Statistics 210 FC Assignment 1 Due: 4PM, March 29$^{\text{th}}$, 2019

1 In a certain city, three newspapers $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are published. Suppose that 60% of the families in the city subscribe to newspaper $\mathcal{A}$, 40% of the families subscribe to newspaper $\mathcal{B}$, and 30% subscribe to newspaper $\mathcal{C}$. Suppose also that 20% of the families subscribe to both $\mathcal{A}$ and $\mathcal{B}$, 10% subscribe to both $\mathcal{A}$ and $\mathcal{C}$, 20% subscribe to both $\mathcal{B}$ and $\mathcal{C}$, and 5% subscribe to all three newspapers $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

The sample space is $\Omega = \{\text{all families in the city}\}$. We can formulate events as follows:

$A = \{\text{families that subscribe to newspaper } \mathcal{A}\}$,

$B = \{\text{families that subscribe to newspaper } \mathcal{B}\}$,

$C = \{\text{families that subscribe to newspaper } \mathcal{C}\}$.

a (6 marks) Write down all the information given above in terms of probability statements. Your answer should contain seven statements, each formulated in exactly the same way as the wording is given in the question.

Hint: The first statement is $\mathbb{P}(A) = 0.6$.

b (3 marks) We define the following events:

$$
\begin{aligned}
E_1 &= A \cap B \cap C, \\
E_2 &= A \cap B \cap \bar{C}, \\
E_3 &= A \cap \bar{B} \cap \bar{C}, \\
E_4 &= A \cap \bar{B} \cap C, \\
E_5 &= \bar{A} \cap B \cap \bar{C}, \\
E_6 &= \bar{A} \cap B \cap C, \\
E_7 &= \bar{A} \cap \bar{B} \cap C.
\end{aligned}
$$

Using these definitions, prove the following identities:

$$
\begin{aligned}
E_1 \cup E_2 \cup E_3 \cup E_4 &= A, \\
E_1 \cup E_2 \cup E_5 \cup E_6 &= B, \\
E_1 \cup E_4 \cup E_6 \cup E_7 &= C.
\end{aligned}
$$

Show your working only for one of the three identities. It is evident that the method of proof is the same for all three identities.

Hint: Have a look at the identities in Section 2.3 of the course book for STATS 125.

c (3 marks) Use the identities in part (b) in order to prove that $A \cap B = E_1 \cup E_2$, $A \cap C = E_1 \cup E_4$ and $B \cap C = E_1 \cup E_6$.

Show your working only for one of the three identities.

d (6 marks) Let $p_i = \mathbb{P}(E_i)$, where $i \in \{1, 2, \ldots, 7\}$. Prove that the following identities hold true:

$$\begin{aligned}
\mathbb{P}(A) &= p_1 + p_2 + p_3 + p_4, \\
\mathbb{P}(B) &= p_1 + p_2 + p_5 + p_6, \\
\mathbb{P}(C) &= p_1 + p_4 + p_6 + p_7, \\
\mathbb{P}(A \cup B \cup C) &= \sum_{i=1}^{7} p_i, \\
\mathbb{P}(A \cap B) &= p_1 + p_2, \\
\mathbb{P}(A \cap C) &= p_1 + p_4, \\
\mathbb{P}(B \cap C) &= p_1 + p_6.
\end{aligned}$$

Show your working only for $\mathbb{P}(C)$, $\mathbb{P}(A \cup B \cup C)$ and $\mathbb{P}(A \cap B)$.

Hint: Use the identities given in part (b) and in part (c).

e (3 marks) Apply the results from part (d) in order to show that

$$\begin{aligned}
\mathbb{P}(A \cup B \cup C) = \; & \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \\
& -[\mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C)] \\
& +\mathbb{P}(A \cap B \cap C).
\end{aligned}$$

f (2 marks) What percentage of the families in the city subscribe to at least one of the three newspapers?

g (6 marks) What percentage of the families in the city subscribe to exactly one of the three newspapers?

Hint: Use the identities given in part (b) and in part (d).

h (2 marks) If a family selected at random from the city subscribes to newspaper $\mathcal{A}$, what is the probability that the family also subscribes to newspaper $\mathcal{B}$?

2 A box contains $r = 2$ red balls and $b = 8$ blue balls. One ball is selected at random and its color is observed. The ball is then returned to the box and $k = 2$ additional balls of the same color are also put into the box. A second ball is then selected at random, its color is observed, and it is returned to the box together with $k = 2$ additional balls of the same color. Each time another ball is selected, the process is repeated. Answer the following questions:

a (5 marks) What is the probability that the second ball will be red?

b (5 marks) If four balls are selected, what is the probability that the first three balls will be red and the fourth ball will be blue?

Hint: Use the formula for the probability of the chains of events (see Section 1.8 of the course book):

3 Suppose that a box contains five coins, and that for each coin there is a different probability that a head will be obtained when the coin is tossed. Let p_i denote the probability of a head when the ith coin is tossed, where $i \in \{1, 2, 3, 4, 5\}$. Suppose that

$$p_1 = 0, \quad p_2 = 1/4, \quad p_3 = 1/2, \quad p_4 = 3/4, \quad p_5 = 1.$$

a (8 marks) Suppose that one coin is selected <u>at random</u> from the box and when it is tossed once, a head is obtained. What is the probability that the ith coin was selected? Note that $i \in \{1, 2, 3, 4, 5\}$.

b (8 marks) If the same coin were tossed again, what would be the probability of obtaining another head?

Hint: We define the following events:
$C^i = \{\text{the } i\text{th coin was selected}, \ i = 1, 2, 3, 4, 5\}$,
$H_j = \{\text{head on the } j\text{th toss}, \ j = 1, 2\}$,
$T_j = \{\text{tail on the } j\text{th toss}, \ j = 1, 2\}$.
In your calculations, you may use the identities written below, after replacing the symbol "?" with the correct quantities.

$$\begin{aligned}
&\mathbb{P}(H_2|H_1) \\
&= \sum_{i=2}^{5} \mathbb{P}(H_2|C^i \cap H_1)\mathbb{P}(C^i|H_1) \text{ [Partition Th. with extra conditioning]} \\
&= \sum_{i=2}^{5} \frac{\mathbb{P}(H_2 \cap H_1 \cap C^i)}{?}\mathbb{P}(C^i|H_1) \text{ [formula cond. prob.]} \\
&= \sum_{i=2}^{5} \frac{? * ?}{\mathbb{P}(H_1|C^i)\mathbb{P}(C^i)}\mathbb{P}(C^i|H_1) \text{ [formula cond. prob.]} \\
&= \sum_{i=2}^{5} \frac{\mathbb{P}(H_2|C^i)\mathbb{P}(H_1|C^i)}{\mathbb{P}(H_1|C^i)}\mathbb{P}(C^i|H_1) \text{ [conditional independence].}
\end{aligned}$$

c (8 marks) If a tail had been obtained on the first toss of the selected coin and the same coin were tossed again, what would be the probability of obtaining a head on the second toss?

Total: 65 marks.