1 The answers for all parts are presented below:

(a) It is clear that

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}(\{s_1, s_2\}) \\
&= \mathbb{P}(\{s_1\} \cup \{s_2\}) \\
&= \frac{1}{4} + \frac{1}{4} \\
&= \frac{1}{2}.
\end{aligned}$$

Similarly, we have: $\mathbb{P}(B) = \mathbb{P}(C) = \frac{1}{2}$.
The following calculations are straightforward:

$$\begin{aligned}
\mathbb{P}(A \cap B) &= \mathbb{P}(\{s_1\}) \\
&= \frac{1}{4}, \\
\mathbb{P}(A \cap B \cap C) &= \mathbb{P}(\{s_1\}) \\
&= \frac{1}{4}.
\end{aligned}$$

(b) From the identities in part (a), we have that $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) = \frac{1}{4}$. Hence, the events A and B are statistically independent.

(c) Using the results in part (a), it is easy to prove that the events A, B and C are pairwise independent. All that remains for deciding if they are mutually independent is to verify if the identity $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$ holds. From part (a), we have that $\mathbb{P}(A \cap B \cap C) = \frac{1}{4}$ and $\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \frac{1}{8}$. Hence, the events A, B and C are *not* mutually independent.

2 The answers for all parts are presented below:

(a) We have $X \sim \text{Binomial}(5, 1/2)$: The number of trials equals the number of children in the family and the trials are statistically independent (from hypothesis). With the understanding that success means “the child has blue eyes”, the probability of success in each trial is the same (1/2).

(b) Using the properties of the Binomial distribution, we get:

$$\begin{aligned}
\mathbb{P}(X \geq 1) &= 1 - \mathbb{P}(X = 0) \\
&= 1 - \binom{5}{0} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0 \\
&= 1 - \left(\frac{1}{2}\right)^5 \\
&= \frac{31}{32}.
\end{aligned}$$

(c) We have the following chain of identities:

$$\begin{aligned}
&\mathbb{P}(X \geq 3) \\
&= \sum_{k=3}^{5} \mathbb{P}(X = k) \quad [X \sim \text{Binomial}(5, 1/2)] \\
&= \frac{1}{2} \sum_{k=3}^{5} \{\mathbb{P}(X = k) + \mathbb{P}(X = 5 - k)\} \quad [\mathbb{P}(X = k) = \mathbb{P}(X = 5 - k)] \\
&= \frac{1}{2} \sum_{k=0}^{5} \mathbb{P}(X = k) \\
&= \frac{1}{2} \times 1 \quad [\text{valid prob. function}] \\
&= \frac{1}{2}.
\end{aligned}$$

(d) We need to calculate:

$$\begin{aligned}
&\mathbb{P}(X \geq 3 | X \geq 1) \\
&= \frac{\mathbb{P}\left(\{X \geq 3\} \cap \{X \geq 1\}\right)}{\mathbb{P}\left(\{X \geq 1\}\right)} \quad [\text{formula cond. probabilities}] \\
&= \frac{\mathbb{P}(X \geq 3)}{\mathbb{P}(X \geq 1)} \\
&= \frac{1/2}{31/32} \quad [\text{from parts (b) -- (c)}] \\
&= \frac{16}{31}.
\end{aligned}$$

3 We define the following events:

$$\begin{aligned}
A &= \{\text{Screen is manufactured by company } A\} \\
B &= \{\text{Screen is manufactured by company } B\} \\
C &= \{\text{Screen is manufactured by company } C\} \\
D &= \{\text{Screen is defective}\}
\end{aligned}$$

Apply Bayes Theorem for computing $\mathbb{P}(A|D)$:

$$\begin{aligned}
&\mathbb{P}(A|D) \\
&= \frac{\mathbb{P}(D|A)\mathbb{P}(A)}{\mathbb{P}(D)} \\
&= \frac{\mathbb{P}(D|A)\mathbb{P}(A)}{\mathbb{P}(D|A)\mathbb{P}(A) + \mathbb{P}(D|B)\mathbb{P}(B) + \mathbb{P}(D|C)\mathbb{P}(C)} \text{ [Partition Theorem]} \\
&= \frac{0.01 \times 0.5}{(0.01 \times 0.5) + (0.02 \times 0.3) + (0.03 \times 0.2)} \\
&= 0.2941.
\end{aligned}$$

4 This question is similar to the examples discussed in Sections 2.2-2.3 of the course book.

(a) We have: $X \sim \text{Binomial}(20, p)$ and

$$\begin{aligned}
H_0 : &\quad p = 0.2, \\
H_1 : &\quad p \neq 0.2 \text{ (two} - \text{sided)}.
\end{aligned}$$

(b) The following calculations are straightforward:

$$\begin{aligned}
\mathbb{P}(X = 6) &= \mathbb{P}(X \leq 6) - \mathbb{P}(X \leq 5) \\
&= F_X(6) - F_X(5) \\
&= 0.9133 - 0.8042 \\
&= 0.1091.
\end{aligned}$$

(c) Under H_0, $X \sim \text{Binomial}(20, 0.2)$. The probability function of X (under the null hypothesis) is represented in Figure 1. Remark that the curve peaks at $\lfloor 20 \times 0.2 + 0.2 \rfloor = 4$.

(d) In order to calculate the p-value, we use the data from Table 1:

$$\begin{aligned}
2 \times \mathbb{P}(X \geq 8) &= 2 \times (1 - \mathbb{P}(X \leq 7)) \\
&= 2 \times (1 - F_X(7)) \\
&= 2 \times (1 - 0.9679) \\
&\approx 0.0642.
\end{aligned}$$

Hence, if the null hypothesis is true, we would have a 6.4% chance of observing something as extreme as 8 defective items. This is slightly unusual, but not very unusual. We conclude that there is no real evidence against the null hypothesis. The observations are compatible with the possibility that $p = 0.2$. However, the observations are also consistent with the the possibility that $p \neq 0.2$. We just do not have enough evidence either way.

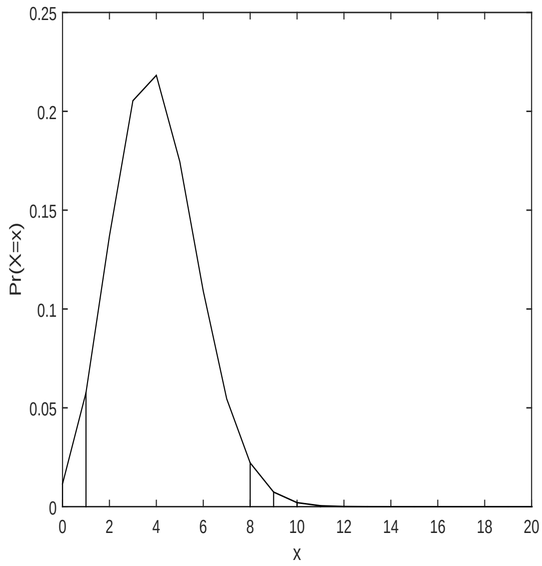

Figure 1: Probability function for Binomial(20,0.2).

5 The calculations involved are very similar to those presented in Section 2.5 of the course book.

(a) $L(p;8) = \mathbb{P}(X = 8)$ when $X \sim \text{Binomial}(20, p)$. Therefore, we have $L(p;8) = \left(\begin{array}{c} 20 \\ 8 \end{array}\right) p^8(1-p)^{12}$, where $0 \leq p \leq 1$.

(b) Differentiate the likelihood with respect to p:

$$\begin{aligned} \frac{dL}{dp} &= \left(\begin{array}{c} 20 \\ 8 \end{array}\right) \left[8p^7(1-p)^{12} - p^8 12(1-p)^{11}\right] \\ &= \left(\begin{array}{c} 20 \\ 8 \end{array}\right) p^7(1-p)^{11} \left[8(1-p) - 12p\right] \\ &= \left(\begin{array}{c} 20 \\ 8 \end{array}\right) p^7(1-p)^{11} \left(8 - 20p\right). \end{aligned}$$

The maximizing value of p satisfies:

$$\begin{aligned} \frac{dL}{dp} &= 0 \\ &\Downarrow \\ \left(\begin{array}{c} 20 \\ 8 \end{array}\right) p^7(1-p)^{11} \left(20 - 8p\right) &= 0 \\ &\Downarrow \\ p &\in \{0, 1, 8/20\}. \end{aligned}$$

(c) Based on the graph plotted in Figure 2 and the answer to part (b), we can conclude that the maximizing value is $p = 8/20 = 0.4$. Therefore, the maximum likelihood estimate is $\hat{p} = 0.4$. This is the value of p at which the observation $X = 8$ is more likely than at any other value of p.

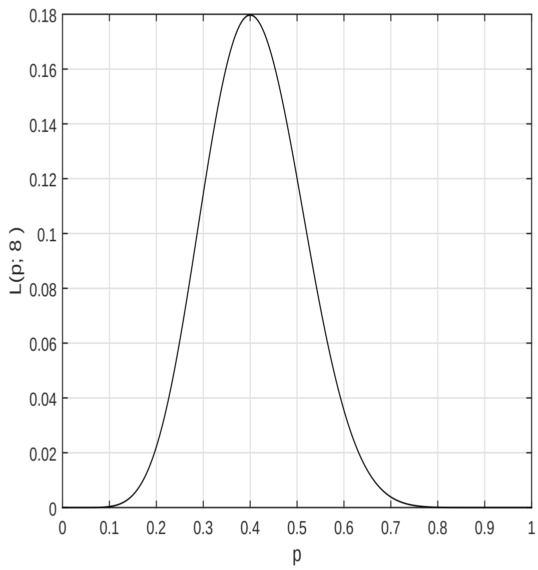

Figure 2: Likelihood function $L(p; 8)$ for the case when the total number of items is 20.

Note: The test is based on questions from M.H. DeGroot and M.J. Schervish, "Probability and statistics", Addison-Wesley, 2002.