Statistics 210 FC **Assignment 4** **Due: 4PM, May 31, 2019**

1 A common daily lottery game involves the drawing of three digits from 0 to 9 independently with replacement and independently from day to day. For example, a possible result of the drawing might be the "number" 010. Lottery watchers often get excited when all three digits are the same, an event called *triples*. It is evident that a possible triple might be 000.

Let p be the probability of obtaining triples on any day when the drawing is done.

Let X be the number of days without triples before the first triple is observed.

Answer the following questions:

- **a** (3 marks) Calculate the value of p.

  Hint: Count how many equally likely daily numbers exist. Count how many different triples exist.
- **b** (2 marks) State the distribution of X, with parameters.
- **c** (3 marks) Find the expected number of days until we observe triples. Note that you need to consider the day when the triples are observed.
- **d** (5 marks) Prove that, for all nonnegative integers k and t, we have:

  $$\mathbb{P}(X = k + t | X \geq k) = \mathbb{P}(X = t).$$

  Show your working.

  Hint: Use the result that you have obtained in part (b). It might be helpful to recall some of the results from Assignment 3, question 4.

2 Suppose that a sequence of independent tosses are made with a coin for which the probability of obtaining a head is 1/30. Answer the following questions:

- **a** (3 marks) Let X be the number of tails obtained before we get five heads. State the distribution of X, with parameters.
- **b** (1 mark) What is the expected number of tails that will be obtained before five heads have been obtained?

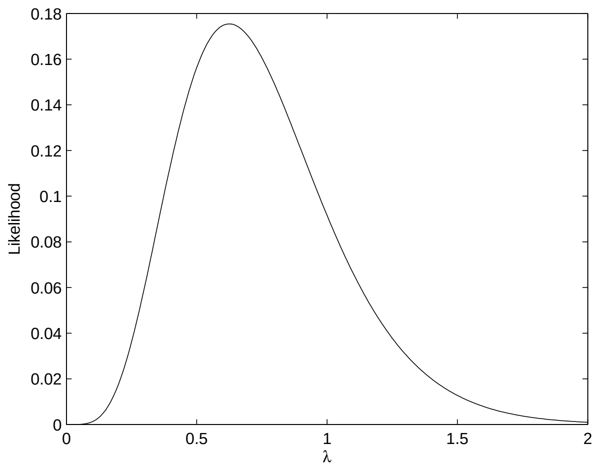

Figure 1: Plot for question 4, part (b)

- **c** (1 mark) What is the variance of the number of tails that will be obtained before five heads have been obtained?
- **d** (2 marks) What is the expected number of tosses that will be obtained before five heads have been obtained?
- **e** (2 marks) What is the variance of the number of tosses that will be obtained before five heads have been obtained?

3 Suppose that n students are selected at random without replacement from a class containing 28 students, of whom 8 are boys and 20 are girls. We assume that $0 < n < 28$.

Let X denote the number of boys that are obtained.

Answer the following questions:

- **a** (4 marks) State the distribution of X, with parameters.
- **b** (1 mark) Write down the possible values of X.
- **c** (1 mark) Express $\mathbb{E}(X)$ in terms of n.
- **d** (4 marks) For what sample size n will $\text{Var}(X)$ be a maximum? Show your working.

  Hint: If you have difficulties to find the answer analytically, calculate all possible values of $\text{Var}(X)$.
- **e** (5 marks) Let $\tilde{X}$ be the *proportion* of boys in the sample. Calculate the mean and the variance of $\tilde{X}$ for the case when $n = 14$.

4 Mary is a waitress in a city centre restaurant. She receives tips from customers at an average rate of λ per hour. During a particular eight hour shift, she receives 5 tips. We wish to estimate λ.

a (2 marks) Let X be the number of tips Mary receives in an 8 hour period. If tips are received according to a Poisson process with rate λ tips per hour, state the distribution of X, with parameters.

b (5 marks) Write down the likelihood function, $L(\lambda; 5)$ for this problem. Remember to state the range of values of λ for which the likelihood is defined. Hence find the maximum likelihood estimate of λ. You should make reference to the graph of the likelihood function, shown in Figure 1. Show all working in your answer.

c (2 marks) In general, if Mary receives X tips in an 8 hour period, what is the maximum likelihood estima<u>tor</u> for λ in terms of X?

d (4 marks) Let $\hat{\lambda}$ be the maximum likelihood estima<u>tor</u> for λ. Find expressions for the variance and estimated variance of $\hat{\lambda}$: $\text{Var}(\hat{\lambda})$ and $\widehat{\text{Var}}(\hat{\lambda})$. Use the result that you have obtained in part (b) for computing $\widehat{\text{Var}}(\hat{\lambda})$.

5 Let X be a <u>continuous</u> random variable with probability density function

$$f_X(x) = \begin{cases} k/x^{2019}, & \text{for } x \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

a (3 marks) Find the value of k.

b (3 marks) Find the cumulative distribution function of X, $F_X(x)$. Remember to give the value of $F_X(x)$ for all $x \in \mathbb{R}$.

6 Suppose that the cumulative distribution function of a <u>continuous</u> random variable X is:

$$F_X(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ kx^2 & \text{for } 0 < x < 2019, \\ 1 & \text{for } x \geq 2019, \end{cases}$$

where k is a constant to be determined.

a (2 marks) Find the value of k.

b (3 marks) Find the probability density function of X, $f_X(x)$.

Total: 61 marks.