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A Markov regime switching jump-diffusion model for the pricing of portfolio credit derivatives

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ABSTRACT

The class of reduced form models is a very important class of credit risk models, and the modeling of the default dependence structure is essential in the reduced form models. This paper proposes a thinning-dependent structure model in the reduced form framework. The intensity process is the jump-diffusion version of the Vasicek model with the coefficients allowed to switch in different regimes. This article will investigate the joint (conditional) survival probability and the pricing formulas of portfolio credit derivatives. The exact analytical expressions are provided.

1. Introduction

The reduced form approach has become a standard tool for modeling credit risk. Within this class of models, there are two types of approaches in portfolio credit risk modeling (Giesecke, 2008): the bottom-up approach and the top-down approach. The former approach builds models for the correlated default times of individual names in the portfolio, see, for example, Brigo et al. (2007) and Yu (2007); and the latter one builds models for the loss of the portfolio without referring to the constituent names; see, for example, Giesecke and Goldberg (2005).

This paper follows the bottom-up approach and introduces the thinning-dependence structure to model default dependence. The idea of using the thinning-dependence structure to model default dependence among the firms is similar to that of Wang and Yuen (2005) who produce certain correlation between claim-number processes through the thinning-dependence structure in their risk model. We assume that each event occurs at time t may lead to a default of firm i with probability $p_i(t)$ for $i = 1, 2, \ldots, n$. We thus introduce the so-called thinning-dependence structure to generate the dependence among the default times. In the proposed model, defaults are driven by exogenous events, whose arrivals are modeled by counting processes. The intensity process is supposed to be the jump-diffusion version of the Vasicek model. The Vasicek model is one of the most classical models to modeling interest rate and intensity rate. Many researchers have introduced different extended Vasicek models to describe jumps in the interest rate dynamics and the intensity rate dynamics. (See, e.g., Chacko and Das, 2002; Durham, 2005; Siu, 2010; Liang et al., 2011.)

In recent years, Markov regime switching models have become important in finance, and we witness diverse applications of Markov regime switching models in finance. Some works include Elliott et al. (2005) for option pricing in a regime switching model, Siu (2010) for bond valuation in a Markov regime switching jump-augmented Vasicek model, Liang and

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Wang (2012) for pricing kth-to-default basket default swap in a Markov regime switching common shock model. Since the dynamics of the intensity process is allowed to switch in different regimes in our model, the model is a Markov regime switching model and a generalization of the exponential model (Giesecke, 2003) to the Markov regime switching jump-diffusion intensity processes setting.

The remainder of the article is organized as follows. Section 2 describes the thinning-dependence structure model and derives joint conditional survival probability and joint survival probability. Section 3 applies the model to the valuations of portfolio credit derivatives and gets the closed-form expressions for the pricing formulas.

2. The model

Suppose that there are n firms concerned, and that the exogenous events' arrival is modeled by a Cox process $N(t)$, whose intensity process $\lambda(t)$ depends on the economic state. The economic state of the world is described by a Markov chain $X = \{X_t,\ t \geq 0\}$ with state space $D = \{e_1, e_2, \dots, e_N\}$, $e_i = (0, \dots, 0, 1, 0, \dots, 0)' \in R^N$. This representation follows the canonical representation of the state of a Markov chain which was adopted by Elliott et al. (1994). The process $\{X_t\}$ has the representation property:

$$X_t = X_0 + \int_0^t AX_u du + M(t), \tag{1}$$

where A is the rate matrix of the time-homogeneous process X_t, and $M = \{M(t),\ t \geq 0\}$ is a martingale with respect to the filtration $\mathcal{G}_t$ generated by X.

We assume that each event occurs at time t may lead to a default of firm i with probability $p_i(t)$ for $i = 1, 2, \dots, n$. For simplicity, we consider the case where each $p_i(t)$ is constant, that is,

$$p_i(t) \equiv p_i, \quad \text{for } i = 1, \dots, n.$$

We denote by $N_i(t)$ the p_i-thinning of $N(t)$. The default time of firm i is given by

$$\tau_i = \inf\{t \geq 0 : N_i(t) > 0\}, \quad i = 1, 2, \dots, n. \tag{2}$$

We thus introduce the so-called thinning-dependence structure among the default times $\tau_1, \tau_2, \dots, \tau_n$.

Since $N(t)$ is a Cox process, there exists a standard Poisson process $\overline{N}(t)$ which is independent of the intensity process $\lambda(t)$, such that $N(t) = \overline{N}(\Lambda_t)$, where

$$\Lambda_t = \int_0^t \lambda(s)ds. \tag{3}$$

See Grandell (1991, p. 35).

For $i = 1, \dots, n,\ N_i(t)$ is also a Cox process with cumulated intensity

$$\Lambda_{it} = \int_0^t p_i\lambda(s)ds = p_i\Lambda_t. \tag{4}$$

We need the following assumption, which is similar to the one in Wang and Yuen (2005).

Assumption 2.1. The processes $\{N_1(t)\}, \{N_2(t)\}, \dots, \{N_n(t)\}$ are assumed to be conditionally independent given the process $\{N(t)\}$.

Our intensity process $\lambda(t)$ is modeled by a jump-diffusion process, which is a combination of a Brownian motion and a jump process in the following form

$$d\lambda(t) = \kappa(\mu(t) - \lambda(t))dt + \sigma(t)dW_t + dJ_t, \tag{5}$$

where κ is a positive constant, $\mu(t)$ and $\sigma(t)$ are supposed to be dependent on the state of Markov chain X_t, which are given by

$$\mu(t) = \mu_j, \quad \text{and} \quad \sigma(t) = \sigma_j, \quad \text{if } X_t = e_j,\ \text{for } j = 1, \dots, N, \tag{6}$$

where all μ_j, σ_j are nonnegative constants. W_t is a standard Brownian motion and J_t is a jump process given by

$$J_t = \sum_{i=1}^{M_t} Y_i,$$

where the process M_t is a Cox process with intensity $\rho(t) = \rho(X_t)$ which is given by

$$\rho(t) = \rho_j, \quad \text{if } X_t = e_j,\ \text{for } j = 1, \dots, N. \tag{7}$$

$\{Y_i\}$ are sequences of i.i.d. random variables with distribution $f(x)$, and these random variables are independent of M_t, X_t. We assume that W_t and J_t are independent, and that X_t is independent of W_t and M_t. We denote by $\mathcal{G}_t^0$ the filtration generated by the process $\lambda(t)$, and set

$$\mathcal{F}_t = \mathcal{G}_t^0 \vee \mathcal{G}_t. \tag{8}$$

We define

$$\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_N), \qquad \vec{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_N),$$
$$\vec{\rho} = (\rho_1, \rho_2, \ldots, \rho_N), \qquad \vec{\varpi} = ((\sigma_1)^2, (\sigma_2)^2, \ldots, (\sigma_N)^2),$$

and then from (6) and (7), we have

$$\mu(t) = \langle \vec{\mu}, X_t \rangle, \qquad \sigma(t) = \langle \vec{\sigma}, X_t \rangle, \tag{9}$$

$$\rho(t) = \langle \vec{\rho}, X_t \rangle, \qquad (\sigma(t))^2 = \langle \vec{\varpi}, X_t \rangle. \tag{10}$$

One can derive the explicit solution of (5) as follows (see Park, 2008)

$$\lambda(t) = e^{-\kappa t}\lambda(0) + \int_0^t \kappa e^{-\kappa(t-s)}\mu(s)ds + \int_0^t e^{-\kappa(t-s)}\sigma(s)dW_s + \int_0^t e^{-\kappa(t-s)}dJ_s;$$

hence

$$\int_0^t \lambda(s)ds = \lambda(0)a(0,t) + \kappa\int_0^t \mu(s)a(s,t)ds + \int_0^t a(s,t)\sigma(s)dW_s + \int_0^t a(s,t)dJ_s,$$

where

$$a(s,t) = \frac{1 - e^{-\kappa(t-s)}}{\kappa}. \tag{11}$$

2.1. Joint conditional survival probability and joint survival probability

The default times $\tau_1, \tau_2, \ldots, \tau_n$ are modeled by using the thinning-dependent framework described in the previous section. In this section, we shall derive joint conditional survival probability and the joint survival probability of the n firms. The following lemmas are needed for this purpose.

Following the idea in the proof of Proposition 2.3.2 of Ross (1996), one can verify that the following Lemma 2.1 holds true.

Lemma 2.1. *If $N(t)$ is a standard Poisson process, and $\{N_i(t)\}$ is the p_i-thinning of Poisson process $N(t)$, $i = 1, 2, \ldots, n$, $0 < t_1 \le t_2$, $l_1 \le l_2$, $l_1, l_2 \in \{0, 1, 2, \ldots\}$, then we have*

$$P(N_i(t_1) = l_1 \mid N(t_2) = l_2) = C_{l_2}^{l_1}\left(\frac{p_i t_1}{t_2}\right)^{l_1}\left(1 - \frac{p_i t_1}{t_2}\right)^{l_2 - l_1}, \quad i = 1, 2, \ldots, n.$$

The following very useful results come from Lemma A.1 of Buffington and Elliott (2002).

Lemma 2.2. *Let X be a Markov chain given by (1), for $t > 0$, and $\vec{\theta} = (\theta_1, \theta_2, \ldots, \theta_N) \in R^N$, we have*

$$E\left(e^{\int_0^t \langle \vec{\theta}, X_s \rangle ds}\right) = \langle e^{(A+\Theta)t} \cdot X_0, \mathbf{1} \rangle,$$
$$E\left(e^{\int_0^t \langle \vec{\theta}, X_s \rangle ds} X_t\right) = e^{(A+\Theta)t} \cdot X_0,$$

where Θ is a diagonal matrix given by $\Theta = \mathrm{diag}(\theta_1, \theta_2, \ldots, \theta_N)$, and $\mathbf{1}$ is a vector given by $\mathbf{1} = (1, 1, \ldots, 1)' \in R^N$.

We extend Lemma 2.2 to the general case as is shown in the following lemma.

Lemma 2.3. *Let X be a Markov chain given by (1), for each $i = 1, \ldots, k$; $k \in \{1, 2, \ldots\}$, $f_i(t)$ is a deterministic function, $\beta_i = (b_1^i, b_2^i, \ldots, b_N^i) \in R^N$, then for $t > 0$, we have*

$$E\left(e^{\int_0^t \sum_{i=1}^k \langle \beta_i, X_s \rangle f_i(s) ds}\right) = \left\langle e^{\int_0^t \left(A + \sum_{i=1}^k B_i f_i(s)\right) ds} \cdot X_0, \mathbf{1} \right\rangle,$$
$$E\left(e^{\int_0^t \sum_{i=1}^k \langle \beta_i, X_s \rangle f_i(s) ds} X_t\right) = e^{\int_0^t \left(A + \sum_{i=1}^k B_i f_i(s)\right) ds} \cdot X_0,$$

where each B_i is a diagonal matrix given by

$$B_i = \mathrm{diag}(b_1^i, b_2^i, \dots, b_N^i),$$

$i = 1, \dots, k$; $k \in \{1, 2, \dots\}$, *and* $\mathbf{1}$ *is a vector given by* $\mathbf{1} = (1, 1, \dots, 1)' \in R^N$.

Proposition 2.1. *If* $0 < t_1 < t_2 < \cdots < t_n$, $t > 0$, *then the joint conditional probability of the variables* $(\tau_1, \tau_2, \dots, \tau_n)$ *is given by*

$$P(\tau_1 > t_1, \tau_2 > t_2, \dots, \tau_n > t_n \mid \mathcal{G}_{t_n}) = E\left(e^{-\left(1-\prod_{i=1}^n\left(1-\frac{p_i\Lambda_{t_i}}{\Lambda_{t_n}}\right)\right)\Lambda_{t_n}} \mid \mathcal{G}_{t_n}\right); \tag{12}$$

therefore,

$$P(\tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n > t \mid \mathcal{G}_t) = E\left(e^{-\int_0^t(1-\prod_{i=1}^n(1-p_i))\lambda(u)du} \mid \mathcal{G}_t\right). \tag{13}$$

Proof. Noting (8), we have the following inclusion among sigma fields:

$$\sigma(X_t) = \mathcal{G}_t \subset \mathcal{G}_t^0 \vee \mathcal{G}_t = \mathcal{F}_t;$$

hence we conclude that

$$P(\tau_1 > t_1, \tau_2 > t_2, \dots, \tau_n > t_n \mid \mathcal{G}_{t_n}) = E(E(1_{\tau_1>t_1,\tau_2>t_2,\dots,\tau_n>t_n} \mid \mathcal{F}_{t_n}) \mid \mathcal{G}_{t_n}). \tag{14}$$

From Assumption 2.1, we get

$$\begin{aligned} E(1_{\tau_1>t_1,\tau_2>t_2,\dots,\tau_n>t_n} \mid \mathcal{F}_{t_n}) &= P(\tau_1 > t_1, \tau_2 > t_2, \dots, \tau_n > t_n \mid \mathcal{F}_{t_n}) \\ &= P(N_1(t_1) = 0, \dots, N_n(t_n) = 0 \mid \mathcal{F}_{t_n}) \\ &= \sum_{n=0}^{+\infty}\prod_{i=1}^{n} P(N_i(t_i) = 0 \mid N(t_n) = n, \mathcal{F}_{t_n})P(N(t_n) = n \mid \mathcal{F}_{t_n}) \\ &= \sum_{n=0}^{+\infty}\prod_{i=1}^{n} P(\bar{N}(\Lambda_{it_i}) = 0 \mid \bar{N}(\Lambda_{t_n}) = n, \mathcal{F}_{t_n})P(\bar{N}(\Lambda_{t_n}) = n \mid \mathcal{F}_{t_n}). \end{aligned}$$

Using Lemma 2.1 and (4), we obtain

$$\begin{aligned} \sum_{n=0}^{+\infty}\prod_{i=1}^{n} P(\bar{N}(\Lambda_{it_i}) = 0 \mid \bar{N}(\Lambda_{t_n}) = n, \mathcal{F}_{t_n})P(\bar{N}(\Lambda_{t_n}) = n \mid \mathcal{F}_{t_n}) &= \sum_{n=0}^{+\infty}\prod_{i=1}^{n}\left(1 - \frac{\Lambda_{it_i}}{\Lambda_{t_n}}\right)^n \frac{(\Lambda_{t_n})^n e^{-\Lambda_{t_n}}}{n!} \\ &= e^{-\left(1-\prod_{i=1}^n\left(1-\frac{p_i\Lambda_{t_i}}{\Lambda_{t_n}}\right)\right)\Lambda_{t_n}}. \end{aligned} \tag{15}$$

From (14) and (15), we have

$$P(\tau_1 > t_1, \tau_2 > t_2, \dots, \tau_n > t_n \mid \mathcal{G}_{t_n}) = E\left(e^{-\left(1-\prod_{i=1}^n\left(1-\frac{p_i\Lambda_{t_i}}{\Lambda_{t_n}}\right)\right)\Lambda_{t_n}} \mid \mathcal{G}_{t_n}\right).$$

From (12) and (3), one can get (13).

This completes the proof. □

In order to derive joint survival probability, we consider $E\left[\exp\{-\int_0^t \lambda(s)ds\}\right]$. Recall that

$$\int_0^t \lambda(s)ds = \lambda(0)a(0,t) + \kappa\int_0^t \mu(s)a(s,t)ds + \int_0^t a(s,t)\sigma(s)dW_s + \int_0^t a(s,t)dJ_s,$$

where $a(s,t)$ is given by (11). Noting that $\rho(t)$ is only dependent on Markov chain X_t, when conditioning on $\mathcal{G}_t$ and using Propositions 3.5 and 3.6 of Cont and Tankov (2003), we have

$$E\left(\exp\left\{-\int_0^t a(s,t)dJ_s\right\} \mid \mathcal{G}_t\right) = \exp\left\{\int_0^t \rho(s)\int_R (e^{-a(s,t)x} - 1)f(x)dxds\right\}.$$

Let

$$\int_R (e^{-a(s,t)x} - 1)f(x)dx = b(s,t), \tag{16}$$

So

$$E\left(\exp\left\{-\int_0^t a(s,t)dJ_s\right\} \mid \mathcal{G}_t\right) = \exp\left\{\int_0^t \rho(s)b(s,t)ds\right\}.$$

Since $\mu(t)$, $\sigma(t)$ are also only dependent on Markov chain X_t, so

$$\begin{aligned}
&E\left(\exp\left\{-\int_0^t \lambda(s)ds\right\} \mid \mathcal{G}_t\right)\\
&\quad= \exp\left\{-\lambda(0)a(0,t) + \int_0^t(-\kappa)\mu(s)a(s,t)ds\right\} E\left(\exp\left\{-\int_0^t a(s,t)[\sigma(s)dW_s + dJ_s]\right\} \mid \mathcal{G}_t\right)\\
&\quad= \exp\left\{-\lambda(0)a(0,t) + \int_0^t\left[\frac{1}{2}(a(s,t)\sigma(s))^2 + \rho(s)b(s,t) - \kappa\mu(s)a(s,t)\right]ds\right\}.
\end{aligned} \tag{17}$$

From (9), (10) and (17), we get

$$\begin{aligned}
E\left[\exp\left\{-\int_0^t \lambda(s)ds\right\}\right] &= E\left[E(\exp\left\{-\int_0^t \lambda(s)ds\right\} \mid \mathcal{G}_t)\right]\\
&= E\left[\exp\left\{-\lambda(0)a(0,t) + \int_0^t \frac{1}{2}(a(s,t)\sigma(s))^2 + \rho(s)b(s,t) - \kappa\mu(s)a(s,t)ds\right\}\right],\\
&= \exp\{-\lambda(0)a(0,t)\}E\left[\exp\left\{\int_0^t \frac{1}{2}(a(s,t))^2\langle\vec{\varpi}, X_s\rangle\right.\right.\\
&\qquad \left.\left. + b(s,t)\langle\vec{\rho}, X_s\rangle - \kappa a(s,t)\langle\vec{\mu}, X_s\rangle ds\right\}\right].
\end{aligned} \tag{18}$$

Proposition 2.2. *If $t > 0$, then the joint survival probability is given by*

$$P(\tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n > t) = \exp\left\{-\left[1 - \prod_{i=1}^n(1-p_i)\right]g(t)\right\}\left\langle e^{\int_0^t A - \left[1-\prod_{i=1}^n(1-p_i)\right]C(s,t)ds} \cdot X_0, \mathbf{1}\right\rangle,$$

where $g(t)$ is a deterministic function given by

$$g(t) = \lambda(0)a(0,t), \tag{19}$$

$\mathbf{1} = (1, 1, \ldots, 1)' \in R^N$, *and $C(s,t)$ is a diagonal matrix given by*

$$C(s,t) = b(s,t)B - \kappa a(s,t)C + \frac{1}{2}(a(s,t))^2 D, \tag{20}$$

where $a(s,t)$ is given by (11), $b(s,t)$ is given by (16) and B, C, D are diagonal matrices given by

$$B = \operatorname{diag}\{\rho_1, \rho_2, \ldots, \rho_N\}, \qquad C = \operatorname{diag}\{\mu_1, \mu_2, \ldots, \mu_N\}, \qquad D = \operatorname{diag}\{(\sigma_1)^2, (\sigma_2)^2, \ldots, (\sigma_N)^2\}.$$

Proof. From (13) in Proposition 2.1 and using similar arguments in (17), we have

$$\begin{aligned}
P(\tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n > t \mid \mathcal{G}_t) &= E\left(e^{-\int_0^t\left[1-\prod_{i=1}^n(1-p_i)\right]\lambda(u)du} \mid \mathcal{G}_t\right)\\
&= \exp\left\{-\left[1 - \prod_{i=1}^n(1-p_i)\right]\left[\lambda(0)a(0,t)\right.\right.\\
&\qquad \left.\left. + \int_0^t \frac{1}{2}(a(s,t)\sigma(s))^2 + \rho(s)b(s,t) - \kappa\mu(s)a(s,t)\right]ds\right\}.
\end{aligned}$$

Noting the arguments in (18) and using Lemma 2.3, one can obtain the result.

This ends the proof. □

3. Pricing of portfolio credit derivatives

In this section, we shall price portfolio credit derivatives using the model described in the previous section. As we know, the most important examples of portfolio credit derivatives are index default swaps and collateralized debt obligations (CDO). These portfolio credit derivatives are obtained by putting together a collection of CDSs with the same maturity T on different names, $1, 2, \ldots, n$, each with notional $\frac{1}{n}$. In these contracts, the protection seller agrees to pay a specified loss, in return for a fixed periodic premium. We denote by $t_i,\ i = 1, 2, \ldots, I(t_I = T)$ the payments dates, $\Delta_i = t_i - t_{i-1},\ (i = 1, 2, \ldots, I)$ the year fractions.

Since a portfolio credit derivative can be modeled as a contingent claim whose payoff is a function of the portfolio loss process $(L_t)_{0\le t\le T}$, the valuation of the portfolio credit derivative requires the distribution of the number of defaults N_t and of the portfolio loss L_t. As shown in Giesecke (2003), the distribution of N_t can be computed directly from the joint survival probabilities, which has been obtained in the previous section, and hence the distribution of the default loss L_t can be obtained accordingly. Now we focus on the distribution of N_t.

Let $\mathbb{J}$ be the set of non-empty subsets of $\{1, 2, \ldots, n\}$. For $J \in \mathbb{J}$, we denote by $P_J(t) = P(\cap_{j\in J}\{\tau_j > t\})$ the $|J|$-dimensional marginal joint survival probability, $P_{J|\mathcal{G}}(t) = P(\cap_{j\in J}\{\tau_j > t\} \mid \mathcal{G}_t)$ the $|J|$-dimensional marginal joint survival conditional probability. For example, if $J = \{2, 3, 4\}$, we have

$$P_J(t) = P(\tau_2 > t, \tau_3 > t, \tau_4 > t),$$

and

$$P_{J|\mathcal{G}}(t) = P(\tau_2 > t, \tau_3 > t, \tau_4 > t \mid \mathcal{G}_t).$$

From Giesecke (2003) we see that the conditional distribution and distribution of N_t can be calculated directly from $P_{J|\mathcal{G}}(t)$ and $P_J(t)$:

$$P(N_t = k \mid \mathcal{G}_t) = \sum_{i=n-k}^{n} \binom{i}{n-k} (-1)^{i-n+k} \sum_{J\subset\{1,\ldots,n\},|J|=i} P_{J|\mathcal{G}}(t), \quad k = 0, 1, \ldots, n,$$

$$P(N_t = k) = \sum_{i=n-k}^{n} \binom{i}{n-k} (-1)^{i-n+k} \sum_{J\subset\{1,\ldots,n\},|J|=i} P_J(t), \quad k = 0, 1, \ldots, n,$$

where $P_{J|\mathcal{G}}(t)$ and $P_J(t)$ are available from Propositions 2.1 and 2.2.

Note that

$$P_{J|\mathcal{G}}(t) = E\left(e^{-\int_0^t\left[1-\prod_{j\in J}(1-p_j)\right]\lambda(u)du} \mid \mathcal{G}_t\right),$$

and

$$P_J(t) = E\left(e^{-\int_0^t\left[1-\prod_{j\in J}(1-p_j)\right]\lambda(u)du}\right).$$

So we have

$$\begin{aligned} P(N_t = k \mid \mathcal{G}_t) &= \sum_{i=n-k}^{n} \binom{i}{n-k} (-1)^{i-n+k} \sum_{J\subset\{1,\ldots,n\},|J|=i} P_{J|\mathcal{G}}(t) \\ &= \sum_{J\subset\{1,\ldots,n\},|J|=i,n-k\le i\le n} C_{kJ} E\left(e^{-\int_0^t c_J\lambda(u)du} \mid \mathcal{G}_t\right), \quad k = 0, 1, \ldots, n, \end{aligned} \tag{21}$$

where C_{kJ} and c_J are given by

$$C_{kJ} = \binom{|J|}{n-k} (-1)^{|J|-n+k}, \qquad c_J = 1 - \prod_{j\in J}(1-p_j), \quad J \subset \{1, \ldots, n\}. \tag{22}$$

Using similar arguments in Proposition 2.2, we have

$$\begin{aligned} P(N_t = k) &= \sum_{J\subset\{1,\ldots,n\},|J|=i,n-k\le i\le n} C_{kJ} E\left(e^{-\int_0^t c_J\lambda(u)du}\right) \\ &= \sum_{J\subset\{1,\ldots,n\},|J|=i,n-k\le i\le n} C_{kJ} e^{-g(t)c_J} \left\langle e^{\int_0^t A-c_J C(s,t)ds} \cdot X_0, \mathbf{1}\right\rangle, \end{aligned} \tag{23}$$

where $g(t)$ is a deterministic function given by (19) and $C(s, t)$ is a diagonal matrix given by (20).

We assume that

$$r(t) = \langle \vec{r}, X_t\rangle, \quad \vec{r} = (r_1, r_2, \ldots, r_N) \in R^N$$

is the instantaneous interest rate depending on the state X of economy, and Υ is a diagonal matrix given by $\Upsilon = \mathrm{diag}(r_1, r_2, \ldots, r_N)$. Now we value the following two types of portfolio credit derivatives: index default swaps and CDO tranches.

3.1. Index default swaps

Index default swaps are the most liquid portfolio derivatives. The protection seller agrees to pay all default losses in the index, and in exchange for the loss payments, a periodic premium with swap rate S_1 is paid to the protection seller until final maturity T. The cash flows are as follows.

- Default leg: the protection seller covers portfolio losses as they occur, in other words, the protection seller pays the increments of the portfolio loss to the protection buyer.
- Premium leg: the protection buyer pays premium on the notional of the remaining obligors at each date t_i, $i = 1, \dots, I (t_I = T)$.

Let

$$\Delta_i = t_i - t_{i-1} \equiv \Delta, \qquad R_i \equiv R, \quad i = 1, 2, \dots, n, \; R \in (0, 1],$$

where R_i is the percentage loss given default of firm i. Denote by N_t the number of defaulted names in the portfolio up to time t. Thus,

$$L_t = R\frac{N_t}{n}. \tag{24}$$

We assume that $L_0 = 0$. Hence the discounted payoff of the two legs at time $t_0 = 0$ are given by

$$V_0^{\text{def}} = E\left(\int_0^T e^{-\int_0^t r(u)du} dL_t\right),$$

$$V_0^{\text{prem}} = E\left(\sum_{i=1}^{I} e^{-\int_0^{t_i} r(u)du} S_1 \Delta \left(1 - \frac{N_{t_i}}{n}\right)\right).$$

The fair spread S_1 is obtained by solving the equation $V_0^{\text{def}} = V_0^{\text{prem}}$:

$$S_1 = \frac{V_0^{\text{def}}}{E\left(\sum_{i=1}^{I} e^{-\int_0^{t_i} r(u)du} \Delta \left(1 - \frac{N_{t_i}}{n}\right)\right)}. \tag{25}$$

Using partial integration we get

$$\begin{aligned} V_0^{\text{def}} &= E\left(\int_0^T e^{-\int_0^t r(u)du} dL_t\right) \\ &= E\left(e^{-\int_0^T r(u)du} L_T + \int_0^T L_t e^{-\int_0^t r(u)du} r(t)dt\right). \end{aligned} \tag{26}$$

Noting (24) and (21), and using Lemma 2.3, we get for the first term

$$\begin{aligned} E\left(e^{-\int_0^T r(u)du} L_T\right) &= \frac{R}{n}\sum_{k=0}^{n} E\left(k e^{-\int_0^T r(u)du} P(N_T = k \mid \mathcal{G}_T)\right) \\ &= \frac{R}{n}\sum_{k=1}^{n} \sum_{J\subset\{1,\dots,n\},|J|=i,n-k\le i\le n} E\left[k C_{kJ} E\left(e^{-\int_0^T r(u)+c_J\lambda(u)du} \mid \mathcal{G}_T\right)\right] \\ &= \frac{R}{n}\sum_{k=1}^{n} \sum_{J\subset\{1,\dots,n\},|J|=i,n-k\le i\le n} k C_{kJ} e^{-g(T)c_J} \left\langle e^{\int_0^T A-\Upsilon-c_J C(s,T)ds} \cdot X_0, \mathbf{1}\right\rangle. \end{aligned} \tag{27}$$

Noting (21), and using Lemma 2.3, we get for the second term

$$\begin{aligned} &E\left(\int_0^T L_t e^{-\int_0^t r(u)du} r(t)dt\right) \\ &= \frac{R}{n}\sum_{k=0}^{n}\int_0^T kE\left[P(N_t = k \mid \mathcal{G}_t) e^{-\int_0^t r(u)du} r(t)\right] dt \\ &= \frac{R}{n}\sum_{k=1}^{n}\int_0^T \sum_{J\subset\{1,\dots,n\},|J|=i,n-k\le i\le n} k C_{kJ} E\left[\left\langle \vec{r}, e^{-\int_0^t r(u)} X_t E\left(e^{-\int_0^t c_J\lambda(u)du} \mid \mathcal{G}_t\right)\right\rangle\right] dt \\ &= \frac{R}{n}\sum_{k=1}^{n} \sum_{J\subset\{1,\dots,n\},|J|=i,n-k\le i\le n} k C_{kJ} \int_0^T \left\langle \vec{r}, e^{-g(t)c_J} e^{\int_0^t A-\Upsilon-c_J C(s,t)ds} \cdot X_0\right\rangle dt. \end{aligned} \tag{28}$$

Hence we get from (26)–(28)

$$V_0^{\text{def}} = \frac{R}{n}\sum_{k=1}^{n}\sum_{J\subset\{1,\dots,n\},|J|=i,n-k\le i\le n} kC_{kJ}\left[e^{-g(T)c_J}\left\langle e^{\int_0^T A-\Upsilon-c_J C(s,T)ds}\cdot X_0,\mathbf{1}\right\rangle\right.$$
$$\left.+\int_0^T\left\langle\vec{r},e^{-g(t)c_J}e^{\int_0^t A-\Upsilon-c_J C(s,t)ds}\cdot X_0\right\rangle dt\right]. \tag{29}$$

In the following computations we use (21) and Lemma 2.3. Therefore

$$\begin{aligned}E\left[\sum_{i=1}^{I}e^{-\int_0^{t_i}r(u)du}\Delta\left(1-\frac{N_{t_i}}{n}\right)\right] &= E\left[\sum_{i=1}^{I}\Delta e^{-\int_0^{t_i}\langle\vec{r},X_u\rangle du}\sum_{k=0}^{n}\left(1-\frac{k}{n}\right)P[N_{t_i}=k\mid\mathcal{G}_T]\right]\\ &=\sum_{i=1}^{I}\sum_{k=0}^{n}\sum_{J\subset\{1,\dots,n\},|J|=i,n-k\le i\le n}\Delta C_{kJ}\left(1-\frac{k}{n}\right)e^{-g(t_i)c_J}\\ &\quad\times\left\langle e^{\int_0^{t_i}A-\Upsilon-c_J C(s,t_i)ds}\cdot X_0,\mathbf{1}\right\rangle.\end{aligned} \tag{30}$$

Substituting (29) and (30) into (25), one can obtain the formula of the fair spread S_1.

3.2. CDO tranches

A tranche swap is specified by a lower attachment point $\underline{K}\in[0,1]$ and an upper attachment point $\overline{K}\in(\underline{K},1]$. The difference $K=\overline{K}-\underline{K}$ is the tranche notional. In the tranche swap with upfront rate G and swap spread S_2, the cash flows are as follows.

- Default leg: the protection seller covers tranche losses as they occur, i.e. the increments of the tranche loss $U_t^{\underline{K},\overline{K}}$.
- Premium leg: the protection buyer pays premium at each date $t_i,\ i=0,1,\dots,I(t_0=0,t_I=T)$. Part of the premium can be paid at inception as an upfront GK and the rate S_2 is paid on the "survived" tranche notional.

The value at time $t_0=0$ of the default leg is

$$V_0^{\text{def}}(\underline{K},\overline{K})=E\left(\int_0^T e^{-\int_0^t r(u)du}dU_t^{\underline{K},\overline{K}}\right),$$

where the tranche loss $U_t^{\underline{K},\overline{K}}$ is given by

$$U_t^{\underline{K},\overline{K}}=\frac{1}{K}[(L_t-\underline{K})1_{\underline{K}<L_t\le\overline{K}}+K1_{L_t>\overline{K}}].$$

The value of the premium leg is

$$V_0^{\text{prem}}(\underline{K},\overline{K})=GK+E\left(\sum_{i=1}^{I}e^{-\int_0^{t_i}r(u)du}S_2\Delta(1-U_{t_i}^{\underline{K},\overline{K}})\right).$$

The fair tranche spread S_2 which equalizes the two legs at inception is given by

$$S_2=\frac{V_0^{\text{def}}(\underline{K},\overline{K})-GK}{E\left(\sum_{i=1}^{I}e^{-\int_0^{t_i}r(u)du}\Delta(1-U_{t_i}^{\underline{K},\overline{K}})\right)}. \tag{31}$$

Now we calculate $V_0^{\text{def}}(\underline{K},\overline{K})$ and $E(\sum_{i=1}^{I}e^{-\int_0^{t_i}r(u)du}\Delta(1-U_{t_i}^{\underline{K},\overline{K}}))$, respectively.

$$\begin{aligned}V_0^{\text{def}}(\underline{K},\overline{K}) &= E\left[e^{-\int_0^T r(u)du}U_T^{\underline{K},\overline{K}}+\int_0^T U_t^{\underline{K},\overline{K}}e^{-\int_0^t r(u)du}r(t)dt\right]\\ &=\frac{1}{K}\sum_{k=[n\underline{K}]+1}^{[n\overline{K}]}E\left[\left(\frac{k}{n}-\underline{K}\right)e^{-\int_0^T r(u)du}P(N_T=k\mid\mathcal{G}_T)\right]\\ &\quad+\frac{1}{K}\sum_{k=[n\overline{K}]+1}^{n}E\left[Ke^{-\int_0^T r(u)du}P(N_T=k\mid\mathcal{G}_T)\right]\end{aligned}$$

$$+\frac{1}{K}\sum_{k=[n\underline{K}]+1}^{[n\overline{K}]}\int_0^T\left(\frac{k}{n}-\underline{K}\right)E\left[P(N_t=k\mid \mathcal{G}_t)e^{-\int_0^t r(u)du}r(t)\right]dt$$
$$+\frac{1}{K}\sum_{k=[n\overline{K}]+1}^{n}\int_0^T(K)E\left[P(N_t=k\mid \mathcal{G}_t)e^{-\int_0^t r(u)du}r(t)\right]dt.$$

Using Lemma 2.3, we get

$$V_0^{\text{def}}(\underline{K},\overline{K}) = \left[\sum_{k=[n\underline{K}]+1}^{[n\overline{K}]}\frac{\left(\frac{k}{n}-\underline{K}\right)}{K}+\sum_{k=[n\overline{K}]+1}^{n}1\right]\sum_{J\subset\{1,\ldots,n\},|J|=i,n-k\le i\le n}C_{kJ}\left[e^{-g(T)c_J}\left\langle e^{\int_0^T A-\Upsilon-c_J C(s,T)ds}\cdot X_0,\mathbf{1}\right\rangle\right.$$
$$\left.+\int_0^T\left\langle\vec{r},e^{-g(t)c_J}e^{\int_0^t A-\Upsilon-c_J C(s,t)ds}\cdot X_0\right\rangle dt\right]. \tag{32}$$

Similarly, we get

$$E\left(\sum_{i=1}^{I}e^{-\int_0^{t_i}r(u)du}\Delta(1-U_{t_i}^{\underline{K},\overline{K}})\right)=\sum_{i=1}^{I}\sum_{k=[n\underline{K}]+1}^{[n\overline{K}]}\sum_{J\subset\{1,\ldots,n\},|J|=i,n-k\le i\le n}\Delta C_{kJ}\left[1-\frac{1}{K}\left(\frac{k}{n}-\underline{K}\right)\right]$$
$$\times e^{-g(t_i)c_J}\left\langle e^{\int_0^{t_i}A-\Upsilon-c_J C(s,t_i)ds}\cdot X_0,\mathbf{1}\right\rangle. \tag{33}$$

Substituting (32) and (33) into (31), one can obtain the formula of the fair spread S_2.

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