**1** The answers for both parts are presented below:

(a) Applying notational conventions which are similar to those used in the course book (see page 6), we write, for example, *bbg* for the case when the first two children are boys and the third one is a girl. Hence, we have:

$$\begin{aligned} A &= \{bbb, ggg\}, \\ B &= \{bgg, gbg, ggb, ggg\}, \\ A \cap B &= \{ggg\}. \end{aligned}$$

As we have probability $1/8$ of getting each of the outcomes *bbb*, *gbb*, *bgb*, *bbg*, *bgg*, *gbg*, *ggb*, *ggg*, it follows that

$$\begin{aligned} \mathbb{P}(A) &= 2 \times \frac{1}{8}, && (1) \\ \mathbb{P}(B) &= 4 \times \frac{1}{8}, && (2) \\ \mathbb{P}(A \cap B) &= 1 \times \frac{1}{8}. && (3) \end{aligned}$$

An alternative proof can be obtained by taking $X$ to be the number of boys in a 3-child family. We have that $X \sim \text{Binomial}(3, 1/2)$ (see page 11 in the course book). Therefore, we get:

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}(X = 0) + \mathbb{P}(X = 3) \\ &= \binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 + \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 \\ &= \frac{1}{8} + \frac{1}{8} \\ &= 2 \times \frac{1}{8}, && (4) \\ \mathbb{P}(B) &= \mathbb{P}(X = 0) + \mathbb{P}(X = 1) \\ &= \binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 + \binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 \\ &= \frac{1}{8} + 3 \times \frac{1}{8} \\ &= 4 \times \frac{1}{8}, && (5) \end{aligned}$$

$$\begin{aligned}\mathbb{P}(A \cap B) &= \mathbb{P}(X = 0) \\ &= \binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 \\ &= \frac{1}{8}. \qquad (6)\end{aligned}$$

From (1)-(3), or equivalently from (4)-(6), we have that $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, which proves that $A$ and $B$ are statistically independent.

(b) Applying the notational conventions from part (a), we get:

$$\begin{aligned}C &= \{gbb, bgb, bbg, bgg, gbg, ggb\}, \\ B \cap C &= \{bgg, gbg, ggb\}.\end{aligned}$$

We note in passing that $C = \overline{A}$. These observations lead to

$$\begin{aligned}\mathbb{P}(C) &= 6 \times \frac{1}{8}, \qquad (7) \\ \mathbb{P}(B \cap C) &= 3 \times \frac{1}{8}. \qquad (8)\end{aligned}$$

The same results can be obtained by using the random variable $X$ defined in part (a):

$$\begin{aligned}\mathbb{P}(C) &= \mathbb{P}(X = 1) + \mathbb{P}(X = 2) \\ &= \binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 + \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 \\ &= 3 \times \frac{1}{8} + 3 \times \frac{1}{8} \\ &= 6 \times \frac{1}{8}, \qquad (9) \\ \mathbb{P}(B \cap C) &= \mathbb{P}(X = 1) \\ &= \binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 \\ &= 3 \times \frac{1}{8}. \qquad (10)\end{aligned}$$

From (2), (7) and (8), or equivalently from (5), (9) and (10), we have that $\mathbb{P}(B \cap C) = \mathbb{P}(B)\mathbb{P}(C)$, which proves that $B$ and $C$ are statistically independent.

**2** This question is similar to Question 3 from Assignment 1. We define the following events:

$$\begin{aligned}B_1 &= \{\text{Box}_1 \text{ is selected}\}, \\ B_2 &= \{\text{Box}_2 \text{ is selected}\}, \\ R &= \{\text{The selected ball is red}\}.\end{aligned} \qquad (11)$$

Hence, we have:

$$\begin{aligned}
\mathbb{P}(B_1) &= \frac{1}{2}, && (12)\\
\mathbb{P}(B_2) &= \frac{1}{2}, && (13)\\
\mathbb{P}(R|B_1) &= \frac{999}{1000}, && (14)\\
\mathbb{P}(R|B_2) &= \frac{1}{1000}. && (15)
\end{aligned}$$

After these preparations, we answer the question:

$$\begin{aligned}
&\mathbb{P}(B_1|R)\\
&= \frac{\mathbb{P}(R|B_1)\mathbb{P}(B_1)}{\mathbb{P}(R)} \text{ [Bayes' Theorem]}\\
&= \frac{\mathbb{P}(R|B_1)\mathbb{P}(B_1)}{\mathbb{P}(R|B_1)\mathbb{P}(B_1)+\mathbb{P}(R|B_2)\mathbb{P}(B_2)} \text{ [Partition Theorem]}\\
&= \frac{\frac{999}{1000}\frac{1}{2}}{\frac{999}{1000}\frac{1}{2}+\frac{1}{1000}\frac{1}{2}} \text{ [see (12) − (15)]}\\
&= \frac{999}{1000}.
\end{aligned}$$

**3** This question is similar to the examples discussed in Sections 2.2-2.3 of the course book.

(a) Let $X$ be the number of heads out of 20 tosses: $X \sim \text{Binomial}(20, p)$ and

$$\begin{aligned}
H_0 :&\quad p = \frac{1}{2},\\
H_1 :&\quad p \neq \frac{1}{2} \text{ (two − sided)}.
\end{aligned}$$

(b) The following calculations are straightforward:

$$\begin{aligned}
\mathbb{P}(X = 15) &= \mathbb{P}(X \leq 15) - \mathbb{P}(X \leq 14)\\
&= F_X(15) - F_X(14)\\
&= 0.9941 - 0.9793\\
&= 0.0148,
\end{aligned}$$

$$
\begin{aligned}
\mathbb{P}(X=4) &= \binom{20}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{16} \\
&= \frac{20!}{4!16!}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{16} \\
&= \frac{20!}{16!4!}\left(\frac{1}{2}\right)^{16}\left(\frac{1}{2}\right)^4 \\
&= \mathbb{P}(X=16) \\
&= \mathbb{P}(X\le 16)-\mathbb{P}(X\le 15) \\
&= F_X(16)-F_X(15) \\
&= 0.9987-0.9941 \\
&= 0.0046.
\end{aligned}
$$

(c) Under $H_0$, $X \sim \text{Binomial}(20, 0.5)$. The probability function of $X$ (under the null hypothesis) is represented in Figure 1. Remark that the curve peaks at $20 \times 0.5 = 10$.

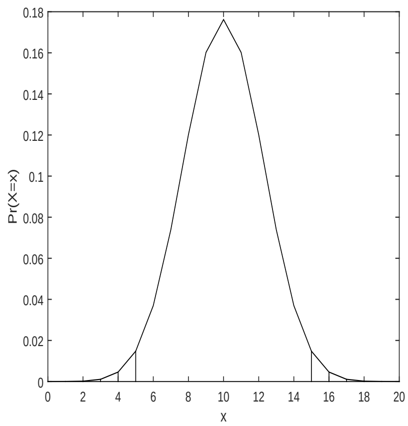


Figure 1: Probability function for Binomial(20,0.5).

(d) In order to calculate the $p$-value, we use the data from Table 1:

$$
\begin{aligned}
2\times\mathbb{P}(X\ge 15) &= 2\times(1-\mathbb{P}(X\le 14)) \\
&= 2\times(1-F_X(14)) \\
&= 2\times(1-0.9793) \\
&\approx 0.0414.
\end{aligned}
$$

Hence, if the null hypothesis is true (coin is fair), we would have a 4.1% chance of observing something as extreme as 15 heads or tails. This is unusual, but not extremely unusual. There is some evidence that the coin is not fair. However, the observations are also consistent with the the possibility that the coin is fair.

(e) Let $Y$ be the number of heads out of 40 tosses: $X \sim \text{Binomial}(40, p)$ and

$$\begin{aligned} H_0 : \quad & p = \frac{1}{2}, \\ H_1 : \quad & p \neq \frac{1}{2} \text{ (two} - \text{sided).} \end{aligned}$$

The calculation of $p$-value is straightforward (see Table 2):

$$\begin{aligned} 2 \times \mathbb{P}(Y \geq 30) &= 2 \times (1 - \mathbb{P}(Y \leq 29)) \\ &= 2 \times (1 - F_Y(29)) \\ &= 2 \times (1 - 0.9989) \\ &\approx 0.0022. \end{aligned}$$

Hence, if the null hypothesis is true (coin is fair), we would have a 0.2% chance of observing something as extreme as 30 heads or tails. The evidence against the null hypothesis is much stronger than in the case when 15 heads are observed in 20 tosses. Dividing the total number of heads by the total number of tosses gives us the same result in both cases, 3/4, but in the second case the sample size is larger.

**4** The calculations involved are very similar to those presented in Section 2.5 of the course book.

(a) $L(p; 30) = \mathbb{P}(Y = 30)$ when $Y \sim \text{Binomial}(40, p)$. Therefore, we have $L(p; 30) = \begin{pmatrix} 40 \\ 30 \end{pmatrix} p^{30}(1-p)^{40-30}$, where $0 \leq p \leq 1$.

(b) Differentiate the likelihood with respect to $p$:

$$\begin{aligned} \frac{dL}{dp} &= \begin{pmatrix} 40 \\ 30 \end{pmatrix} \left[30p^{29}(1-p)^{10} - p^{30}10(1-p)^{9}\right] \\ &= \begin{pmatrix} 40 \\ 30 \end{pmatrix} p^{29}(1-p)^{9} \left[30(1-p) - 10p\right] \\ &= \begin{pmatrix} 40 \\ 30 \end{pmatrix} p^{29}(1-p)^{9} \left(30 - 40p\right). \end{aligned}$$

The maximizing value of $p$ satisfies:

$$\begin{aligned} \frac{dL}{dp} &= 0 \\ &\Downarrow \\ \binom{40}{30} p^{29}(1-p)^{9}\,(40-30p) &= 0 \\ &\Downarrow \\ p &\in \{0, 1, 30/40\}. \end{aligned}$$

(c) Based on the graph plotted in Figure 2 and the answer to part (b), we can conclude that the maximizing value is $p = 30/40 = 3/4$. Therefore, the maximum likelihood estimate is $\hat{p} = 3/4$. This is the value of $p$ at which the observation $Y = 30$ is more likely than at any other value of $p$.

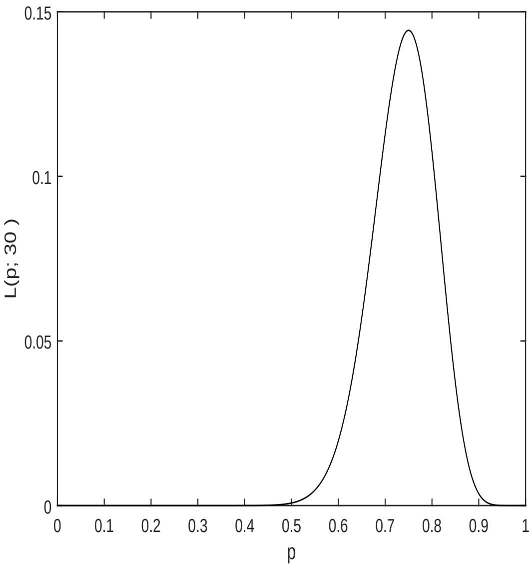


Figure 2: Likelihood function $L(p; 30)$ for the case when the total number of tosses equals 40.

(d) We know the general result (see page 62 in the course book): If we make a single observation $X = x$ when the distribution of $X$ is Binomial(n,p) and $n$ is known, then the maximum likelihood estimate of $p$ is given by $x/n$. It follows that, in our case, the maximum likelihood estimate of $p$ is $15/20 = 3/4$.