1 The following calculations are straightforward:

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}(s_1) + \mathbb{P}(s_2) = 1/2, & (1)\\
\mathbb{P}(B) &= \mathbb{P}(s_2) + \mathbb{P}(s_3) = 1/2, & (2)\\
\mathbb{P}(A \cap B) &= \mathbb{P}(s2) = 1/2.
\end{aligned}$$

Because $\left(A \cap \overline{B}\right) \cap \left(\overline{A} \cap B\right) = \emptyset$, we have:

$$\begin{aligned}
&\mathbb{P}\left(\left(A \cap \overline{B}\right) \cup \left(\overline{A} \cap B\right)\right)\\
&= \mathbb{P}\left(A \cap \overline{B}\right) + \mathbb{P}\left(\overline{A} \cap B\right)\\
&= [\mathbb{P}(A) - \mathbb{P}(A \cap B)] + [\mathbb{P}(B) - \mathbb{P}(A \cap B)] \text{ [Partition Theorem]}\\
&= 0 \text{ [see (1) and (2)]}.
\end{aligned}$$

2 (a) It is easy to observe that

$$\begin{aligned}
\mathbb{P}(\Omega) &= \mathbb{P}(HH) + \mathbb{P}(HT) + \mathbb{P}(TH) + \mathbb{P}(TT)\\
&= p^2 + 2pq + q^2\\
&= (p+q)^2\\
&= 1.
\end{aligned}$$

(b) Obviously, we have:

$$\begin{aligned}
A &= \{HH, HT\},\\
B &= \{HH, TH\},\\
A \cap B &= \{HH\}.
\end{aligned}$$

So,

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}(HH) + \mathbb{P}(HT)\\
&= p^2 + p \times q\\
&= p \times (p+q)\\
&= p,\\
\mathbb{P}(B) &= \mathbb{P}(HH) + \mathbb{P}(TH)\\
&= p^2 + p \times q\\
&= p \times (p+q)\\
&= p,\\
\mathbb{P}(A \cap B) &= \mathbb{P}(HH)\\
&= p^2.
\end{aligned}$$

From the calculations above we have that $\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B)$, which leads to the conclusion that A and B are statistically independent.

3 (a) Define:

$$\begin{aligned} C_1 &= \{\text{You pick a regular coin}\}, \\ C_2 &= \{\text{You pick the fake two} - \text{headed coin}\}, \\ H &= \{\text{The coin lands heads up}\}. \end{aligned}$$

Hence, we have (see Partition Theorem):

$$\begin{aligned} \mathbb{P}(H) &= \mathbb{P}(H|C_1)\mathbb{P}(C_1) + \mathbb{P}(H|C_2)\mathbb{P}(C_2) \\ &= \frac{1}{2}\frac{2}{3} + \frac{1}{1}\frac{1}{3} \\ &= \frac{2}{3}, \end{aligned}$$

where we used the fact that all coins are selected from the box with equal probability.

(b) Apply Bayes' Theorem:

$$\begin{aligned} \mathbb{P}(C_2|H) &= \frac{\mathbb{P}(H|C_2)\mathbb{P}(C_2)}{\mathbb{P}(H)} \\ &= \frac{\frac{1}{1}\frac{1}{3}}{\frac{2}{3}} \\ &= \frac{1}{2}. \end{aligned}$$

4 (a) When rolling the dice, we get a pair of numbers, say (x, y), where $x, y \in \{1, 2, \ldots, 6\}$. From the total of 36 possible pairs, there are exacty 6 for which $x + y = 7$: $(1, 6)$, $(2, 5)$, $(3, 4)$, $(4, 3)$, $(5, 2)$ and $(6, 1)$. It is clear that the number of pairs for which $x + y \neq 7$ is $36 - 6 = 30$. It follows that $\mathbb{P}(F_1) = \mathbb{P}(F_2) = \cdots = \mathbb{P}(F_{10}) = 30/36 = 5/6$.

(b) We can write

$$\begin{aligned} \mathbb{P}(A) &= 1 - \mathbb{P}(\overline{A}) \\ &= 1 - \mathbb{P}\left(F_1 \cap F_2 \cap \cdots \cap F_{10}\right) \\ &= 1 - \mathbb{P}(F_1) \times \mathbb{P}(F_2) \times \cdots \times \mathbb{P}(F_{10}) \\ &= 1 - (5/6)^{10}, \end{aligned}$$

where we have used the fact that $F_1, F_2, \ldots, F_{10}$ are statistically independent.

(a) Let X be the number of boys out of 36 offspring: $X \sim Binomial(36, p)$ and

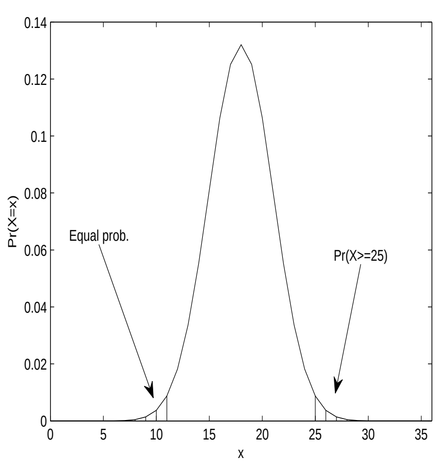
$$\begin{aligned} H_0 : &\quad p = \frac{1}{2}, \\ H_1 : &\quad p \neq \frac{1}{2} \text{ (two − sided)}. \end{aligned}$$

(b) The following calculations are straightforward:

$$\begin{aligned} p_1 &= f_X(24) \\ &= \mathbb{P}(X = 24) \\ &= \mathbb{P}(X \leq 24) - \mathbb{P}(X \leq 23) \\ &= F_X(24) - F_X(23) \\ &= 0.9856 - 0.9674 \\ &= 0.0182, \end{aligned}$$

$$\begin{aligned} c_1 &= F_X(22) \\ &= \mathbb{P}(X \leq 22) \\ &= \mathbb{P}(X \leq 21) + \mathbb{P}(X = 22) \\ &= F_X(21) + f_X(22) \\ &= 0.8785 + 0.0552 \\ &= 0.9337. \end{aligned}$$

(c) Under H_0, $X \sim Binomial(36, 0.5)$. The probability function of X (under the null hypothesis) is represented below. Remark that the curve peaks at $36 \times 0.5 = 18$.



(d) For the p-value, we use the data from the table in order to calculate

$$\begin{aligned} 2 \times \mathbb{P}(X \geq 25) &= 2 \times (1 - \mathbb{P}(X \leq 24)) \\ &= 2 \times (1 - F_X(24)) \\ &= 2 \times (1 - 0.9856) \\ &\approx 0.029. \end{aligned}$$

Hence, if the null hypothesis is true (the members of the STATS Department are equally likely to have sons and daughters), we would have a 2.9% chance of observing something as extreme as 25 boys or girls. This is unusual, but not extremely unusual. There is some evidence against the null hypothesis. However, the observations are also consistent with the the possibility that the members of the STATS Department are equally likely to have sons and daughters.