1 The answers for all parts are presented below:

a It is easy to see that $p = 10^{-2}$ because there are 10 different triples among the 1000 equally likely daily numbers.

b X is the number of failures before the first success in a series of Bernoulli trials with $\mathbb{P}(\text{success}) = p$. Hence, $X \sim \text{Geometric}(p)$, where $p = 10^{-2}$.

c The expected number of days until we observe triples is

$$\begin{aligned}
\mathbb{E}(X+1) &= \mathbb{E}(X) + 1 \text{ [properties of } \mathbb{E}(\cdot)] \\
&= 1 + \frac{1-p}{p} \text{ [course book page 94]} \\
&= 1 + \frac{1-0.01}{0.01} \\
&= 1 + 99 \\
&= 100.
\end{aligned}$$

d Let k and t be nonnegative integers. Using the formula for the conditional probabilities, we get:

$$\begin{aligned}
&\mathbb{P}(X = k + t | X \geq k) \\
&= \frac{\mathbb{P}\left(\{X = k+t\} \cap \{X \geq k\}\right)}{\mathbb{P}\left(\{X \geq k\}\right)} \\
&= \frac{\mathbb{P}(X = k+t)}{\mathbb{P}(X \geq k)} \text{ [because } t \geq 0] \\
&= \frac{\mathbb{P}(X = k+t)}{\mathbb{P}(X = k) + \mathbb{P}(X > k)} \\
&= \frac{p(1-p)^{k+t}}{p(1-p)^k + (1-p)^{k+1}} \text{ [see Assignment 3, Question 4, part (a)]} \\
&= \frac{p(1-p)^{k+t}}{(1-p)^k(p+1-p)} \\
&= \frac{p(1-p)^{k+t}}{(1-p)^k} \\
&= p(1-p)^t \\
&= \mathbb{P}(X = t).
\end{aligned}$$

2 The answers for all parts are presented below:

a X is the number of failures before the 5th success in a series of Bernoulli trials with $\mathbb{P}(\text{success}) = 1/30$. Hence, $X \sim \text{NegBin}(k, p)$, where $k = 5$ and $p = 1/30$.

b According to the course book (page 99), we have:

$$\begin{aligned}\mathbb{E}(X) &= \frac{k(1-p)}{p} \\ &= \frac{5(1-1/30)}{1/30} \\ &= 145.\end{aligned}$$

c Again, according to the course book (page 99), we have:

$$\begin{aligned}\text{Var}(X) &= \frac{k(1-p)}{p^2} \\ &= \frac{5(1-1/30)}{(1/30)^2} \\ &= 4350.\end{aligned}$$

d We need to calculate

$$\begin{aligned}\mathbb{E}(X+k) &= \mathbb{E}(X) + k \text{ [properties } \mathbb{E}(\cdot)] \\ &= 145 + 5 \text{ [see part (b)]} \\ &= 150.\end{aligned}$$

e Similarly, we have:

$$\begin{aligned}\text{Var}(X+k) &= \text{Var}(X) \text{ [properties Var}(\cdot)] \\ &= 4350 \text{ [see part (c)]}\end{aligned}$$

3 The answers for all parts are presented below:

a We have $N = 28$ students ("objects"); $M = 8$ of the N "objects" are boys ("special"); the other $N - M$ "objects" are girls ("not special"). We remove n "objects" at random without replacement. Hence, X is the number of the n removed objects that are "special". It follows that $X \sim \text{Hypergeometric}(N, M, n)$.

b The possible values of X are: $\max(0, n-20), \ldots, \min(n, 8)$.

c According to the course book (page 102), we have:

$$\begin{aligned}\mathbb{E}(X) &= n\frac{M}{N} \\ &= \frac{8}{28}n \\ &= \frac{2}{7}n.\end{aligned}$$

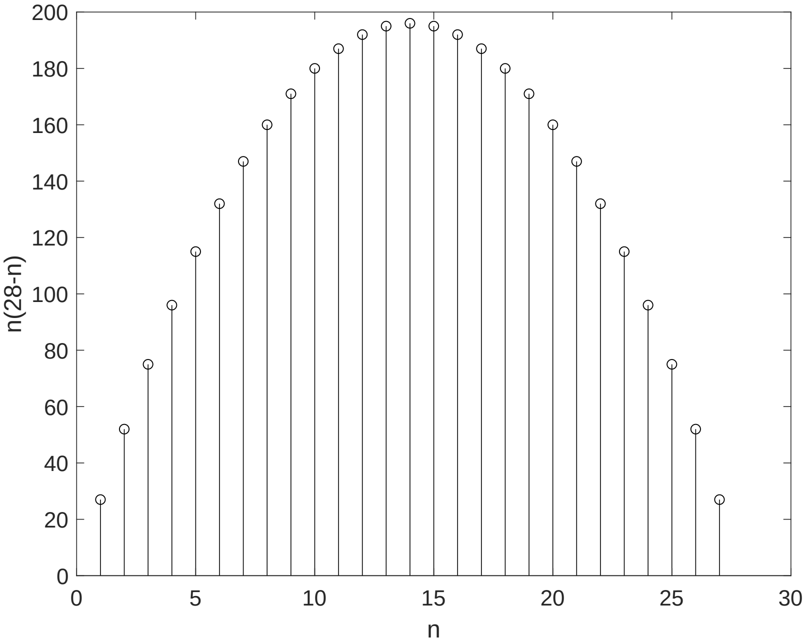

Figure 1: The values of the product $n(28-n)$ for $n \in \{1, 2, \ldots, 27\}$.

d Again, according to the course book (page 102), we have:

$$\begin{aligned}
\mathrm{Var}(X) &= n\frac{M}{N}\left(1-\frac{M}{N}\right)\frac{N-n}{N-1} \\
&= n\frac{8}{28}\left(1-\frac{8}{28}\right)\frac{28-n}{27} \\
&= \frac{10}{1323}n(28-n).
\end{aligned}$$

We need to find the values of n for which $n(28-n) > (n-1)[28-(n-1)]$ when $n \in \{2, \ldots, 27\}$. It is easy to see that the inequality is equivalent to $2n < 29$. It follows that the variance attains its maximum when $n = 28/2 = 14$. This can be also seen in Figure 1.

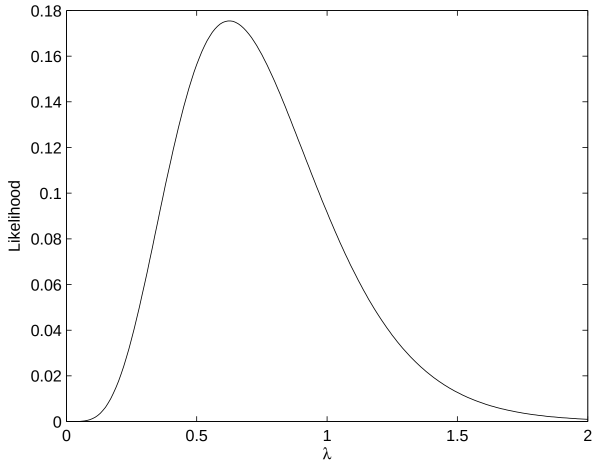

Figure 2: Plot for question 4, part (b)

e We have that $\tilde{X} = \frac{X}{n}$. Hence, we get:

$$\begin{aligned}\mathbb{E}(\tilde{X}) &= \mathbb{E}\left(\frac{X}{n}\right)\\ &= \frac{1}{n}\mathbb{E}(X) \text{ [properties } \mathbb{E}(\cdot)]\\ &= \frac{1}{14}\frac{2}{7}14 \text{ [see part (b)]}\\ &= \frac{2}{7},\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(\tilde{X}) &= \mathrm{Var}\left(\frac{X}{n}\right)\\ &= \frac{1}{n^2}\mathrm{Var}(X) \text{ [properties Var}(\cdot)]\\ &= \frac{1}{14^2}\frac{10}{1323}14^2\\ &= \frac{10}{1323}.\end{aligned}$$

4 The answers for all parts are presented below:

a We have $X \sim \text{Poisson}(8\lambda)$.

b The expression of the likelihood function is:

$$L(\lambda; 5) = \mathbb{P}(X = 5) = \frac{(8\lambda)^5}{5!}e^{-8\lambda}, \text{ where } \lambda > 0.$$

Simple calculations lead to

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}\lambda} &= \frac{8^5}{5!}\left[5\lambda^4 e^{-8\lambda} - 8\lambda^5 e^{-8\lambda}\right] \\ &= \frac{8^5}{5!}\lambda^4 e^{-8\lambda}\left(5 - 8\lambda\right).\end{aligned}$$

We have:

$$\left.\frac{\mathrm{d}L}{\mathrm{d}\lambda}\right|_{\lambda=\hat{\lambda}} = 0 \;\Rightarrow\; \hat{\lambda} = \frac{5}{8} \text{ (see also Figure 2)}$$

c Replace the observation $x = 5$ with the random variable X in part (b) for obtaining the maximum likelihood estimator $\hat{\lambda} = \dfrac{X}{8}$.

d We have:

$$\begin{aligned}\mathrm{Var}(\lambda) &= \mathrm{Var}\left(\frac{X}{8}\right) \\ &= \frac{1}{64}\mathrm{Var}(X) \text{ [properties Var}(\cdot)] \\ &= \frac{8\lambda}{64} \text{ [because } X \sim \text{Poisson}(8\lambda)] \\ &= \frac{\lambda}{8}.\end{aligned}$$

It is clear that $\widehat{\mathrm{Var}}(\hat{\lambda}) = \dfrac{\hat{\lambda}}{8}$. Using the result from part (b), we get:

$$\widehat{\mathrm{Var}}(\hat{\lambda}) = \frac{5/8}{8} = \frac{5}{64}.$$

5 The answers for both parts are presented below:

a Total area under the curve is 1:

$$\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = 1. \tag{1}$$

We also have:

$$\begin{aligned}\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x &= \int_{-\infty}^{1} f_X(x)\mathrm{d}x + \int_{1}^{\infty} f_X(x)\mathrm{d}x \\ &= 0 + k\int_{1}^{\infty} x^{-2019}\mathrm{d}x \\ &= k\left(-\frac{1}{2018}\right)\left[x^{-2018}\right]_1^{\infty} \\ &= -\frac{k}{2018}(0-1) \\ &= \frac{k}{2018}. \qquad (2)\end{aligned}$$

From (1) and (2), we get $k = 2018$.

b For $x \geq 1$, we have:

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(u)\mathrm{d}u \\ &= \int_{1}^{x} \frac{2018}{u^{2019}}\mathrm{d}u \\ &= \left[-\frac{1}{u^{2018}}\right]_1^x \\ &= -\left(\frac{1}{x^{2018}} - 1\right) \\ &= 1 - \frac{1}{x^{2018}}. \end{aligned}$$

Hence, we get:

$$F_X(x) = \begin{cases} 0, & \text{for } x < 1 \\ 1 - \frac{1}{x^{2018}}, & \text{for } x \geq 1 \end{cases}$$

6 The answers for both parts are presented below:

a We have that $F_X(x) = 1$ at $x = 2019$, hence $k * 2019^2 = 1$. It follows that $k = 1/2019^2$.

b We know that $f_X(x) = \dfrac{\mathrm{d}F_X(x)}{\mathrm{d}x}$ for all x.

Because $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x^2}{2019^2}\right) = \dfrac{2x}{2019^2}$, we get:

$$f_X(x) = \begin{cases} \frac{2x}{2019^2}, & \text{for } 0 < x < 2019 \\ 0, & \text{otherwise} \end{cases}$$

Note: Questions 1-3 are from M.H. DeGroot and M.J. Schervish, "Probability and statistics", Addison-Wesley, 2002.