1.(a) Let 'success' = 'successful serve'. Then $\mathbb{P}(\text{success}) = 0.64$.
Y is the number of failures before the first success.
So $Y \sim \text{Geometric}(p = 0.64)$. (2)

(b) For $Y \sim \text{Geometric}(p = 0.64)$, the probability function is

$$\mathbb{P}(Y = y) = 0.64(1 - 0.64)^y = 0.64 \times (0.36)^y \quad \text{for } y = 0, 1, 2, \ldots.$$

So

$$\begin{aligned}
\mathbb{P}(Y = 0) &= 0.64 \times (0.36)^0 = 0.64. \\
\mathbb{P}(Y = 1) &= 0.64 \times (0.36)^1 = 0.230. \qquad \text{(3d.p.)} \\
\mathbb{P}(Y = 2) &= 0.64 \times (0.36)^2 = 0.083. \qquad \text{(3d.p.)}
\end{aligned}$$

(3)

(c) We need $\mathbb{P}(Y < 2)$. (2)

(d)

$$\mathbb{P}(Y < 2) = \mathbb{P}(Y = 0) + \mathbb{P}(Y = 1) = 0.64 + 0.230 = 0.870.$$

(2)

2.(a) Let 'success' = 'wins point'. Then $\mathbb{P}(\text{success}) = p$.
X is the number of failures before the 4th success.
So $X \sim \text{NegBin}(k = 4, p)$. (3)

(b) X is the number of points lost, so $X = 2$. (1)

(c) The Negative Binomial probability function is:

$$\mathbb{P}(X = x) = \binom{k + x - 1}{x} p^k (1 - p)^x.$$

So the likelihood is:

$$\begin{aligned}
L(p\,; 2) &= \binom{4 + 2 - 1}{2} p^4 (1 - p)^2 \\
&= \binom{5}{2} p^4 (1 - p)^2 \\
&= 10\, p^4 (1 - p)^2 \quad \text{for } 0 < p < 1.
\end{aligned}$$

(2)

(d)

$$\begin{aligned}\frac{dL}{dp} &= 10\Big(\, 4p^3(1-p)^2 - 2p^4(1-p)\Big)\\ &= 10\,p^3(1-p)\Big(\, 4(1-p) - 2p\Big)\\ &= 10\,p^3(1-p)(4-6p).\end{aligned}$$

The maximum likelihood estimate of p satisfies:

$$\left.\frac{dL}{dp}\right|_{p=\widehat{p}} = 0 \quad\Rightarrow\quad \widehat{p}=0, \quad \widehat{p}=1, \quad \widehat{p}=\frac{4}{6}.$$

From the likelihood graph we can see that the maximum does not occur at 0 or 1. Thus the MLE is

$$\widehat{p} = \frac{4}{6} = \frac{2}{3}.$$

(4)

3.(a)

$$\begin{aligned}\mathbb{P}(X_1=2\,,X_2=5) &= \mathbb{P}(X_1=2)\mathbb{P}(X_2=5) \quad \text{when } X_1,\, X_2 \text{ are independent}\\ &= \binom{5}{2}p^4(1-p)^2 \times \binom{4+5-1}{5}p^4(1-p)^5\\ &= 10\,p^4(1-p)^2\times 56\,p^4(1-p)^5\\ &= 560\,p^8(1-p)^7.\end{aligned}$$

(3)

(b) $L(p\,;\,2,\,5) = 560\,p^8(1-p)^7$ for $0<p<1$. (2)

(c)

$$\begin{aligned}\frac{dL}{dp} &= 560\Big(\, 8p^7(1-p)^7 - 7p^8(1-p)^6\Big)\\ &= 560\,p^7(1-p)^6\Big(\, 8(1-p) - 7p\Big)\\ &= 560\,p^7(1-p)^6(8-15p).\end{aligned}$$

The maximum likelihood estimate of p satisfies:

$$\left.\frac{dL}{dp}\right|_{p=\widehat{p}} = 0 \quad\Rightarrow\quad \widehat{p}=0, \quad \widehat{p}=1, \quad \widehat{p}=\frac{8}{15}.$$

From the likelihood graph we can see that the maximum does not occur at 0 or 1. Thus the MLE is

$$\widehat{p} = \frac{8}{15} \quad \text{as stated.}$$

(4)

(d) We knew that we would estimate the success probability as $\widehat{p} = \frac{8}{15}$, because in total we had seven failures $(5+2)$ and 8 successes $(4+4)$, giving a total of 8 successes out of 15 trials. (2)

Total: 30