1.(a) $\displaystyle\sum_{k=1}^{n} pk^2 = p\sum_{k=1}^{n} k^2.$ TRUE. k is dummy. p and n are free.

(b) $\displaystyle\sum_{k=1}^{n} kn^2 = k\sum_{k=1}^{n} n^2.$ FALSE. k is dummy. n is free.

(c) $\displaystyle\sum_{x=1}^{n} npx = np\sum_{x=1}^{n} x$ TRUE. x is dummy. n and p are free.

(d) $\displaystyle\int_0^\infty yx^2\,dx = y\int_0^\infty x^2\,dx.$ TRUE. x is dummy. y is free.

(e) $\displaystyle\int_0^x xy\,dy = x\int_0^x y\,dy.$ TRUE. x is free. y is dummy.

(f) $\displaystyle\sum_{i=1}^{n} i^2 = \sum_{j=1}^{n} j^2.$ TRUE. i and j are dummy, n is free.

2.(a) By the Product Rule, using $g(p) = u(p)\times v(p)$ where $u(p) = p^2$ and $v(p) = (p-3)$:

$$\begin{aligned}
\frac{dg}{dp} &= u\frac{dv}{dp} + v\frac{du}{dp} \\
&= p^2 \times 1 + (p-3)\times 2p \\
&= p\Big\{\, p + 2(p-3)\Big\} \\
&= p\Big\{\, p + 2p - 6\Big\} \\
&= p(3p-6),
\end{aligned}$$

as required.

(b)

$$\begin{aligned}
\frac{dg}{dp} = 0 \quad &\Rightarrow \quad p(3p-6) = 0 \\
&\Rightarrow \quad p = 0 \quad \text{or} \quad 3p - 6 = 0 \\
&\Rightarrow \quad p = 0 \quad \text{or} \quad p = \tfrac{6}{3} = 2.
\end{aligned}$$

Thus $g(p)$ has a local maximum or minimum at $p = 0$ and $p = 2$.

(c) By looking at the graph, we can see that the local maximum occurs when $p = 0$.

(d) (i) By the Product Rule, using $\frac{dg}{dp} = u(p) \times v(p)$ where $u(p) = p$ and $v(p) = (3p - 6)$:

$$\begin{aligned} \frac{d^2g}{dp^2} &= u\frac{dv}{dp} + v\frac{du}{dp} \\ &= p \times 3 + (3p - 6) \times 1 \\ &= 6p - 6. \end{aligned}$$

(ii) Alternatively, expanding $\frac{dg}{dp} = p(3p - 6) = 3p^2 - 6p$, we get:

$$\frac{dg}{dp} = 3p^2 - 6p \quad \Rightarrow \quad \begin{aligned} \frac{d^2g}{dp^2} &= 3 \times 2p - 6 \\ &= 6p - 6 \quad \text{as before.} \end{aligned}$$

(e)

$$\text{For } p = 0: \quad \frac{d^2g}{dp^2} = 6 \times 0 - 6 = -6.$$

$$\text{For } p = 2: \quad \frac{d^2g}{dp^2} = 6 \times 2 - 6 = 6.$$

The local maximum requires $\dfrac{d^2g}{dp^2} < 0$, so the local maximum occurs at $p = 0$ as in (c).

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3. We need to find $\int_0^\infty xe^{-2x}dx$. Split the integrand up into $(x) \times (e^{-2x})$. We can integrate the second term and differentiate the first, so use integration by parts.

In general,

$$\int_0^\infty u\,\frac{dv}{dx}\,dx = \Big[uv\Big]_0^\infty - \int_0^\infty v\,\frac{du}{dx}\,dx\,.$$

Let

$$u = x\,, \qquad \text{so} \quad \frac{du}{dx} = 1\,;$$

and let

$$\frac{dv}{dx} = e^{-2x}, \qquad \text{so} \quad v = -\frac{1}{2}e^{-2x}\,.$$

Then,

$$\begin{aligned} \int_0^\infty xe^{-2x}dx &= \int_0^\infty u\,\frac{dv}{dx}\,dx \\ &= \Big[uv\Big]_0^\infty - \int_0^\infty v\,\frac{du}{dx}\,dx \\ &= \left[x\left(-\frac{1}{2}e^{-2x}\right)\right]_0^\infty - \int_0^\infty \left(-\frac{1}{2}e^{-2x}\right)dx \\ &= (0 - 0) + \left[-\frac{1}{4}e^{-2x}\right]_0^\infty \\ &= \frac{1}{4}\,. \end{aligned}$$