Product Rule for Differentiation and Integration by Parts

Product Rule for Differentiation

Let $u = u(x)$ and let $v = v(x)$. Then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}. \tag{1}$$

Integration by Parts

Derive this by manipulating equation (1):

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\,,$$

so (rearranging terms),

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}.$$

Integrate both sides:

$$\int_a^b \left(u\frac{dv}{dx}\right)\,dx = \int_a^b \left(\frac{d}{dx}(uv)\right)dx - \int_a^b \left(v\frac{du}{dx}\right)dx.$$

But $\int_a^b \left(\frac{d}{dx}(uv)\right)dx = \Big[\,uv\,\Big]_a^b$, so

$$\int_a^b u\,\frac{dv}{dx}\,dx \quad = \quad \Big[\,uv\,\Big]_a^b \; - \; \int_a^b v\,\frac{du}{dx}\,dx\,. \tag{2}$$

To integrate by parts, therefore:

1. Write the function to be integrated as $u\,\frac{dv}{dx}$: that is, decide which part is to be u and which part is to be $\frac{dv}{dx}$.
2. Write $u = \ldots,$ so $\frac{du}{dx} = \ldots$.
3. Write $\frac{dv}{dx} = \ldots,$ so $v = \ldots$ (i.e. integrate it).
4. Then all the ingredients are there to apply the formula (2): simply substitute u, $\frac{du}{dx}$, v, and $\frac{dv}{dx}$ into formula (2) and finish off.

Example of integration by parts

Find $\int_0^\infty xe^{-x}\,dx$.

Following the instructions overleaf:

1. Let $xe^{-x} = u\dfrac{dv}{dx}$. Either term can be integrated or differentiated easily. However, if we let $u = x$, then $\dfrac{du}{dx}$ in formula (2) will be $= 1$ and we will be able to find the integral in (2) easily.

   So let $u = x$ and let $\dfrac{dv}{dx} = e^{-x}$.
2. $u = x$, so $\dfrac{du}{dx} = 1$.
3. $\dfrac{dv}{dx} = e^{-x}$, so $v = -e^{-x}$.
4. By formula (2),

$$\int_0^\infty xe^{-x}\,dx = \Big[x \times (-e^{-x})\Big]_0^\infty - \int_0^\infty -e^{-x} \times 1\,dx.$$

More informal method:

$$\int_0^\infty xe^{-x}\,dx \quad = \quad \underbrace{\Big[x \times (-e^{-x})\Big]_0^\infty}_{\substack{\text{leave one alone, } x,\\ \text{and integrate the}\\ \text{other, } -e^{-x}.}} \quad - \quad \underbrace{\int_0^\infty (-e^{-x}) \times 1\,dx}_{\substack{\text{leave alone the integrated}\\ \text{one, } (-e^{-x}) \text{, and differentiate}\\ \text{the other, } \frac{dx}{dx} = 1.}}$$

Finishing off:

$$\begin{aligned} &= (0-0) + \int_0^\infty e^{-x}\,dx \\ &= 0 + \Big[-e^{-x}\Big]_0^\infty \\ &= 0 + -(0 - e^0) \\ &= 1. \end{aligned}$$