1.(a) We need $\int_0^6 f_X(x)\,dx = 1$.

$$\begin{aligned} 1 &= \int_0^6 k(6x - x^2)\,dx \\ &= k\left[\frac{6x^2}{2} - \frac{x^3}{3}\right]_0^6 \\ &= k\left(\frac{6 \times 6^2}{2} - \frac{6^3}{3} - 0\right) \\ &= 36k \\ \Rightarrow \quad k &= \frac{1}{36}. \end{aligned}$$

(3)

(b)
$$\begin{aligned} \mathbb{P}(3 < X < 4) &= \int_3^4 \frac{1}{36}(6x - x^2)\,dx \\ &= \frac{1}{36}\left[\frac{6x^2}{2} - \frac{x^3}{3}\right]_3^4 \\ &= \frac{1}{36}\left\{\left(\frac{6 \times 4^2}{2} - \frac{4^3}{3}\right) - \left(\frac{6 \times 3^2}{2} - \frac{3^3}{3}\right)\right\} \\ &= \frac{13}{54} \quad (0.241). \end{aligned}$$

(3)

(c) Clearly,

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x > 6. \end{cases}$$

For $0 \leq x \leq 6$ we have:

$$\begin{aligned} F_X(x) &= \int_0^x f_X(u)\,du \quad \text{for } 0 \leq x \leq 6 \\ &= \int_0^x \frac{1}{36}(6u - u^2)\,du \\ &= \frac{1}{36}\left[\frac{6u^2}{2} - \frac{u^3}{3}\right]_0^x \\ &= \frac{1}{36}\left(3x^2 - \frac{x^3}{3}\right). \end{aligned}$$

Thus for all x, the expression is:

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0, \\ \frac{1}{36}\left(3x^2 - \frac{x^3}{3}\right) & \text{for } 0 \le x \le 6, \\ 1 & \text{for } x > 6. \end{cases}$$

(3)

(d)

$$F_X(6) = \frac{1}{36}\left(3 \times 6^2 - \frac{6^3}{3}\right) = 1, \quad \text{as required.}$$

(1)

2.(a) We require $\mathbb{P}(X < 30)$:

$$\mathbb{P}(X < 30) \;=\; F_X(30) \;=\; 1 - \left(1 + \frac{30}{20}\right) e^{-30/20} \;=\; 0.442.$$

(2)

(b)

$$\mathbb{P}(X > 100) \;=\; 1 - F_X(100) \;=\; 1 - 1 + \left(1 + \frac{100}{20}\right) e^{-100/20} \;=\; 0.040.$$

(2)

(c)

$$\mathbb{P}(30 < X < 50) \;=\; F_X(50) - F_X(30) \;=\; 1 - \left(1 + \frac{50}{20}\right) e^{-50/20} - 0.442 \;=\; 0.271.$$

(2)

(d)

$$\begin{aligned} f_X(x) &= F'_X(x) \\ &= 0 - \frac{d}{dx}\left\{\left(1 + \frac{x}{20}\right) e^{-x/20}\right\} \qquad \text{for } 0 < x < \infty \\ &= -\left\{\frac{1}{20}e^{-x/20} - \frac{1}{20}e^{-x/20}\left(1 + \frac{x}{20}\right)\right\} \qquad \text{(product rule)} \\ &= -\frac{1}{20}e^{-x/20}\left\{1 - 1 - \frac{x}{20}\right\} \\ &= \frac{x}{400}e^{-x/20} \qquad \text{for } 0 < x < \infty. \end{aligned}$$

(4)

Total: 20