using weight limit w
- Case 2: OPT selects item i.
  - new weight limit = w – w_i
  - OPT selects best of { 1, 2, …, i–1 } using this new weight limit

$$OPT(i,w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1,w) & \text{if } w_i > w \\ \max\{\, OPT(i-1,w),\ \ v_i + OPT(i-1, w-w_i)\,\} & \text{otherwise} \end{cases}$$

Knapsack Problem: Bottom-Up

Knapsack. Fill up an n-by-W array.

```
Input: n, w1,…,wN, v1,…,vN

for w = 0 to W
   M[0, w] = 0

for i = 1 to n
   for w = 1 to W
      if (wi > w)
         M[i, w] = M[i-1, w]
      else
         M[i, w] = max {M[i-1, w], vi + M[i-1, w-wi ]}

return M[n, W]
```

Knapsack Algorithm

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT: { 4, 3 }
value = 22 + 18 = 40

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Knapsack Problem: Running Time

Running time. $\Theta(n\ W)$.

- Not polynomial in input size!
- "Pseudo-polynomial."
- Decision version of Knapsack is NP-complete. [Chapter 8]

Knapsack approximation algorithm. There exists a polynomial algorithm that produces a feasible solution that has value within 0.01% of optimum. [Section 11.8]

Optimal Binary Search Trees

- Estruturas de Dados e Seus Algoritmos
  - Lilian Markezon
  - Jayme Szwarcfyter

Optimal Binary Search Trees

Problem

- Given sequence $K = k_1 < k_2 < \cdots < k_n$ of n sorted keys, with a search probability p_i for each key k_i.
- Want to build a binary search tree (BST) with minimum expected search cost.
- Actual cost = # of items examined.
- For key k_i,

$$\text{cost} = \text{depth}_T(k_i) + 1,$$

where $\text{depth}_T(k_i)$ = depth of k_i in BST T. (root is at depth 0)

Expected Search Cost

$$E[\text{search cost in } T]$$

$$= \sum_{i=1}^{n} (\text{depth}_T(k_i) + 1) \cdot p_i$$

$$= \sum_{i=1}^{n} \text{depth}_T(k_i) \cdot p_i + \sum_{i=1}^{n} p_i$$

Sum of probabilities is 1.

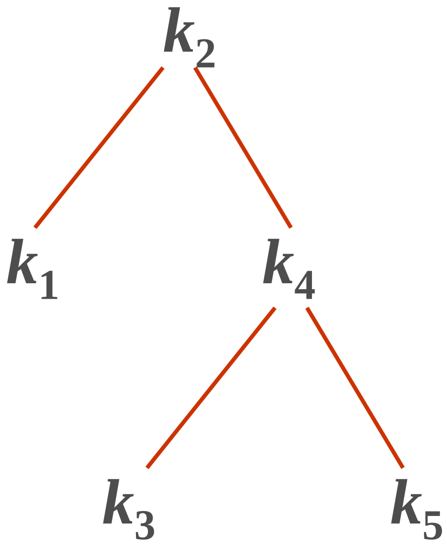
$$= 1 + \sum_{i=1}^{n} \text{depth}_T(k_i) \cdot p_i$$

Identity (1)

Example

Consider 5 keys with these search probabilities:

$p_1 = 0.25$, $p_2 = 0.2$, $p_3 = 0.05$, $p_4 = 0.2$, $p_5 = 0.3$.

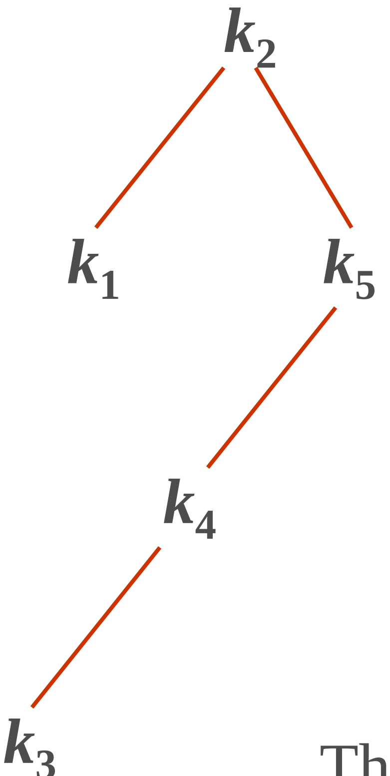


i	$\text{depth}_T(k_i)$	$\text{depth}_T(k_i) \cdot p_i$
1	1	0.25
2	0	0
3	2	0.1
4	1	0.2
5	2	0.6
		1.15

Therefore, E[search cost] = 2.15.

Example

$p_1 = 0.25$, $p_2 = 0.2$, $p_3 = 0.05$, $p_4 = 0.2$, $p_5 = 0.3$.



i	$\text{depth}_T(k_i)$	$\text{depth}_T(k_i) \cdot p_i$
1	1	0.25
2	0	0
3	3	0.15
4	2	0.4
5	1	0.3
		1.10

Therefore, E[search cost] = 2.10.

This tree turns out to be optimal for this set of keys.

Example

Observations:

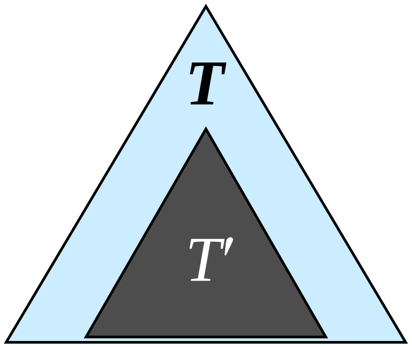
- Optimal BST may not have smallest height.
- Optimal BST may not have highest-probability key at root.

Build by exhaustive checking?

- Construct each n-node BST.
- For each,
  assign keys and compute expected search cost.
- But there are $\Omega(4^n/n^{3/2})$ different BSTs with n nodes.

Optimal Substructure

Any subtree of a BST contains keys in a contiguous range k_i, ..., k_j for some $1 \leq i \leq j \leq n$.



If T is an optimal BST and

T contains subtree T' with keys k_i, ... ,k_j ,

then T' must be an optimal BST for keys k_i, ..., k_j.

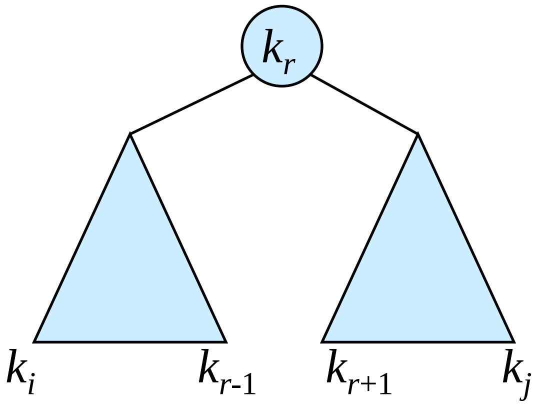
Proof: Cut and paste.

Optimal Substructure

One of the keys in $k_i, \ldots, k_j$, say k_r, where $i \le r \le j$,
must be the root of an optimal subtree for these keys.

Left subtree of k_r contains $k_i, \ldots, k_{r-1}$.

Right subtree of k_r contains $k_r+1, \ldots, k_j$.



To find an optimal BST:

- Examine all candidate roots k_r , for $i \le r \le j$
- Determine all optimal BSTs containing $k_i, \ldots, k_{r-1}$ and containing $k_{r+1}, \ldots, k_j$

Recursive Solution

Define $e[i, j]$ = expected search cost of optimal BST for $k_i,\ldots,k_j$.

If $j = i-1$, then $e[i, j] = 0$.

If $j \geq i$,

- Select a root k_r, for some $i \leq r \leq j$.
- Recursively make an optimal BSTs
  - for $k_i,..,k_{r-1}$ as the left subtree, and
  - for $k_{r+1},..,k_j$ as the right subtree.

Recursive Solution

When the OPT subtree becomes a subtree of a node:

- Depth of every node in OPT subtree goes up by 1.
- Expected search cost increases by

$$w(i, j) = \sum_{l=i}^{j} p_l$$

from Identity (1)

If k_r is the root of an optimal BST for $k_i,..,k_j$:

- $e[i, j] = p_r + (e[i, r-1] + w(i, r-1)) + (e[r+1, j] + w(r+1, j))$

  $= e[i, r-1] + e[r+1, j] + w(i, j).$ (because $w(i, j) = w(i, r-1) + p_r + w(r+1, j)$)

But, we don't know k_r. Hence,

$$e[i, j] = \begin{cases} 0 & \text{if } j = i-1 \\ \min\limits_{i \le r \le j} \{e[i, r-1] + e[r+1, j] + w(i, j)\} & \text{if } i \le j \end{cases}$$

Computing an Optimal Solution

For each subproblem (i,j), store:

expected search cost in a table $e[1..n+1, 0..n]$

- Will use only entries $e[i, j]$, where $j \geq i-1$.

$root[i, j]$ = root of subtree with keys $k_i,..,k_j$, for $1 \leq i \leq j \leq n$.

$w[1..n+1, 0..n]$ = sum of probabilities

- $w[i, i-1] = 0$ for $1 \leq i \leq n$.
- $w[i, j] = w[i, j-1] + p_j$ for $1 \leq i \leq j \leq n$.

Pseudo-code

```
OPTIMAL-BST(p, q, n)
for i ← 1 to n + 1
    do e[i, i– 1] ← 0
       w[i, i– 1] ← 0
for l ← 1 to n
    do for i ← 1 to n–l + 1
        do j ←i + l–1
            e[i, j ]←∞
            w[i, j ] ← w[i, j–1] + p_j
            for r ←i to j
                do t ← e[i, r–1] + e[r + 1, j ] + w[i, j ]
                    if t < e[i, j ]
                        then e[i, j ] ← t
                             root[i, j ] ←r
return e and root
```

Consider all trees with *l* keys.

Fix the first key.

Fix the last key

Determine the root of the optimal (sub)tree

Time: $O(n^3)$

Remark: It can be improved to $O(n^2)$ by using Knuth principle

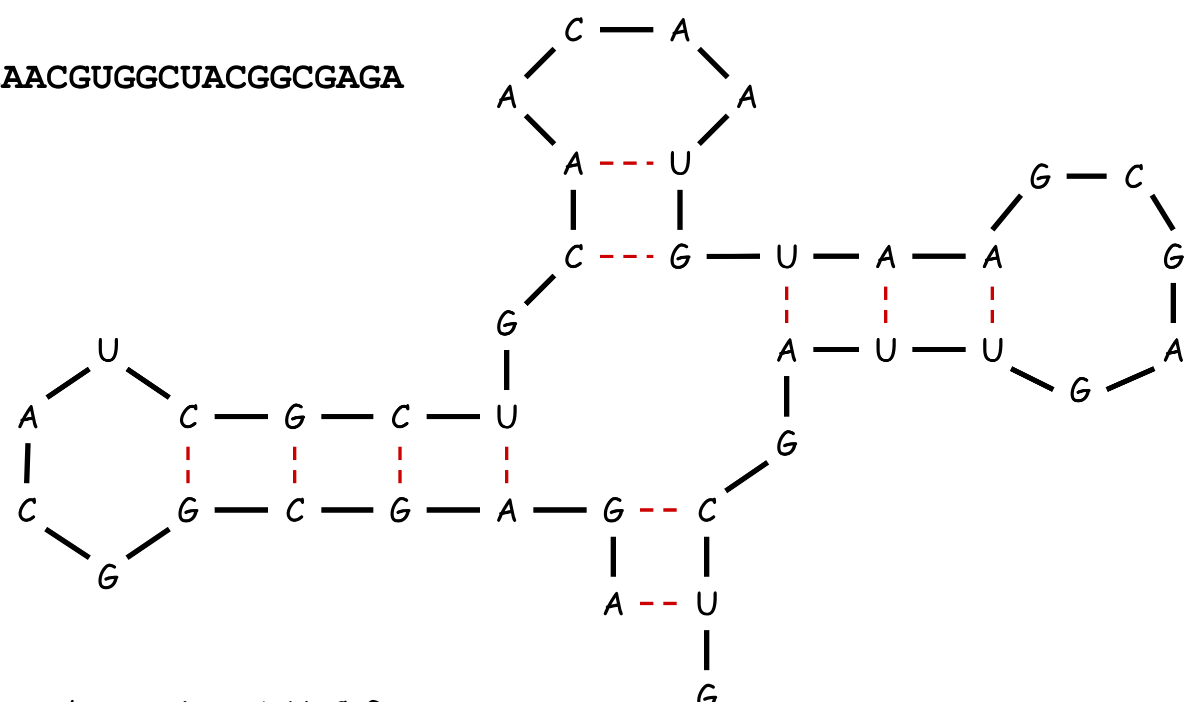
6.5 RNA Secondary Structure

RNA Secondary Structure

RNA. String $B = b_1 b_2 \ldots b_n$ over alphabet { A, C, G, U }.

Secondary structure. RNA is single-stranded so it tends to loop back and form base pairs with itself. This structure is essential for understanding behavior of molecule.

Ex: **GUCGAUUGAGCGAAUGUAACAACGUGGCUACGGCGAGA**



complementary base pairs: A-U, C-G

RNA Secondary Structure

Secondary structure. A set of pairs $S = \{ (b_i, b_j) \}$ that satisfy:

- [Watson-Crick.] S is a matching and each pair in S is a Watson-Crick complement: A-U, U-A, C-G, or G-C.
- [No sharp turns.] The ends of each pair are separated by at least 4 intervening bases. If $(b_i, b_j) \in S$, then $i < j - 4$.
- [Non-crossing.] If (b_i, b_j) and (b_k, b_l) are two pairs in S, then we cannot have $i < k < j < l$.

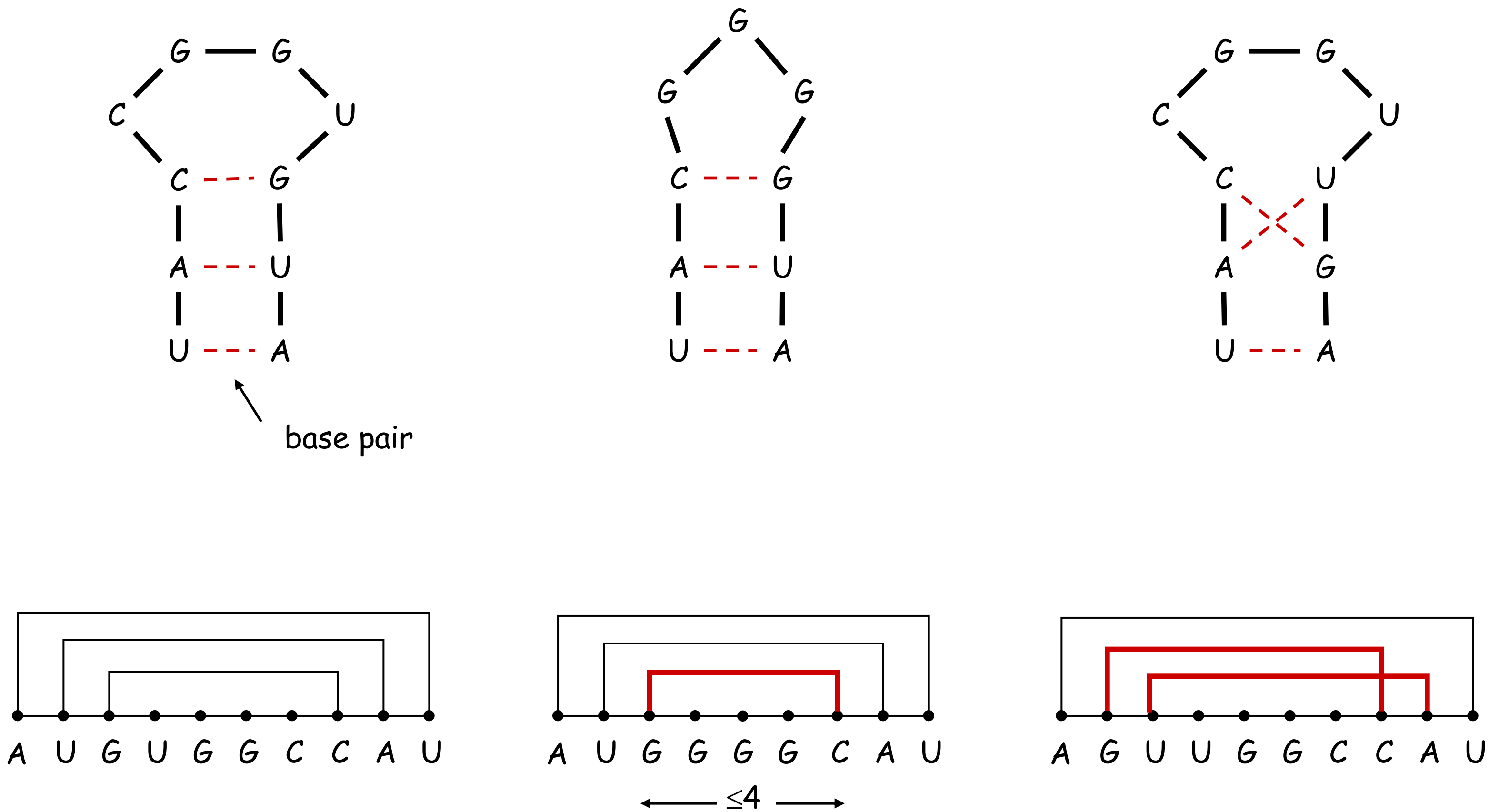
Free energy. Usual hypothesis is that an RNA molecule will form the secondary structure with the optimum total free energy.

approximate by number of base pairs

Goal. Given an RNA molecule $B = b_1 b_2 \ldots b_n$, find a secondary structure S that maximizes the number of base pairs.

RNA Secondary Structure: Examples

Examples.



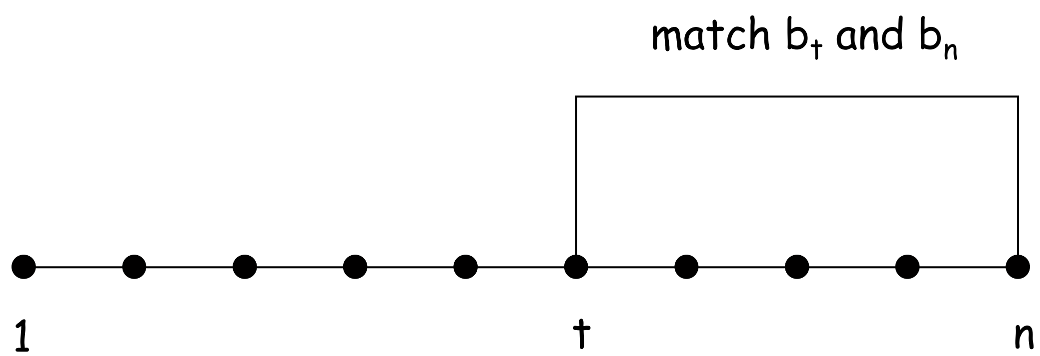
ok

sharp turn

crossing

RNA Secondary Structure: Subproblems

First attempt. $OPT(j)$ = maximum number of base pairs in a secondary structure of the substring $b_1 b_2 \ldots b_j$.



Difficulty. Results in two sub-problems.

- Finding secondary structure in: $b_1 b_2 \ldots b_{t-1}$. ← $OPT(t-1)$
- Finding secondary structure in: $b_{t+1} b_{t+2} \ldots b_{n-1}$. ← need more sub-problems

Dynamic Programming Over Intervals

Notation. $OPT(i, j)$ = maximum number of base pairs in a secondary structure of the substring $b_i b_{i+1} \ldots b_j$.

- Case 1. If $i \geq j - 4$.
  - $OPT(i, j) = 0$ by no-sharp turns condition.
- Case 2. Base b_j is not involved in a pair.
  - $OPT(i, j) = OPT(i, j-1)$
- Case 3. Base b_j pairs with b_t for some $i \leq t < j - 4$.
  - non-crossing constraint decouples resulting sub-problems
  - $OPT(i, j) = 1 + \max_t \{ OPT(i, t-1) + OPT(t+1, j-1) \}$

take max over t such that $i \leq t < j-4$ and b_t and b_j are Watson-Crick complements

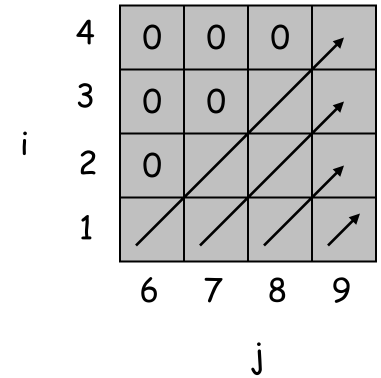
Remark. Same core idea in CKY algorithm to parse context-free grammars.

Bottom Up Dynamic Programming Over Intervals

Q. What order to solve the sub-problems?
A. Do shortest intervals first.

```
RNA(b1,…,bn) {
   for k = 5, 6, …, n-1
      for i = 1, 2, …, n-k
         j = i + k
         Compute M[i, j]

   return M[1, n]
}
```

using recurrence



Running time. $O(n^3)$.

Dynamic Programming Summary

Recipe.

- Characterize structure of problem.
- Recursively define value of optimal solution.
- Compute value of optimal solution.
- Construct optimal solution from computed information.

Dynamic programming techniques.

- Binary choice: weighted interval scheduling.
- Multi-way choice: segmented least squares.
- Adding a new variable: knapsack.
- Dynamic programming over intervals: RNA secondary structure.

Viterbi algorithm for HMM also uses DP to optimize a maximum likelihood tradeoff between parsimony and accuracy

CKY parsing algorithm for context-free grammar has similar structure

Top-down vs. bottom-up: different people have different intuitions.

6.6 Sequence Alignment

String Similarity

How similar are two strings?

- `ocurrance`
- `occurrence`

o	c	u	r	r	a	n	c	e	-
o	c	c	u	r	r	e	n	c	e

6 mismatches, 1 gap

o	c	-	u	r	r	a	n	c	e
o	c	c	u	r	r	e	n	c	e

1 mismatch, 1 gap

o	c	-	u	r	r	-	a	n	c	e
o	c	c	u	r	r	e	-	n	c	e

0 mismatches, 3 gaps

Edit Distance

Applications.

- Basis for Unix diff.
- Speech recognition.
- Computational biology.

Edit distance. [Levenshtein 1966, Needleman-Wunsch 1970]

- Gap penalty δ; mismatch penalty α_{pq}.
- Cost = sum of gap and mismatch penalties.

C	T	G	A	C	C	T	A	C	C	T
C	C	T	G	A	C	T	A	C	A	T

$\alpha_{TC} + \alpha_{GT} + \alpha_{AG} + 2\alpha_{CA}$

-	C	T	G	A	C	C	T	A	C	C	T
C	C	T	G	A	C	-	T	A	C	A	T

$2\delta + \alpha_{CA}$

Sequence Alignment

Goal: Given two strings $X = x_1\, x_2 \ldots x_m$ and $Y = y_1\, y_2 \ldots y_n$ find alignment of minimum cost.

Def. An alignment M is a set of ordered pairs x_i-y_j such that each item occurs in at most one pair and no crossings.

Def. The pair x_i-y_j and $x_{i'}$-$y_{j'}$ cross if $i < i'$, but $j > j'$.

$$\text{cost}(M) = \underbrace{\sum_{(x_i, y_j) \in M} \alpha_{x_i y_j}}_{\text{mismatch}} + \underbrace{\sum_{i\,:\,x_i \text{ unmatched}} \delta + \sum_{j\,:\,y_j \text{ unmatched}} \delta}_{\text{gap}}$$

Ex: CTACCG vs. TACATG.

Sol: $M = x_2\text{-}y_1,\ x_3\text{-}y_2,\ x_4\text{-}y_3,\ x_5\text{-}y_4,\ x_6\text{-}y_6$.

x_1	x_2	x_3	x_4	x_5		x_6
C	T	A	C	C	-	G
-	T	A	C	A	T	G
	y_1	y_2	y_3	y_4	y_5	y_6

Sequence Alignment: Problem Structure

Def. OPT(i, j) = min cost of aligning strings $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_j$.

- Case 1: OPT matches x_i-y_j.
  - pay mismatch for x_i-y_j + min cost of aligning two strings $x_1 x_2 \ldots x_{i-1}$ and $y_1 y_2 \ldots y_{j-1}$
- Case 2a: OPT leaves x_i unmatched.
  - pay gap for x_i and min cost of aligning $x_1 x_2 \ldots x_{i-1}$ and $y_1 y_2 \ldots y_j$
- Case 2b: OPT leaves y_j unmatched.
  - pay gap for y_j and min cost of aligning $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_{j-1}$

$$OPT(i,\ j) = \begin{cases} j\delta & \text{if } i = 0 \\ \min \begin{cases} \alpha_{x_i y_j} + OPT(i-1,\ j-1) \\ \delta + OPT(i-1,\ j) \\ \delta + OPT(i,\ j-1) \end{cases} & \text{otherwise} \\ i\delta & \text{if } j = 0 \end{cases}$$

Sequence Alignment: Algorithm

```
Sequence-Alignment(m, n, x₁x₂...xₘ, y₁y₂...yₙ, δ, α) {
   for i = 0 to m
      M[i, 0] = iδ
   for j = 0 to n
      M[0, j] = jδ

   for i = 1 to m
      for j = 1 to n
         M[i, j] = min(α[xᵢ, yⱼ] + M[i-1, j-1],
                       δ + M[i-1, j],
                       δ + M[i, j-1])
   return M[m, n]
}
```

Analysis. $\Theta(mn)$ time and space.

English words or sentences: $m, n \leq 10$.

Computational biology: $m = n = 100{,}000$. 10 billions ops OK, but 10GB array?

6.7 Sequence Alignment in Linear Space

Sequence Alignment: Linear Space

Q. Can we avoid using quadratic space?

Easy. Optimal value in O(m + n) space and O(mn) time.
- Compute OPT(i, •) from OPT(i-1, •).
- No longer a simple way to recover alignment itself.

Theorem. [Hirschberg 1975] Optimal alignment in O(m + n) space and O(mn) time.
- Clever combination of divide-and-conquer and dynamic programming.
- Inspired by idea of Savitch from complexity theory.

Sequence Alignment: Value of OPT with Linear Space

```
Sequence-Alignment(m, n, x1x2...xm, y1y2...yn, δ, α) {

   for i = 0 to m
      CURRENT[i] = jδ

   for j = 1 to n
      LAST←CURRENT ( vector copy)
      CURRENT[0]← jδ
      for i = 1 to m
         CURRENT[i] ← min(α[xi, yj] + LAST[i-1],
                          δ + LAST[i],
                          δ + CURRENT[i-1] )
return CURRENT[m]
}
```

- Two vectors of of n positions: LAST e CURRENT
- $O(mn)$ time and $O(m+n)$ space

Sequence Alignment: Value of OPT with Linear Space

LAST CURRENT

		T	A	C	A	T	G
C							
T							
A							
C							
C							
G							

Sequence Alignment: Algorithm for recovering the sequence

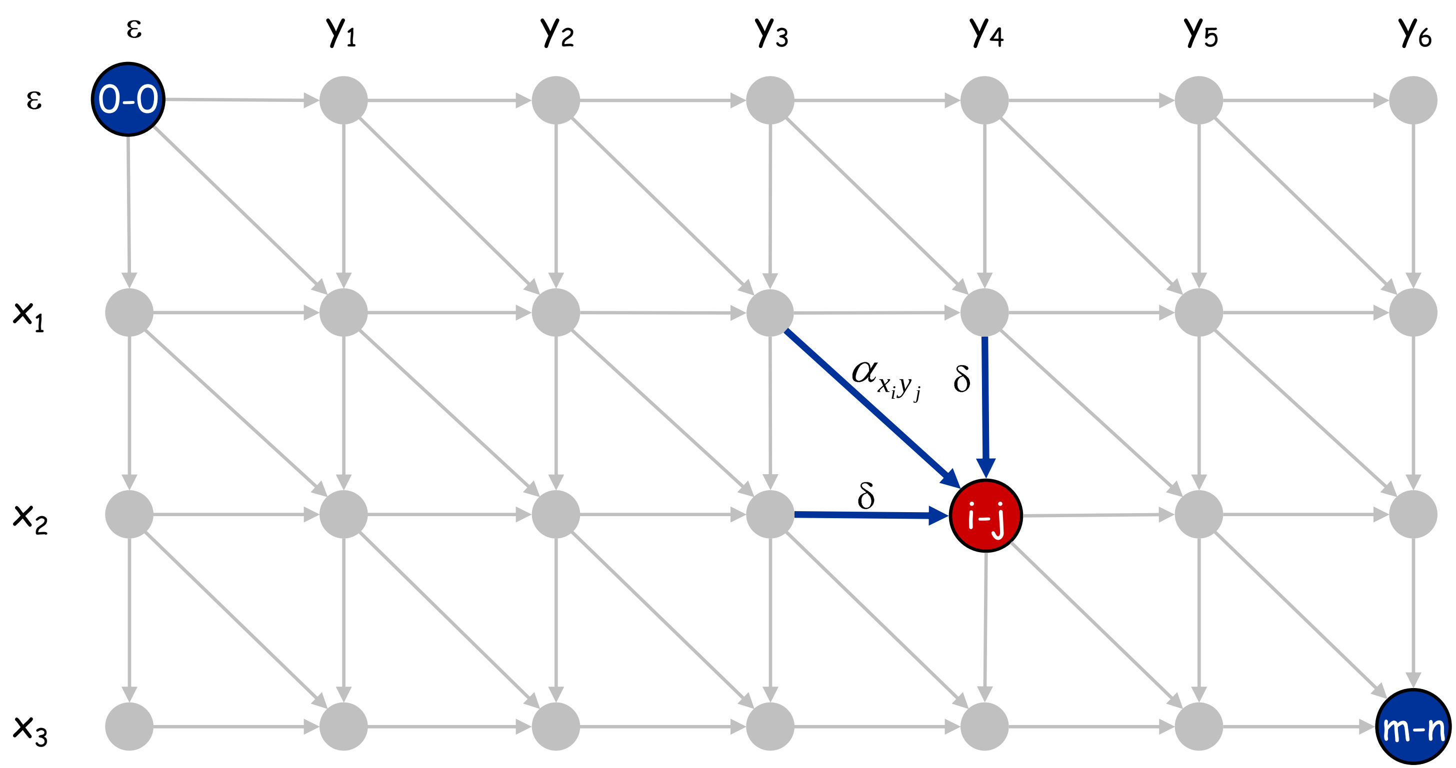
```
Find_Sequence (i, j, x₁x₂...xₘ, y₁y₂...yₙ, δ, α) {
    If i=0 or j=0 return
    Else
        If M[i, j] = α[xᵢ, yⱼ] + M[i-1, j-1]
            Add pair xᵢ - yⱼ to the solution
            Return Find_Sequence(i-1,j-1)
        Else If M[i,j]= δ + M[i-1, j]
             Return Find_sequence(i-1,j)
        Else return Find_sequence(i,j-1)
}
```

Analysis. $\Theta(mn)$ space and $O(m+n)$ time

Sequence Alignment: Linear Space

Edit distance graph.

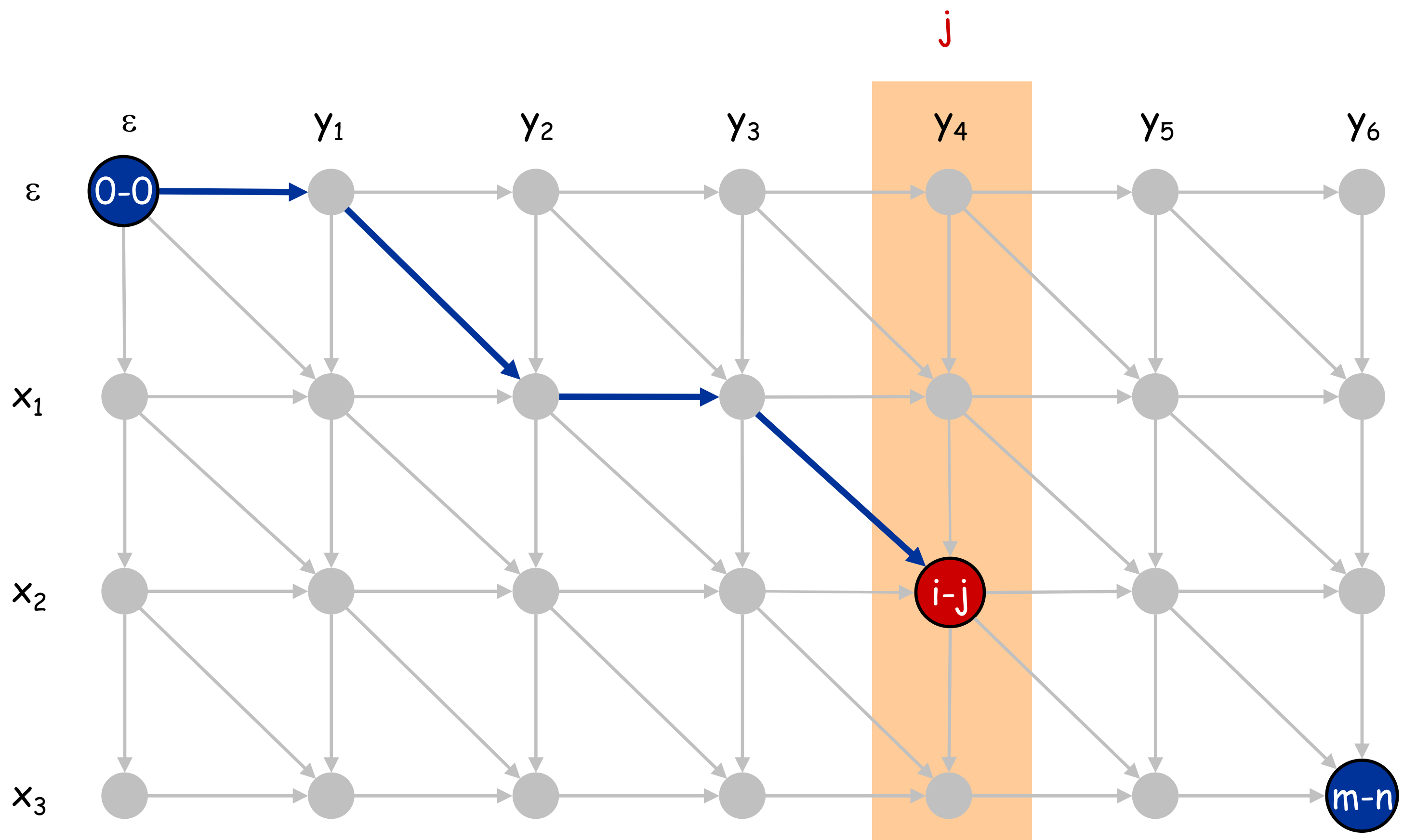
- Let f(i, j) be shortest path from (0,0) to (i, j).
- Observation: f(i, j) = OPT(i, j).



Sequence Alignment: Linear Space

Edit distance graph.

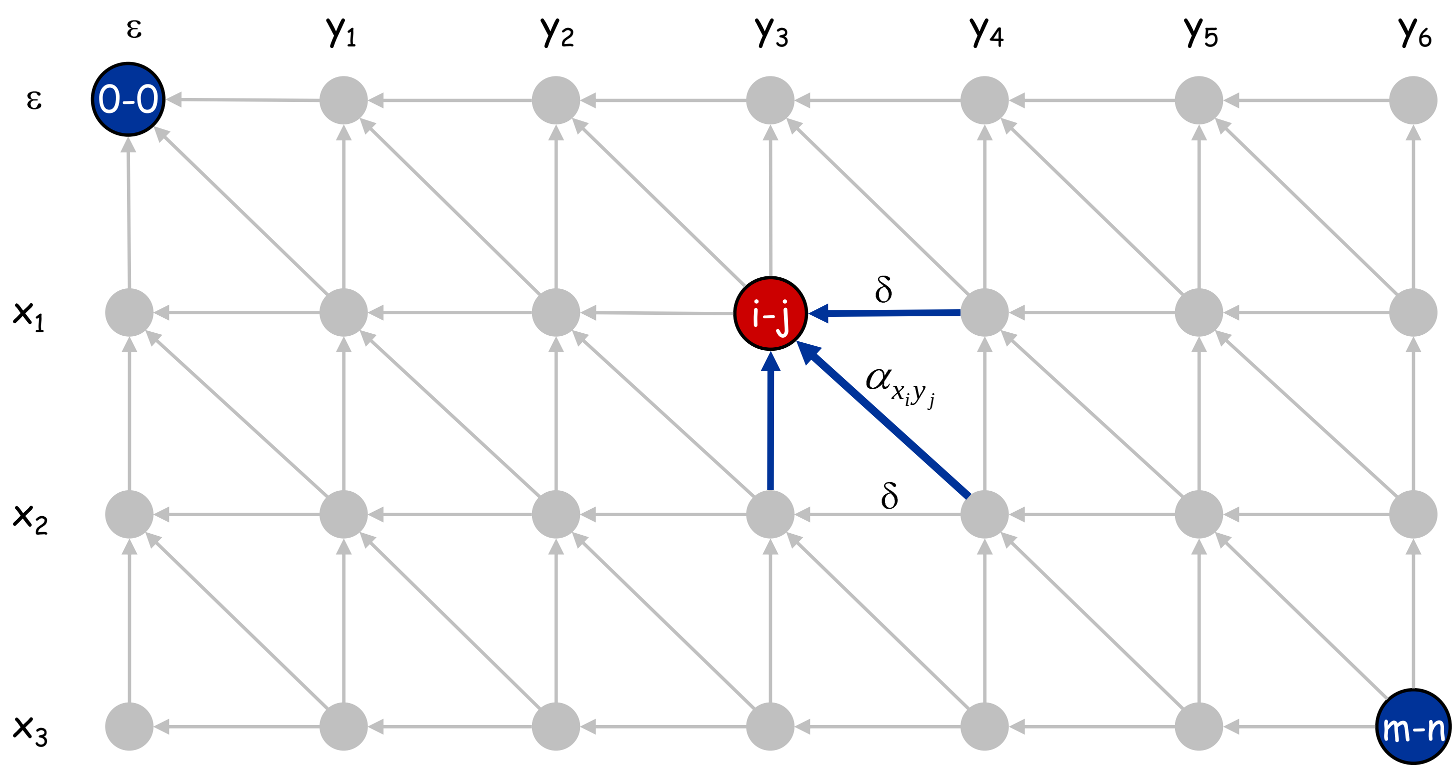
- Let $f(i, j)$ be shortest path from (0,0) to (i, j).
- Can compute $f(\cdot, j)$ for any j in $O(mn)$ time and $O(m + n)$ space.



Sequence Alignment: Linear Space

Edit distance graph.

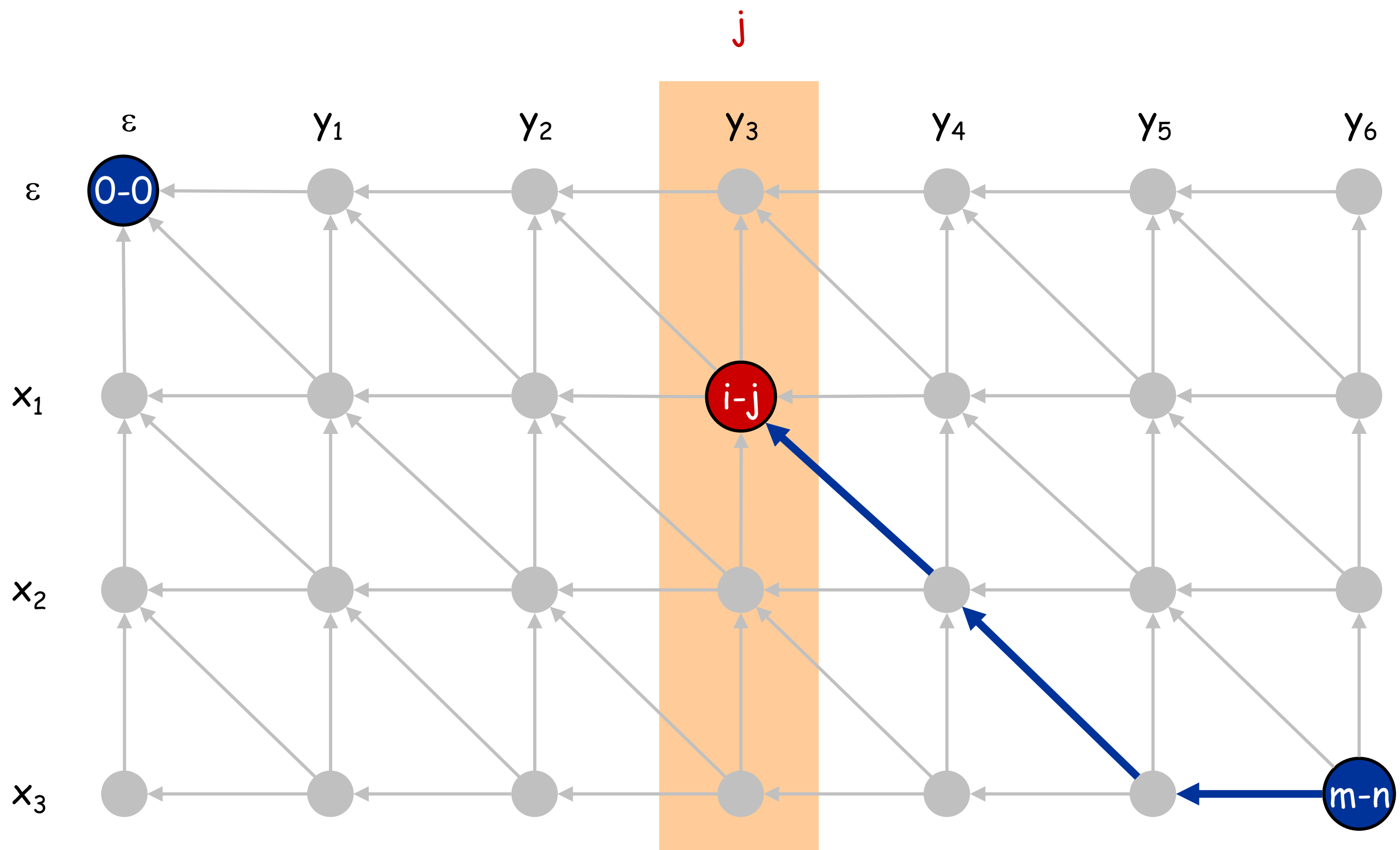
- Let $g(i, j)$ be shortest path from (i, j) to (m, n).
- Can compute by reversing the edge orientations and inverting the roles of $(0, 0)$ and (m, n)



Sequence Alignment: Linear Space

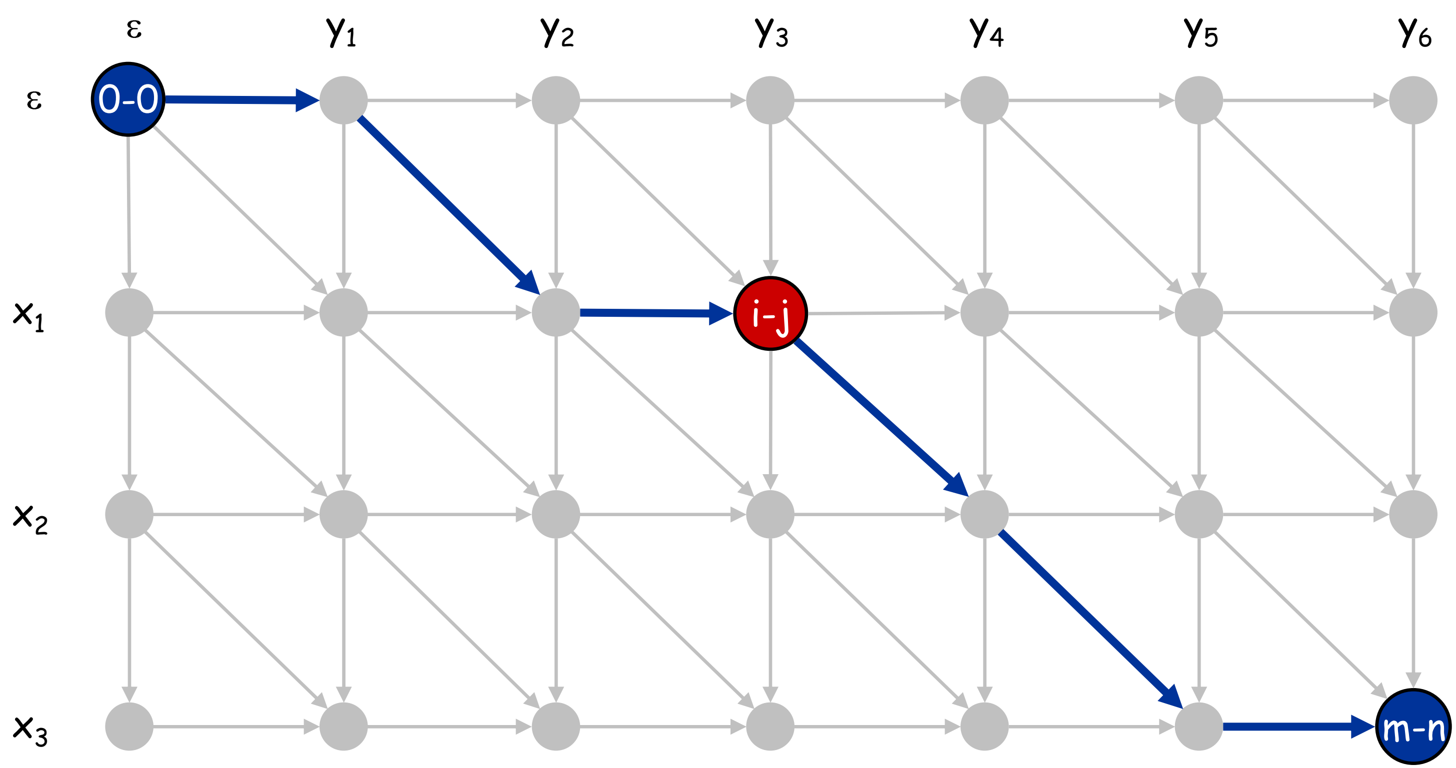
Edit distance graph.

- Let $g(i, j)$ be shortest path from (i, j) to (m, n).
- Can compute $g(\bullet, j)$ for any j in $O(mn)$ time and $O(m + n)$ space.



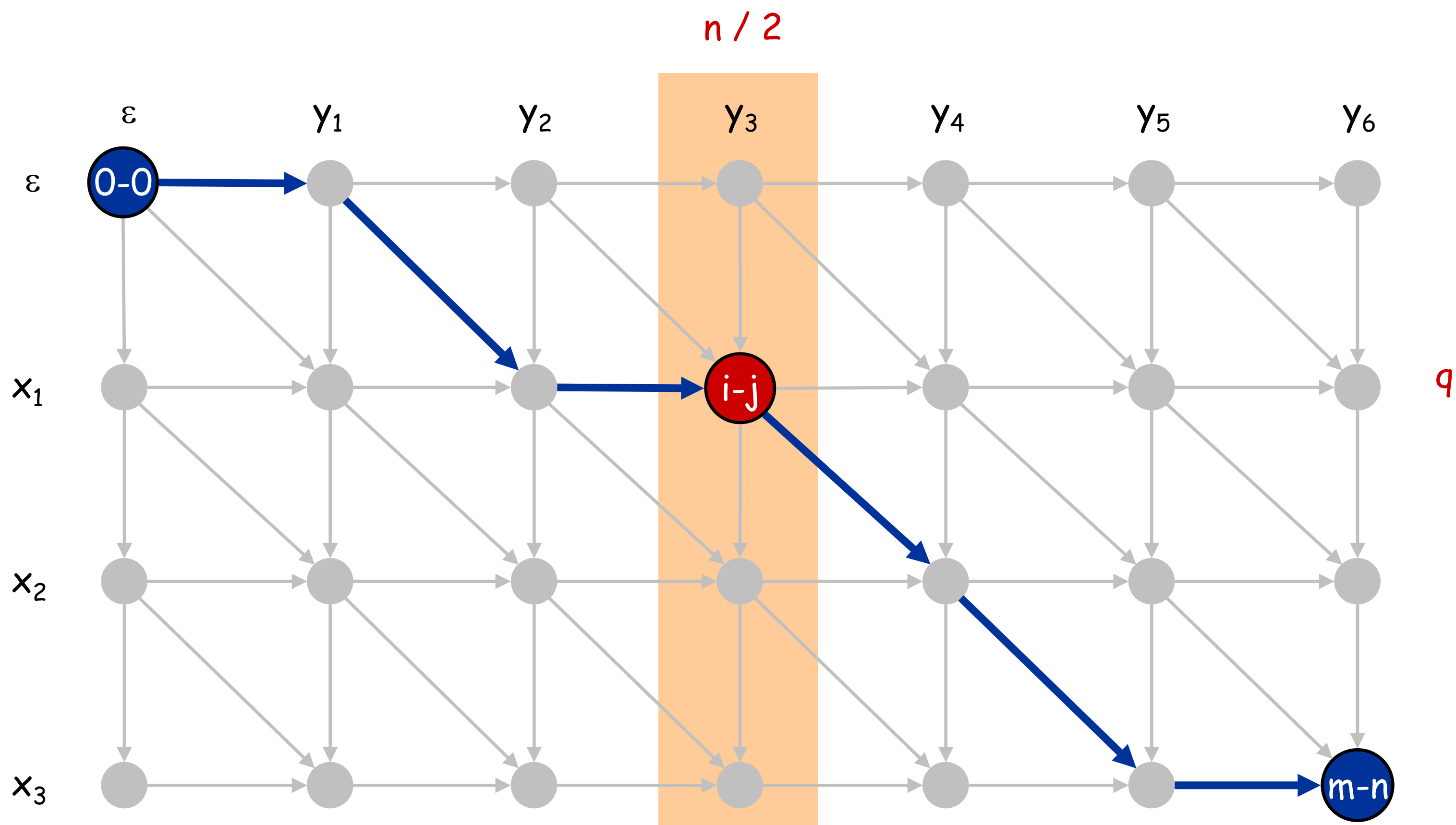
Sequence Alignment: Linear Space

Observation 1. The cost of the shortest path that uses (i, j) is $f(i, j) + g(i, j)$.



Sequence Alignment: Linear Space

Observation 2. let q be an index that minimizes f(q, n/2) + g(q, n/2). Then, the shortest path from (0, 0) to (m, n) uses (q, n/2).

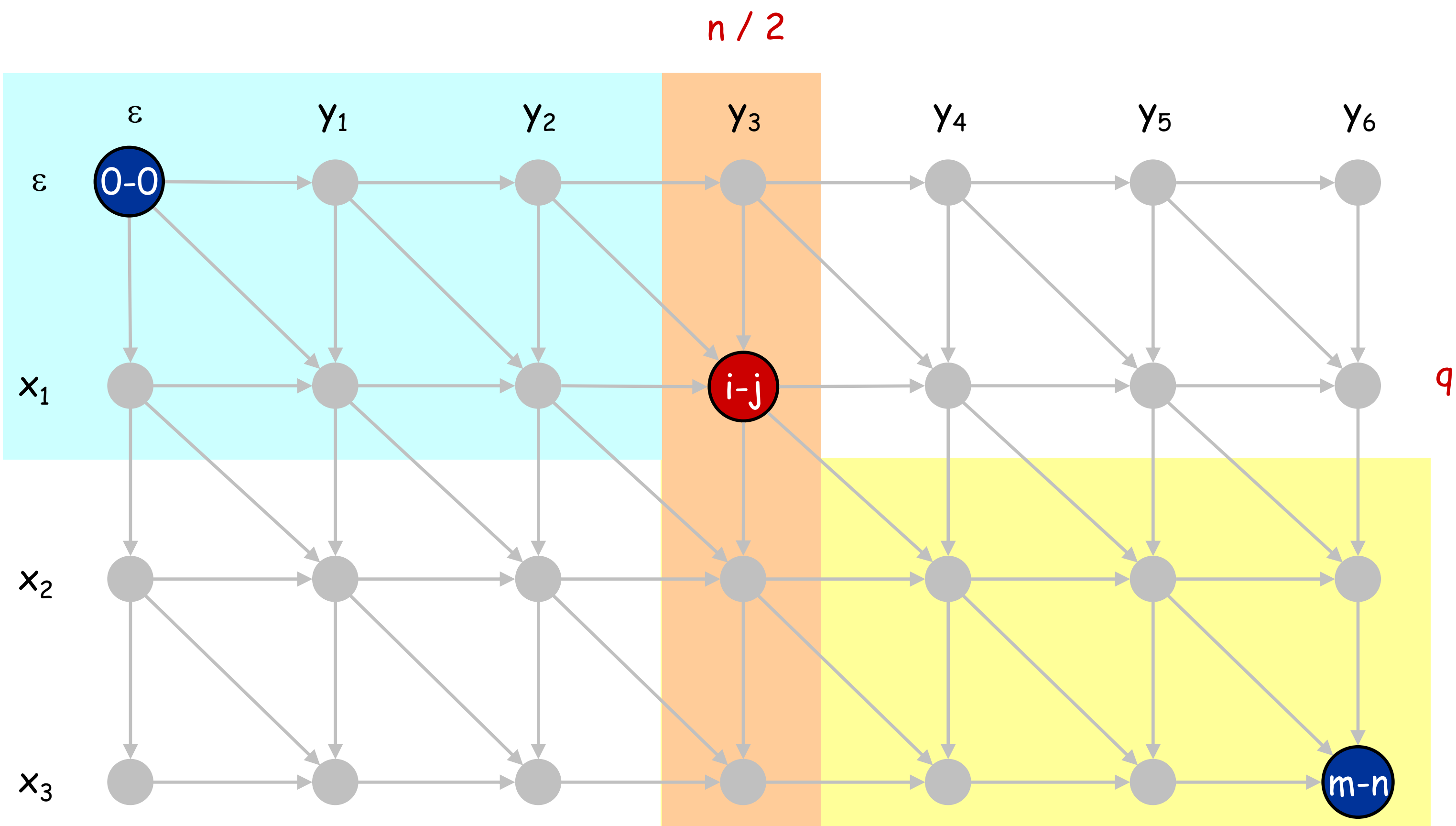


Sequence Alignment: Linear Space

Divide: find index q that minimizes f(q, n/2) + g(q, n/2) using DP.

- Align x_q and $y_{n/2}$.

Conquer: recursively compute optimal alignment in each piece.



Sequence Alignment: Running Time Analysis Warmup

Theorem. Let T(m, n) = max running time of algorithm on strings of length at most m and n. T(m, n) = O(mn log n).

$$T(m,n) \;\le\; 2T(m,\, n/2) + O(mn) \;\;\Rightarrow\;\; T(m,n) \;=\; O(mn \log n)$$

Remark. Analysis is not tight because two sub-problems are of size (q, n/2) and (m - q, n/2). In next slide, we save log n factor.

Sequence Alignment: Running Time Analysis

Theorem. Let T(m, n) = max running time of algorithm on strings of length m and n. T(m, n) = O(mn).

Pf. (by induction on n)

- O(mn) time to compute f(•, n/2) and g (•, n/2) and find index q.
- T(q, n/2) + T(m - q, n/2) time for two recursive calls.
- Choose constant c so that:

$$\begin{aligned} T(m,\ 2) &\leq cm \\ T(2,\ n) &\leq cn \\ T(m,\ n) &\leq cmn + T(q,\ n/2) + T(m-q,\ n/2) \end{aligned}$$

- Base cases: m = 2 or n = 2.
- Inductive hypothesis: T(m, n) $\leq$ 2cmn.

$$\begin{aligned} T(m,n) &\leq T(q,n/2) + T(m-q,n/2) + cmn \\ &\leq 2cqn/2 + 2c(m-q)n/2 + cmn \\ &= cqn + cmn - cqn + cmn \\ &= 2cmn \end{aligned}$$