Statistics 210 FC **Assignment 3** **Due: 4PM, May 17, 2019**

1 Suppose that X and Y are independent random variables for which $\mathrm{Var}(X) = \mathrm{Var}(Y) = 1$.

a (2 marks) Calculate $\mathrm{Var}(X - Y)$. Show your working.

b (2 marks) Calculate $\mathrm{Var}(2X - 3Y + 1)$. Show your working.

2 (5 marks) Let n be an integer with the property that $n \geq 2$. The probability function of the random variable X is:

$$f_X(x) = \mathbb{P}(X = x) = \frac{1}{n}, \text{ for } x = 1, 2, \ldots, n.$$

Compute the variance of X.

Hint: You may use the following identities without proving them:

$$\begin{aligned} \sum_{k=1}^{n} k &= \frac{n(n+1)}{2}, \\ \sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

3 Three men, A, B, C shoot a target. Suppose that A shoots three times and the probability that he will hit the target on any given shot is $1/8$, B shoots five times and the probability that he will hit the target on any given shot is $1/4$, and C shoots twice and the probability that he will hit the target on any given shot is $1/2$. Assume that all shots at the target are independent. Answer the following questions:

a (3 marks) Let X be the number of times that A will hit the target (out of three shots). Similarly, let Y be the number of times that B will hit the target (out of five shots) and let Z be the number of times that C will hit the target (out of two shots). State the distributions of X, Y and Z (with parameters).

b (3 marks) What is the expected number of times that the target will be hit?

c (3 marks) What is the variance of the number of times that the target will be hit?

4 Let $X \sim \text{Geometric}\left(\dfrac{1}{3}\right)$ and $Y \sim \text{Geometric}\left(\dfrac{1}{5}\right)$. We assume that the random variables X and Y are statistically independent. Answer the following questions:

a (3 marks) For all $x \in \{0, 1, 2, \ldots\}$, show that

$$\mathbb{P}(X > x) \quad = \quad \left(\frac{2}{3}\right)^{x+1}.$$

Similarly, for all $y \in \{0, 1, 2, \ldots\}$, show that

$$\mathbb{P}(Y > y) \quad = \quad \left(\frac{4}{5}\right)^{y+1}.$$

Show your working <u>only for one of the two identities</u> that are presented above.

Hint: You may use the following identity without proving it.
For any non-negative integer ℓ, we have:

$$q^0 + q^1 + \cdots + q^\ell = \frac{1 - q^{\ell+1}}{1 - q}.$$

Note that, in the cases of interest for us, $q \in (0, 1)$.

b (5 marks) Let $Z = \min(X, Y)$. Prove that $Z \sim \text{Geometric}\left(\dfrac{7}{15}\right)$.
Show your working.

Hint: It is evident that Z can only take the values $0, 1, 2, \ldots$. For $z \in \{0, 1, 2, \ldots\}$, we get $Z = z$ only if one of the three situations occur: (i) $X = z$ and $Y > z$; (ii) $X > z$ and $Y = z$; (iii) $X = z$ and $Y = z$.

Another possible approach is to use the results in part (a) in order to calculate $\mathbb{P}(Z > z)$ for $z \in \{0, 1, 2, \ldots\}$. Then show that $\mathbb{P}(Z \leq z) = \mathbb{P}(T \leq z)$ for all $z \in \{0, 1, 2, \ldots\}$, where $T \sim \text{Geometric}\left(\dfrac{7}{15}\right)$.

c (6 marks) Every Sunday morning two children, Craig and Jill, independently try to launch their model airplanes. On each Sunday, Craig has probability $1/3$ of a successful launch, and Jill has probability $1/5$ of a successful launch. Determine the expected number of Sundays required until at least one of the two children has a successful launch. Assume that all launches are independent. Note that your answer should consider the Sunday when the successful launch happens.

Hint: Use the result obtained in part (b).

5 Assume that we have three independent observations:

$$X_1 = 2, \quad X_2 = 5, \quad X_3 = 3,$$

where $X_i \sim \text{Binomial}(n = 7, p)$ for $i \in \{1, 2, 3\}$. The value of $p \in (0, 1)$ is not known.

When we have observations like this from different, independent random variables, we can find joint probabilities by multiplying together the individual probabilities. For example,

$$\mathbb{P}(X_1 = x_1, X_2 = x_2, X_3 = x_3) = \mathbb{P}(X_1 = x_1)\mathbb{P}(X_2 = x_2)\mathbb{P}(X_3 = x_3).$$

This should remind you the discussion on statistical independence of random variables that can be found in the course book (see page 22).

Answer the following questions:

a (4 marks) Consider that X_1, X_2, X_3 are independent $\text{Binomial}(7, p)$ random variables, as above.
Write down the likelihood function, $L(p; 2, 5, 3)$. State the range of values of p for which the likelihood function is defined.

b (4 marks) Show that

$$\frac{dL}{dp} = Kp^9(1 - p)^{10}(10 - 21p),$$

where K is a constant that you should specify (but not calculate).

c (3 marks) The graph of the likelihood function for $0 < p < 1$ is given in Figure 1. Find the maximum likelihood estimate of p. Remember to make reference to the graph in justifying your answer.

d (1 mark) Write a sentence in English to explain what the maximum likelihood estimate from (c) represents (see page 57 in the course book).

e (4 marks) Let m be an integer with the property that $m \geq 2$. Consider that $X_1, X_2, \ldots, X_m$ are independent $\text{Binomial}(n, p)$ random variables, where n is *known* and p is *unknown*. Note that $p \in (0, 1)$. Write down the expression of the likelihood function

$$L(p; x_1, \ldots, x_m) = \mathbb{P}(X_1 = x_1, X_2 = x_2 \ldots, X_m = x_m).$$

We assume that $\min(x_1, \ldots, x_m) < n$ and $\max(x_1, \ldots, x_m) > 0$.

f (5 marks) Find $\dfrac{dL}{dp}$, and give all possible solutions to the equation $\dfrac{dL}{dp} = 0$. Show that the maximum likelihood estimat<u>or</u> for p is

$$\hat{p} = \frac{X_1 + X_2 + \cdots + X_m}{mn}.$$

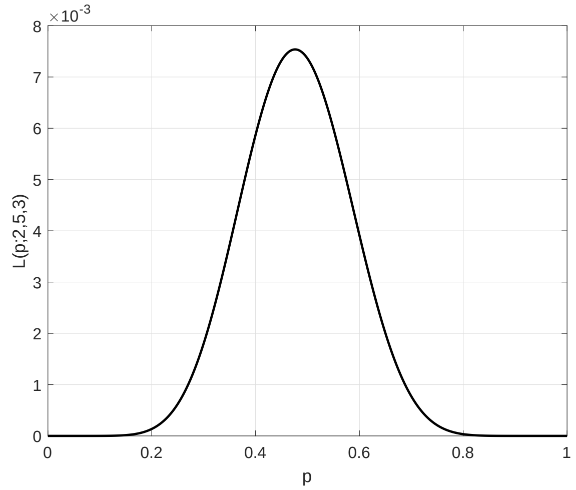

Figure 1: Plot for question 5, part (c).

You may assume that the likelihood attains a maximum in the range $0 < p < 1$.

g (1 mark) Find the maximum likelihood estimat<u>e</u> of p by using the data: $X_1 = 2$, $X_2 = 5$, $X_3 = 3$. Note that $m = 3$. Compare the result with the one that you have obtained in part (c).

h (8 marks) Using the expression for $\hat{p}$ obtained in part (f), find $\mathbb{E}(\hat{p})$ and $\text{Var}(\hat{p})$, fully justifying all your working. Is $\hat{p}$ an unbiased estimator of p? Explain your answer.

Total: 62 marks.