Free variables and dummy variables.

Look at the expression below:

$$\sum_{x=1}^{10} ax^2$$

There is an important difference between the variables x and a in this expression:

- a is a ***free variable.*** This means:
  - (i) The expression would still make sense if a were replaced by a number (any number), e.g. replacing $a = 5$ would give $\sum_{x=1}^{10} 5x^2$, which is still a valid expression.
  - (ii) Because a is a free variable, it can be taken outside the sum:

$$\sum_{x=1}^{10} ax^2 = a\sum_{x=1}^{10} x^2 \qquad \checkmark \text{ Correct}$$

- x is a ***dummy variable***, also called a ***bound variable.*** This means:
  - (i) The expression would ***not*** make sense if x were replaced by a number, e.g. replacing $x = 5$ would give $\sum_{5=1}^{10} a5^2$. This is ***not*** a valid expression: writing $5 = 1$ makes no sense at all!
  - (ii) Because x is a dummy variable, or bound variable, it can ***not*** be taken outside the sum:

$$\sum_{x=1}^{10} ax^2 = x^2\sum_{x=1}^{10} a \ ? \qquad ✗ \text{ Wrong!}$$

Because x cannot be taken outside the sum, it is *bound* inside the sum, which is why it is called a bound variable. The word *dummy* is used because x could be swapped for any other symbol (y, i, j, k, etc), without changing the meaning of the expression.

It is important to know the difference between free and dummy variables. Follow the simple test below:

> Substitute a number (e.g. 5) in place of the variable.
> If the expression still makes sense, it is a **free variable.**
> If not, it is a **dummy variable** or **bound variable.**

And the rules:

> 1. Free variables can be taken outside a sum or integral; dummy variables can't.
> 2. We can change the symbol for a dummy variable, but not for a free variable:
> e.g. $\sum_{x=1}^{n} ax^2 = \sum_{y=1}^{n} ay^2$ is true (changing dummy x on LHS to dummy y on RHS), but we CAN'T change the symbol a between LHS and RHS, because a is a free variable.

1. For each of the following expressions, state whether they are **true** or **false**. For the variables given, state whether they are **free** or **dummy**.

EXAMPLE: $\sum_{k=1}^{n} 6k^2 = 6\sum_{k=1}^{n} k^2$? Variables k, n.

ANSWER: True. k is dummy, n is free.

(a) $\sum_{k=1}^{n} pk^2 = p\sum_{k=1}^{n} k^2$? Vars p, k, n.

(b) $\sum_{k=1}^{n} kn^2 = k\sum_{k=1}^{n} n^2$? Vars k, n.

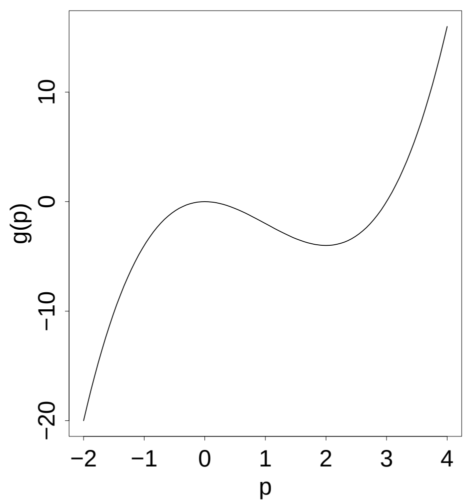
(c) $\sum_{x=1}^{n} npx = np\sum_{x=1}^{n} x$? Vars x, n, p.

(d) $\int_0^\infty yx^2\,dx = y\int_0^\infty x^2\,dx$? Vars x, y.

(e) $\int_0^x xy\,dy = x\int_0^x y\,dy$? Vars x, y.

(f) $\sum_{i=1}^{n} i^2 = \sum_{j=1}^{n} j^2$? Vars i, j, n.

2. **Product rule for differentiation, and maximizing a function.**
Let $g(p) = p^2(p-3)$ be a function of p for $-\infty < p < \infty$. The graph of $g(p)$ is below:



(a) Use the product rule for differentiation to show that $\frac{dg}{dp} = p(3p-6)$.

(b) By solving $\frac{dg}{dp} = 0$, find the values of p at which $g(p)$ has a local maximum or minimum.

(c) Using the graph, state which value of p from (b) corresponds to a local maximum of $g(p)$.

(d) Find $\frac{d^2g}{dp^2}$, by two methods:

(i) use the product rule for differentiating $p(3p-6)$;

(ii) expand the brackets in $p(3p-6)$, and differentiate the result directly.

(e) Using your answer to (d), find $\frac{d^2g}{dp^2}$ for the two values of p that you named in part (b).

Hence verify that your answer to (c) is correct. [Hint: the maximum satisfies $\frac{d^2g}{dp^2} < 0$.]

3. **Integration by Parts.** Using integration by parts, find $\int_0^\infty xe^{-2x}dx$.