1 The answers for all parts are presented below:

(a) We have:
$\mathbb{P}(A) = 0.6$, $\mathbb{P}(B) = 0.4$, $\mathbb{P}(C) = 0.3$,
$\mathbb{P}(A \cap B) = 0.2$, $\mathbb{P}(A \cap C) = 0.1$, $\mathbb{P}(B \cap C) = 0.2$,
$\mathbb{P}(A \cap B \cap C) = 0.05$.

(b) Using that (i) C and $\bar{C}$ form a partition of the sample space and (ii) B and $\bar{B}$ form a partition of the sample space, the following calculations are straightforward:

$$\begin{aligned} & E_1 \cup E_2 \cup E_3 \cup E_4 \\ &= \big[(A \cap B \cap C) \cup (A \cap B \cap \bar{C})\big] \cup \big[(A \cap \bar{B} \cap \bar{C}) \cup (A \cap \bar{B} \cap C)\big] \\ &= (A \cap B) \cup (A \cap \bar{B}) \qquad (1) \\ &= A. \end{aligned}$$

Similarly, we have:

$$\begin{aligned} & E_1 \cup E_2 \cup E_5 \cup E_6 \\ &= \big[(A \cap B \cap C) \cup (A \cap B \cap \bar{C})\big] \cup \big[(\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap B \cap C)\big] \\ &= (A \cap B) \cup (\bar{A} \cap B) \\ &= B, \end{aligned}$$

$$\begin{aligned} & E_1 \cup E_4 \cup E_6 \cup E_7 \\ &= \big[(A \cap B \cap C) \cup (A \cap \bar{B} \cap C)\big] \cup [(\bar{A} \cap B \cap C) \cup (\bar{A} \cap \bar{B} \cap C)] \\ &= (A \cap C) \cup (\bar{A} \cap C) \qquad (2) \\ &= C. \end{aligned}$$

(c) From (1), we have that $E_1 \cup E_2 = A \cap B$. Similarly, we have that $E_1 \cup E_4 = A \cap C$ from (2). Using the fact that A and $\bar{A}$ form a partition of the sample space, together with the definitions of E_1 and E_6, we get $E_1 \cup E_6 = B \cap C$.
Assuming that the identities for A, B and C in part (b) have not been obtained as presented above, another possible approach for proving the identities in part (c) is described below.
The key observation is that

$$E_i \cap E_j = \emptyset \text{ for all } i, j \in \{1, 2, \ldots, 7\} \text{ with property } i \neq j. \qquad (3)$$

In order to write the equations in a more compact form, we define $E_{i,j} = E_i \cup E_j$ for all $i, j \in \{1, 2, \ldots, 7\}$ with property that $i < j$. Hence, we have:

$$\begin{aligned} & A \cap B \\ = \; & (E_{1,2} \cup E_{3,4}) \cap (E_{1,2} \cup E_{5,6}) \text{ [associative law]} \\ = \; & [(E_{1,2} \cup E_{3,4}) \cap E_{1,2}] \cup [(E_{1,2} \cup E_{3,4}) \cap E_{5,6}] \text{ [distributive law]} \\ = \; & E_{1,2} \cup \emptyset \\ = \; & E_{1,2} \\ = \; & E_1 \cup E_2, \end{aligned}$$

$$\begin{aligned} & A \cap C \\ = \; & (E_{1,4} \cup E_{2,3}) \cap (E_{1,4} \cup E_{6,7}) \text{ [associative law]} \\ = \; & [(E_{1,4} \cup E_{2,3}) \cap E_{1,4}] \cup [(E_{1,4} \cup E_{2,3}) \cap E_{6,7}] \text{ [distributive law]} \\ = \; & E_{1,4} \cup \emptyset \\ = \; & E_{1,4} \\ = \; & E_1 \cup E_4, \end{aligned}$$

$$\begin{aligned} & B \cap C \\ = \; & (E_{1,6} \cup E_{2,5}) \cap (E_{1,6} \cup E_{4,7}) \text{ [associative law]} \\ = \; & [(E_{1,6} \cup E_{2,5}) \cap E_{1,6}] \cup [(E_{1,6} \cup E_{2,5}) \cap E_{4,7}] \text{ [distributive law]} \\ = \; & E_{1,6} \cup \emptyset \\ = \; & E_{1,6} \\ = \; & E_1 \cup E_6. \end{aligned}$$

(d) We have:

$$\begin{aligned} \mathbb{P}(C) &= \mathbb{P}(E_1 \cup E_4 \cup E_6 \cup E_7) \text{ [see part (b)]} \\ &= p_1 + p_4 + p_6 + p_7 \text{ [see (3)]}, \end{aligned}$$

$$\begin{aligned} \mathbb{P}(A \cup B \cup C) &= \mathbb{P}(\cup_{i=1}^{7} E_i) \text{ [see part (b)]} \\ &= \sum_{i=1}^{7} p_i \text{ [see (3)]}, \end{aligned}$$

$$\begin{aligned} \mathbb{P}(A \cap B) &= \mathbb{P}(E_1 \cup E_2) \text{ [see part (c)]} \\ &= p_1 + p_2 \text{ [see (3)]}. \end{aligned}$$

(e) Based on the results in part (d), we get: $\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - [\mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C)] = \sum_{i=2}^{7} p_i$. All that remains is to observe that $\mathbb{P}(A \cap B \cap C) = p_1$ and $\mathbb{P}(A \cup B \cup C) = \sum_{i=1}^{7} p_i$ (see again part (d)).

(f) We need to calculate $\mathbb{P}(A \cup B \cup C)$. Combining the results from part (a) with those from part (e), we have: $\mathbb{P}(A \cup B \cup C) = (0.6 + 0.4 + 0.3) - (0.2 + 0.1 + 0.2) + 0.05 = 0.85$. Hence, 85% of the families in the city subscribe to at least one of the three newspapers.

(g) We need to calculate $\mathbb{P}(E_3 \cup E_5 \cup E_7)$. We have:

$$
\begin{aligned}
&\mathbb{P}(E_3 \cup E_5 \cup E_7) \\
&\quad = \quad p_3 + p_5 + p_7 \text{ [see (3)]} \\
&\quad = \quad (\sum_{i=1}^{7} p_i) - p_1 - (p_2 + p_4 + p_6) \\
&\quad = \quad (\sum_{i=1}^{7} p_i) - p_1 - [\mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C) - 3p_1] \text{ [see part (d)]} \\
&\quad = \quad (\sum_{i=1}^{7} p_i) + 2p_1 - [\mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C)] \\
&\quad = \quad (\sum_{i=1}^{7} p_i) + 2\mathbb{P}(A \cap B \cap C) - [\mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C)] \\
&\quad = \quad \mathbb{P}(A \cup B \cup C) + 2\mathbb{P}(A \cap B \cap C) \\
&\qquad\qquad -[\mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) + \mathbb{P}(B \cap C)] \text{ [see part (d)]} \\
&\quad = \quad 0.85 + 2 * 0.05 - (0.2 + 0.1 + 0.2) \\
&\quad = \quad 0.45.
\end{aligned}
$$

Hence, 45% of the families in the city subscribe to exactly one of the three newspapers.

(h) Apply the formula for conditional probabilities:

$$
\begin{aligned}
\mathbb{P}(B|A) &= \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} \\
&= \frac{0.2}{0.6} \\
&= \frac{1}{3}.
\end{aligned}
$$

2 We define the following events for $j \geq 1$:

$R_j = \{\text{a red ball is obtained on the } j\text{th draw}\}$,

$B_j = \{\text{a blue ball is obtained on the } j\text{th draw}\}$.

(a) We have:

$$\begin{aligned}
\mathbb{P}(R_2) &= \mathbb{P}(R_1 \cap R_2) + \mathbb{P}(B_1 \cap R_2) \\
&= \mathbb{P}(R_1)\mathbb{P}(R_2|R_1) + \mathbb{P}(B_1)\mathbb{P}(R_2|B_1) \text{ [formula cond. prob.]} \\
&= \frac{r}{r+b}\frac{r+k}{r+b+k} + \frac{b}{r+b}\frac{r}{r+b+k} \\
&= \frac{r}{r+b}\left[\frac{r+k}{r+b+k} + \frac{b}{r+b+k}\right] \\
&= \frac{r}{r+b} \\
&= 0.2.
\end{aligned}$$

(b) We apply the formula for the probability of the chains of events:

$$\begin{aligned}
&\mathbb{P}(R_1 \cap R_2 \cap R_3 \cap B_4) \\
&\quad = \mathbb{P}(R_1)\mathbb{P}(R_2|R_1)\mathbb{P}(R_3|R_1 \cap R_2)\mathbb{P}(B_4|R_1 \cap R_2 \cap R_3) \\
&\quad = \frac{r}{r+b}\frac{r+k}{r+b+k}\frac{r+2k}{r+b+2k}\frac{b}{r+b+3k} \\
&\quad = 0.0143.
\end{aligned}$$

3 We define the following events:

$C^i = \{\text{the } i\text{th coin was selected}, \ i = 1, 2, 3, 4, 5\}$,

$H_j = \{\text{head on the } j\text{th toss}, \ j = 1, 2\}$,

$T_j = \{\text{tail on the } j\text{th toss}, \ j = 1, 2\}$.

(a) For $i \in \{1, 2, 3, 4, 5\}$, we have:

$$\begin{aligned}
\mathbb{P}(C^i|H_1) &= \frac{\mathbb{P}(H_1|C^i)\mathbb{P}(C^i)}{\mathbb{P}(H_1)} \text{ [Bayes Theorem]} \\
&= \frac{\mathbb{P}(H_1|C^i)\mathbb{P}(C^i)}{\sum_{i=1}^{5} \mathbb{P}(H_1|C^i)\mathbb{P}(C^i)} \text{ [Partition Theorem].}
\end{aligned}$$

Simple calculations lead to

$$\begin{aligned}
\sum_{i=1}^{5} \mathbb{P}(H_1|C^i)\mathbb{P}(C^i) &= \frac{1}{5}\left[\frac{0}{4} + \frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4}\right] \\
&= \frac{10}{20} = \frac{1}{2}.
\end{aligned}$$

Hence, we get:

$$\begin{aligned}
\mathbb{P}(C^1|H_1) &= \frac{0 * (1/5)}{1/2} = 0, \\
\mathbb{P}(C^2|H_1) &= \frac{(1/4) * (1/5)}{1/2} = \frac{1}{10}, \\
\mathbb{P}(C^3|H_1) &= \frac{(1/2) * (1/5)}{1/2} = \frac{2}{10}, \\
\mathbb{P}(C^4|H_1) &= \frac{(3/4) * (1/5)}{1/2} = \frac{3}{10}, \\
\mathbb{P}(C^5|H_1) &= \frac{1 * (1/5)}{1/2} = \frac{4}{10}.
\end{aligned}$$

(b) We have:

$$\begin{aligned}
&\mathbb{P}(H_2|H_1) \\
&= \sum_{i=2}^{5} \mathbb{P}(H_2|C^i \cap H_1)\mathbb{P}(C^i|H_1) \text{ [Partition Th. with extra conditioning]} \\
&= \sum_{i=2}^{5} \frac{\mathbb{P}(H_2 \cap H_1 \cap C^i)}{\mathbb{P}(H_1 \cap C_i)}\mathbb{P}(C^i|H_1) \text{ [formula cond. prob.]} \\
&= \sum_{i=2}^{5} \frac{\mathbb{P}(H_2 \cap H_1|C^i)\mathbb{P}(C^i)}{\mathbb{P}(H_1|C^i)\mathbb{P}(C^i)}\mathbb{P}(C^i|H_1) \text{ [formula cond. prob.]} \\
&= \sum_{i=2}^{5} \frac{\mathbb{P}(H_2|C^i)\mathbb{P}(H_1|C^i)}{\mathbb{P}(H_1|C^i)}\mathbb{P}(C^i|H_1) \text{ [conditional independence]} \\
&= \sum_{i=2}^{5} \mathbb{P}(H_2|C^i)\mathbb{P}(C^i|H_1) \\
&= \frac{1}{4}\frac{1}{10} + \frac{2}{4}\frac{2}{10} + \frac{3}{4}\frac{3}{10} + \frac{4}{4}\frac{4}{10} \\
&= \frac{30}{40} = \frac{3}{4}.
\end{aligned}$$

Another possibility is to do the calculations as follows:

$$\begin{aligned}
&\mathbb{P}(H_2|H_1) \\
&= \frac{\mathbb{P}(H_1 \cap H_2)}{\mathbb{P}(H_1)} \text{ [formula cond. prob.]} \\
&= \frac{1}{\mathbb{P}(H_1)} \sum_{i=2}^{5} \mathbb{P}(H_1 \cap H_2|C^i)\mathbb{P}(C^i) \text{ [Partition Theorem]} \\
&= \frac{1}{\mathbb{P}(H_1)} \sum_{i=2}^{5} \mathbb{P}(H_1|C^i)\mathbb{P}(H_2|C^i)\mathbb{P}(C^i) \text{ [cond. indep.]} \qquad (4) \\
&= \sum_{i=2}^{5} \mathbb{P}(H_2|C^i)\frac{\mathbb{P}(H_1|C^i)\mathbb{P}(C^i)}{\mathbb{P}(H_1)} \\
&= \sum_{i=2}^{5} \mathbb{P}(H_2|C^i)\mathbb{P}(C^i|H_1) \text{ [Bayes Theorem]}
\end{aligned}$$

(c) The formula is similar to the one used in part (b):

$$\mathbb{P}(H_2|T_1) = \sum_{i=1}^{4} \mathbb{P}(H_2|C^i)\mathbb{P}(C^i|T_1).$$

For $i \in \{1, 2, 3, 4\}$, we need to calculate

$$\mathbb{P}(C^i|T_1) = \frac{\mathbb{P}(T_1|C^i)\mathbb{P}(C^i)}{\mathbb{P}(T_1)} \text{ [Bayes Theorem].}$$

Because $\mathbb{P}(H_1) = 1/2$, we have $\mathbb{P}(T_1) = 1/2$. Hence, we get:

$$\begin{aligned}
\mathbb{P}(C^1|T_1) &= \frac{(4/4) * (1/5)}{1/2} = \frac{4}{10}, \\
\mathbb{P}(C^2|T_1) &= \frac{(3/4) * (1/5)}{1/2} = \frac{3}{10}, \\
\mathbb{P}(C^3|T_1) &= \frac{(1/2) * (1/5)}{1/2} = \frac{2}{10}, \\
\mathbb{P}(C^4|T_1) &= \frac{(1/4) * (1/5)}{1/2} = \frac{1}{10}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbb{P}(H_2|T_1) &= \frac{0}{4}\frac{4}{10} + \frac{1}{4}\frac{3}{10} + \frac{2}{4}\frac{2}{10} + \frac{3}{4}\frac{1}{10} \\
&= \frac{10}{40} = \frac{1}{4}.
\end{aligned}$$

Another possibility is to use the formula that it is similar to (4):

$$
\begin{aligned}
&\mathbb{P}(H_2|T_1)\\
&= \frac{1}{\mathbb{P}(T_1)}\sum_{i=1}^{4}\mathbb{P}(T_1|C^i)\mathbb{P}(H_2|C^i)\mathbb{P}(C^i)\\
&= \frac{1/5}{1/2}\left[(1*0)+\left(\frac{3}{4}*\frac{1}{4}\right)+\left(\frac{1}{2}*\frac{1}{2}\right)+\left(\frac{1}{4}*\frac{3}{4}\right)\right]\\
&= \frac{2}{5}\left(\frac{3}{16}+\frac{4}{16}+\frac{3}{16}\right)\\
&= \frac{1}{4}.
\end{aligned}
$$

Note: Questions are from M.H. DeGroot and M.J. Schervish, “Probability and Statistics”, Addison-Wesley, 2002.