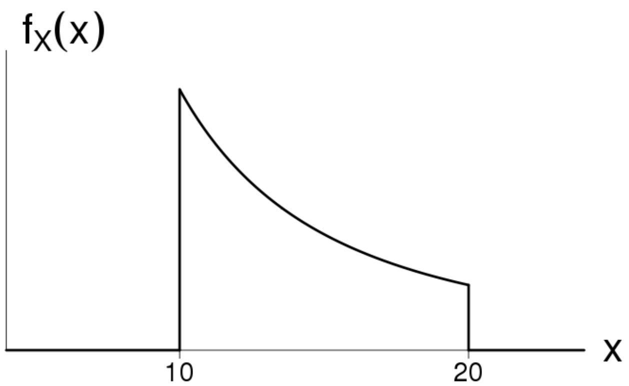
1(a) We need $\int_{10}^{20} f_X(x)\,dx = 1$.

$$\begin{aligned} 1 &= \int_{10}^{20} kx^{-2}\,dx \\ &= k\left[\frac{x^{-1}}{-1}\right]_{10}^{20} \\ &= -k\left(\frac{1}{20} - \frac{1}{10}\right) \\ &= \frac{k}{20} \\ \Rightarrow \quad k &= 20. \end{aligned}$$

(b)

(2)



(2)

(c)

$$\begin{aligned} \mathbb{P}(X > 15) &= \int_{15}^{20} 20x^{-2}\,dx \\ &= 20\left[\frac{x^{-1}}{-1}\right]_{15}^{20} \\ &= -20\left(\frac{1}{20} - \frac{1}{15}\right) \\ &= \frac{1}{3} \end{aligned}$$

(2)

(d) Clearly,

$$F_X(x) = \begin{cases} 0 & \text{for } x \leq 10, \\ 1 & \text{for } x \geq 20. \end{cases}$$

For $10 < x < 20$ we have:

$$\begin{aligned} F_X(x) &= \int_{10}^{x} f_X(u)\,du \qquad \text{for } 10 < x < 20 \\ &= 20\left[\frac{u^{-1}}{-1}\right]_{10}^{x} \\ &= -20\left(\frac{1}{x} - \frac{1}{10}\right) \\ &= 2 - \frac{20}{x}\,. \end{aligned}$$

Thus for all x, the expression is:

$$F_X(x) = \begin{cases} 0 & \text{for } x \leq 10, \\ 2 - \frac{20}{x} & \text{for } 10 < x < 20, \\ 1 & \text{for } x \geq 20. \end{cases}$$

(2)

(e)

$$\mathbb{P}(X > 15) = 1 - F_X(15) = 1 - \left(2 - \frac{20}{15}\right) = \frac{1}{3}, \quad \text{as above.}$$

$$\mathbb{P}(12 < X < 14) = F_X(14) - F_X(12) = \left(2 - \frac{20}{14}\right) - \left(2 - \frac{20}{12}\right) = 0.238.$$

(2)

(f)

$$\begin{aligned} \mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx &= \int_{10}^{20} \frac{20x}{x^2}\,dx \\ &= 20\int_{10}^{20} x^{-1}\,dx \\ &= 20\Big[\log(x)\Big]_{10}^{20} \qquad \text{(natural log)} \\ &= 20\Big(\log(20) - \log(10)\Big) \\ &= 13.86 \text{ minutes.} \end{aligned}$$

(2)

(g)

$$\begin{aligned}
\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx &= \int_{10}^{20} \frac{20x^2}{x^2}\,dx \\
&= 20\left[x\right]_{10}^{20} \\
&= 20\,(20-10) \\
&= 200.
\end{aligned}$$

Thus

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = 200 - 13.86^2 = 7.82.$$

(2)

(h) Description B is correct.

This is because each window constitutes a *separate* random number, so some of them may be large while the others are small and overall they balance each other to some extent.

By contrast, Description A takes only one random number and multiplies it.

The variance under Description A is $\mathrm{Var}(10X) = 100\mathrm{Var}(X)$.
This is much greater than the variance under Description B,
$\mathrm{Var}(X_1 + \ldots + X_{10}) = 10\mathrm{Var}(X)$.
This occurs because of the balancing of big X_i's with small X_i's in Description B. (2)

(i)

$$\begin{aligned}
\mathbb{E}(T) &= \mathbb{E}(X_1) + \mathbb{E}(X_2) + \ldots + \mathbb{E}(X_{10}) && \text{— does NOT need independence} \\
&= 10\mathbb{E}(X) \\
&= 138.6 \text{ minutes.} && \text{using } \mathbb{E}(X) = 13.86 \text{ from part (f).}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(T) &= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}(X_{10}) && \text{— DOES need independence} \\
&= 10\mathrm{Var}(X) \\
&= 78.2 && \text{using } \mathrm{Var}(X) = 7.82 \text{ from part (g).}
\end{aligned}$$

(2)

2(a) For a Poisson process, the waiting time is Exponential.
So $T \sim \text{Exponential}(\lambda = 20)$. (2)

(b) $\mathbb{E}(T) = \dfrac{1}{\lambda} = \dfrac{1}{20} = 0.05.$ (2)

(c) 5 minutes is 5/60 hours, i.e. 1/12 hours. From memory, the CDF of the Exponential(20) distribution is $F_T(t) = 1 - e^{-20t}$.

$$\mathbb{P}\left(T > \frac{1}{12}\right) = 1 - F_T\left(\frac{1}{12}\right) = e^{-20/12} = 0.189.$$

(2)

(d) Let $T_1, \ldots, T_8$ be the times for the eight passengers. Each T_i has the same distribution as T, so $\mathbb{E}(T_i) = \mathbb{E}(T) = 0.05$ for $i = 1, \ldots, 8$.

$\mathbb{E}(T_1 + \ldots + T_8) = \mathbb{E}(T_1) + \ldots + \mathbb{E}(T_8) = 8 \times 0.05 = 0.4$ hours (24 minutes).

(2)

Total: 26