1 The answers for both parts are given below:

(a) We have the following chain of identities:

$$\begin{aligned}
\mathrm{Var}(X-Y) &= \mathrm{Var}(X+Z), \text{ where } Z=-Y \\
&= \mathrm{Var}(X)+\mathrm{Var}(Z) \text{ [statistical independence]} \\
&= \mathrm{Var}(X)+\mathrm{Var}[(-1)Y] \\
&= \mathrm{Var}(X)+(-1)^2\mathrm{Var}(Y) \text{ [Theorem 2.9B]} \\
&= \mathrm{Var}(X)+\mathrm{Var}(Y) \\
&= 2.
\end{aligned}$$

(b) Using the properties of the variance, we get:

$$\begin{aligned}
\mathrm{Var}(2X-3Y+1) &= \mathrm{Var}(2X-3Y) \text{ [Theorem 2.9B]} \\
&= \mathrm{Var}(T+Z), \text{ where } T=2X,\ Z=-3Y \\
&= \mathrm{Var}(T)+\mathrm{Var}(Z) \text{ [statistical independence]} \\
&= 2^2\mathrm{Var}(X)+(-3)^2\mathrm{Var}(Y) \text{ [Theorem 2.9B]} \\
&= 13.
\end{aligned}$$

2 Based on the Theorem 2.9A, we have:

$$\mathrm{Var}(X) = \mathbb{E}(X^2)-\mu_X^2,$$

where

$$\begin{aligned}
\mu_X &= \sum_{x=1}^{n} x f_X(x) \\
&= \frac{1}{n}\sum_{x=1}^{n} x \\
&= \frac{1}{n}\frac{n(n+1)}{2} \text{ [see Hint]} \\
&= \frac{n+1}{2}.
\end{aligned}$$

Applying the formula for the expected value of a transformed random variable, we get:

$$\mathbb{E}(X^2) = \sum_{x=1}^{n} x^2 f_X(x)$$

$$\begin{aligned}
&= \frac{1}{n}\sum_{x=1}^{n} x^2 \\
&= \frac{1}{n}\frac{n(n+1)(2n+1)}{6} \text{ [see Hint]} \\
&= \frac{(n+1)(2n+1)}{6}.
\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}
\operatorname{Var}(X) &= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} \\
&= \frac{n+1}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right] \\
&= \frac{n+1}{2}\frac{n-1}{6} \\
&= \frac{n^2-1}{12}.
\end{aligned}$$

3 The answers for all parts are presented below.

(a) We have $X \sim \text{Binomial}\left(3, \frac{1}{8}\right)$, $Y \sim \text{Binomial}\left(5, \frac{1}{4}\right)$, $Z \sim \text{Binomial}\left(2, \frac{1}{2}\right)$.

(b) The expected number of times that the target will be hit is given by

$$\begin{aligned}
&\mathbb{E}(X+Y+Z) \\
&= \mathbb{E}(X) + \mathbb{E}(Y) + \mathbb{E}(Z) \text{ [apply properties of } \mathbb{E}(\cdot)] \\
&= \frac{3}{8} + \frac{5}{4} + \frac{2}{2} \text{ [see page 92 in the course book]} \\
&= \frac{21}{8} \\
&= 2.625.
\end{aligned}$$

(c) The variance of the number of times that the target will be hit is given by

$$\begin{aligned}
&\operatorname{Var}(X+Y+Z) \\
&= \operatorname{Var}(X) + \operatorname{Var}(Y) + \operatorname{Var}(Z) \text{ [statistical independence]} \\
&= 3\frac{1}{8}\frac{7}{8} + 5\frac{1}{4}\frac{3}{4} + 2\frac{1}{2}\frac{1}{2} \text{ [see page 92 in the course book]} \\
&= \frac{113}{64} \\
&= 1.7656.
\end{aligned}$$

4 The answers for all parts are presented below:

(a) For $x \in \{0, 1, 2, \ldots\}$, we have:

$$\begin{aligned}
&\mathbb{P}(X > x) \\
&\quad = \ 1 - \mathbb{P}(X \leq x) \ \text{[valid prob. function]} \\
&\quad = \ 1 - [\mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \cdots + \mathbb{P}(X = x)] \ \text{[discrete distribution]} \\
&\quad = \ 1 - \frac{1}{3}\left[\left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^1 + \cdots + \left(\frac{2}{3}\right)^x\right] \ \text{[course book page 94]} \\
&\quad = \ 1 - \frac{1}{3}\frac{1 - (2/3)^{x+1}}{1 - (2/3)} \ \text{[see Hint]} \\
&\quad = \ \left(\frac{2}{3}\right)^{x+1}.
\end{aligned}$$

Similarly, for $y \in \{0, 1, 2, \ldots\}$, we have:

$$\begin{aligned}
&\mathbb{P}(Y > y) \\
&\quad = \ 1 - \mathbb{P}(Y \leq y) \ \text{[valid prob. function]} \\
&\quad = \ 1 - [\mathbb{P}(Y = 0) + \mathbb{P}(Y = 1) + \cdots + \mathbb{P}(Y = y)] \ \text{[discrete distribution]} \\
&\quad = \ 1 - \frac{1}{5}\left[\left(\frac{4}{5}\right)^0 + \left(\frac{4}{5}\right)^1 + \cdots + \left(\frac{4}{5}\right)^y\right] \ \text{[course book page 94]} \\
&\quad = \ 1 - \frac{1}{5}\frac{1 - (4/5)^{y+1}}{1 - (4/5)} \ \text{[see Hint]} \\
&\quad = \ \left(\frac{4}{5}\right)^{y+1}.
\end{aligned}$$

(b) Using the Hint, we have the following chain of identities for $z \in \{0, 1, 2, \ldots\}$:

$$\begin{aligned}
&\mathbb{P}(Z = z) \\
&\quad = \ \mathbb{P}(X = z, Y > z) + \mathbb{P}(X > z, Y = z) + \mathbb{P}(X = z, Y = z) \\
&\quad = \ \mathbb{P}(X = z)\mathbb{P}(Y > z) + \mathbb{P}(X > z)\mathbb{P}(Y = z) + \mathbb{P}(X = z)\mathbb{P}(Y = z) \ \text{[stat. indep.]} \\
&\quad = \ \left(\frac{2}{3}\right)^z \frac{1}{3}\left(\frac{4}{5}\right)^{z+1} + \left(\frac{2}{3}\right)^{z+1}\left(\frac{4}{5}\right)^z \frac{1}{5} \ \text{[see part (a)]} \\
&\qquad\quad + \left(\frac{2}{3}\right)^z \frac{1}{3}\left(\frac{4}{5}\right)^z \frac{1}{5} \\
&\quad = \ \left(\frac{2}{3}\frac{4}{5}\right)^z \left[\frac{1}{3}\frac{4}{5} + \frac{2}{3}\frac{1}{5} + \frac{1}{3}\frac{1}{5}\right] \\
&\quad = \ \left(\frac{8}{15}\right)^z \frac{7}{15},
\end{aligned}$$

which demonstrates that $Z \sim \text{Geometric}\left(\dfrac{7}{15}\right)$.

As suggested in the Hint, an alternative approach is the following one. Let $z \in \{0, 1, 2, \ldots\}$. We have:

$$\begin{aligned}
\mathbb{P}(Z > z) &= \mathbb{P}\left(\min(X, Y) > z\right) \\
&= \mathbb{P}(X > z, Y > z) \\
&= \mathbb{P}(X > z)\mathbb{P}(Y > z) \text{ [statistical independence]} \\
&= \left(\frac{2}{3}\right)^{z+1}\left(\frac{4}{5}\right)^{z+1} \\
&= \left(\frac{8}{15}\right)^{z+1}.
\end{aligned}$$

This leads to the identity:

$$\mathbb{P}(Z \leq z) = 1 - \left(\frac{8}{15}\right)^{z+1}. \tag{1}$$

If $T \sim \text{Geometric}\left(\dfrac{7}{15}\right)$, then we get:

$$\begin{aligned}
\mathbb{P}(T \leq z) &= \mathbb{P}(T = 0) + \mathbb{P}(T = 1) + \cdots + \mathbb{P}(T = z) \\
&= \frac{7}{15}\left[\left(\frac{8}{15}\right)^0 + \left(\frac{8}{15}\right)^1 + \cdots + \left(\frac{8}{15}\right)^z\right] \text{ [course book page 94]} \\
&= \frac{7}{15}\frac{1 - (8/15)^{z+1}}{7/15} \\
&= 1 - \left(\frac{8}{15}\right)^{z+1}. \qquad (2)
\end{aligned}$$

From (1) and (2), we may conclude that $Z \sim \text{Geometric}\left(\dfrac{7}{15}\right)$.

(c) We define the following random variables:

X is the number of Sundays until the first successful launch of Craig (without considering the Sunday when the successful launch happens);

Y is the number of Sundays until the first successful launch of Jill (without considering the Sunday when the successful launch happens).

It is clear that $X \sim \text{Geometric}\left(\dfrac{1}{3}\right)$ and $Y \sim \text{Geometric}\left(\dfrac{1}{5}\right)$. Additionally, we define $Z = \min(X, Y)$. It follows from part (b) that $Z \sim \text{Geometric}\left(\dfrac{7}{15}\right)$.

We have to calculate:

$$\begin{aligned}
&\mathbb{E}[\min(X,Y)+1] \\
&= \mathbb{E}[\min(X,Y)]+1 \text{ [Properties } \mathbb{E}(\cdot)] \\
&= \mathbb{E}(Z)+1 \\
&= \frac{1-(7/15)}{7/15}+1 \text{ [page 94 in the course book]} \\
&= \frac{15}{7}.
\end{aligned}$$

5 The answers for all parts are presented below:

(a) The likelihood function is:

$$\begin{aligned}
&L(p;2,5,3) \\
&= \mathbb{P}(X_1=2, X_2=5, X_3=3) \\
&= \mathbb{P}(X_1=2)\mathbb{P}(X_2=5)\mathbb{P}(X_3=3) \quad \text{[statistical independence]} \\
&= \left[\binom{7}{2}p^2(1-p)^5\right]\left[\binom{7}{5}p^5(1-p)^2\right] \\
&\quad \times \left[\binom{7}{3}p^3(1-p)^4\right] \quad \text{[Binomial distribution]} \\
&= \underbrace{\left[\binom{7}{2}\binom{7}{5}\binom{7}{3}\right]}_{K} p^{10}(1-p)^{11} \text{ for } p \in (0,1).
\end{aligned}$$

(b) Using the result above, we get:

$$\begin{aligned}
\frac{dL}{dp} &= K\left[10p^9(1-p)^{11} - 11p^{10}(1-p)^{10}\right] \\
&= Kp^9(1-p)^{10}\left[10(1-p)-11p\right] \\
&= Kp^9(1-p)^{10}(10-21p) \text{ as required.}
\end{aligned}$$

Note that $K = \binom{7}{2}\binom{7}{5}\binom{7}{3}$.

(c) The maximizing value of p occurs when $\frac{dL}{dp}=0$. This gives

$$\frac{dL}{dp} = Kp^9(1-p)^{10}(10-21p) = 0,$$

which implies that $p=0$ or $p=1$ or $p=\frac{10}{21}=0.4762$. In Figure 1, we can see that the likelihood function attains its maximum when $p=0.4762$. Hence, the maximum likelihood estimate is $\hat{p}=0.4762$.

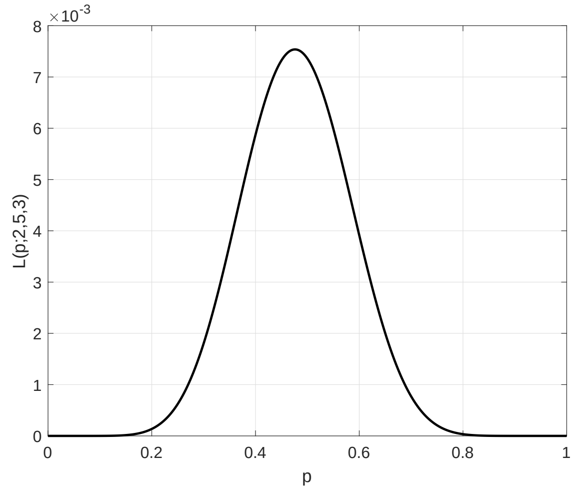

Figure 1: Plot for question 5, part (c).

(d) The maximum likelihood estimate $\hat{p} = 0.4762$ is the value of p at which the observations $(X_1 = 2, X_2 = 5, X_3 = 3)$ are more likely than at any other value of p.

(e) For $0 < p < 1$, we have:

$$\begin{aligned}
&L(p; x_1, x_2, \ldots, x_m)\\
&= \mathbb{P}(X_1 = x_1, X_2 = x_2, \ldots, X_m = x_m)\\
&= \mathbb{P}(X_1 = x_1)\mathbb{P}(X_2 = x_2)\cdots\mathbb{P}(X_m = x_m) \quad \text{[statistical independence]}\\
&= \left[\binom{n}{x_1} p^{x_1}(1-p)^{n-x_1}\right]\left[\binom{n}{x_2} p^{x_2}(1-p)^{n-x_2}\right] \times \cdots\\
&\qquad \times \left[\binom{n}{x_n} p^{x_m}(1-p)^{n-x_m}\right] \quad \text{[Binomial distribution]}\\
&= \underbrace{\left[\binom{n}{x_1}\binom{n}{x_2}\cdots\binom{n}{x_m}\right]}_{K} p^{m\bar{x}_m}(1-p)^{mn-m\bar{x}_m},
\end{aligned}$$

where $\bar{x}_m = \dfrac{x_1 + x_2 + \cdots + x_m}{m}$.

(f) The maximizing value of p occurs when $\dfrac{dL}{dp} = 0$.

We have:

$$\begin{aligned}
&\frac{dL}{dp}\\
&= K\left[m\bar{x}_m p^{m\bar{x}_m - 1}(1-p)^{mn-m\bar{x}_m} - (mn - m\bar{x}_m)p^{m\bar{x}_m}(1-p)^{mn-m\bar{x}_m-1}\right]\\
&= Kp^{m\bar{x}_m-1}(1-p)^{mn-m\bar{x}_m-1} m\left[\bar{x}_m(1-p) - (n-\bar{x}_m)p\right]
\end{aligned}$$

$$= \quad Kp^{m\bar{x}_m-1}(1-p)^{mn-m\bar{x}_m-1}m\left(\bar{x}_m-np\right).$$

This gives

$$\frac{dL}{dp} = Kp^{m\bar{x}_m-1}(1-p)^{mn-m\bar{x}_m-1}m\left(\bar{x}_m-np\right) = 0,$$

which implies that $p = 0$ or $p = 1$ or $p = \frac{\bar{x}_m}{n}$. If p were be zero, then $x_1 = x_2 = \cdots = x_m = 0$, which is not the case because $\max(x_1, x_2, \ldots, x_m) > 0$. If p were be one, then $x_1 = x_2 = \cdots = x_m = n$, which is not true because $\min(x_1, x_2, \ldots, x_m) < n$.

Hence, $\hat{p} = \frac{\bar{x}_m}{n}$, which implies that the maximum likelihood estimat<u>or</u> is

$$\begin{aligned} \hat{p} &= \frac{(X_1 + X_2 + \cdots + X_m)/m}{n} \\ &= \frac{X_1 + X_2 + \cdots + X_m}{mn}. \end{aligned}$$

(g) All we need to do is to use the data for evaluating the expression of $\hat{p}$ that we have obtained in part (f):

$$\begin{aligned} \hat{p} &= \frac{x_1 + x_2 + x_3}{3n} \\ &= \frac{2+5+3}{21} \\ &= \frac{10}{21} \\ &= 0.4762. \end{aligned}$$

Obviously, the result is the same as the one computed in part (c).

(h) For computing $\mathbb{E}(\hat{p})$, we write the following chain of identities:

$$\begin{aligned} &\mathbb{E}(\hat{p}) \\ &= \mathbb{E}\left[\frac{X_1 + X_2 + \cdots + X_m}{mn}\right] \text{ [see part (f)]} \\ &= \frac{\mathbb{E}(X_1) + \mathbb{E}(X_2) + \cdots + \mathbb{E}(X_m)}{mn} \text{ [apply properties of } \mathbb{E}(\cdot)] \\ &= \frac{(np) + (np) + \cdots + (np)}{mn} \text{ [course book page 92]} \\ &= p. \end{aligned}$$

Because $\mathbb{E}(\hat{p}) = p$, $\hat{p}$ is an unbiased estimator of p.

Similarly, we have:

$\mathrm{Var}(\hat{p})$

$$
\begin{aligned}
&= \frac{\operatorname{Var}(X_1 + X_2 + \cdots + X_m)}{(mn)^2} \text{ [apply properties of Var}(\cdot)] \\
&= \frac{\operatorname{Var}(X_1) + \operatorname{Var}(X_2) + \cdots + \operatorname{Var}(X_m)}{(mn)^2} \text{ [statistical independence]} \\
&= \frac{(npq) + (npq) + \cdots + (npq)}{(mn)^2} \text{ [Note :} q = 1 - p, \text{course book page 92]} \\
&= \frac{pq}{mn}.
\end{aligned}
$$

Note: Questions 1-4 are from from M.H. DeGroot and M.J. Schervish, "Probability and statistics", Addison-Wesley, 2002.