1 The answers for both parts are presented below:

(a) The following calculations are straightforward:

$$\begin{aligned}
\mathbb{P}(A \cap B|C) &= \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(C)} \\
&= \frac{\mathbb{P}\left(\{s_7\}\right)}{\mathbb{P}\left(\{s_0, s_1, \ldots, s_7\}\right)} \\
&= \frac{1/11}{8/11} \\
&= \frac{1}{8}, \\
\mathbb{P}(A|C) &= \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} \\
&= \frac{\mathbb{P}\left(\{s_0, s_7\}\right)}{8/11} \\
&= \frac{2/11}{8/11} \\
&= \frac{1}{4}, \\
\mathbb{P}(B|C) &= \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)} \\
&= \frac{\mathbb{P}\left(\{s_3, s_4, s_6, s_7\}\right)}{8/11} \\
&= \frac{4/11}{8/11} \\
&= \frac{1}{2}.
\end{aligned}$$

It is clear that $\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C)\mathbb{P}(B|C)$. We also have that $\mathbb{P}(A) = 2/11$, $\mathbb{P}(B) = 4/11$ and $\mathbb{P}(A \cap B) = 1/11$. Therefore, $\mathbb{P}(A \cap B) \neq \mathbb{P}(A)\mathbb{P}(B)$.

(b) Similarly, we have:

$$\begin{aligned}
\mathbb{P}(A \cap B|C) &= \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(C)} \\
&= \frac{\mathbb{P}(\{s_7\})}{\mathbb{P}(\{s_0, s_2, s_3, s_6, s_7\})} \\
&= \frac{1/8}{5/8} \\
&= \frac{1}{5}, \\
\mathbb{P}(A|C) &= \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} \\
&= \frac{\mathbb{P}(\{s_0, s_7\})}{5/8} \\
&= \frac{2/8}{5/8} \\
&= \frac{2}{5}, \\
\mathbb{P}(B|C) &= \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)} \\
&= \frac{\mathbb{P}(\{s_2, s_3, s_6, s_7\})}{5/8} \\
&= \frac{4/8}{5/8} \\
&= \frac{4}{5}.
\end{aligned}$$

It follows that $\mathbb{P}(A \cap B|C) \neq \mathbb{P}(A|C)\mathbb{P}(B|C)$. Because $\mathbb{P}(A \cap B) = 1/8$, $\mathbb{P}(A) = 2/8$ and $\mathbb{P}(B) = 4/8$, we get $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

2 The answers for all parts are presented below:

(a) As X takes only values that are greater or equal to one, it is clear that $F_X(x) = \mathbb{P}(X \leq x) = 0$ when $x \in (-\infty, 1)$. For the case when $x \in [1, 9]$, we have:

$$\begin{aligned}
F_X(x) &= \mathbb{P}(X \leq x) \\
&= \mathbb{P}(X \leq \lfloor x \rfloor) \text{ [}X\text{ takes only integer values]} \\
&= \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \cdots + \mathbb{P}(X = \lfloor x \rfloor) \\
&= \log_{10}\left(\frac{2}{1}\right) + \log_{10}\left(\frac{3}{2}\right) + \cdots + \log_{10}\left(\frac{\lfloor x \rfloor + 1}{\lfloor x \rfloor}\right) \\
&= \log_{10}\left(\frac{2}{1}\frac{3}{2}\cdots\frac{\lfloor x \rfloor + 1}{\lfloor x \rfloor}\right) \text{ [properties } \log_{10}(\cdot)\text{]} \\
&= \log_{10}\left(\frac{\lfloor x \rfloor + 1}{1}\right) \\
&= \log_{10}\left(\lfloor x \rfloor + 1\right).
\end{aligned}$$

For $x > 9$, we get: $F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X \leq 9) = F_X(9) = 1$ (see the equations above).

(b) The computation of $F_Y(y)$ for $y \in (-\infty, 1) \cup (9, \infty)$ is similar to the calculation of $F_X(x)$ for $x \in (-\infty, 1) \cup (9, \infty)$. We focus on the case when $y \in [1, 9]$:

$$\begin{aligned}
F_Y(y) &= \mathbb{P}(Y \leq y) \\
&= \mathbb{P}(Y \leq \lfloor y \rfloor) \text{ [}Y\text{ takes only integer values]} \\
&= \mathbb{P}(Y = 1) + \mathbb{P}(Y = 2) + \cdots + \mathbb{P}(Y = \lfloor y \rfloor) \\
&= \underbrace{\frac{1}{9} + \frac{1}{9} + \cdots + \frac{1}{9}}_{\text{no. of terms}=\lfloor y \rfloor} \\
&= \frac{1}{9}\lfloor y \rfloor.
\end{aligned}$$

3 The answers for all parts are presented below:

(a) It is clear that $X \sim \text{Binomial}(n, p)$, where $n = 6$ and $p = 1/2$.

(b) Events:

- The event that the family has at least one girl: $\{X \geq 1\}$.
- The event that the family has at least one boy: $\{n - X \geq 1\}$.

(c) Apply the formula for conditional probabilities:

$$\begin{aligned}
&\mathbb{P}(n - X \geq 1 | X \geq 1) \\
&= \frac{\mathbb{P}\left((n - X \geq 1) \cap (X \geq 1)\right)}{\mathbb{P}(X \geq 1)} \\
&= \frac{\mathbb{P}(1 \leq X \leq n-1)}{\mathbb{P}(X \geq 1)} \\
&= \frac{1 - \mathbb{P}(X = 0) - \mathbb{P}(X = n)}{1 - \mathbb{P}(X = 0)} \text{ [valid prob. function]} \\
&= \frac{1 - 2\left(\frac{1}{2}\right)^n}{1 - \left(\frac{1}{2}\right)^n} \text{ [prob. function Binomial distrib.]} \\
&= \frac{1 - \left(\frac{1}{2}\right)^{n-1}}{1 - \left(\frac{1}{2}\right)^n} \\
&\approx 0.984.
\end{aligned}$$

4 The answers for all parts are presented below:

(a) Observe that Z can take only two values: 0 and 1. We have:

$$\begin{aligned}
\mathbb{P}(Z = 0) &= \mathbb{P}(X = 0, Y = 0) + \mathbb{P}(X = 1, Y = 1) \text{ [definition } \oplus\text{]} \\
&= \mathbb{P}(X = 0)\mathbb{P}(Y = 0) + \mathbb{P}(X = 1)\mathbb{P}(Y = 1) \text{ [statistical indep.]} \\
&= (1 - p)\frac{1}{2} + p\frac{1}{2} \text{ [distributional properties } X, Y\text{]} \\
&= \frac{1}{2}.
\end{aligned}$$

It is clear that $\mathbb{P}(Z = 1) = 1/2$. For sake of completeness, we also present the calculations for $\mathbb{P}(Z = 1)$:

$$\begin{aligned}
\mathbb{P}(Z = 1) &= \mathbb{P}(X = 0, Y = 1) + \mathbb{P}(X = 1, Y = 0) \text{ [definition } \oplus\text{]} \\
&= \mathbb{P}(X = 0)\mathbb{P}(Y = 1) + \mathbb{P}(X = 1)\mathbb{P}(Y = 0) \text{ [statistical indep.]} \\
&= (1 - p)\frac{1}{2} + p\frac{1}{2} \text{ [distributional properties } X, Y\text{]} \\
&= \frac{1}{2}.
\end{aligned}$$

Hence, $Z \sim \text{Bernoulli}(1/2)$.

(b) We calculate $\mathbb{P}(Y = y, Z = z)$ for all $y, z \in \{0, 1\}$:

$$\begin{aligned}
\mathbb{P}(Y = 0, Z = 0) &= \mathbb{P}(X = 0, Y = 0) \text{ [definition } \oplus\text{]} \\
&= \mathbb{P}(X = 0)\mathbb{P}(Y = 0) \text{ [statistical indep.]} \\
&= \frac{1 - p}{2},
\end{aligned}$$

$$\begin{aligned}\mathbb{P}(Y=0,Z=1) &= \mathbb{P}(X=1,Y=0) \text{ [definition } \oplus\text{]}\\ &= \mathbb{P}(X=1)\mathbb{P}(Y=0) \text{ [statistical indep.]}\\ &= \frac{p}{2},\end{aligned}$$

$$\begin{aligned}\mathbb{P}(Y=1,Z=0) &= \mathbb{P}(X=1,Y=1) \text{ [definition } \oplus\text{]}\\ &= \mathbb{P}(X=1)\mathbb{P}(Y=1) \text{ [statistical indep.]}\\ &= \frac{p}{2},\end{aligned}$$

$$\begin{aligned}\mathbb{P}(Y=1,Z=1) &= \mathbb{P}(X=0,Y=1) \text{ [definition } \oplus\text{]}\\ &= \mathbb{P}(X=0)\mathbb{P}(Y=1) \text{ [statistical indep.]}\\ &= \frac{1-p}{2}.\end{aligned}$$

From part (a), we have that

$$\begin{aligned}\mathbb{P}(Y=0)\mathbb{P}(Z=0) &= \frac{1}{4},\\ \mathbb{P}(Y=0)\mathbb{P}(Z=1) &= \frac{1}{4},\\ \mathbb{P}(Y=1)\mathbb{P}(Z=0) &= \frac{1}{4},\\ \mathbb{P}(Y=1)\mathbb{P}(Z=1) &= \frac{1}{4}.\end{aligned}$$

If $p \neq 1/2$, then $\mathbb{P}(Y=y,Z=z) \neq \mathbb{P}(Y=y)\mathbb{P}(Z=z)$ for all $y,z \in \{0,1\}$. Hence, the random variables Y and Z are not statistically independent.

(c) In the case when $p = 1/2$, it follows from the calculations in part (b) that $\mathbb{P}(Y=y,Z=z) = \mathbb{P}(Y=y)\mathbb{P}(Z=z)$ for all $y,z \in \{0,1\}$. Hence, the random variables Y and Z are statistically independent.

5 The answers for all parts are presented below:

(a) $X \sim \text{Binomial}(464, 0.5)$.

(b) $X \sim \text{Binomial}(464, p)$ and

$$\begin{aligned}H_0 : &\quad p = 0.5,\\ H_1 : &\quad p \neq 0.5 \text{ (two} - \text{sided)}.\end{aligned}$$

(c) See Fig. 1.

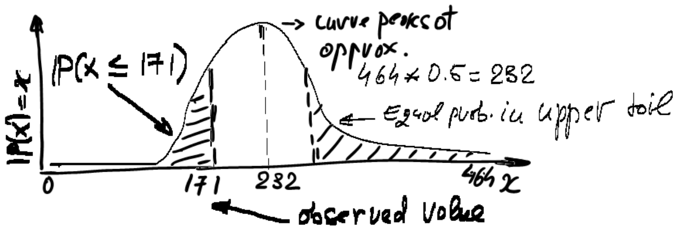

Figure 1: Plot for question 4, part (c).

(d) The following calculations are straightforward:

$$
\begin{aligned}
p\text{-value} &= 2\mathbb{P}(X \leq 171) \\
&= 2 * \text{pbinom}(171, 464, 0.5) \\
&\approx 1.6 * 10^{-8}.
\end{aligned}
$$

We conclude that it is extremely unlikely that this observation could have occurred by chance if the letters from the sets $\mathcal{V}$ and $\mathcal{C}$ had equal probabilities to appear in the text. We have very strong evidence that the letters from $\mathcal{V}$ are less likely than the letters from $\mathcal{C}$ to appear in the text. More in-depth studies of this type have been already done by linguists.

Note: Questions 2 and 4 are from J.K. Blitzstein and J. Hwang, "Introduction to probability", CRC Press, 2014. Question 3 is from M.H. DeGroot and M.J. Schervish, "Probability and Statistics", Addison-Wesley, 2002.