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STABILITY AND BIFURCATION - MEMORIAL ISSUE DEDICATED TO JORGE SOTOMAYOR

Gutierrez–Sotomayor flows: isolating blocks and homotopical cancellations

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Abstract

Peixoto's stability theorem stands as a cornerstone in the global dynamical examination of flows on smooth two-manifolds, a significant landmark in Dynamical Systems research. This theorem has served as a blueprint for subsequent global classification theorems within the field. Building upon Peixoto's foundational work, Gutierrez and Sotomayor introduced a compelling generalization and their contribution extends Peixoto's conditions for structural stability of C^1-vector fields on smooth surfaces to encompass singular two-manifolds M. Furthermore, generalizing this classical theorem to varied and richer topological configurations such as these non-smooth surfaces which feature singular loci comprising cones ($\mathcal{C}$), cross-caps ($\mathcal{W}$), double ($\mathcal{D}$), and triple points ($\mathcal{T}$) marks a milestone for research in Singular Dynamics. In homage to their contributions, we have named this class of dynamical systems as Gutierrez–Sotomayor flows, GS flows for short. It is our intent, in this article to produce a survey of the state of the art for GS flows which have garnered significant attention in the past years. Our interest is two-fold: firstly present a local and global analysis of GS flows φ on singular surfaces M and secondly describe the effects of homotopical deformations on (φ, M) which are in correspondence to a spectral sequence of an associated chain complex for φ. Herein we address the far reaching results that are obtained by using Spectral Sequence Theory which has yielded several homotopical cancellation theorems within the dynamics.

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1 Introduction

The qualitative study of dynamical systems has provided deep insights into the behavior of intricate systems across various disciplines. Among the myriad of dynamical phenomena, the Gutierrez–Sotomayor flows stand out as a fascinating class of dynamical systems. Originating from the seminal work of Gutierrez and Sotomayor in the late twentieth century, [1], these flows have garnered significant attention due to their rich topological structure with the presence of cone, cross caps, double and triple point singularities which display elaborate global dynamical behavior. See [2–4].

In this work, we provide an overview on the fundamental nature of Gutierrez–Sotomayor singularities, both locally by describing their isolating blocks, as well as globally, by exploring how they build up a GS flow on a closed singular 2-manifold M.

We begin our paper by elucidating this notion of isolating blocks within Gutierrez–Sotomayor flows, a fundamental concept that underpins their dynamical behavior. Isolating blocks serve as building blocks for understanding the more complicated structure of these flows on M, providing crucial insights into their long-term behavior.

Subsequently, we seek to highlight the interplay of the GS singularities on M and the dynamics underlying their formation, their evolution and potential demise through homotopical cancellation. We also shed light on the role played by trapping regions of the flow such as folds and stable and unstable manifolds in a global flow homotopical deformation mechanism.

It is the description of this mechanism which lies at the heart of our investigation and will embody the beautiful connections between topology, dynamics and algebra. This is done by establishing a dynamical homotopical cancellation theory for GS-flows, φ, on M. This framework is established by defining a GS-chain complex for (M, φ) and computing its spectral sequence (E^r, d^r). See [4].

The remarkable phenomena behind this endeavour is that there exists a family of flows φ_r associated to the dynamical homotopical cancellation of GS-singularities which are in correspondence to the algebraic cancellation of the modules in E^r of a filtered spectral sequence of the corresponding chain complex for φ. Moreover, the quest for understanding ever more deeply the dynamical information obtained from the algebra led us to prove that the convergence of the spectral sequence corresponded to an *irreducible* GS-flow $\varphi_{\bar{r}}$ on $M_{\bar{r}}$, for some $\bar{r}$, with the property that $\varphi_{\bar{r}}$ admits no further dynamical homotopical cancellation of GS-singularities.

Our paper is structured in a manner that facilitates a comprehensive understanding of Gutierrez–Sotomayor flows, beginning with an exposition of foundational concepts and gradually progressing towards the more advanced topics. We leverage a diverse array of mathematical techniques and methodologies, drawing upon concepts from topology, algebra, and dynamical systems theory to prove the cancellation theorems for Gutierrez–Sotomayor flows.

In summary, we hope to offer an encompassing survey of this fascinating class of dynamical systems, shedding light on their fundamental properties, structural characteristics, and profound algebraic topological implications in the realm of Singular Dynamics. Through our investigation, we aim to provide both novice and seasoned mathematicians with a deeper appreciation for the rich topological and dynamical phenomena embodied by Gutierrez–Sotomayor flows on singular surfaces, paving the way for further exploration and discovery in this captivating field.

This survey is based on joint work with H. Montufar, [2], S. Raminelli, [4], N. Grulha, [3].

2 Background

2.1 Gutierrez–Sotomayor vector fields

In [1], Gutierrez and Sotomayor introduced 2-*manifolds with simple singularities*[1]. They arise when the regularity conditions in the definition of smooth surfaces of $\mathbb{R}^3$, in terms of implicit functions and immersions, are not satisfied but there is still the presence of certain stability [5]. The net result is the increase of the types of admissible local charts. A map $f : K \subset \mathbb{R}^m \to \mathbb{R}^n$ is said to be of *class* C^r, where $1 \leq r \leq \infty$, if it admits an extension $\hat{f}$ of class C^r to an open neighbourhood of K. A homeomorphism $f : K_1 \subset \mathbb{R}^m \to K_2 \subset \mathbb{R}^n$ is called a *diffeomorphism of class* C^r if both f and f^{-1} are of class C^r.

Definition 1 A subset $\mathbf{M} \subset \mathbb{R}^l$ is called a 2-**manifold with simple singularities**, or a **GS** 2-**manifold**, if for every point $p \in \mathbf{M}$ there is a neighbourhood V_p of p in $\mathbf{M}$ and a C^∞-diffeomorphism $\Psi : V_p \to \mathcal{P}$ such that $\Psi(p) = 0$ and $\mathcal{P}$ is one of the following subsets of $\mathbb{R}^3$:

i) $\mathcal{R} = \{(x, y, z) : z = 0\}$, plane;
ii) $\mathcal{C} = \{(x, y, z) : z^2 - y^2 - x^2 = 0\}$, cone;
iii) $\mathcal{W} = \{(x, y, z) : zx^2 - y^2 = 0, z \geq 0\}$, Whitney's umbrella[2];
iv) $\mathcal{D} = \{(x, y, z) : xy = 0\}$, double crossing;
v) $\mathcal{T} = \{(x, y, z) : xyz = 0\}$, triple crossing;

[1] In this work we follow the terminology given by Gutierrez and Sotomayor by calling these singularities as *simple* and remark that there is a different notion of simple singularities in the classical theory of singularities of mappings (see [5]).

[2] The locus of the subset $\mathcal{W}$ is called cross-cap. In this paper, we chose to keep the nomenclature used by Gutierrez and Sotomayor in [1].

The subsets $\mathbf{M}(\mathcal{P}) \subset \mathbf{M}$ of the points of $\mathbf{M}$ which admit local charts of type $\mathcal{P}$, where $\mathcal{P} = \mathcal{R}, \mathcal{C}, \mathcal{W}, \mathcal{D}$, or $\mathcal{T}$, provide a decomposition of $\mathbf{M}$, where the regular part $\mathbf{M}(\mathcal{R})$ of $\mathbf{M}$ is a 2-dimensional manifold, $\mathbf{M}(\mathcal{D})$ is a 1-dimensional manifold, while $\mathbf{M}(\mathcal{C})$, $\mathbf{M}(\mathcal{W})$ and $\mathbf{M}(\mathcal{T})$ are discrete sets. Moreover, the collection $\{\mathbf{M}(\mathcal{P}), \mathcal{P}\}$ is a stratification of M in the sense of Thom, [6], hence

$$\mathbf{M} = \bigcup_{\mathcal{P}} \mathbf{M}(\mathcal{P}), \text{ where } \mathcal{P} = \mathcal{R}, \mathcal{C}, \mathcal{W}, \mathcal{D}, \mathcal{T}.$$

Definition 2 A vector field $\mathbf{X}$ of class C^r on $\mathbb{R}^l$ is **tangent to a manifold $\mathbf{M} \subset \mathbb{R}^l$ with simple singularities** if it is tangent to the smooth submanifolds $\mathbf{M}(\mathcal{P})$, for all $\mathcal{P} = \mathcal{R}, \mathcal{C}, \mathcal{W}, \mathcal{D}, \mathcal{T}$. The space of such vector fields is denoted by $\mathfrak{X}^{\mathbf{r}}(\mathbf{M})$.

A flow X_t associated to a vector field $\mathbf{X} \in \mathfrak{X}^{\mathbf{r}}(\mathbf{M})$ is called a Gutierrez–Sotomayor flow (GS flow, for short) on $\mathbf{M}$.

Definition 3 Denote by $\Sigma^r(\mathbf{M})$ the set of all vector fields $\mathbf{X} \in \mathfrak{X}^{\mathbf{r}}(\mathbf{M})$ satisfying:

- X has finitely many hyperbolic fixed points and hyperbolic periodic orbits;
- the singular limit cycles of X are simple and X has no saddle connections;
- the α-limit and ω-limit sets of every trajectory of X are either fixed points, periodic orbits or singular cycles.

In this paper we consider GS flows associated to vector fields in $\Sigma^r(\mathbf{M})$ without periodic orbits and singular cycles. The set of such vector fields is denoted by $\Sigma_0^r(\mathbf{M})$. Locally some of these singularities are depicted in Fig. 1 and by considering the reverse flow, one obtains the complete set.

Definition 4 Let $\mathbf{M}$ be a GS 2-manifold and $\mathbf{X} \in \mathfrak{X}^{\mathbf{r}}(\mathbf{M})$ a vector field on $\mathbf{M}$. The **set of folds on $\mathbf{M}$**, denoted by $\mathcal{F}(\mathbf{M})$, is defined as:

$$\mathcal{F}(\mathbf{M}) = \mathbf{M}(\mathcal{D}) \setminus \mathcal{S}_{\mathcal{D}},$$

where $\mathcal{S}_{\mathcal{D}}$ is the set of double crossing points on $\mathbf{M}$ which are singularities (stationary points) of $\mathbf{X}$.

A GS flow defined on a fold F has the property that, given any point $x \in F$, the limit sets, $\alpha(x)$ and $\omega(x)$, are either a Whitney, double crossing or a triple crossing singularity.

2.2 Conley index and isolating blocks

Consider a continuous flow $X_t : \mathbf{M} \to \mathbf{M}$ on $\mathbf{M}$. A few definitions are in order:

1. the *maximal invariant set* of $N \subset \mathbf{M}$, denoted by Inv (N) is:

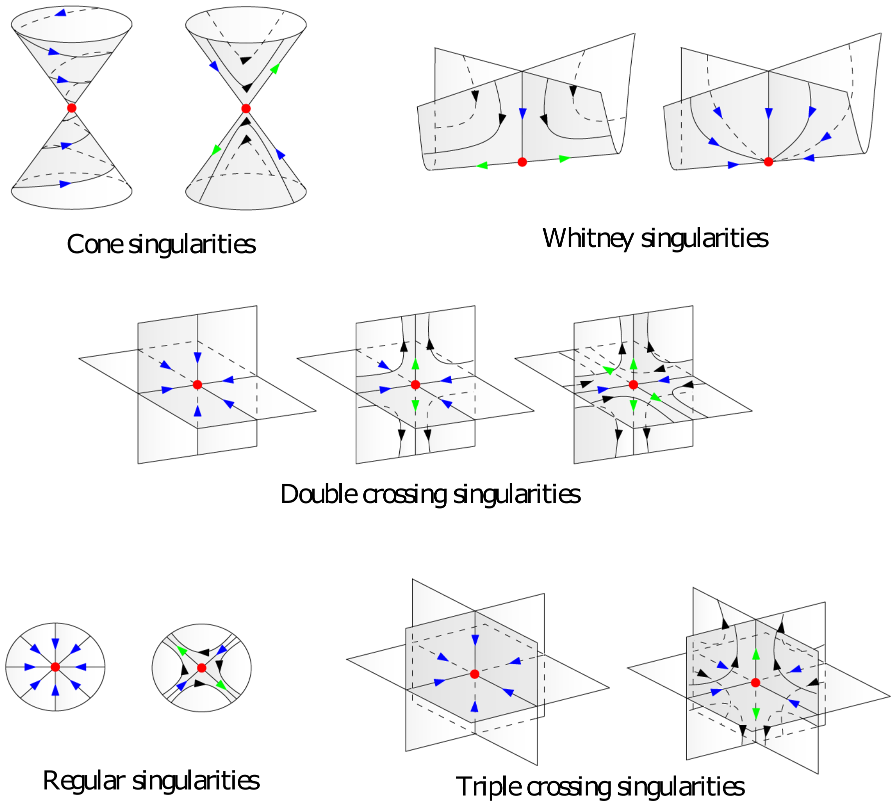


Fig. 1 Local types of GS-singularities

$$\mathrm{Inv}(N) = \{x \in N \mid X_t(x) \in N, \forall t \in \mathbb{R}\}.$$

2. a subset $S \subset \mathbf{M}$ is an *isolated invariant set* with respect to the flow X_t if there exists an *isolating neighborhood* N for S, i.e., a compact set N such that $S = \mathrm{Inv}(N) \subset \mathrm{int}(N)$.
3. an *index pair* for an isolated invariant set S is a pair of compact sets $L \subset N$ satisfying the following:

   (a) $\overline{N \backslash L}$ is an isolating neighborhood of S in M;
   (b) L is positively invariant relative to N, that it, given $x \in L$ such that $X_t(x) \in N$ for all $t \in [0, t_0]$, then $X_t(x) \in L$ for all $t \in [0, t_0]$;
   (c) L is the exiting set of the flow in N, that is, given $x \in N$ and $t_1 > 0$ such that $X_{t_1}(x) \notin N$, then there exists $t_0 \in [0, t_1]$ such that $X_t(x) \in N$, for $t \in [0, t_0]$ and $X_{t_0}(x) \in L$.

That every isolated invariant set S has an index pair (N, L) for S and that the Conley index $(N/L, [L])$ is homotopically invariant are foundational theorems proved by Conley. See [7].

Definition 5 Let (N, L) be an index pair for an isolated invariant set S with respect to a flow $X_t : \mathbf{M} \to \mathbf{M}$.

i) The **homotopy Conley index** of S is the homotopy type of the pointed space $(N/L, [L])$.
ii) The **homology Conley index** $CH_k(S)$ of S is defined as the k-th reduced homology group of $(N/L, [L])$.
iii) The **numerical Conley index** h_k of S is the rank of the homology Conley index $CH_k(S)$.

In [2], the homotopy and homology Conley indices for each type of singularities of a GS flow associated to a vector field in $\Sigma_0^r(\mathbf{M})$ are computed.

Definition 6 An **isolating block** is an isolating neighborhood N such that its entering and exiting sets, given respectively by

$$N^+ = \{x \in N \mid \phi([0,T),x) \nsubseteq N,\ \forall T < 0\}$$
$$N^- = \{x \in N \mid \phi([0,T),x) \nsubseteq N,\ \forall T > 0\},$$

are both closed, with the additional property that the flow is transversal to the boundary of N.

The existence of GS isolating blocks follows directly from the existence of Lyapunov functions for GS flows without periodic orbits and singular cycles, proved in [2]. Recall that if p is a singularity of a GS flow and f is a Lyapunov function with $f(p) = c$, then given $\epsilon > 0$ such that their is no critical value of f in $[c - \epsilon, c + \epsilon]$, the connected component N of $f^{-1}([c - \epsilon, c + \epsilon])$ which contains p is an isolating block for p. In this case, the exiting set is $N^- = N \cap f^{-1}(c - \epsilon)$.

3 The building blocks for Gutierrez–Sotomayor flows on singular surfaces

The GS isolating blocks are a very useful tool in the construction of *GS* flows. In fact, given a list of GS isolating blocks, we can successively glue a connected component of the entering set of an isolating block with a connected component of the exiting set of another block, provided they are homeomorphic. This should be done until all boundaries of isolating blocks, with the exception of repellor isolating blocks, have been used creating an attractor isolating block A with a set of entering boundaries. The boundary of A should be homeomorphic to the union of the boundaries of the repellors. Thus, a perfect match is realized creating a singular surface for a GS flow.

In order to analyze the conditions for which this gluing is possible, in what follows, we consider some topological information on the connected components of the boundary N^+ and N^-, which are necessary conditions derived from the Poincaré–Hopf equality. These will be exhibited on the edges of the Lyapunov semigraphs associated to GS isolating blocks as weights on the edges. These weights correspond to the first Betti numbers of the boundary. The problem is that these Betti numbers of the branched one manifold boundaries are not sufficient to distinguish homeomorphism classes. These topological obstructions are addressed in the section dealing with realizations of Lyapunov semigraphs.

3.1 Lyapunov functions and graphs

Given a flow X_t associated to a vector field $\mathbf{X} \in \Sigma_0^r(\mathbf{M})$, there is an **associated Lyapunov function** $f : \mathbf{M} \to \mathbb{R}$ such that $f(p) \neq f(q)$ if p and q are different singularities of X_t, and for each stratum $\mathbf{M}(\mathcal{P})$ of $\mathbf{M}$ it follows that:

- $f|_{\mathbf{M}(\mathcal{P})}$ is a smooth function, with f continuous in $\mathbf{M}$,
- The critical points of $f|_{\mathbf{M}(\mathcal{P})}$ are non degenerate and coincide with the singularities of X_t,
- $\dfrac{d}{dt}(f|_{\mathbf{M}(\mathcal{P})}(X_t x)) < 0$, if x is not a singularity of X_t.

Note that the Lyapunov function need not be globally smooth, only continuous. The existence of this function is proven in [2].

Essentially, if $f : M \to \mathbb{R}$ serves as a Lyapunov function linked to a flow X_t on M, then a *Lyapunov graph*, denoted as L_f, is constructed by collapsing each connected component of $f^{-1}(c)$ to a single point. Singularities of X_t are identified as *vertices* within this graph and are annotated with the numerical Conley indices associated with the respective GS singularity. The remaining connected components form the *edges* of the graph, where each edge is assigned a *weight* corresponding to the first Betti number of its one-dimensional level sets on M.

Given the ability to construct a Lyapunov graph from a GS flow X_t on M, a natural inquiry arises: can we reverse this process? Our primary objective lies in delineating local and global algebraic-topological criteria on a Lyapunov graph, and then questioning its realizability both locally as an isolating block as in Sect. 3.3 and also globally as a GS flow on a singular 2-manifold as in Sect. 4.

To be more precise, when presented with an abstract graph L, what conditions are both necessary and sufficient for L to represent the Lyapunov graph L_f of a GS flow X_t on M? Upon discovering these conditions, we deem L as "realizable", with X_t being its corresponding *realization*. Consequently, theorems furnishing adequate conditions for such realizations serve as a route for generating GS flows. Of course, numerous topological impediments may arise, and our objective is to delineate the scenarios in which they can be surmounted.

Definition 7 We say that a GS graph L is **realizable** if there exists a triple ($\mathbf{M}$, X_t ,f), where $\mathbf{M}$ is a closed GS 2-manifold, X_t is a flow associated to a vector field $\mathbf{X} \in \Sigma_0^r(\mathbf{M})$ and $f : \mathbf{M} \to \mathbb{R}$ is a Lyapunov function associated to X_t, such that L is the Lyapunov graph of ($\mathbf{M}$, X_t,f). In this case, we say that the flow X_t defined on $\mathbf{M}$ is a **realization** of L.

3.1.1 Lyapunov semigraphs and graphs for GS flows

In this work we make use of Lyapunov graphs and semigraphs as a combinatorial tool that keeps track of topological and dynamical data of a flow on $\mathbf{M}$. This information on a GS flow X_t on $\mathbf{M}$ or on an isolating block N is transferred to a Lyapunov graph, respectively, semigraph associated to X_t and f as follows:

Definition 8 Let f be a Lyapunov function for a GS flow X_t associated to $\mathbf{X} \in \Sigma_0^r(\mathbf{M})$. Two points x and y on $\mathbf{M}$ are **equivalent** if and only if they belong to the same connected component of a level set of f. In this case, denote $x \sim_f y$. The relation $\sim_f$ defines an **equivalence relation** on $\mathbf{M}$. The quotient space $\mathbf{M}/ \sim_f$ is called a **Lyapunov graph of M asssociated to X_t and f**. For some isolating block $N \subset \mathbf{M}$, the quotient space $N/ \sim_f$ is called a **Lyapunov semigraph of N asssociated to X_t and f**. The quocient space satisfies:

i) each connected component of a level set $f^{-1}(c)$ colapses to a point. Thus, as the value c varies and $f^{-1}(c)$ contains no singularities, $f^{-1}(c)/ \sim_f$ describes a finite set of **edges**, each of which is labelled with a **weight**, which corresponds to the first Betti number of the respective connected component;
ii) if c_0 is such that, a connected component of $f^{-1}(c_0)$ contains a singularity of X_t, then this connected component is a **vertex** v in the quotient space labelled with the numerical Conley index of the singularity,[3] $(h_0, h_1, h_2)_{\mathcal{P}}$, together with its type $\mathcal{P} = \mathcal{R}, \mathcal{C}, \mathcal{W}, \mathcal{D}$ or $\mathcal{T}$;
iii) the number of positively (resp. negatively) incident edges on the vertex v, i.e., the indegree (resp. outdegree) of v, is denoted by $e_v^+ (resp. e_v^-)$.

Moreover, a Lyapunov graph of $\mathbf{M}$ as well as a Lyapunov semigraph of N associated to X_t and f is finite, directed and with no oriented cycles. For more details, see [2].

3.2 Poincaré–Hopf equality and lyapunov semigraphs

Let X_t be a GS flow associated to a vector field $\mathbf{X} \in \Sigma_0^r(\mathbf{M})$, such that p is a singularity of X_t, and N is an isolating block for p. Then (N, N^-) constitutes an index pair for $inv(N) = \{p\}$. In this case, the next result which is referred to as the *Poincaré-Hopf equality* for GS flows, see [2], follows:

[3] Equivalently, the numerical Conley index will be, at times, substituted by the nature of the singularity.

Theorem 1 (Poincaré-Hopf Equality) *Let (N_1, N_0) be an index pair for a singularity p on a GS 2-manifold, M. Let $\mathbf{X} \in \Sigma_0^r(\mathbf{M})$ and (h_0, h_1, h_2) be the numerical Conley indices for p. Then*:

$$(h_2 - h_1 + h_0) - (h_2 - h_1 + h_0)^* = e^+ - \mathcal{B}^+ - e^- + \mathcal{B}^- \tag{1}$$

where *denotes the indices of the reverse flow, e^+ (resp., e^-) is the number of connected components of the entering (resp., exiting) set of N_1 and

$$\mathcal{B}^+ = \sum_{k=1}^{e_k^+} b_k^+ \left(\text{resp., } \mathcal{B}^- = \sum_{k=1}^{e_k^-} b_k^- \right),$$

where b_k^+ (resp., b_k^-) is the first Betti number of the k-th connected component of the entering (resp., exiting) set of N_1.

Note that the boundary of N is nonempty, i.e. $\partial N = N^+ \cup N^- \neq \emptyset$. Hence $H_2(N) = 0$. Since N is connected, $\tilde{H}_0(N) = 0$. Consequently, from the long exact sequence

$$0 \longrightarrow CH_2(p) \xrightarrow{\partial_2} H_1(N^-) \xrightarrow{i_1} H_1(N) \xrightarrow{p_1} CH_1(p) \xrightarrow{\partial_1} \tilde{H}_0(N^-) \longrightarrow 0$$

one has

$$dim(\tilde{H}_0(N^-)) \leq dim(CH_1(p)) = h_1. \tag{2}$$

Analogously, considering the reverse flow, one has:

$$dim(\tilde{H}_0(N^+)) \leq dim(CH_1(p^*)) = h_1^*. \tag{3}$$

Hence, (1), (2) and (3) are necessary conditions on the number of entering e^+ and exiting e^- boundaries of an isolating block, as well as the first Betti number of the incoming, b_1^+, and outgoing, b_1^-, boundary components.

From (1), (2) and (3), one can describe how the GS isolating blocks look like in terms of their Lyapunov semigraphs. It is easy to see that the necessary conditions transfer over to the semigraphs. More specifically, let v be a vertex on the Lyapunov semigraph of an isolating block for a GS singularity p, then the positively incident edges on v (e_v^+) and the negatively incidents edges on v (e_v^-) satisfy:

$$dim(\tilde{H}_0(N^-) = dim(H_0(N^-)) - 1 = e_v^- - 1 \leq h_1 \tag{4}$$

$$dim(\tilde{H}_0(N^+) = dim(H_0(N^+)) - 1 = e_v^+ - 1 \leq h_1^*. \tag{5}$$

□

Therefore, the Poincaré-Hopf conditions (1)are necessary conditions on a Lyapunov semigraph L_v. Hence, these conditions are naturally transferred to GS isolating blocks for each type and nature of a GS singularity. See [2].

Therefore it is crucial to introduce the concept of the *nature of a singularity*, as it directly influences both the numerical homology Conley index and, consequently the Poincaré–Hopf condition.

Definition 9 We define the **nature of a singularity** p as follows.

1. Let p be an attractor (resp. repeller). If p has type $\mathcal{R}, \mathcal{C}, \mathcal{W}$, we say it has attracting (resp. repelling) nature, or nature **a** (resp. nature **r**), for short. If p a singularity of type $\mathcal{D}$, since it has two attracting (resp. repelling) natures, we say it has nature $\mathbf{a^2}$ (resp. $\mathbf{r^2}$). Similarly, in the case of a singularity of type $\mathcal{T}$, since it has three attracting (resp. repelling) natures, we say it has nature $\mathbf{a^3}$ (resp. $\mathbf{r^3}$).
2. A singularity of type $\mathcal{R}$ or $\mathcal{C}$ which is neither an attractor or a repeller is said to have saddle nature, for short, nature **s**.
3. A regular saddle on a disk identified along the stable (resp. unstable) manifolds so as to produce a Whitney chart is of nature $\mathbf{s_s}$ (resp. $\mathbf{s_u}$). Two regular saddles on two disjoint disks identified along their stable (resp. unstable) manifold so as to produce a double crossing chart is of type $\mathbf{ss_s}$ (resp. $\mathbf{ss_u}$).
4. A regular saddle and an attracting (resp. repelling) singularity on two disjoint disks identified along a stable (resp. unstable) direction so as to produce a double crossing chart is of nature **sa** (resp. **sr**).
5. Similarly, two regular saddles and an attracting (resp. repelling) singularity on three disjoint disks identified as follows: the two saddle disks identify along their unstable (resp. stable) manifolds and subsequently the attracting (resp., repelling) disk identified along stable (resp. unstable) directions so as to produce a triple crossing chart is of nature **ssa** (resp. **ssr**).

We now ennunciate theorems that delineate realizable graphs and introduce the table of semigraphs meeting these topological criteria, refraining from making assertions about their realizations.

In Table 1 these conditions are presented for all GS singularities with natures a, a^2, a^3 s, s_s, sa, ss_s and ssa. By reserving the flow, one obtains the conditions on Lyapunov semigraphs for GS singularities with natures r, r^2, r^3, s, s_u, sr, ss_u and ssr, respectively. Also the corresponding Poincaré-Hopf condition is given by exchanging $\mathcal{B}^+$ by $\mathcal{B}^-$ and vice versa.

Definition 10 For each semigraph in Table 1, considering that $b_i^{\pm}$ is greater or equal to one, we define the respective **Lyapunov semigraph with minimal weights** to be the one with smallest positive integers satisfying the Poincaré-Hopf condition.

In the next section we consider the realizability of minimal abstract Lypaunov semigraph satisfying the Poincaré-Hopf condition as given in Table 2 as isolating blocks.

Table 1 Collection of Lyapunov semigraphs and their respective Poincaré-Hopf condition

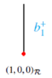
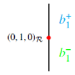
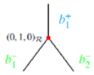
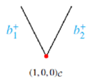
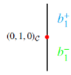
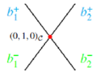
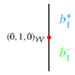
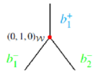
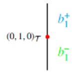
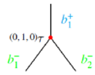
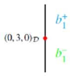
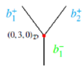
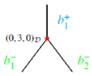
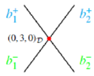
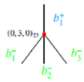
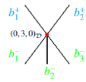
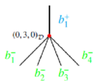
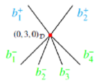
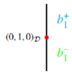
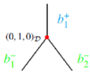
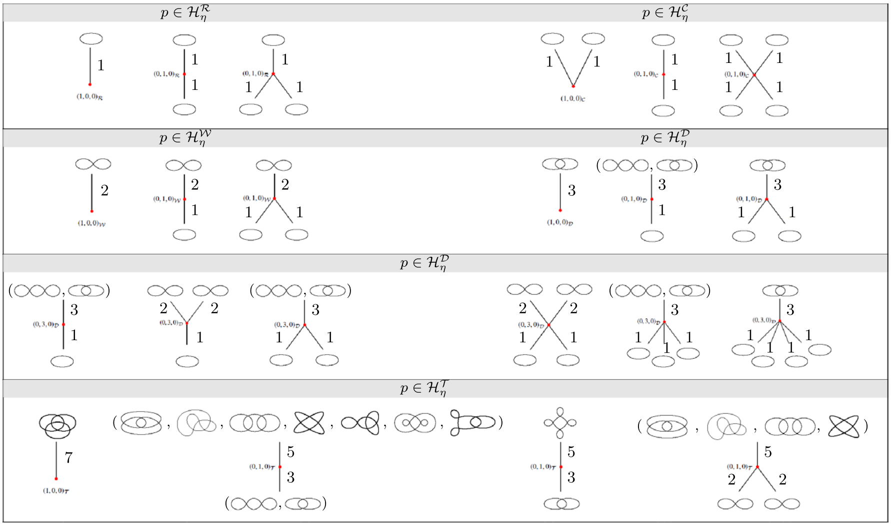
Type	$\mathcal{R}$			$\mathcal{C}$		
Nature	a	s	s	a	s	s
Lyapunov semigraph						
Poincaré-Hopf condition	$b_1^+ = 1$	$b_1^+ = b_1^-$	$b_1^+ = \mathcal{B}^- - 1$	$b_1^+ = b_2^+ = 1$	$b_1^+ = b_1^-$	$\mathcal{B}^+ = \mathcal{B}^-$
Type	$\mathcal{W}$			$\mathcal{T}$		
Nature	a	s	s	a^3	ssa	ssa
Lyapunov semigraph						
Poincaré-Hopf condition	$b_1^+ = 2$	$b_1^+ = b_1^- + 1$	$\mathcal{B}^+ = \mathcal{B}^-$	$b_1^+ = 7$	$b_1^+ = b_1^- + 2$	$\mathcal{B}^+ = \mathcal{B}^- + 1$
Type	$\mathcal{D}$					
Nature	ss_s	ss_s	ss_s	ss_s	ss_s	ss_s
Lyapunov semigraph						
Poincaré-Hopf condition	$b_1^+ = b_1^- + 2$	$\mathcal{B}^+ - 3 = \mathcal{B}^-$	$\mathcal{B}^+ = \mathcal{B}^- + 1$	$\mathcal{B}^+ = \mathcal{B}^- + 2$	$\mathcal{B}^+ = \mathcal{B}^-$	$\mathcal{B}^+ = \mathcal{B}^- + 1$
Nature	ss_s	ss_s	a^2	sa	sa	
Lyapunov semigraph						
Poincaré-Hopf condition	$\mathcal{B}^+ = \mathcal{B}^- - 1$	$\mathcal{B}^+ = \mathcal{B}^-$	$b_1^+ = 3$	$b_1^+ = b_1^- + 2$	$\mathcal{B}^+ = \mathcal{B}^- + 1$	

Table 2 Admissible boundaries on minimal GS isolating blocks for each Lyapunov semigraph



3.3 Constructing isolating blocks via a handle theory approach

It is evident from the topological analyses of weights relevant to the Poincaré–Hopf condition outlined in the preceding section, that first Betti numbers are not confined to being labelled with the integer one as in the smooth case. In the context of GS singularities, these first Betti numbers are integers greater than or equal to one. Consequently, the boundaries of isolating blocks evolve from simple circles to potentially distinguished branched 1-manifolds.

Definition 11 A **distinguished branched** 1-**manifold** is a topological space containing at most 4 connected components each of which has a finite number of **branched charts**, where each branched chart is an intersection of two transverse arcs. This intersection of two transverse arcs is called a **branch point**.

In [2], the construction of GS isolating blocks was approached akin to the method for building compact manifolds using classical Handle Theory. A *GS handle* of types $\mathcal{R}, \mathcal{C}, \mathcal{W}, \mathcal{D}$, or $\mathcal{T}$ is a subset of $\mathbb{R}^3$ that is homeomorphic to a chart as defined in Definition 1, featuring a well-defined local GS flow.

Given a GS handle $\mathcal{H}_\eta^{\mathcal{P}}$ containing a singularity p of type $\mathcal{P}$ and nature η, we explore the attachment of the handle's gluing region onto a *distinguished branched* 1-*manifold*. See Table 2. Further details can be found in [2] and the complete analysis is done in [3].

Theorem 2 *Let $\mathcal{H}_\eta^{\mathcal{P}}$ be a GS handle for a singularity p of type $\mathcal{P}$ and nature n and N a minimal GS isolating block for p. Then all possible distinguished branched* 1-*manifolds that form the connected components of the entering and exiting sets of N are described in Table* 2, *in terms of the Lyapunov semigraph of N. This characterization is up to flow reversal.*

In what follows, a schematic description of the steps in the construction of GS isolating blocks N for a singularity $p \in \mathcal{H}_\eta^{\mathcal{P}}$ is presented.

Let $p \in \mathcal{H}_a^{\mathcal{P}}$ (resp. $p \in \mathcal{H}_r^{\mathcal{P}}$) be a GS singularity of attracting (resp. repelling) nature of type $\mathcal{R}, \mathcal{C}, \mathcal{W}, \mathcal{D}$ or $\mathcal{T}$. Then, p admits a unique minimal GS isolating block up to homeomorphism. See Table 3.

Let us now consider saddle type singularities, i.e, that has at least one nature equal to s. The first step is to identify the attaching region. The *attaching region*, denoted by A_k, is defined as the unstable part of $\mathcal{H}_\eta^{\mathcal{P}}$. This region is analogous to the attaching sphere found in the traditional Handle Theory. One follows the steps:

Table 3 Minimal GS isolating block for each attracting singularity

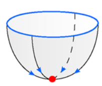
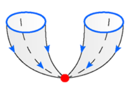
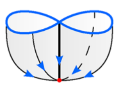
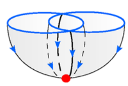
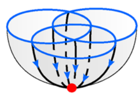
$p \in \mathcal{H}_a^{\mathcal{R}}$	$p \in \mathcal{H}_a^{\mathcal{C}}$	$p \in \mathcal{H}_a^{\mathcal{W}}$	$p \in \mathcal{H}_a^{\mathcal{D}}$	$p \in \mathcal{H}_a^{\mathcal{T}}$
				

Table 4 A Cone handle $\mathcal{H}_s^{\mathcal{C}}$

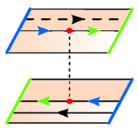
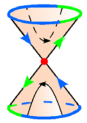
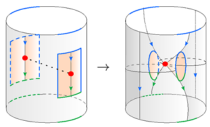
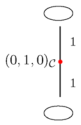
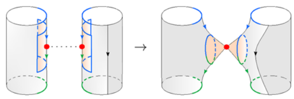
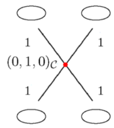
Pre quotient	Local chart	Attaching region
		

Table 5 Minimal isolating blocks for $p \in \mathcal{H}_s^{\mathcal{C}}$

Minimal isolating block	Lyapunov graph	Minimal isolating block	Lyapunov graph
			

1. choose the distinguished branched 1-manifold N^-. To ensure the isolating block remains connected, choose a branched 1-manifold, denoted by N^-, that does not exceed the number of connected components in A_k.
2. one needs to glue A_k in $N^- \times [0, 1]$. This is done by attaching the handle $\mathcal{H}_\eta^{\mathcal{P}}$ to a collar of $N^- \times [0, 1]$ using an embedding $f : A_k \to N^- \times \{1\}$. This embedding should map at least one connected component of A_k to each connected component of $N^- \times \{1\}$, resulting in N.
3. stretch N along the direction of the time-reversed flow.

We exemplify this construction with Tables 4 through 9, excerpted from [3]. For a comprehensive classification, refer to [3].

Example 1 Let $p \in \mathcal{H}_s^{\mathcal{C}}$ be a GS singularity of saddle nature of type $\mathcal{C}$. A cone handle $\mathcal{H}_s^{\mathcal{C}}$ is created by gluing two discs, each with a tubular flow centered on a degenerate singularity. Identifying these centers one obtains a cone, see Table 4. The possible minimal isolating blocks for p are shown in Table 5.

Example 2 Let $p \in \mathcal{H}_s^{\mathcal{W}}$ be a GS singularity of saddle nature of type $\mathcal{W}$. A Whitney handle $\mathcal{H}_{s_s}^{\mathcal{W}}$ corresponds to a regular handle $\mathcal{H}_s^{\mathcal{R}}$ followed by the identification of the two stable orbits, see Table 6. The possible minimal isolating blocks for p are shown in Table 7.

Example 3 Let $p \in \mathcal{H}_{sa}^{\mathcal{D}}$ be a GS singularity of attracting-saddle nature of type $\mathcal{D}$. A double handle $\mathcal{H}_{sa}^{\mathcal{D}}$ corresponds to two regular handles $\mathcal{H}_s^{\mathcal{R}}$ and $\mathcal{H}_a^{\mathcal{R}}$ in which two pairs of stable orbits are identified, as shown in Tables 8 and 9.

Table 6 A Whitney handle $\mathcal{H}^{\mathcal{W}}_{s_s}$

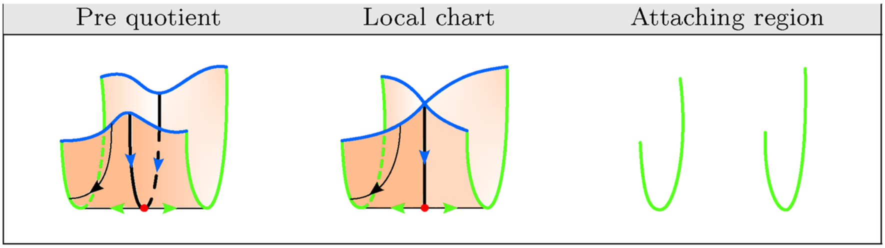


Table 7 Minimal isolating blocks for $p \in \mathcal{H}^{\mathcal{W}}_{s_s}$

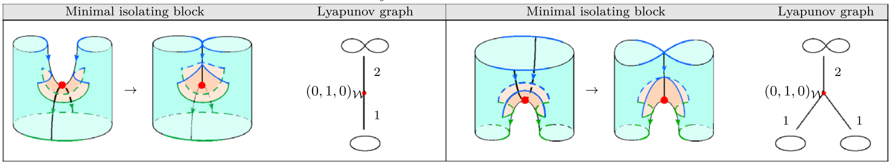


Table 8 A Double handle $\mathcal{H}^{\mathcal{D}}_{sa}$

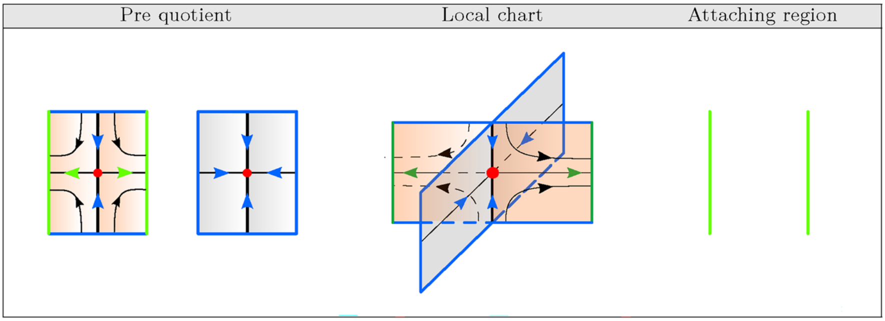
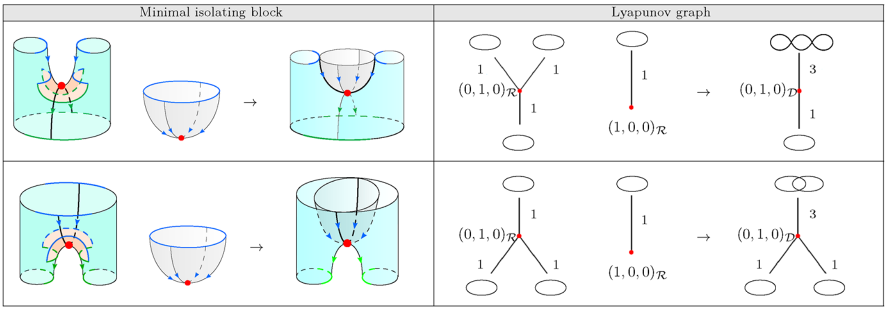


Table 9 Minimal isolating blocks for $p \in \mathcal{H}^{\mathcal{D}}_{sa}$



In [3] the existence of non-realizable semigraphs as minimal isolating blocks is established. More specifically, not all semigraphs for double singularities in Table 1 are realizable.

3.3.1 Passageways in isolating blocks

To realize isolating blocks with greater topological intricacy at their boundaries-i.e., higher Betti numbers-introduce passageways.

Consider a GS flow X_t associated with a vector field $\mathbf{X} \in \Sigma_0^r(\mathbf{M})$, where p denotes a singularity of X_t and N represents an isolating block for p. It's worth noting that each branched point on the entering boundary of N^+ (or the exiting boundary N^-) signifies a fold within the isolating block.

In the context of the Lyapunov semigraph, $b_i^+ - 1$ (or $b_i^- - 1$) denotes the count of folds that enter (or exit) through the corresponding connected component of the block.

For a minimal GS isolating block N, the total number F of folds entering and exiting N is: 0 if $p \in \mathbf{M}(\mathcal{R}) \cup \mathbf{M}(\mathcal{C})$; 1 if $p \in \mathbf{M}(\mathcal{W})$; 2 if $p \in \mathbf{M}(\mathcal{D})$; 6 if $p \in \mathbf{M}(\mathcal{T})$.

Furthermore, in the minimal case, the ω-limit (resp. α-limit) of all folds that enter (resp., exit) through N^+ (resp. N^-) is the singularity $p \in N$.

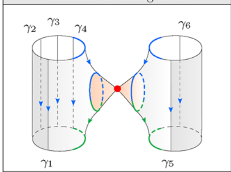
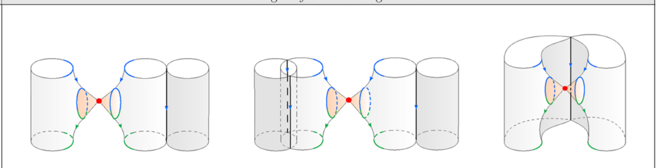
Definition 12 Let N be a GS isolating block for a singularity p of a vector field $\mathbf{X} \in \Sigma_0^r(\mathbf{M})$. N is a GS isolating block with **passageways** if there exists at least one fold in N for which p is neither the α-limit nor ω-limit.

It follows directly from the definition that GS singularities of attracting or repelling nature do not admit a GS isolating block with passageways.

Creating passageways on minimal blocks is relatively straightforward. Let N be a minimal GS isolating block for $p \in \mathcal{H}^{\mathcal{P}}\eta$. Suppose we have a collection $\Gamma = (\gamma_1, \gamma_2), (\gamma_3, \gamma_4), \dots, (\gamma_{2k-1}, \gamma_{2k})$ of pairs of orbits in the regular part of N, with $\gamma_i \neq \gamma_j$ if $i \neq j$, and such that p is neither the α-limit nor the ω-limit of any γ_i in Γ. Then, by identifying each pair of orbits $\gamma_{2i-1} \sim \gamma_{2i}$, for $i = 1, \dots, k$, the quotient space $N/\sim$ forms an isolating block for $p \in \mathcal{H}_\eta^{\mathcal{P}}$ with k passageways.

For instance, consider a minimal GS isolating block for a cone type singularity with saddle nature as in the figure on the left in Table 10. One can construct three examples of GS isolating blocks with passageways by identifying the following pairs of orbits: $\Gamma_1 = \{(\gamma_5, \gamma_6)\}$, $\Gamma_2 = \{(\gamma_1, \gamma_3), (\gamma_2, \gamma_4), (\gamma_5, \gamma_6)\}$, and $\Gamma_3 = \{(\gamma_3, \gamma_6)\}$.

Table 10 Adding passageways to a minimal GS isolating block for $p \in \mathcal{H}_s^{\mathcal{C}}$

Minimal isolating block	Passageways in isolating blocks
	

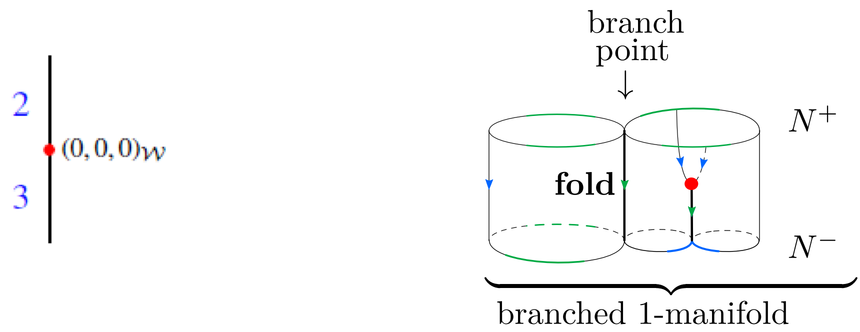


Fig. 2 A realization of an abstract Lyapunov semigraph

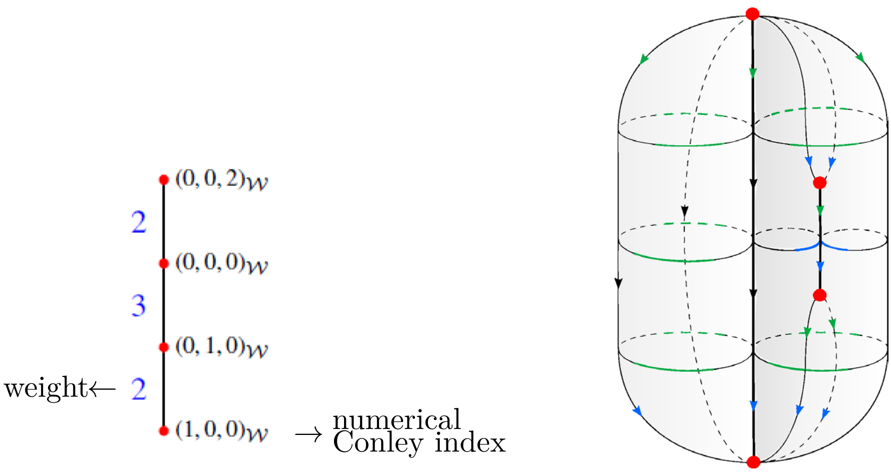


Fig. 3 An abstract Lyapunov graph and an associated realization

4 Global realization of GS flows on singular closed surfaces

It is clear that the realization of an abstract graph L as a GS flow X_t on M can be broken down into the realization of the semigraphs ℓ_v formed by a vertex v of L and its incident edges. By a realization of ℓ_v we mean a GS flow on an isolating block N with e_v^+ connected components for its entering boundary N^+ and e_v^- connected components for its exiting boundary N^-, where e_v^+ is the indegree of ℓ_v and e_v^- is the outdegree of ℓ_v. The first Betti numbers of these boundary components must be equal to the weights on the corresponding edges of ℓ_v. See Figs. 2 and 3.

A natural approach to global realization of a GS Lyapunov graph is to realize ℓ_v as a GS flow on an isolating block for every v of L. However, as Example 4 in Fig. 4 shows, local realization is no guarantee that we are able to glue these blocks according to L in order to obtain a global realization.

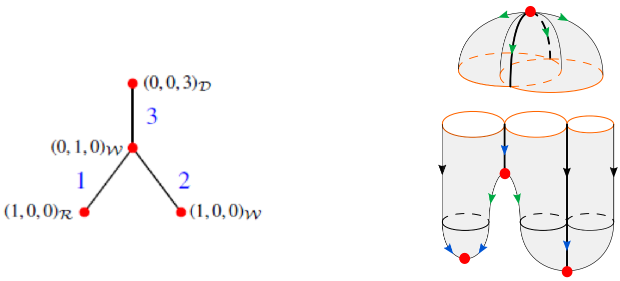


Fig. 4 An example of a non-realizable GS graph

4.1 Local realization is not sufficient in general for global realization

On the other hand, given that ℓ_v admits a realization as a GS flow on an isolating block for every v of L, we must be able to glue these blocks according to L in order to obtain a global realization. We glue blocks by identifying homeomorphic connected components of their boundaries in correspondence with the same edge of L. However, it may occur that the branched 1-manifolds that make up the boundary of the blocks are non-homeomorphic, preventing the gluing of the blocks. The number of non-homeomorphic branched 1-manifolds increases with respect to the number of branch points. This topological obstruction is the main difficulty in the realization of Lyapunov graphs with arbitrary weights. The following theorem presents necessary and sufficient conditions for a Lyapunov GS semigraph to be realized as an isolating block.

Theorem 3 (Local realization characterization) *A Lyapunov semigraph ℓ_v with a single vertex v labelled with a GS singularity is associated to a GS flow on a GS isolating block N if and only if*:

i) *the Poincaré-Hopf condition is satisfied*;
ii) *if v is labelled with a singularity of type $\mathcal{C}$ and nature s, with $e_v^- = e_v^+ = 2$, then b_1^+ is equal to either b_1^- or b_2^-*;
iii) *if v is labelled with a singularity of type $\mathcal{D}$ and nature ss_s (resp. ss_u), with $e_v^+ = 2$, (resp., $e_v^- = 2$) then $e_v^- \leq 2$ (resp., $e_v^+ \leq 2$). Moreover, if ℓ_v is minimal then $b_1^+ = b_2^+$ (resp., $b_1^- = b_2^-$).*
iv) *If v is labelled with a singularity of type $\mathcal{T}$ with $e_v^- = 2$ (resp., $e_v^+ = 2$) and ℓ_v is minimal then $b_1^- = b_2^-$ (resp., $b_1^+ = b_2^+$).*

Consider the following example where all semigraphs are realizable and there exists no choice of appropriate branched one manifolds as boundaries that makes the Lyapunov GS graph realizable.

Example 4 (*Non-realizable GS graph*) Local realization is not sufficient for global realization.

Example 4 illustrates that the question of gluing isolating blocks or put more globally, the question of realizing a GS Lyapunov graph depends firstly on satisfying the local realization Theorem 3 for Lyapunov semigraphs ℓ_v. However, as the example above shows, local realization of all ℓ_v does not imply in global realization of L.

Secondly, one can attempt to circumvent topological obstructions that may occur with inappropriate choices of branched one manifold as boundary or obstructions caused by the presence of singular orbits, i.e. *folds*. In other words, are there other choices of branched one manifolds in Example 4 which realize ℓ_v and whose isolating blocks can all be glued in order to realize L? In this example, there are no other choices since all semigraphs ℓ_v containing a unique vertex of L admit only one realization as a GS isolating block.

However, it may occur that the branched 1-manifolds that make up the boundary of the blocks are non-homeomorphic as in Example 4, preventing the gluing of the blocks and that no other choice is possible. It is clear that number of non-homeomorphic branched 1-manifolds increases with respect to the number of branch points and so does the complexity of determining the homeomorphism classes.

Thus, the global realization problem in Example 4 is reduced to identifying homeomorphic connected components of the boundaries of isolating blocks in correspondence with the edges of L. This topological obstruction is the main difficulty in the realization of Lyapunov graphs with arbitrary weights.

4.1.1 Distinguished families of branched one manifolds and global realization theorems

The global realization challenge can be framed as the assignment of distinguished branched 1-manifolds to the edges of a graph L, as detailed in Table 2. Our aim is to accomplish realizations for both minimal GS graphs and those with varied weights, taking into consideration the inclusion of folds.

Exploring families of distinguished branched 1-manifolds with multiple assignment choices for each weight might appear to offer stronger global results. However, it's essential to acknowledge that the number of non-homeomorphic distinguished branched 1-manifolds increases swiftly with each successive weight. Determining whether two distinguished branched 1-manifolds with arbitrary weights are homeomorphic or not presents a challenging problem. Consequently, establishing sufficient conditions for the global realization of Lyapunov graphs becomes significantly more intricate in this context.

The sufficient conditions for the global realization theorems presented in this section originate from two families of distinguished branched 1-manifolds. These families are structured to meticulously control the increase of branch points. Moreover, each family uniformly assigns the same distinguished branched 1-manifold to every edge of the graph with a common weight. This strategic assignment ensures that boundary components with matching weights of two isolating blocks are identical, thereby circumventing the homeomorphism problem. As a result, all boundaries of the isolating blocks are glued according to the graph, ensuring global realization.

We have chosen to present in this section some of the global results presented in [3] which encompass singularities of types $\mathcal{R}, \mathcal{C}, \mathcal{W}, \mathcal{D}$, and $\mathcal{T}$, with the caveat that the $\mathcal{T}$-singularities have precisely two boundary components on their isolating block. This restriction on $\mathcal{T}$-singularities is due to the inherent complexity of the branched 1-manifold boundaries of their isolating blocks in the other cases. For other stronger realization theorems, see [3].

Except for the realization of graphs with minimal weights on all edges, as proven in Theorem 4, determining sufficient conditions for the global realization of graphs involving $\mathcal{R}, \mathcal{C}, \mathcal{W}, \mathcal{D}$, and $\mathcal{T}$ singularities remains an open question.

The realization theorems discussed in this section come with certain simplifications. These might include limitations on the types of singularities assigned to the graph, constraints on the weights of the edges, or restrictions on the degrees of the vertices. Without such simplifications, the increase of edge weights would lead to a multitude of choices for distinguished branched 1-manifolds. Moreover, relying solely on local information from the graph may not suffice to ensure the proper gluing of isolating blocks as dictated by the graph.

Additionally, in Theorem 2, we offer a comprehensive characterization of the branched 1-manifolds constituting the boundary of these blocks, assuming the weights on the edges of ℓ_v adhere to the minimal criteria satisfying the Poincaré–Hopf condition. Furthermore, we explore a controlled approach to introducing folds into an isolating block to facilitate the realization of semigraphs ℓ_v with non-minimal weights. This is feasible because folds inherently increase the first Betti number of the branched 1-manifold boundaries, leading to a rise in the weights on the corresponding edges of ℓ_v.

In this section we show that the sufficient conditions for the local realizability of ℓ_v turn out to be sufficient to guarantee the global realizability of an abstract graph L as a GS flow on a closed singular two-manifold in the following cases:

1. the vertices of L are labelled with the numerical Conley indices of singularities $\mathcal{R}, \mathcal{C}, \mathcal{W}, \mathcal{D}$ or $\mathcal{T}$ and all weights are minimal, as proved in Theorem 4;
2. the vertices of L are labelled with the numerical Conley indices of singularities $\mathcal{R}, \mathcal{C}$ or $\mathcal{W}$ and with no weight restrictions, as proved in Theorem 7;
3. the vertices of L are labelled with the numerical Conley indices of singularities $\mathcal{R}, \mathcal{C}, \mathcal{W}$ or $\mathcal{D}$ and L is a linear graph, as proved in Theorem 5.

Definition 13 In the case that the weights on the edges of a GS semigraph are the smallest satisfying Poincaré-Hopf it will be called a **minimal GS semigraph**. Hence, we say that a Lyapunov graph L is a **GS graph** (resp., **minimal GS graph**) if for all vertices v of L the semigraph formed by v and its incident edges is a GS semigraph (resp., **minimal GS semigraph**).

In the next section, we investigate under what conditions a GS graph is realizable as a GS flow on a closed 2-dimensional singular manifold.

4.1.2 Realization theorems

The realization of a GS graph L involves the assignment of distinguished branched 1-manifolds to the edges of L in such a way that ensures the semigraph of each vertex v of L can be realized as a GS isolating block, with these chosen distinguished branched 1-manifolds serving as boundary components.

In the subsequent discussion, our endeavor to determine these assignments will be presented as global realization theorems. The complexity of these assignments naturally varies depending on the type of singularities and their isolating blocks. Therefore, most of the results outlined in this section-excluding Theorem 4-will concentrate on GS flows devoid of triple crossing singularities due to the intricate nature of the distinguished branched 1-manifolds constituting the boundary components of their isolating blocks. The homeomorphism problems concerning branched 1-manifolds can swiftly become intractable.

The first realization theorem will impose weight restrictions and the second will impose degree restriction on the vertices.

Theorem 4 (Minimal case) *Let L be a minimal GS graph containing singularities of type $\mathcal{R}, \mathcal{C}, \mathcal{W}, \mathcal{D}$ and $\mathcal{T}$. L admits a realization.*

Proof Since L is minimal, the possible weights on the edges of L are 1 2, 3, 5 or 7. For each edge, consider the assignment of distinguished branched 1-manifolds given in Table 11.

This selection is based on the assortment of minimal GS isolating blocks outlined in Theorem 2 (refer to Table 2). It's worth noting that for weights 1, 2, and 7, there exists a unique choice of distinguished branched 1-manifolds. However, for weight 3, there are two possible options.

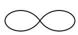
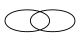
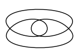
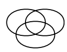
However, the selected choice is consistent across all GS semigraphs with weight 3, making it always realizable. For weight 5, there are four potential choices of distinguished branched 1-manifolds common to all GS semigraphs, any of which can be selected.

□

Example 5 In Fig. 5, a minimal GS graph is realized as a singular flow on a GS 2-manifold, as asserted in Theorem 4.

Remark 1 The sole instance of a minimal GS graph featuring a vertex labeled with a singularity of type $\mathcal{T}$, whether of attracting or repelling nature, is represented by the

Table 11 Distinguished branched 1-manifolds for a realization of a minimal graph

weight (w)	$w = 1$	$w = 2$	$w = 3$	$w = 5$	$w = 7$
N^-					

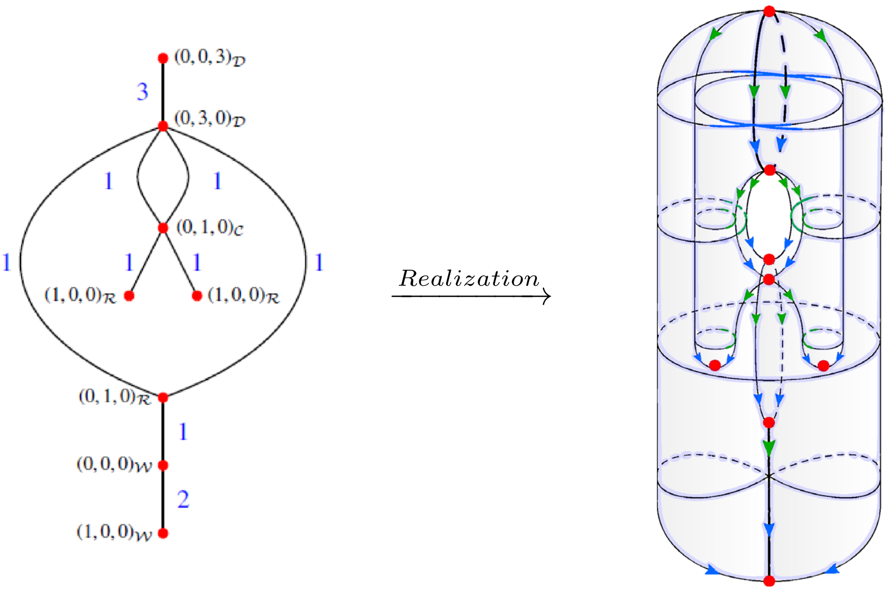


Fig. 5 A minimal GS graph and an associated realization as a GS flow on a closed manifold

repeller-attractor pair whose configuration entails a flow composed of two singularities of type $\mathcal{T}$, one being an attractor and the other a repeller.

The upcoming theorem will introduce a degree constraint on a GS graph. Remarkably, GS graphs devoid of bifurcation vertices are always realizable.

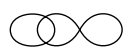
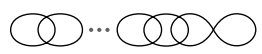
Theorem 5 (Linear Graph) *Let L be a GS graph such that vertices of L labelled with singularities of type $\mathcal{R}$, $\mathcal{C}$, $\mathcal{W}$ or $\mathcal{D}$ have degree less than or equal to 2 and vertices labelled with singularities of type $\mathcal{T}$ have degree 2. Then L is realizable.*

Proof To achieve the realization of L, one can select distinguished branched 1-manifolds for each edge, based on the weights specified in Table 12.

Let N^+ represent a distinguished branched 1-manifold, serving as the entering boundary of a minimal GS isolating block N. Consider the sequence of identifications of a pair of points illustrated in Fig. 6. These points in N^+ lie on orbits within N, with the property that their ω-limit does not reside within N.

With each identification, a branch point is generated. After executing k identifications, the GS isolating block will feature k passageways. It's straightforward to

Table 12 A family of distinguished branched 1-manifolds useful for linear graphs

weight (w)	$w=1$	$w=2$	$w=3$	$w=4$	...	$w=2k-1$	$w=2k$
N^-							

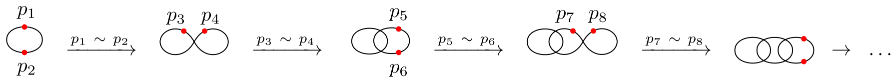


Fig. 6 Identification sequence

confirm that the exiting boundaries of any isolating block formed in this manner will have as boundary components the distinguished branched 1-manifolds outlined in Table 12.

Each GS semigraph ℓ_v with a single vertex $v \in L$ can be realized using the aforementioned process, beginning from the corresponding minimal GS semigraph. This corresponding minimal GS semigraph shares the same indegree, outdegree, and labeling as the semigraph ℓ_v.

As each boundary with weight n corresponds to a unique distinguished branched 1-manifold realized on the boundary of isolating blocks, the process of attaching one block to another becomes straightforward. Consequently, L is realizable. $\square$

Example 6 In Fig. 7, a GS graph (on the left) satisfying Theorem 5 is realized as a GS flow (on the right).

It's worth noting that bifurcation vertices, particularly those with incident edges not labeled with minimal weights, introduce intricate combinatorial considerations regarding the selection of distinguished branched 1-manifolds as boundary components of isolating blocks. Example 4 illustrates this complexity.

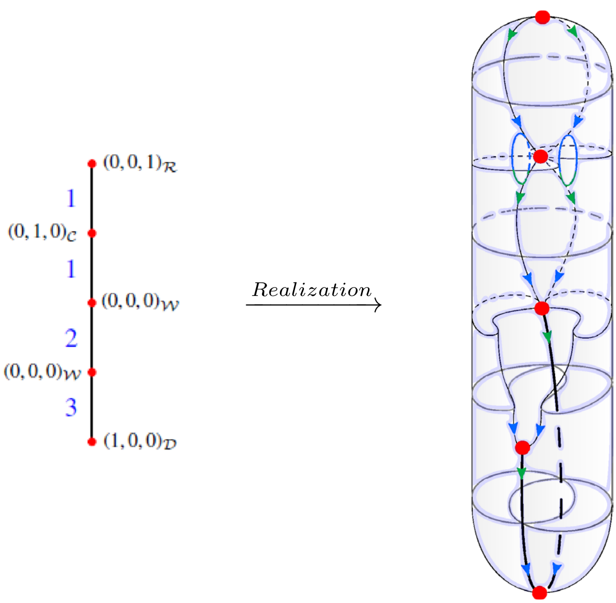


Fig. 7 A realization of a linear GS graph as a GS flow

Table 13 A family of distinguished branched 1-manifolds useful for the $\mathcal{RCW}$-case

weight (w)	$w=1$	$w=2$	$w=3$	$w=4$	...	$w=n$
N^-						

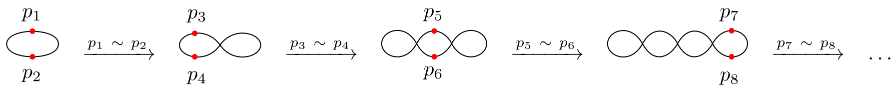


Fig. 8 Identification sequence

The following theorem aims to address the issue highlighted in Example 4. It seeks to reconcile the findings from Theorems 4 and 5 by enforcing minimal weights exclusively on the incident edges of bifurcation vertices.

Theorem 6 *Let L be a GS graph containing singularities of types $\mathcal{R}, \mathcal{C}, \mathcal{W}, \mathcal{D}$ and $\mathcal{T}$. Suppose that all vertices labelled with a singularity of type $\mathcal{T}$ have degree* 2 *and all incident edges of vertices v of L with degree greater than or equal to* 3 *have minimal weights. Then L is realizable.*

The main idea behind the proof is that bifurcation vertices of the graph break it down into linear semi-graphs. One constructs a realization for each linear semi-graph using Theorem 5 and then uses the minimality of the bifurcation vertices to glue them all together, which is always possible according to Theorem 4. We refer the reader to [3] for a more detailed and technical proof of this theorem.

Theorem 7 ($\mathcal{RCW}$-Case) *All GS graphs labelled only with singularities of type $\mathcal{R}, \mathcal{C}$ and $\mathcal{W}$ are realizable.*

Proof Consider the choice of distinguished branched 1-manifolds for the edges of L labelled with weights w given in Table 13.

Each GS semigraph L_v, formed by a single vertex $v \in L$ and its incident edges, can be realized based on the realization of the corresponding minimal GS semigraph ℓ_v. To elaborate, initiating from a minimal GS isolating block N_0 with boundary components outlined in Table 13, which realizes ℓ_v, a series of identifications of pairs of points in N^+ can be executed. This process, illustrated in Fig. 8, involves pairs of points on orbits in N whose ω-limit sets lie outside N.

These identifications must be performed until the weights on the boundaries of the block align with the weights on the edges of L_v.

Notice that each identification on the boundary N^+ results in a branch point, creating a fold where a cylinder is attached to the block. It's worth noting that for a vertex of degree 2, the positioning of these cylinders relative to the minimal GS block may lead to topologically distinct realizations of the same semigraph.

Finally, the realizations of all GS semigraphs L_v can be glued together because all boundaries with weight w share the same distinguished branched 1-manifold. □

Example 7 In Fig. 9, we present two topologically non equivalent realizations of a GS graph satisfying Theorem 7.

Although Theorem 7 was derived under the assumption of excluding vertices labeled with singularities of type $\mathcal{D}$, a closer examination of the family of distinguished branched 1-manifolds in Table 13 reveals that it's possible to realize nearly every GS semigraph with a single vertex labeled with a singularity of type $\mathcal{D}$ using this family of distinguished branched 1-manifolds as boundary components. The exceptions are the GS semigraphs of degree 1 labeled with natures a or r, the GS semigraphs of degree 3 labeled with natures sa or sr, and the GS semigraphs of degree 5 labeled with natures ss_s or ss_u.

However, even if we cannot guarantee the existence of a realization, i.e., a triple $(\mathbf{M}, X_t, f)$, we still maintain control over the Euler characteristic of a potential realization on M. This control stems from the fact that the Euler characteristic can be computed based on the natures and types of GS singularities.

Proposition 8 *Let L be a GS graph. If $\mathbf{M}$ is a realization of L, then*:

$$\mathcal{X}(\mathbf{M}) = a - s + r + \frac{W}{2} + T,$$

where W and T are the number of vertices in L labelled with singularities of type $\mathcal{W}$ and $\mathcal{T}$, respectively, while a, s and r are, respectively, the total number of attracting, saddle and repelling natures of the singularities with which the vertices are labeled.

This proposition is proved by using the Conley numerical index and the top Morse-Conley equality:

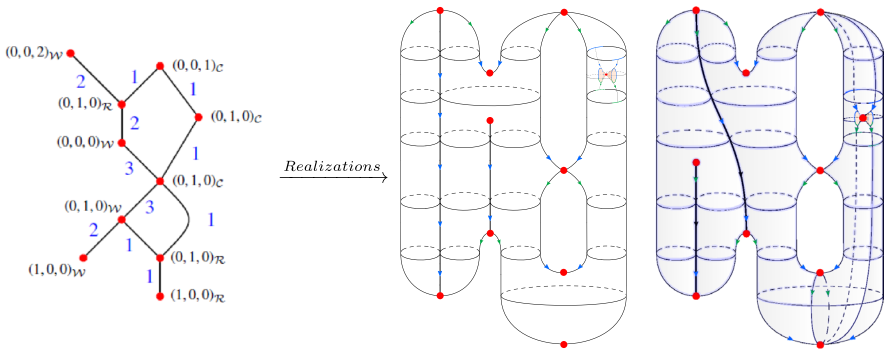


Fig. 9 Two topologically non-equivalent realizations of an $\mathcal{RCW}$-case

$$\mathcal{X}(\mathbf{M}) = \sum_{p_i \in Sing(\mathbf{M})} (h_0^i - h_1^i + h_2^i). \tag{6}$$

Example 8 Let $\mathbf{M}$ be the GS 2-manifold of Example 5. One has that, $\mathcal{X}(\mathbf{M}) = 3 - 5 + 2 + \frac{2}{2} + 0 = 1$. In the case $\mathbf{M}$ is the GS 2-manifold of Example 6, one has that $\mathcal{X}(\mathbf{M}) = 2 - 3 + 1 + \frac{2}{2} + 0 = 1$. On the other hand, if $\mathbf{M}$ is the GS 2-manifold of Example 7, then $\mathcal{X}(\mathbf{M}) = 2 - 6 + 2 + \frac{4}{2} + 0 = 0$. Lastly, if $\mathbf{M}$ is the GS 2-manifold of Fig. 3, it follows that $\mathcal{X}(\mathbf{M}) = 1 - 2 + 1 + \frac{4}{2} + 0 = 2$.

5 The dynamics of morsifications and the Gutierrez–Sotomayor chain complex

In [8, 9], global cancellation theorems were obtained for Morse-Smale flows with no periodic orbits on smooth manifolds M^n, $n > 5$ and $n = 2$, by considering a spectral sequence analysis of the associated Morse chain complex. In [4] these techniques were explored in a new context, specifically by considering Gutierrez–Sotomayor singular flows.

This approach, previously fruitful in [8–10], led to the derivation of cancellation theorems in smooth settings, particularly for gradient flows of Morse functions and circle-valued Morse functions. In [4], this analysis was pushed one step ahead by scrutinizing global GS flows on closed singular surfaces. Therein, the interconnections among flow lines of GS flows were explored through a spectral sequence analysis of a corresponding chain complex.

An aspect of Smale's work stands out and deserves attention here. Smale established a local cancellation theorem for hyperbolic singularities of consecutive indices, p and q, on M^n, where $n > 5$ or $n = 2$. In essence, when the algebraic intersection number of p and q is ± 1, this pair of singularities can be cancelled within a neighborhood U, resulting in the emergence of a flow φ' that coincides with φ_X outside of U. Moreover, the flow φ' restricted to U is topologically equivalent to a tubular flow. For further details, we refer the reader to [11].

However, adapting the smooth theory to the singular setting, particularly for GS flows, posed a significant challenge in [4]. The aim is to uphold the foundational principles that govern the smooth setting within the singular framework. For the theory to remain coherent and constitute a valid generalization, the definitions and principles of the singular setting had to encompass those of the smooth setting. Consequently, defining intersection numbers turned out to be problematic in the absence of differentiability, as well as the establishment of a chain complex generated by GS singularities.

In this section, we present how the authors in [4] constructed a chain complex that describes the dynamics of a GS flow. Their approach involved a Morsification process of both the flow and the manifold to adapt the smooth theory to the singular setting.

5.1 Morsification techniques for GS flow analysis

In this section, we present how the authors in [4] establish a regularization process for the GS singularities, yielding a smooth 2-manifold $\widetilde{M}$ along with a smooth flow featuring regular singularities. We refer to this regularization process as the Morsification of GS singularities.

In [4], the authors were interested in working with vector fields that possess only one type of GS singularities in addition to regular singularities. Hence, we adopt the following notation for subsets of $\mathfrak{M}(\mathcal{GS})$ and $\Sigma^r_{\mathcal{G}}\mathcal{S}(M)$: $\mathfrak{M}(\mathcal{GC})$ (resp., $\mathfrak{M}(\mathcal{GW})$, $\mathfrak{M}(\mathcal{GD})$, $\mathfrak{M}(\mathcal{GT})$) denotes the set of singular 2-manifolds with GS singularities of regular and cone (resp., Whitney, double crossing, triple crossing) types; $\Sigma_{\mathcal{GC}}(M)$ (resp., $\Sigma_{\mathcal{GC}}(W)$, $\Sigma_{\mathcal{GC}}(D)$, $\Sigma_{\mathcal{GC}}(T)$) denotes the set of all vector fields on $M \in \mathfrak{M}(\mathcal{GC})$ (resp., $\mathfrak{M}(\mathcal{GW})$, $\mathfrak{M}(\mathcal{GD})$, $\mathfrak{M}(\mathcal{GT})$) which only possess regular and cone (resp., Whitney, double crossing, triple crossing) singularities.

From now on we fix a compact singular 2-manifold $M \in \mathfrak{M}(\mathcal{GS})$ and a GS vector field $X \in \Sigma_{\mathcal{GS}}(M)$ on M, where $\mathcal{S} = \mathcal{C}$, $\mathcal{W}$, $\mathcal{D}$, or $\mathcal{T}$. Consider the Gutierrez–Sotomayor flow φ_X on M associated with X. Define the ***singular part*** of M, denoted by $\mathscr{SP}(M)$, as the union of all non-regular singularities and folds, i.e., $\mathscr{SP}(M) = M(\mathcal{C}) \cup M(\mathcal{W}) \cup M(\mathcal{D}) \cup M(\mathcal{T})$.

Definition 14 Let $M \in \mathfrak{M}(\mathcal{GS})$ be a compact singular 2-manifold, $X \in \Sigma_{\mathcal{GS}}(M)$ a GS vector field on M and φ_X the GS flow associated to X, where $\mathcal{S} = \mathcal{C}$, $\mathcal{W}$, $\mathcal{D}$, or $\mathcal{T}$. An isolating block (N, φ_X) admits a **Morsification** if there exists a quadruple $(\widetilde{N}, \varphi_{\widetilde{X}}, \mathfrak{h}, \mathfrak{p})$ such that

1. $\widetilde{N}$ is a smooth 2-manifold;
2. $\widetilde{\varphi}$ is a smooth flow on $\widetilde{N}$ having only hyperbolic regular singularities and no saddle connections;
3. $\mathfrak{h} : N \to \widetilde{N}$ is a multivalued map such that $\mathfrak{h}$ restricted to

$$N \setminus \{\mathscr{SP}(N) \cup \{x \in N \mid \omega(x) = p \text{ or } \alpha(x) = p, \text{where } p \text{ is a saddle cone singularity}\}\}$$

   is a homeomorphism;
4. $\mathfrak{p} : \widetilde{N} \to N$ is the projection map and $\mathfrak{h} \circ \mathfrak{p} = id|_{\widetilde{N}}$.

In this case, one says that (N, φ_X) admits a Morsification to $(\widetilde{N}, \varphi_{\widetilde{X}})$, or that $(\widetilde{N}, \varphi_{\widetilde{X}})$ is a Morsification of (N, φ).

The next theorem, proved in [4], guarantees the existence of Morsifications for GS isolating blocks.

Theorem 9 *Let $M \in \mathfrak{M}(\mathcal{GS})$ be a singular 2-manifold, $X \in \Sigma_{\mathcal{GS}}(M)$ a GS vector field on M and φ_X the GS flow associated to X, where $\mathcal{S} = \mathcal{C}, \mathcal{W}, \mathcal{D}$ or $\mathcal{T}$. Given a GS singularity p and an isolating block (N, φ_X) for p, there exists a Morsification $(\widetilde{N}, \varphi_{\widetilde{X}})$, where $\widetilde{N}$ is an isolating block w.r.t. the regularized flow $\varphi_{\widetilde{X}}$.*

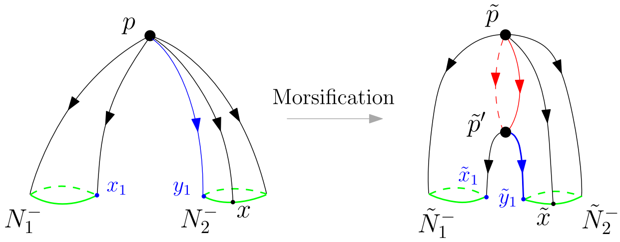


Fig. 10 Morsification of an isolating block for a repelling cone singularity

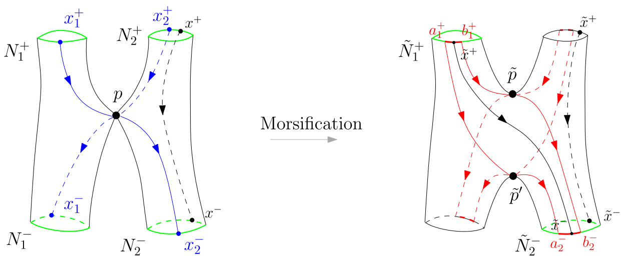


Fig. 11 Morsification of an isolating block for a saddle cone singularity

The proof of Theorem 9 is carried out through a meticulous case-by-case analysis, examining each type of singularity individually. The overarching concept in all instances is to introduce the quadruple $(\widetilde{N}, \varphi_{\widetilde{X}}, \mathfrak{h}, \mathfrak{p})$ which defines the morsification. This is accomplished by delineating the neighborhood $\widetilde{N}$ and the associated maps $\mathfrak{h}, \mathfrak{p}$. However, for brevity, we defer the technical details and the more intricate proofs to the original paper [4], opting instead for a more illustrative and constructive exposition of the smooth flow $\varphi_{\widetilde{X}}$.

5.1.1 Morsification of cone singularities

1. Let p be a repelling (resp. attracting) cone singularity. Define the multivalued map $\mathfrak{h} : N \to \widetilde{N}$ which maps the singularity p to the cycle that contains the singularities $\tilde{p}', \tilde{p}$ and their connecting orbits in the morsified flow. See Fig. 10.
2. Let p be a saddle cone singularity and N its isolating block. There are two cases to consider, the first being the case where the boundary of the exit and entering sets of N are disconnected and the second where they are connected. Define the multivalued map $\mathfrak{h} : N \to \widetilde{N}$ which maps the singularity p to the singularities $\tilde{p}'_i, \tilde{p}$. Also it maps the two orbits that have p as ω-limit (resp. α-limit) to four orbits: two orbits that have $\tilde{p}'_i$ as ω-limit (resp. α-limit) and two orbits that have $\tilde{p}$ as ω-limit (resp. α-limit). See Figs. 11 and 12.

Combinatorially the isolating blocks for cone singularities together with its Morsification can be seen as the Lyapunov semigraphs in Fig. 13.

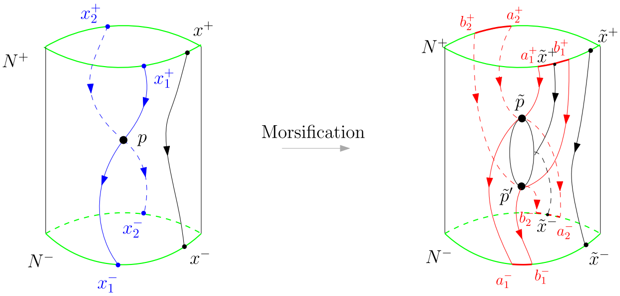


Fig. 12 Morsification of an isolating block for a saddle cone singularity and its Morsification

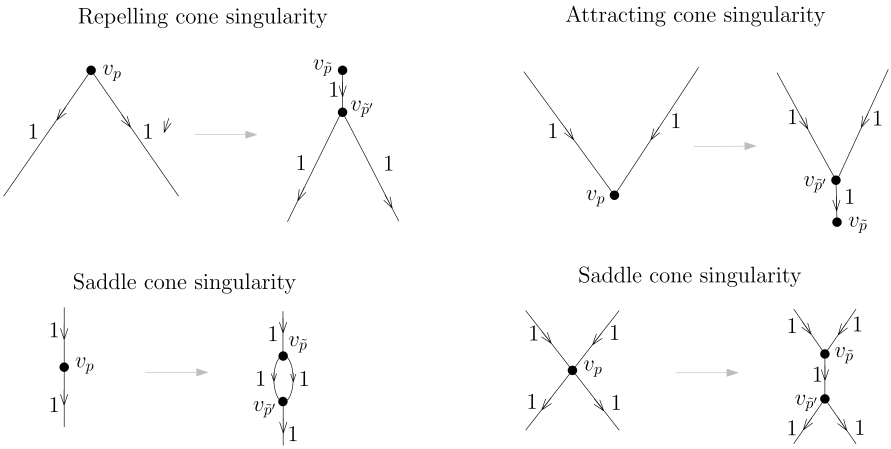


Fig. 13 Morsification of a Lyapunov semigraph with a vertex associated to a cone singularity

5.1.2 Morsification of Whitney singularities

1. Let p be a repelling (resp. attracting) Whitney singularity and N its isolating block. Define the multivalued map $\mathfrak{h} : N \to \widetilde{N}$ which maps the singularity p to a regular singularity $\tilde{p}$ and the orbit along the fold, that has p as ω-limit (resp. α-limit), to two orbits having $\tilde{p}$ as ω-limit (resp. α-limit) in the morsified flow. See Fig. 14.
2. Let p be a Whitney cone singularity and N its isolating block. There are two cases to consider, the first being the case where the boundary of either the exit or entering set of N are disconnected and the second where they are both connected. In the first case, define the multivalued map $\mathfrak{h} : N \to \widetilde{N}$ which maps the singularity p to the singularities $\tilde{p}$ and the orbit on the fold, that has p as ω-limit (resp. α-limit), to two orbits having $\tilde{p}$ as ω-limit (resp. α-limit). See Fig. 15. In the second case, define the multivalued map $\mathfrak{h} : N \to \widetilde{N}$ which maps the singularity p to the singularities $\tilde{p}'_i, \tilde{p}$. Also it maps the two orbits that have p as ω-limit (resp. α-limit)

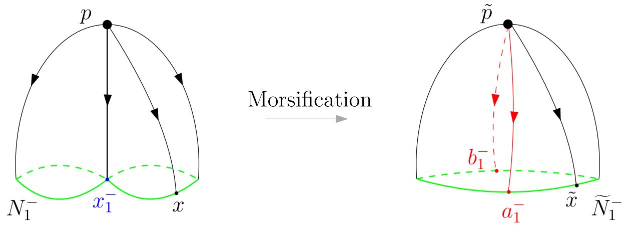


Fig. 14 Morsification of an isolating block for a repelling Whitney singularity

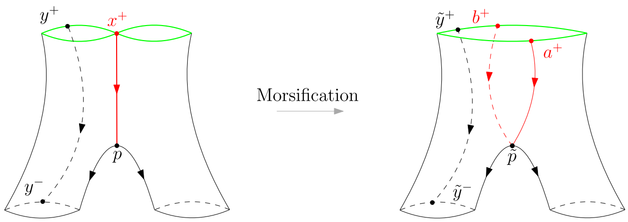


Fig. 15 Morsification of an isolating block for a saddle Whitney singularity

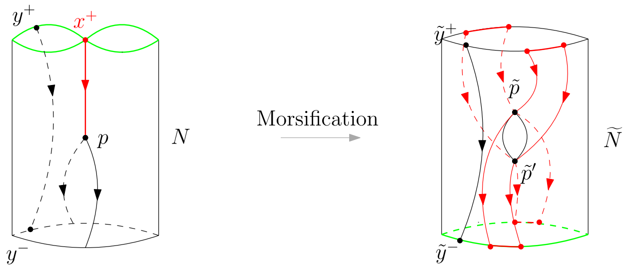


Fig. 16 Morsification of an isolating block for a saddle Whitney singularity

to four orbits: two orbits that have $\tilde{p}'_i$ as ω-limit (resp. α-limit) and two orbits that have $\tilde{p}$ as ω-limit (resp. α-limit). See Fig. 16.

Combinatorially the isolating blocks for Whitney singularities together with its Morsification can be seen as the Lyapunov semigraphs in Fig. 17.

5.1.3 Morsification of double point singularities

1. Let p be a repelling (resp., attracting) double crossing singularity. Define the multivalued map $\mathfrak{h} : N \to \widetilde{N}$ which maps the singularity p to the regular repeller singularities $\tilde{p}_1, \tilde{p}_2$. Also it maps the two orbits on the folds, that has p as α-limit

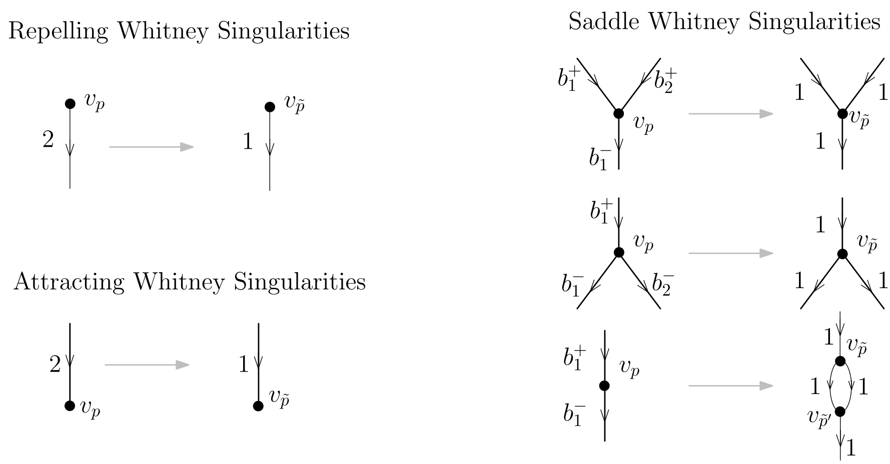


Fig. 17 Morsification of a Lyapunov semigraph with vertex associated to a Whitney singularity

(resp. ω-limit), to four orbits: two orbits having $\tilde{p}_1$ as α-limit (resp. ω-limit) and two orbits having $\tilde{p}_2$ as α-limit (resp. ω-limit). See Fig. 18.

2. Let p be a saddle double crossing singularity of *sa* (resp., *sr*) nature. Define the multivalued map $\mathfrak{h} : N \to \widetilde{N}$ which maps the singularity p to the regular repeller singularities $\tilde{p}_1, \tilde{p}_2$. Also it maps the two orbits on the folds, that has p as ω-limit (resp. α-limit), to four orbits: two orbits having $\tilde{p}_1$ as ω-limit (resp. α-limit) and two orbits having $\tilde{p}_2$ as ω-limit (resp. α-limit). See Figs. 19 and 20.
3. Let p be a double crossing saddle singularity of ss_s-nature (ss_u-nature). Define the multivalued map $\mathfrak{h} : N \to \widetilde{N}$ which maps the singularity p to the regular repeller singularities $\tilde{p}_1, \tilde{p}_2$. Also it maps the two orbits on the folds, that has p as ω-limit (resp. α-limit), to four orbits: two orbits having $\tilde{p}_1$ as ω-limit (resp. α-limit) and two orbits having $\tilde{p}_2$ as ω-limit (resp. α-limit). See Figs. 21 and 22. Combinatorially the isolating blocks for double singularities together with its Morsification can be seen as the Lyapunov semigraphs in Fig. 23. By considering the opposite direction on the graphs in Fig. 23, we obtain the graphs for attracting double

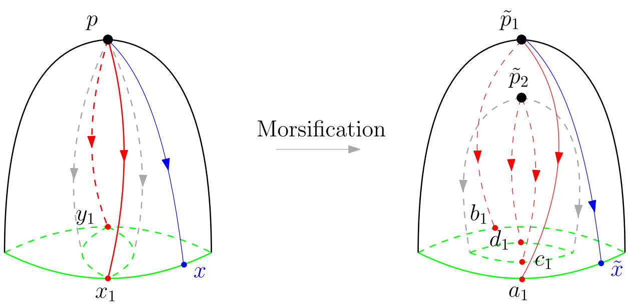


Fig. 18 Morsification of an isolating block for a repelling double crossing singularity

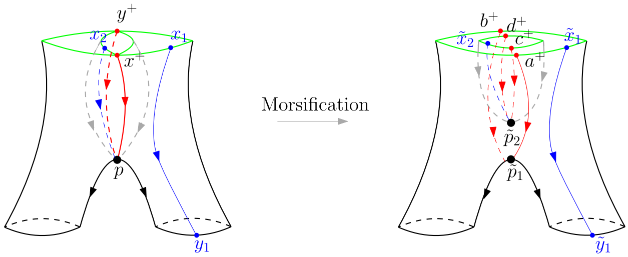


Fig. 19 Morsification of an isolating block for a saddle double crossing singularity of *sa*-nature

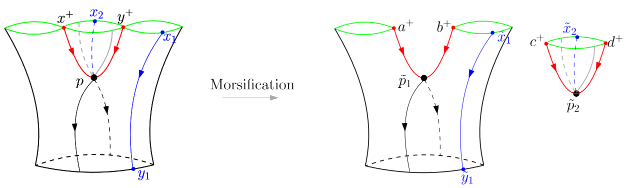


Fig. 20 Morsification of an isolating block for a saddle double crossing singularity of *sa*-nature

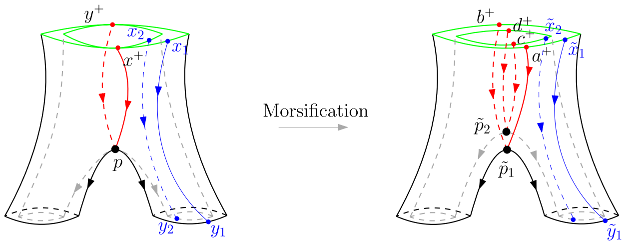


Fig. 21 Morsification of an isolating block for a saddle double crossing singularity of ss_s-nature

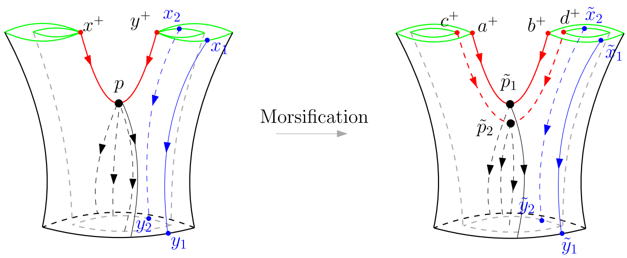


Fig. 22 Morsification of an isolating block for a saddle double crossing singularity of ss_s-nature

crossing singularities. There are other isolating blocks for saddle double crossing singularities which are not considered in this analysis.

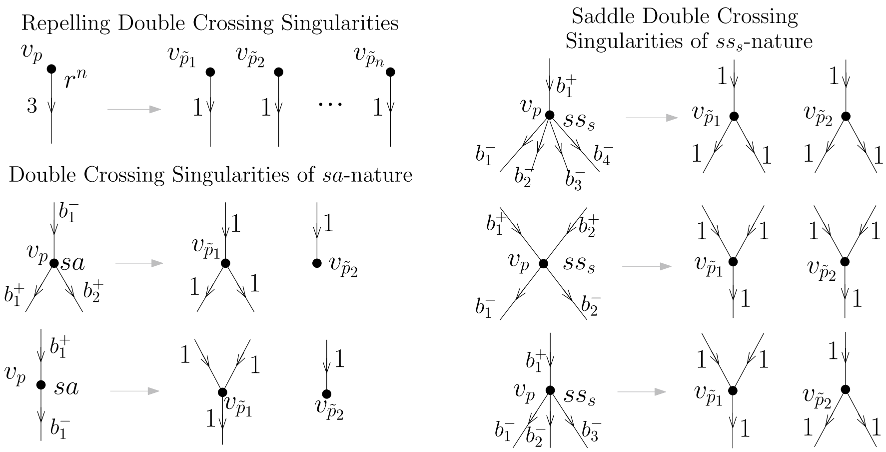


Fig. 23 Morsification of a Lyapunov semigraph with vertex associated to a double crossing singularity

5.1.4 Morsification of triple point singularities

Combinatorially the isolating blocks for triple singularities together with its Morsification can be seen as the Lyapunov semigraphs in Fig. 24.

For technical details, we refer to [4].

5.2 Gutierrez–Sotomayor chain complex

It was established in [4] a chain complex that captures the behavior of a given GS flow on a closed singular 2-manifold M, belonging to $\mathfrak{M}(\mathcal{GS})$. This was achieved by employing the Morsification process, described in the last section, which allows us to define GS intersection numbers, which allow for the counting (with sign) of flow lines between successive GS singularities, as detailed in Sect. 5.2.2. Additionally,

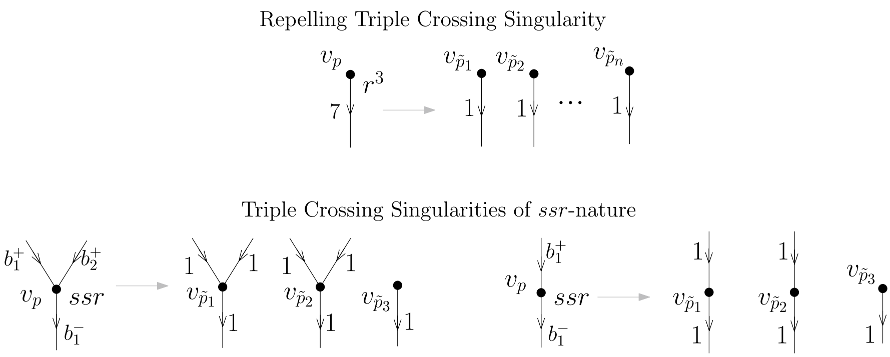


Fig. 24 Morsification of a Lyapunov semigraph with vertex associated to a triple crossing singularity

Sect. 5.2.1 provides a concise overview of Morse chain complexes associated with Morse functions on closed manifolds.

5.2.1 A brief overview of morse chain complex

Consider M as a smooth, closed n-manifold and let $f : M \to \mathbb{R}$ be a *Morse function*, meaning every critical point of f is nondegenerate. The Morse index $ind_f(p)$ of a critical point p is defined as the dimension of the largest subspace on which the Hessian H_p^f of f at p is negative definite. Moreover, because M is closed, the set of critical points for any Morse function on M is finite. Fix a Riemannian metric g on M. The identity $g(\nabla f, \cdot) = df(\cdot)$ uniquely determines a gradient vector field ∇f on M. Denote the flow associated to $-\nabla f$ by φ_f, which is called the negative gradient flow. Note that the singularities of t $-\nabla f$ correspond to the critical points of f.

Unless stated otherwise, f is assumed to satisfy the Morse-Smale transversality condition, the unstable manifolds of φ_f at x, $W^u(x)$, and the stable manifold of φ_f at y, $W^s(y)$, intersect transversely, for all $x, y \in Crit(f)$.

Given $x, y \in Crit(f)$, the *connecting manifold* of x and y is $\mathcal{M}_{xy} := W^u(x) \cap W^s(y)$. The *moduli space* between x and y is given by $\mathcal{M}_y^x := \mathcal{M}_{xy} \cap f^{-1}(a)$, where a is a regular value between $f(x)$ and $f(y)$, representing orbits from x to y. For a Morse-Smale function f, these spaces are orientable closed submanifolds of M with dimensions $\dim(\mathcal{M}_{xy}) = ind_f(x) - ind_f(y)$, and $\dim(\mathcal{M}_y^x) = ind_f(x) - ind_f(y) - 1$.

Once orientations are chosen for $W^u(x)$ and $W^u(y)$, these induce an orientation on $\mathcal{M}_{xy}$ denoted by $[\mathcal{M}_{xy}]_{ind}$, for $x, y \in Crit(f)$. The procedure given in [12] to obtain this orientation is:

(1) If $ind_f(y) > 0$, then

1. Let $\mathcal{V}_{\mathcal{M}_{xy}} W^s(y)$ be the normal bundle of $W^s(y)$ restricted to $\mathcal{M}_{xy}$. Consider the fiber $\mathcal{V}_y W^s(y)$ with an orientation given by the isomorphism
$$T_y W^u(y) \oplus T_y W^s(y) \simeq T_y M \simeq \mathcal{V}_y W^s(y) \oplus T_y W^s(y).$$
The orientation on the fiber at y determines an orientation on the normal bundle $\mathcal{V}_{\mathcal{M}_{xy}} W^s(y)$ restricted to the submanifold $\mathcal{M}_{xy}$.
2. The orientation on $\mathcal{M}_{xy}$ is determined by the isomorphism $T_{\mathcal{M}_{xy}} W^u(x) \simeq T\mathcal{M}_{xy} \oplus \mathcal{V}_{\mathcal{M}_{xy}} W^s(y)$.

(2) If $ind_f(y) = 0$, then $\mathcal{V}_y W^s(y) = 0$. Hence, $T_{\mathcal{M}_{xy}} W^u(x) \simeq T\mathcal{M}_{xy}$.

Note that there are no restrictions on the orientability of the manifold M.

Given $x, y \in Crit(f)$ with $ind_f(x) - ind_f(y) = 1$, let $u \in \mathcal{M}_y^x$. The *characteristic sign* n_u of the orbit $\mathcal{O}(u)$ through u is defined via the identity $[\mathcal{O}(u)]_{ind} = n_u[\dot{u}]$, where $[\dot{u}]$ and $[\mathcal{O}(u)]_{ind}$ denote the orientations on $\mathcal{O}(u)$ induced by the flow and by $\mathcal{M}_{xy}$, respectively. The *intersection number* of x and y is defined by

$$n(x, y) = \sum_{u \in \mathcal{M}_y^x} n_u.$$

The intersection number between x and y counts, with sign, the flow lines from x to y.

Fix an arbitrary orientation for the unstable manifolds $W^u(x)$, for each $x \in Crit(f)$, and denote by Or the set of these choices. The *Morse graded group* $C = \{C_k(f)\}$ is defined as the free abelian groups generated by the critical points of f and graded by their Morse index, i.e.,

$$C_k(f) := \bigoplus_{x \in Crit_k(f)} \mathbb{Z}\langle x \rangle,$$

where $\langle x \rangle$ denotes the pair consisting of the critical point x of f and the orientation chosen on $W^u(x)$. The *Morse boundary operator* $\partial_k(x) : C_k(f) \longrightarrow C_{k-1}(f)$ is given on a generator x of $C_k(f)$ by

$$\partial_k\langle x \rangle := \sum_{y \in Crit_{k-1}(f)} n(x, y)\langle y \rangle, \tag{7}$$

and it is extended by linearity to general chains. The pair $(C_*(f), \partial_*)$ is a chain complex, that is, ∂ is of degree -1 and $\partial \circ \partial = 0$. This chain complex is called a *Morse chain complex*.

The *Morse homology groups* with integer coefficients are defined by

$$HM_k(M,f);\mathbb{Z}) = \frac{\ker\ \partial_k}{\operatorname{Im}\ \partial_{k+1}}, \quad \forall k \in \mathbb{Z}.$$

In [12], it was proved that, these homology groups do not depend on the Riemannian metric neither on the choice of orientations. Moreover, the Morse homology of M is isomorphic to the singular homology of M, i.e. $HM_*(M;\mathbb{Z}) \cong H^{sing}(M;\mathbb{Z})$.

5.2.2 Constructing a chain complex for Gutierrez–Sotomayor flows

Let $M \in \mathfrak{M}(\mathcal{GS})$ be a compact singular 2-manifold, $X \in \Sigma_{\mathcal{GS}}(M)$ a GS vector field on M and φ_X the Gutierrez–Sotomayor flow on M associated to X. This section presents the GS chain complex for a given GS flow defined in [4] in an analogous manner to the Morse chain complex of a Morse-Smale flow. First, we recall how to obtain the characteristic signs of the flow lines on M from the characteristic signs of the flow lines on the smooth surface $\widetilde{M}$ coming from a Morsification process. Subsequently, we present the GS chain group and a GS boundary map, as defined in [4].

Denote by $Sing(X)$ the set of singularities of a vector field $X \in \Sigma_{\mathcal{GS}}(M)$.

Definition 15 Let $x, y \in Sing(X)$.

a) The ***connecting manifold*** of x and y is defined as

$$\mathcal{M}_{xy} := W^u(x) \cap W^s(y),$$

where W^s, W^u denoted the stable and unstable sets of the corresponding singularity, respectively.

b) The ***moduli space*** between x and y is defined as the quotient of $\mathcal{M}_{xy}$ by the natural action of $\mathbb{R}$ on the flow lines, i.e.,

$$\mathcal{M}_y^x := \mathcal{M}_{xy}/\mathbb{R}.$$

In other words, the connecting manifold $\mathcal{M}_{xy}$ consists of points $p \in M$ where $\omega(p) = y$ and $\alpha(p) = x$.

Given a singularity $p \in Sing(X)$, we denote by $\eta_k(p)$ the k-th **nature number** of p, where:

$$\begin{cases} k = 2, \text{ represents the repelling nature } r; \\ k = 1, \text{ represents the saddle nature } s; \\ k = 0, \text{ represents the attracting nature } a. \end{cases}$$

One says that two singularities x and y are ***consecutive*** if $\eta_k(x)$ and $\eta_{k-1}(y)$ are both non zero, for some $k = 1, 2$. For example, if p is a triple crossing singularity of ssr nature, one has $\eta_2(p) = 1$, $\eta_1(p) = 2$ and $\eta_0(p) = 0$; if p is a saddle of cone nature, then $\eta_2(p) = 0$, $\eta_1(p) = 1$ and $\eta_0(p) = 0$.

Throughout this section, we restrict our attention to GS flows associated to vector fields X in $\Sigma_{\mathcal{GC}}(M)$, $\Sigma_{\mathcal{GW}}(M)$, $\Sigma_{\mathcal{GD}}(M)$ and $\Sigma_{\mathcal{GT}}(M)$.

We now outline the method for transferring characteristic signs from the Morse framework to the GS framework presented in [4].

(1) GS Characteristic signs of flows lines of $\Sigma_{\mathcal{GC}}(M)$. Consider $x, y \in Sing(X)$ singularities of consecutive natures, where $X \in \Sigma_{\mathcal{GC}}(M)$. Given a flow line $u \in \mathcal{M}_{xy}$, if it has exactly two corresponding orbits $\tilde{u}, \tilde{u}'$ in the Morsified flow, we define

$$n_u := \begin{cases} n_{\tilde{u}}, \text{ if } n_{\tilde{u}} = n_{\tilde{u}'} \\ 0, \text{ if } n_{\tilde{u}} \neq n_{\tilde{u}'} \end{cases}$$

Otherwise, we define $n_u := n_{\tilde{u}}$.

(2) GS Characteristic signs of flows lines of $\Sigma_{\mathcal{GW}}(M)$. Consider $x, y \in Sing(X)$ singularities of consecutive natures and $X \in \Sigma_{\mathcal{GW}}(M)$. Given a flow line $u \in \mathcal{M}_{xy}$, if u belongs to the singular part of M, define $n_u = 0$; if u does not belong to the singular part of M, define $n_u = n_{\tilde{u}}$, where $\tilde{u}$ is the corresponding orbits in the Morsified flow.

(3) GS Characteristic signs of flows lines of $\Sigma_{\mathcal{GD}}(M)$. Consider $x, y \in Sing(X)$ singularities of consecutive natures and $X \in \Sigma_{\mathcal{GD}}(M)$. Given a flow line $u \in \mathcal{M}_{xy}$,

if u does not belong to the singular part of M, define $n_u = n_{\tilde{u}}$; if u belongs to the singular part of M, define n_u to be the pair $n_u = (n^e_u, n^i_u) := (n_{\tilde{u}^e}, n_{\tilde{u}^i})$, where $n_{\tilde{u}^e}$ and $n_{\tilde{u}^i}$ are the characteristic signs of the flow lines $\tilde{u}^e$ and $\tilde{u}^i$, corresponding to u in the Morsified flow.

(4) Characteristic signs of flows lines of $\Sigma_{\mathcal{GT}}(M)$. Consider $x, y \in Sing(X)$ as singularities of consecutive natures and $X \in \Sigma_{\mathcal{GT}}(M)$. Given a flow line $u \in \mathcal{M}_{xy}$, if u does not belong to the singular part of M, define $n_u = n_{\tilde{u}}$; if u belongs to the singular part of M, define n_u to be the triple $n_u = (n^e_u, n^m_u, n^i_u) := (n_{\tilde{u}^e}, n_{\tilde{u}^m}, n_{\tilde{u}^i})$, where $n_{\tilde{u}^e}$, $n_{\tilde{u}^m}$ and $n_{\tilde{u}^i}$ are the characteristic signs of the flow lines $\tilde{u}^e$, $\tilde{u}^m$ and $\tilde{u}^i$ corresponding to u in the Morsified flow.

Once the characteristic signs are well defined for the flow lines of φ_X with $X \in \Sigma_{\mathcal{GS}}(M)$, the ***GS intersection number*** between consecutive singularities x and y can now be defined.

- (Cone and Whitney case) If x, y are consecutive singularities of a vector field $X \in \Sigma_{\mathcal{GS}}(M)$, where $\mathcal{S} = \mathcal{C}, \mathcal{W}$, then

$$n(x, y) := \sum_{u \in \mathcal{M}^x_y} n_u.$$

- (Double crossing case) If x, y are consecutive singularities of a vector field $X \in \Sigma_{\mathcal{GD}}(M)$, then the intersection number between the generator $h^j_k(x)$ and $h^\ell_{k-1}(y)$ is defined as

$$n(h^j_k(x), h^\ell_{k-1}(y)) = \sum n_{\tilde{u}},$$

where the sum is over the flow lines $\tilde{u} \in \widetilde{\mathcal{M}}^{h^j_k(x)}_{h^\ell_{k-1}(y)}$, for $k = 1, 2$, $j = 1, \ldots, \eta_k(x)$ and $\ell = 1, \ldots, \eta_{k-1}(y)$.
- (Triple crossing case) If x, y are consecutive singularities of a vector field $X \in \Sigma_{\mathcal{GT}}(M)$, then the intersection number between the generators $h^j_k(x)$ and $h^\ell_{k-1}(y)$ as

$$n(h^j_k(x), h^\ell_{k-1}(y)) = \sum n_{\tilde{u}},$$

where the sum is over flow lines $\tilde{u} \in \widetilde{\mathcal{M}}^{h^j_k(x)}_{h^\ell_{k-1}(y)}$, for $k = 1, 2$, $j = 1, \ldots, \eta_k(x)$ and $\ell = 1, \ldots, \eta_{k-1}(y)$.

As previously, given a GS singularity x, one can consider a set

$$\{h^i_k(x) \mid \; i = 1, \ldots, \eta_k(x), k = 0, 1, 2\},$$

where $h^i_k(x)$ represents a generator of the k-nature of x. Note that if x is a singularity of type $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{W}$, then there is only one generator associated to x, which can be denoted by x, as well.

Definition 16 Given a GS flow φ_X, where $X \in \Sigma_{\mathcal{GC}}(M)$, the **Gutierrez–Sotomayor chain group** $C_k^{\mathcal{GS}}(M, X)$ with integer coefficients graded by the nature of the singularities is the free abelian group generated by the set of GS singularities $Sing(X)$ of the vector field X, i.e.:

$$C_k^{\mathcal{GS}}(M, X) := \bigoplus_{x \in Sing(X)} \left(\bigoplus_{i=1}^{\eta_k(x)} \mathbb{Z}\langle h_k^i(x) \rangle \right), \quad k \in \mathbb{Z},$$

where $h_k^i(x)$ denotes a generator associated to the k-nature of the singularity x. The k-th **Gutierrez–Sotomayor boundary map**, $\Delta_k^{\mathcal{GS}} : C_k^{\mathcal{GS}}(M, X) \to C_{k-1}^{\mathcal{GS}}(M, X)$, is given on a generator $h_k^i(x)$ by

$$\Delta_k^{\mathcal{GS}} \langle h_k^i(x) \rangle := \sum_{y \in Sing(X)} \left(\sum_{j=1}^{\eta_{k-1}(y)} n(h_k^i(x), h_{k-1}^j(y)) \langle h_{k-1}^j(y) \rangle \right),$$

and it is extended by linearity to general chains.

In [4] it was proven that the pair $(C_*^{\mathcal{GS}}(M, X), \Delta_*^{\mathcal{GS}})$ is a chain complex which we refer to as a ***Gutierrez–Sotomayor chain complex***.

5.2.3 Examples

Example 9 Consider the GS flow φ_X on a singular manifold M and its Morsification $(\widetilde{M}, \varphi_{\widetilde{X}})$ as in Fig. 25, where the characteristic sign transfer process is illustrated. The GS characteristic signs on the orbits of φ_X are obtained from this choice as shown in Fig. 25.

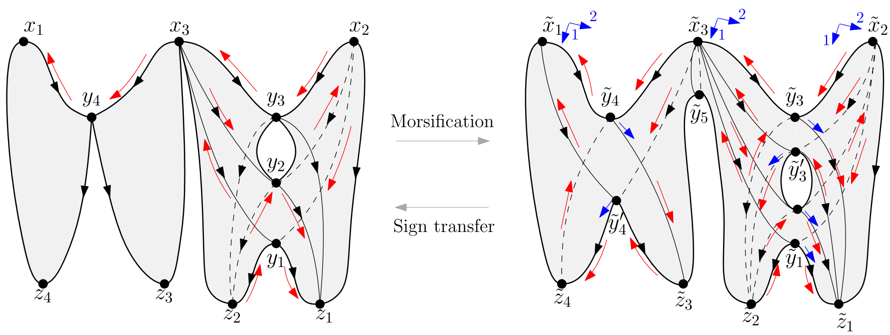


Fig. 25 Sign transfer process in a GS flow with cone singularities

The GS chain groups are given by:

$$\begin{cases} C_2^{\mathcal{GC}}(M,X) = \mathbb{Z}\langle x_1\rangle \oplus \mathbb{Z}\langle x_2\rangle \oplus \mathbb{Z}\langle x_3\rangle \\ C_1^{\mathcal{GC}}(M,X) = \mathbb{Z}\langle y_1\rangle \oplus \mathbb{Z}\langle y_2\rangle \oplus \mathbb{Z}\langle y_3\rangle \oplus \mathbb{Z}\langle y_4\rangle \\ C_0^{\mathcal{GC}}(M,X) = \mathbb{Z}\langle z_1\rangle \oplus \mathbb{Z}\langle z_2\rangle \oplus \mathbb{Z}\langle z_3\rangle \oplus \mathbb{Z}\langle z_4\rangle \\ C_k^{\mathcal{GC}}(M) = 0, k \neq 0,1,2. \end{cases}$$

The GS boundary maps $\Delta_2^{\mathcal{GC}} : C_2 \to C_1$, $\Delta_1^{\mathcal{GC}} : C_1 \to C_0$ and $\Delta_0^{\mathcal{GC}} : C_0 \to 0$ are well-defined by their values on the generators, as follows: $\Delta_2^{\mathcal{GS}}(x_1) = -\langle y_4\rangle$, $\Delta_2^{\mathcal{GS}}(x_2) = \langle y_1\rangle - \langle y_2\rangle + \langle y_3\rangle$, $\Delta_2^{\mathcal{GC}}(x_3) = -\langle y_1\rangle + \langle y_2\rangle - \langle y_3\rangle + \langle y_4\rangle$ $\Delta_1^{\mathcal{GC}}(y_1) = \langle z_1\rangle - \langle z_2\rangle$, $\Delta_1^{\mathcal{GC}}(y_2) = \langle z_1\rangle - \langle z_2\rangle$, $\Delta_1^{\mathcal{GC}}(y_3) = \Delta_1^{\mathcal{GC}}(y_4) = 0$, $\Delta_0^{\mathcal{GC}}(z_i) = 0$, para todo $i = 1, \ldots, 4$.

Example 10 Consider a GS flow φ_X defined on a singular manifold $M \in \mathfrak{M}(\mathcal{GW})$ and its Morsification $(\widetilde{M}, \varphi_{\widetilde{X}})$, as in Fig. 26. Note that z_2, y_1, x_1, x_3 are regular singularities and z_1, y_2, y_3, x_2 are Whitney singularities. Considering a choice of orientations on the unstable manifolds of the critical points of $\widetilde{M}$, the GS characteristic signs on the orbits of φ_X are obtained.

The GS chain groups are given by:

$$\begin{cases} C_2^{\mathcal{GW}}(M) = \mathbb{Z}\langle x_1\rangle \oplus \mathbb{Z}\langle x_2\rangle \oplus \mathbb{Z}\langle x_3\rangle \\ C_1^{\mathcal{GW}}(M) = \mathbb{Z}\langle y_1\rangle \oplus \mathbb{Z}\langle y_2\rangle \oplus \mathbb{Z}\langle y_3\rangle \\ C_0^{\mathcal{GW}}(M) = \mathbb{Z}\langle z_1\rangle \oplus \mathbb{Z}\langle z_2\rangle \\ C_k^{\mathcal{GW}}(M) = 0, \quad k \neq 0,1,2 \end{cases}$$

The GS boundary maps $\Delta_2^{\mathcal{GW}} : C_2 \to C_1$, $\Delta_1^{\mathcal{GW}} : C_1 \to C_0$ and $\Delta_0^{\mathcal{GW}} : C_0 \to 0$ are well-defined by their values on the generators, as follows: $\Delta_2^{\mathcal{GW}}(x_1) = -\langle y_2\rangle$,

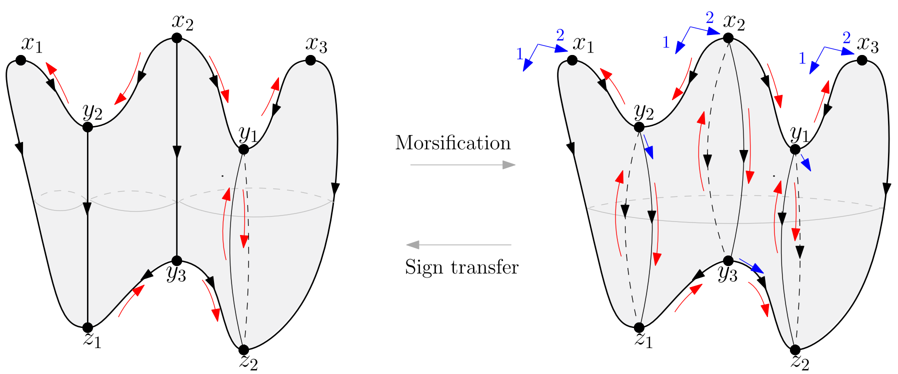


Fig. 26 Sign transfer process in a GS flow with Whitney singularities

$\Delta_2^{\mathcal{GW}}(x_2) = \langle y_1 \rangle + \langle y_2 \rangle$, $\quad \Delta_2^{\mathcal{GW}}(x_3) = -\langle y_1 \rangle$, $\quad \Delta_1^{\mathcal{GW}}(y_1) = 0$, $\quad \Delta_1^{\mathcal{GW}}(y_2) = 0$, $\Delta_1^{\mathcal{GW}}(y_3) = \langle z_2 \rangle - \langle z_1 \rangle$.

Example 11 Consider a GS flow φ_X defined on a singular manifold M and its Morsification $(\widetilde{M}, \varphi_{\widetilde{X}})$ as in Fig. 27. Considering a choice of orientations on the unstable manifolds of the critical points of $\widetilde{M}$, the GS characteristic signs on the orbits of $(\widetilde{M}, \varphi_{\widetilde{X}})$ are obtained. Hence, one gets the GS characteristic signs of the orbits in φ_X.

The groups of the GS chain complex are given by

$$\begin{cases} C_2^{\mathcal{GD}}(M) = \bigoplus_{i=1}^{5} \mathbb{Z}\langle x_i \rangle \oplus \mathbb{Z}\langle y_1^2 \rangle \\ C_1^{\mathcal{GD}}(M) = \mathbb{Z}\langle y_1^1 \rangle \oplus \mathbb{Z}\langle y_2^1 \rangle \oplus \mathbb{Z}\langle y_2^2 \rangle \oplus \mathbb{Z}\langle y_3 \rangle \\ C_0^{\mathcal{GD}}(M) = \mathbb{Z}\langle z_1^1 \rangle \oplus \mathbb{Z}\langle z_1^2 \rangle \oplus \mathbb{Z}\langle z_2^1 \rangle \oplus \mathbb{Z}\langle z_2^2 \rangle \\ C_k^{\mathcal{GD}}(M) = 0, \quad k \neq 0, 1, 2 \end{cases}$$

Therefore, the GS boundary operator $\Delta_2^{\mathcal{GD}} : C_2 \to C_1$, $\Delta_1^{\mathcal{GD}} : C_1 \to C_0$ and $\Delta_0^{\mathcal{GD}} : C_0 \to 0$ are defined on the generators as follows: $\Delta_2^{\mathcal{GD}}(x_1) = -\langle y_2^1 \rangle$, $\Delta_2^{\mathcal{GD}}(x_2) = -\langle y_1^1 \rangle + \langle y_2^1 \rangle$, $\Delta_2^{\mathcal{GD}}(x_3) = -\langle y_1^1 \rangle$, $\Delta_2^{\mathcal{GD}}(x_4) = -\langle y_2^2 \rangle$, $\Delta_2^{\mathcal{GD}}(x_5) = -\langle y_2^2 \rangle$, $\Delta_1^{\mathcal{GD}}(y_3) = \langle z_1^1 \rangle - \langle z_2^1 \rangle, \Delta_1^{\mathcal{GD}}(y_1^1) = \Delta_1^{\mathcal{GD}}(y_2^1) = \Delta_1^{\mathcal{GD}}(y_2^2) = 0, \Delta_0^{\mathcal{GD}}(z_1) = \Delta_0^{\mathcal{GD}}(z_2) = 0$.

6 The dynamics of homotopical cancellations via spectral sequences analysis

When considering the global aspect of cancellations of singularities within a closed manifold, it entails exploring a series of homotopical deformations of the phase space. These deformations may trigger both the death and birth of singularities, along with the formation of new connecting orbits. Alternatively, it can be viewed as a continuation of a flow-a one-parameter family- that undergoes cancellations as the parameter evolves. Regardless of how these cancellations unfold within the manifold, they often set off a chain reaction. However, the crucial question arises: is

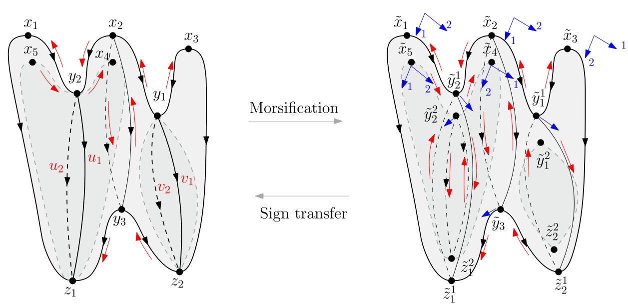


Fig. 27 Sign transfer process in a GS flow with cone singularities double crossing

the final outcome independent of the pairing and order in which cancellations take place?

We aim to delve into a more specific form of cancellation intricately linked with a spectral sequence derived from a filtered chain complex associated with the flow. While numerous results exist in the smooth case, [8–10], the singular setting has been relatively unexplored until [4], which marks the first instance of addressing such a question within this context, namely Gutierrez–Sotomayor singular flows.

The primary contribution of this line of work lies in the introduction of a generalization of these concepts, resulting in the proof of several global homotopical cancellation theorems for GS flows. What we find most remarkable in these theorems is the synchronization between the dynamical homotopical cancellations within the flow on a singular closed surface and the algebraic cancellations observed in the unfolding of the associated spectral sequence-akin to the turning of pages.

In classical cancellation, the manifold before and after the cancellation always remain homeomorphic. However, in homotopical cancellation, the singular manifold before and after the cancellation retain the same homotopy type, but they may not necessarily be homeomorphic.

In this study, we focus exclusively on GS flows characterized by a single type of non-regular singularity. These flows can take the form of a GS cone flow, GS cross cap flow, double point GS flow, or triple point GS flow. Regular singularities are allowed to coexist within any of these classes.

In essence, the concept behind a dynamical homotopical cancellation involves considering a set of three singularities x, x', and y, along with the flow lines u and u' connecting them within a neighborhood U. Through a homotopy, this neighborhood U is transformed into another neighborhood $\bar{U}$ containing a GS singularity $\overline{x}'$. Importantly, the regions U and $\bar{U}$ share the same homotopy type, and this homotopy preserves the number of singular regions (droplets and folds) present in U (Fig. 28).

In this study, we delve into homotopical cancellations occurring within an isolating block housing the maximal invariant set consisting of three GS singularities: one saddle and two attractors (or repellers), along with their connecting orbits. Through this homotopy, a super attractor (or super repeller) singularity emerges. See Fig. 29.

To ensure the well-definedness of these homotopies, in [4] extended their consideration to a broader class of GS singularities, encompassing n-sheet cones, Whitney, double, and triple attractors and repellers. For the sake of simplicity,

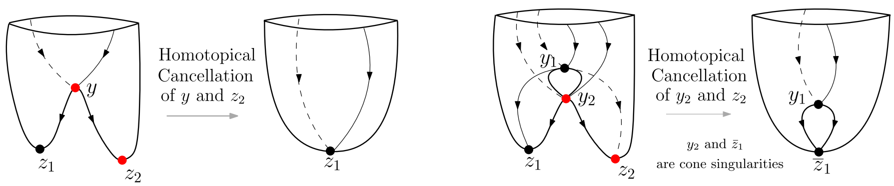


Fig. 28 Dynamical homotopical cancellations on a smooth (left) and on a singular (right) manifold

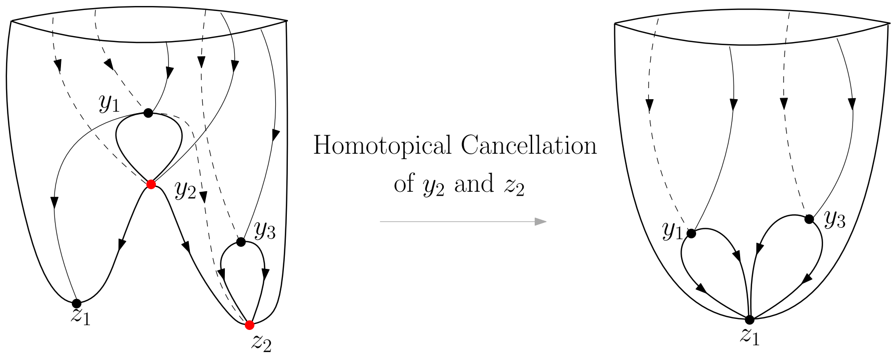


Fig. 29 Homotopical cancellation of a saddle cone and an attracting cone singularities

moving forward, we'll continue to designate these as GS singularities in the context of GS flows.

6.1 Super attractors and super repellers

Let $D^A \subseteq \mathbb{R}^2$ ($D^R \subseteq \mathbb{R}^2$) be the unit disc of center $p = (0, 0)$ and X the attracting radial (resp., repelling) vector field on D with attracting (resp., repelling) singularity p.

Definition 17 A generalized GS singularity p is a ***super attractor*** (resp., ***super repeller***) of type:

a) n-sheet cone of attracting (resp. repelling) nature when obtained by identifying the center points p_i of n discs D_i, $i = 2, \ldots, n$, where D_i has defined on it an attracting (resp., repelling) radial vector field.
b) n-sheet Whitney of attracting (resp., repelling) nature when obtained by identifying the center points p_i and some radii of the n discs D_i^A (resp., D_i^R), $i = 1, \ldots, n$, where D_i^A (resp., D_i^R) has defined on it an attracting (resp. repelling) radial vector field. Moreover, $n - 2$ discs D_i^A (resp., D_i^R) have the property that exactly two radii

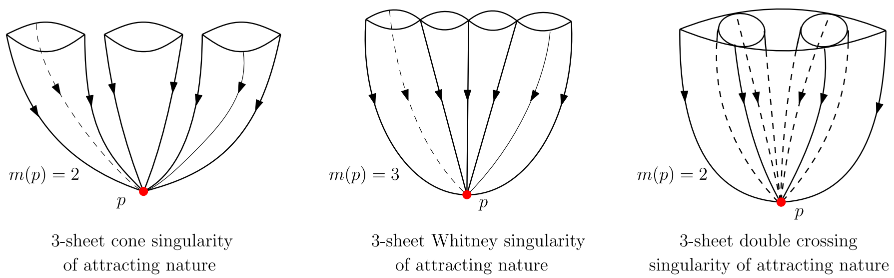


Fig. 30 Super attractor GS singularities

are identified to raddi of two distinct discs. The remaining discs have the property that exactly one radius is identified to a radius of another disc. See Fig. 30.

c) n-sheet double crossing of attracting (resp. repelling) nature, $n = 2, 3, \ldots$, when obtained by identifying the center points p_i of n discs D_i^A, $i = 0, \ldots, n-1$, where each D_i^A (resp., D_i^R) is defined as above. Moreover, we identify exactly one diameter of each disc D_i, $i = 1, \ldots, n-1$ to distinct diameters d_i of the disc D_0, i.e. $D_i \cap D_j \setminus \{p\} = \emptyset$, and $D_i \cap D_0 = d_i$, $i \neq j, i,j = 1, \ldots n-1$. See Fig. 30.

d) n-sheet triple crossing of attracting (resp., repelling) nature, $n = 2k + 1$, when obtained by identifying the center points p_i of n discs D_0, D_i^1, D_i^2, $i = 1, \ldots, n$, where each disc is defined as above. Moreover, consider the sets of distinct diameters $\{d_{0,i}^1, d_{0,i}^2\}$ in D_0, $\{d_i^1, \partial_i^1\}$ in D_i^1 and $\{d_i^2, \partial_i^2\}$ in D_i^2, $i = 1, \ldots, n$. We identify the diameters ∂_i^1 and ∂_i^2, the diameters d_i^1 and $d_{0,i}^1$, and the diameters d_i^2 and $d_{0,i}^2$, so that all of discs D_i^1, D_i^2 are pairwise disjoint, i.e., $(D_i^1 \cup D_i^2) \cap (D_j^1 \cup D_j^2) = \emptyset$, $i \neq j, j = 1, \ldots n$. See Fig. 30.

For an n-sheet generalized GS singularity p, one defines the ***singularity type number*** $m(p)$ as follows:

$$\begin{cases} n-1, & \text{if } p \text{ is of } \mathcal{C}\text{-type or } \mathcal{D}\text{-type} \\ n, & \text{if } p \text{ is of } \mathcal{W}\text{-type} \\ k, & \text{if } p \text{ is of } \mathcal{T}\text{-type where } n = 2k+1 \end{cases}$$

It's noteworthy that a $\mathcal{C}$-type (resp., $\mathcal{W}$, $\mathcal{D}$, $\mathcal{T}$-type) singularity of saddle nature has a type number equal to 1. Additionally, a regular singularity always has a type number equal to zero.

Definition 18 Let $p \in M$ be a super attractor or a super repeller singularity. Denote its **nature** by a (resp., r) if p is an attracting (resp., repelling) n-sheet cone or Whitney; a^n (resp., r^n) if p is an attracting (resp., repelling) n-sheet double or triple crossing.

Definition 19 Denote by $\mathfrak{M}(\mathcal{GS})$ the set of two-dimensional singular manifold with generalized GS singularities. Given $M \in \mathfrak{M}(\mathcal{GS})$, define the set $\Sigma_{\mathcal{GS}}^r(M)$ of generalized GS vector fields on M so that for each $X \in \Sigma_{\mathcal{GS}}^r(M)$ the following conditions are satisfied:

1. X has finitely many generalized GS singularities;
2. X has no periodic orbits nor limit cycles;
3. The α and ω- limit set of every trajectory of X is a generalized GS singularity;
4. There are no saddle connections in the regular part of M.

The corresponding flow φ_X associated to a GS vector field $X \in \Sigma_{\mathcal{GS}}^r(M)$ is called a ***generalized Gutierrez–Sotomayor flow*** on M, or ***generalized GS flow*** for short.

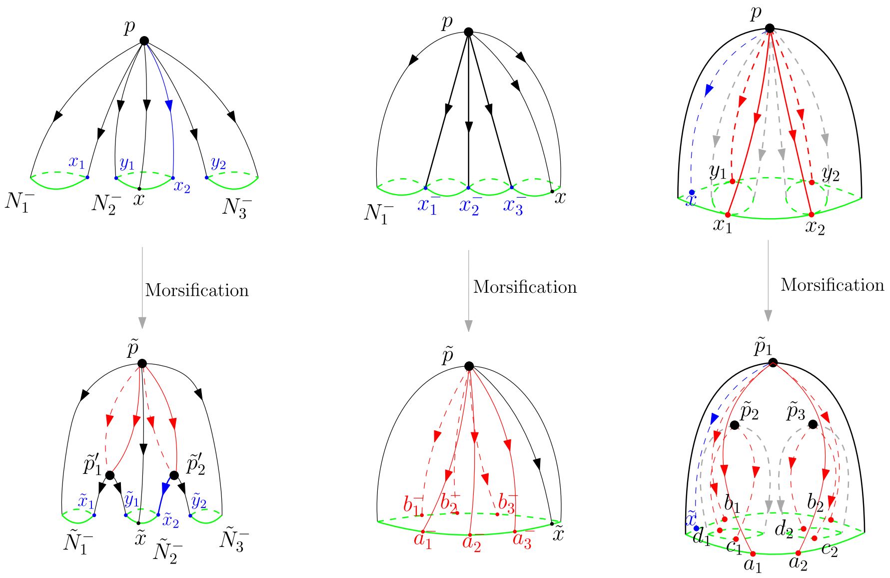


Fig. 31 Some isolating blocks for super repellers and their Morsifications

It is easy to see that the Morsification process can be naturally extended to the super attractors and super repellers GS singularities, see Fig. 31. The GS chain complex can be naturally extended to encompass these super attractor and super repeller singularities, for more details see [4].

6.2 Dynamical homotopical cancellations of GS singularities

In this section we address a homotopical cancellation theorem for consecutive singularities of a Gutierrez–Sotomayor vector field, first announced in [4].

In the homotopical cancellation process of singularities in $\mathcal{S} = \mathcal{C}, \mathcal{W}, \mathcal{D}$ or $\mathcal{T}$, three singularities in $\mathcal{S}$, one of saddle nature y and two of attracting (resp., repelling) nature z_1, z_2 (resp., x_1, x_2), give rise to a new singularity of attracting nature $\bar{z}$ (resp., repelling nature $\bar{x}$). Droplets or folds associated to these singularities are topological invariants and are registered in the singularity type number. After the homotopical cancellation of y and z_i (resp., x_i), $i = 1$ or 2, $\bar{z}$ is the new singularity (resp., $\bar{x}$) and the type number of $\bar{z}$ is related to the types numbers $m(y)$, $m(z_1)$ and $m(z_2)$ (resp. $m(y)$, $m(x_1)$ and $m(x_2)$) of y, z_1 and z_2 (resp., y, x_1 and x_2). We say that $\bar{z}$ (resp., $\bar{x}$) ***inherits the type numbers*** $m(y)$, $m(z_1)$ and $m(z_2)$ as follows:

$$m(\bar{z}) = m(y) + m(z_1) + m(z_2) \quad (\text{resp., } m(\bar{x}) = m(y) + m(x_1) + m(x_2))$$

for $\mathcal{S} = \mathcal{C}, \mathcal{W}, \mathcal{D}$ or $\mathcal{T}$. Recall, that the type number of a regular singularity is always zero. Hence, as a consequence, whenever z_1 and z_2 are regular $\bar{z}$ will inherit a type number equal to zero.

For example, in Fig. 29, the three singularities: z_1 a regular type singularity of attracting nature, y_2 a cone type singularity of saddle nature and z_2 a 2-sheet cone type singularity of attracting nature, are involved in the homotopical cancellation process. More specifically, they give rise to $\bar{z}_1$ which inherits the types of z_1, z_2 and y_2, i.e., z_1 contributes zero, y_2 contributes one and z_2 contributes 2, hence $\bar{z}_1$ is a 3-sheet cone type singularity of attracting nature. For more examples, see Sect. 6.4.

Definition 20 Let $X \in \Sigma_{\mathcal{GS}}(M)$ be a Gutierrez–Sotomayor vector field on $M \in \mathfrak{M}(\mathcal{GS})$, for $\mathcal{S} = \mathcal{C}, \mathcal{W}, \mathcal{D}$ or $\mathcal{T}$. Let $p, q \in Sing(X)$ be consecutive singularities of k and $k-1$ nature numbers, respectively. One says that p and q are **dynamically homotopically cancelled** and that together with q' give rise to $\bar{q}'$ if there is a singular 2-manifold M' of the same homotopy type as M and there exists a vector field $X' \in \Sigma_{\mathcal{GS}}(M')$ which is topologically equivalent to X outside of a neighborhood V of the flow lines u_1 and u_2 connecting p with q and p with q'.

Note that p and q together with q' give rise to an n-sheet $\mathcal{S}$-singularity $\bar{q}' \in \Sigma_{\mathcal{GS}}(M')$ which inherits the type numbers of p, q' and q. For example, in Fig. 32, the three singularities: z_1 and z_2 are 2-sheet double crossing type singularities of attracting nature and y is a regular type singularity of saddle nature; are involved in the dynamical homotopical cancellation process. More specifically, they give rise to $\bar{z}_2$ which inherits the types of z_1, z_2 and y, i.e., z_1 contributes 1, y contributes zero and z_2 contributes 1, hence $\bar{z}_2$ is a 3-sheet double crossing type singularity of attracting nature. For more examples, see Sect. 6.4.

If $X \in \Sigma_{\mathcal{GS}}(M)$ is a GS vector field on $M \in \mathfrak{M}(\mathcal{GS})$ with only regular singularities, then the GS homotopical cancellation of consecutive singularities coincides with the notion of cancellation of critical points established by Smale in the Morse setting. This follows, since Morse critical points have a unique generator and hence $q' = \overline{q}'$ and $p' = \overline{p}'$.

Consider $M \in \mathfrak{M}(\mathcal{GS})$ a closed 2-manifold and a GS flow φ_X on M associated to a GS vector field $X \in \Sigma_{\mathcal{GS}}(M)$, where $\mathcal{S} = \mathcal{C}, \mathcal{W}, \mathcal{D}$ or $\mathcal{T}$. Given consecutive singularities x and y, suppose that:

- the GS intersection number $n(x, y)$ is ± 1, when $\mathcal{S} = \mathcal{C}$ or $\mathcal{W}$;
- one of the coordinates of the GS intersection number $n(x, y)$ is equal to ± 1, when $\mathcal{S} = \mathcal{D}$ or $\mathcal{T}$.

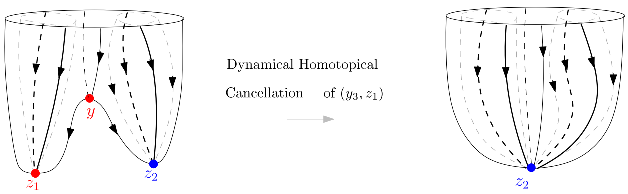


Fig. 32 Dynamical Homotopical Cancellation of the singularities y and z_1

Next, we state two theorems from [4] which guarantee that under these conditions it is always possible to dynamically homotopically cancel the singularities with GS intersection number equal to ± 1.

Theorem 10 (Dynamical Homotopical Cancellation Theorem for GS flows—Cases $\mathcal{C}$ and $\mathcal{W}$) *Let $X \in \Sigma_{\mathcal{GS}}(M)$ be a Gutierrez–Sotomayor vector field on $M \in \mathfrak{M}(\mathcal{GS})$, for $\mathcal{S} = \mathcal{C}$ or $\mathcal{W}$. Let $p, q \in Sing(X)$ be consecutive singularities of k and $k-1$ nature numbers, respectively. If $n(p, q)$ is non zero, then p and q dynamically homotopically cancelled.*

Theorem 11 (*Dynamical Homotopical Cancellation Theorem for GS flows—Cases $\mathcal{D}$ and $\mathcal{T}$*) *Let $X \in \Sigma_{\mathcal{GS}}(M)$ be a Gutierrez–Sotomayor vector field on $M \in \mathfrak{M}(\mathcal{GS})$, for $\mathcal{S} = \mathcal{D}$ or $\mathcal{T}$. Let $p, q \in Sing(X)$ be consecutive singularities, $h_k^j(p)$ and $h_{k-1}^{\ell}(q)$ be the respective consecutive generators of their natures, where $j \in \{1, \ldots, \eta_k(p)\}$ and $\ell \in \{1, \ldots, \eta_{k-1}(q)\}$. If $n(h_k^j(p), h_{k-1}^{\ell}(q))$ is non zero for some j and ℓ, then p and q are dynamically homotopically cancelled.*

6.3 Detecting dynamical homotopical cancellations through spectral sequences

Spectral sequences are a powerful mathematical tool that has been used as a guide for understanding more complex algebraic and geometric configurations, for instance, see [11, 13, 14]. In dynamical systems, spectral sequences has been extensively explored in the works [4, 8–10] as well as in computational topology (see [15, 16]). We aim to explore their application in analyzing the dynamics of gradient-like flows, particularly in identifying orbits that are responsible for the death and birth of connections through the cancellation of critical points.

6.3.1 Spectral sequence

A spectral sequence is a sequence of chain complexes where each chain complex is the homology module of the preceding one. Moroever, as the sequence progresses, it approximates an associated limit module. More specifically, a *k*-***spectral sequence*** E over a principal ideal domain R is a sequence $\{E^r, \partial^r\}_{r \geq k}$, such that

(1) E^r is a bigraded module over R, i.e., an indexed collection of R-modules $E^r_{p,q}$, for all $p, q \in \mathbb{Z}$;
(2) d^r is a differential of bidegree $(-r, r-1)$ on E^r, i.e., an indexed collection of homomorphisms $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$, for all $p, q \in \mathbb{Z}$, and $(d^r)^2 = 0$;
(3) for all $r \geq k$, there exists an isomorphism $H(E^r) \cong E^{r+1}$, where

$$H_{p,q}(E^r) = \frac{\text{Ker } d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}}{\text{Im } d^r : E^r_{p+r,q-r+1} \to E^r_{p,q}}.$$

Define $Z^k_{p,q} = Ker(d^k_{p,q} : E^k_{p,q} \to E^k_{p,q-1})$ and $B^k_{p,q} = Im(d^k_{p,q+1} : E^k_{p,q+1} \to E^k_{p,q})$, then $E^{k+1} = Z^k/B^k$ and

$$B^k \subseteq B^{k+1} \subseteq \cdots \subseteq B^r \subseteq \cdots \subseteq Z^r \subseteq \cdots \subseteq Z^{k+1} \subseteq Z^k.$$

Consider the bigraded modules $Z^\infty = \cap_r Z^r, B^\infty = \cup_r B^r$ and $E^\infty = Z^\infty/B^\infty$. The latter module is called the ***limit of the spectral sequence***. A spectral sequence $E = \{E^r, \partial^r\}$ is ***convergent*** if given p, q there is $r(p,q) \geq k$ such that for all $r \geq r(p,q)$, $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is trivial.

A particular example of spectral sequence is obtained when one considers a chain complex endowed with a convergent and bounded below filtration. An ***increasing filtration*** F on a chain complex (C, ∂) is a sequence of submodules F_pC of C such that $F_pC \subset F_{p+1}C$, for all integer p, and the filtration is compatible with the gradation of C, i.e. F_pC is a chain subcomplex of C consisting of $\{F_pC_q\}$. Such a filtration is ***convergent*** if $\cap_p F_pC = 0$ and $\cup_p F_pC = C$; ***finite*** if there are $p, p' \in \mathbb{Z}$ such that $F_pC = 0$ and $F_{p'}C = C$; ***bounded below*** if for any q there is $p(q)$ such that $F_{p(q)}C_q = 0$.

Given a filtration on C, the associated bigraded module $G(C)$ is defined as

$$G(C)_{p,q} = \frac{F_pC_{p+q}}{F_{p-1}C_{p+q}}.$$

A theorem by Spanier, [17], shows that a convergent and bounded below filtration on a chain complex C leads to a convergent spectral sequence such that

$$E^0_{p,q} = \frac{F_pC_{p+q}}{F_{p-1}C_{p+q}} = G(C)_{p,q} \qquad \text{and} \qquad E^1_{p,q} \approx H_{p+q}\left(\frac{F_pC_{p+q}}{F_{p-1}C_{p+q}}\right),$$

with E^∞ isomorphic to the bigraded module $GH_*(C)$ associated with the induced filtration on $H_*(C)$. This result is proved using algebraic formulas for the modules E^r, which are given by

$$E^r_{p,q} = \frac{Z^r_{p,q}}{Z^{r-1}_{p-1,q+1} + \partial Z^{r-1}_{p+r-1,q-r+2}},$$

where $Z^r_{p,q} = \{c \in F_pC_{p+q} \mid \partial c \in F_{p-r}C_{p+q-1}\}$.

Even though E^∞ doesn't fully determine $H_*(C)$, it does determine the bigraded module $GH_*(C)$, i.e. $E^\infty_{p,q} \cong GH_*(C)_{p,q}$. Also, if $GH_*(C)p, q$ is free and the filtration is bounded, then

$$\bigoplus_{p+q=k} GH_*(C)_{p,q} \cong H_{p+q}(C). \tag{8}$$

In this section, our main motivation is to study how the algebraic cancellations in a spectral sequence of a filtered GS chain complex affects dynamical homotopical cancellations within a GS flow.

To keep track of all modules E^r and differentials d^r of a spectral sequence (E^r, d^r) associated to a finite chain complex over $\mathbb{Z}$ with a finest filtration, in [18] it was introduced an algorithm, called the Spectral Sequence Sweeping Algorithm (SSSA). The SSSA constitutes a more efficient alternative to keep track of all information of the spectral sequence. We refer the reader to [18, 19] for more details on SSSA.

6.3.2 Algebraic cancellation and dynamical homotopical cancellation

Our goal in this section is to present a global homotopical cancellation result for GS flows which follows closely the unfolding of its spectral sequence, first announced in [4].

Theorem 12 (Algebraic Cancellation and Dynamical Homotopical Cancellation via Spectral Sequence)*Let* $(C^{\mathcal{GS}}(M,X), \Delta^{\mathcal{GS}})$ *be the GS chain complex associated to a GS flow* φ_X *on a singular 2-manifold* $M \in \mathfrak{M}(\mathcal{GS})$*, where* $X \in \Sigma_{\mathcal{GS}}(M)$ *and* $\mathcal{S} = \mathcal{C}, \mathcal{W}, \mathcal{D}$ *or* $\mathcal{T}$*. Let* (E^r, d^r) *be the associated spectral sequence for a finest filtration* $F = \{F_p C^{\mathcal{GS}}\}$ *on the chain complex.*

1. *If* $X \in \Sigma_{\mathcal{GC}}(M)$ *or* $X \in \Sigma_{\mathcal{GW}}(M)$*, then the algebraic cancellations of the modules* E^r *of the spectral sequence are in one-to-one correspondence with dynamical homotopical cancellations of the singularities of* X*.*
2. *If* $X \in \Sigma_{\mathcal{GD}}(M)$ *or* $X \in \Sigma_{\mathcal{GT}}(M)$*, then the algebraic cancellations of the modules* E^r *of the spectral sequence are in one-to-one correspondence with dynamical homotopical cancellations of the natures of the singularities of* X*.*

Moreover, the order of homotopical cancellation occurs as the gap r increases with respect to the filtration F.

We want to associate the data of the spectral sequence (E^r, d^r) with a dynamical continuation of the initial flow, by means of homotopical cancellations of the singularities and using as guide the family of matrices $\{\Delta^r\}_{\mathcal{GS}}$ produced by the SSSA, which codifies all data related to the modules and differentials of (E^r, d^r). However, it is easy to see that the matrices $\{\Delta^r\}_{\mathcal{GS}}$ are not in general realized as the GS boundary operator associated to a GS flow. Moreover, the changes of basis caused by pivots in row $j-r$ reflect all the changes in connecting orbits caused by the cancellation of the consecutive generators h_k^j and h_{k-1}^{j-r}. When we cancel the pair of generators h_k^j and h_{k-1}^{j-r}, then

- all the flow lines between the corresponding singularities associated to generators of k-nature and h_{k-1}^{j-r} are immediately removed and new connections are born;
- also all the flow lines between h_k^j and singularities associated to generators of $(k-1)$-nature are immediately removed.

Theorem 13 (Family of GS Flows via Spectral Sequences) *Let* $(C^{\mathcal{GS}}(M,X),\Delta^{\mathcal{GS}})$ *be the GS chain complex associated to a GS flow* φ_X *on a singular* 2*-manifold* $M \in \mathfrak{M}(\mathcal{GS})$, *where* $X \in \Sigma_{\mathcal{GS}}(M)$ *and* $\mathcal{S} = \mathcal{C}, \mathcal{W}, \mathcal{D}$ *or* $\mathcal{T}$. *The RCA for the GS boundary map* $\Delta^{\mathcal{GS}}$ *produces a family of GS flows* $\{\varphi^1 = \varphi_X, \varphi^2, \dots, \varphi^\omega\}$ *where* φ^r *continues to* φ^{r+1} *by cancelling pairs of singularities of* φ^r *having gap* r *with respect to the filtration* F. *Moreover, the flow* φ^ω *is a minimal GS flow in the sense that there is no more possible homotopical cancellations to be done.*

6.4 Examples

To fully appreciate the elegance of these results, we conclude this paper by presenting illustrative examples from the realms of GS flows featuring cone, Whitney, and double crossing singularities.

Example 12 Consider the singular manifold $M \in \mathfrak{M}(\mathcal{GC})$ and the GS flow φ_X associated to a vector field $X \in \Sigma_{\mathcal{GW}}(C)$ as in Fig. 25. The GS chain complex associated with this flow is presented in Example 9.

Consider the finest filtration on the GS chain complex $(C_*^{\mathcal{GC}}(M,X),\Delta_*^{\mathcal{GC}})$, determented bu the following order of critical points:

$$z_1,\ z_2,\ z_3,\ z_4,\ y_1,\ y_2,\ y_3,\ y_4,\ x_1,\ x_2,\ x_3.$$

The spectral sequence associated to $(C_*^{\mathcal{GC}}(M,X),\Delta_*^{\mathcal{GC}})$ enriched with the filtration F is shown in Fig. 33.

Theorem 12 guarantees that the algebraic cancellations that occur in the spectral sequence determine dynamical homotopical cancellations of Gs singularities, as follows: the algebraic cancellation of E_8^1 and E_7^1 determines the dynamical homotopical cancellation of (x_1, y_4); the algebraic cancellation of E_9^2 and E_1^2 determines the dynamical homotopical cancellation of (x_2, y_3); the algebraic cancellation of E_4^2 and E_1^2 determines the dynamical homotopical cancellation of (y_1, z_2).

Figure 34 shows the dynamical homotopical cancellations of the pair of singularities (x_1, y_4), (x_2, y_3) and (y_1, z_2), respectively.

$E^0:$	$\mathbb{Z}[z_1]$	$\mathbb{Z}[z_2]$	$\mathbb{Z}[z_3]$	$\mathbb{Z}[z_4]$	$\mathbb{Z}[y_1]$	$\mathbb{Z}[y_2]$	$\mathbb{Z}[y_3]$	$\mathbb{Z}[y_4]$	$\mathbb{Z}[x_1]$	$\mathbb{Z}[x_2]$	$\mathbb{Z}[x_3]$
								$\xleftarrow{d_8^1}$			
$E^1:$	$\mathbb{Z}[z_1]$	$\mathbb{Z}[z_2]$	$\mathbb{Z}[z_3]$	$\mathbb{Z}[z_4]$	$\mathbb{Z}[y_1]$	$\mathbb{Z}[y_2]$	$\mathbb{Z}[y_3]$	$\mathbb{Z}[y_4]$	$\mathbb{Z}[x_1]$	$\mathbb{Z}[x_2]$	$\mathbb{Z}[x_3]$
$E^2:$	$\mathbb{Z}[z_1]$	$\mathbb{Z}[z_2]$	$\mathbb{Z}[z_3]$	$\mathbb{Z}[z_4]$	$\mathbb{Z}[y_1]$	$\mathbb{Z}[y_2]$	$\mathbb{Z}[y_3]$	0	0	$\mathbb{Z}[x_2]$	$\mathbb{Z}[x_3]$
				$\xleftarrow{d_4^3}$					$\xleftarrow{d_9^3}$		
$E^3:$	$\mathbb{Z}[z_1]$	$\mathbb{Z}[z_2]$	$\mathbb{Z}[z_3]$	$\mathbb{Z}[z_4]$	$\mathbb{Z}[y_1]$	$\mathbb{Z}[y_2]$	$\mathbb{Z}[y_3]$	0	0	$\mathbb{Z}[x_2]$	$\mathbb{Z}[x_3]$
$E^4:$	$\mathbb{Z}[z_1]$	0	$\mathbb{Z}[z_3]$	$\mathbb{Z}[z_4]$	0	$\mathbb{Z}[y_2]$	0	0	0	0	$\mathbb{Z}[x_3]$

Fig. 33 The spectral sequence for $(C_*^{\mathcal{GC}}(M,X),\Delta_*^{\mathcal{GC}})$ with filtration F

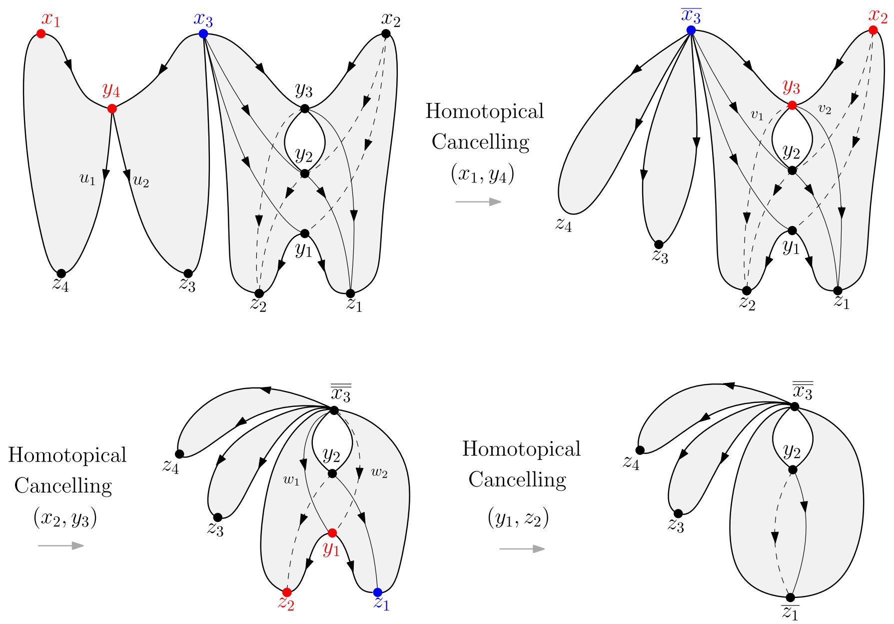


Fig. 34 Homotopical Cancellation of the pairs (x_1, y_4), (x_2, y_3) and (y_1, z_2), sucessively

E^0 :	E^0_0	E^0_1	E^0_2	E^0_3	E^0_4	E^0_5	E^0_6	E^0_7
E^1 :	$\mathbb{Z}[z_1]$	$\mathbb{Z}[z_2]$	$\mathbb{Z}[y_1]$	$\mathbb{Z}[y_2]$	$\mathbb{Z}[y_3]$	$\mathbb{Z}[x_1]$	$\mathbb{Z}[x_2]$	$\mathbb{Z}[x_3]$
E^2 :	$\mathbb{Z}[z_1]$	$\mathbb{Z}[z_2]$	$\mathbb{Z}[y_1]$	$\mathbb{Z}[y_2]$	$\mathbb{Z}[y_3]$	$\mathbb{Z}[x_1]$	$\mathbb{Z}[x_2]$	$\mathbb{Z}[x_3]$
E^3 :	$\mathbb{Z}[z_1]$	$\mathbb{Z}[z_2]$	$\mathbb{Z}[y_1]$	0	$\mathbb{Z}[y_3]$	0	$\mathbb{Z}[x_2]$	$\mathbb{Z}[x_3]$
E^4 :	$\mathbb{Z}[z_1]$	0	$\mathbb{Z}[y_1]$	0	0	0	$\mathbb{Z}[x_2]$	$\mathbb{Z}[x_3]$
E^5 :	$\mathbb{Z}[z_1]$	0	0	0	0	0	0	$\mathbb{Z}[x_3]$

d^2_5 d^3_4 d^4_6

Fig. 35 The spectral sequence for $(C^{\mathcal{GW}}_*(M, X), \Delta^{\mathcal{GW}}_*)$ with filtration F

Example 13 Consider the singular manifold $M \in \mathfrak{M}(\mathcal{GW})$ and the GS flow φ_X associated to a vector field $X \in \Sigma_{\mathcal{GW}}(M)$ as in Fig. 26. The GS chain complex associated to this flow is presented in Example 10. Consider the finest filtration on $(C^{\mathcal{GW}}_*(M, X), \Delta^{\mathcal{GW}}_*)$, induced by the following order on the set of singularities:

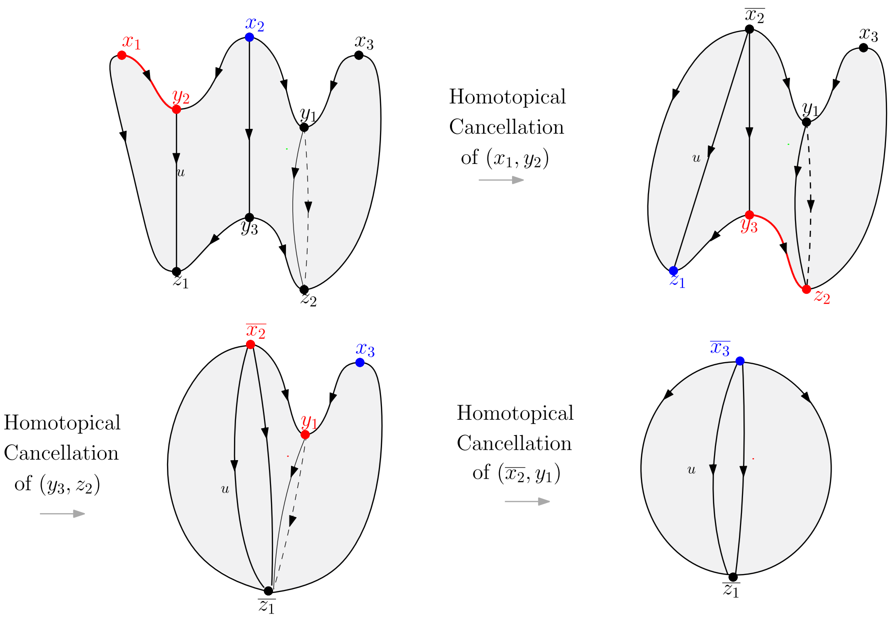


Fig. 36 Homotopical cancellation of the pairs (x_1, y_2), (y_3, z_2) and (x_2, y_1), sucessively

$$z_1,\ z_2,\ y_1,\ y_2,\ y_3,\ x_1,\ x_2,\ x_3.$$

The spectral sequence associated to $(C_*^{\mathcal{GW}}(M, X), \Delta_*^{\mathcal{GW}})$ enriched with the filtration F is shown in Fig. 35.

The algebraic cancellations are associated to dynamical homotopical cancellations as follows: the algebraic cancellation of E_5^1 and E_3^1 determines the dynamical homotopical cancellation of (x_1, y_2); the algebraic cancellation of E_4^3 and E_1^3 determines the dynamical homotopical cancellation of (y_3, z_2); the algebraic cancellation of E_6^4 and E_2^4 determines the dynamical homotopical cancellation of (x_2, y_1).

Figure 36 shows the dynamical cancellations of the pair of singularities (x_1, y_2), (y_3, z_2) and (x_2, y_1), respectively.

Example 14 Consider the singular manifold $M \in \mathfrak{M}(\mathcal{GD})$ and the GS flow φ_X associated to a vector field $X \in \Sigma_{\mathcal{GD}}(M)$ as in Fig. 27. The GS chain complex associated to (M, X) is presented in Example 11.

Consider the finest filtration on $(C_*^{\mathcal{GD}}(M, X), \Delta_*^{\mathcal{GD}})$ induced by the following order on the set of singularities:

$$z_1^e,\ z_1^i,\ z_2^e,\ z_2^i,\ y_1^e,\ y_1^i,\ y_2^e,\ y_2^i,\ y_3,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5.$$

The spectral sequence associated to $(C_*^{\mathcal{GD}}(M, X), \Delta_*^{\mathcal{GD}})$ enriched with the filtration F is shown in Fig. 33.

$E^0, E^1, E^2:$	$\mathbb{Z}[z_1^e]$	$\mathbb{Z}[z_1^i]$	$\mathbb{Z}[z_2^e]$	$\mathbb{Z}[z_e^i]$	$\mathbb{Z}[y_1^e]$	$\mathbb{Z}[y_1^i]$	$\mathbb{Z}[y_2^e]$	$\mathbb{Z}[y_2^i]$	$\mathbb{Z}[y_3]$	$\mathbb{Z}[x_1]$	$\mathbb{Z}[x_2]$	$\mathbb{Z}[x_3]$	$\mathbb{Z}[x_4]$	$\mathbb{Z}[x_5]$
$E^3:$	$\mathbb{Z}[z_1^e]$	$\mathbb{Z}[z_1^i]$	$\mathbb{Z}[z_2^e]$	$\mathbb{Z}[z_e^i]$	$\mathbb{Z}[y_1^e]$	$\mathbb{Z}[y_1^i]$	$\mathbb{Z}[y_2^e]$	$\mathbb{Z}[y_2^i]$	$\mathbb{Z}[y_3]$	$\mathbb{Z}[x_1]$	$\mathbb{Z}[x_2]$	$\mathbb{Z}[x_3]$	$\mathbb{Z}[x_4]$	$\mathbb{Z}[x_5]$
$E^4:$	$\mathbb{Z}[z_1^e]$	$\mathbb{Z}[z_1^i]$	$\mathbb{Z}[z_2^e]$	$\mathbb{Z}[z_e^i]$	$\mathbb{Z}[y_1^e]$	$\mathbb{Z}[y_1^i]$	0	$\mathbb{Z}[y_2^i]$	$\mathbb{Z}[y_3]$	0	$\mathbb{Z}[x_2]$	$\mathbb{Z}[x_3]$	$\mathbb{Z}[x_4]$	$\mathbb{Z}[x_5]$
$E^5:$	$\mathbb{Z}[z_1^e]$	$\mathbb{Z}[z_1^i]$	$\mathbb{Z}[z_2^e]$	$\mathbb{Z}[z_e^i]$	$\mathbb{Z}[y_1^e]$	$\mathbb{Z}[y_1^i]$	0	$\mathbb{Z}[y_2^i]$	$\mathbb{Z}[y_3]$	0	$\mathbb{Z}[x_2]$	$\mathbb{Z}[x_3]$	$\mathbb{Z}[x_4]$	$\mathbb{Z}[x_5]$
$E^6:$	$\mathbb{Z}[z_1^e]$	$\mathbb{Z}[z_1^i]$	$\mathbb{Z}[z_2^e]$	$\mathbb{Z}[z_e^i]$	$\mathbb{Z}[y_1^e]$	$\mathbb{Z}[y_1^i]$	$\mathbb{Z}[y_2^e]$	0	$\mathbb{Z}[y_3]$	$\mathbb{Z}[x_1]$	$\mathbb{Z}[x_2]$	$\mathbb{Z}[x_3]$	0	$\mathbb{Z}[x_5]$
$E^6:$	$\mathbb{Z}[z_1^e]$	$\mathbb{Z}[z_1^i]$	0	$\mathbb{Z}[z_e^i]$	0	$\mathbb{Z}[y_1^i]$	$\mathbb{Z}[y_2^e]$	0	0	$\mathbb{Z}[x_1]$	0	$\mathbb{Z}[x_3]$	0	$\mathbb{Z}[x_5]$

Fig. 37 The spectral sequence for $(C_*^{\mathcal{GD}}(M,X), \Delta_*^{\mathcal{GD}})$ with filtration F

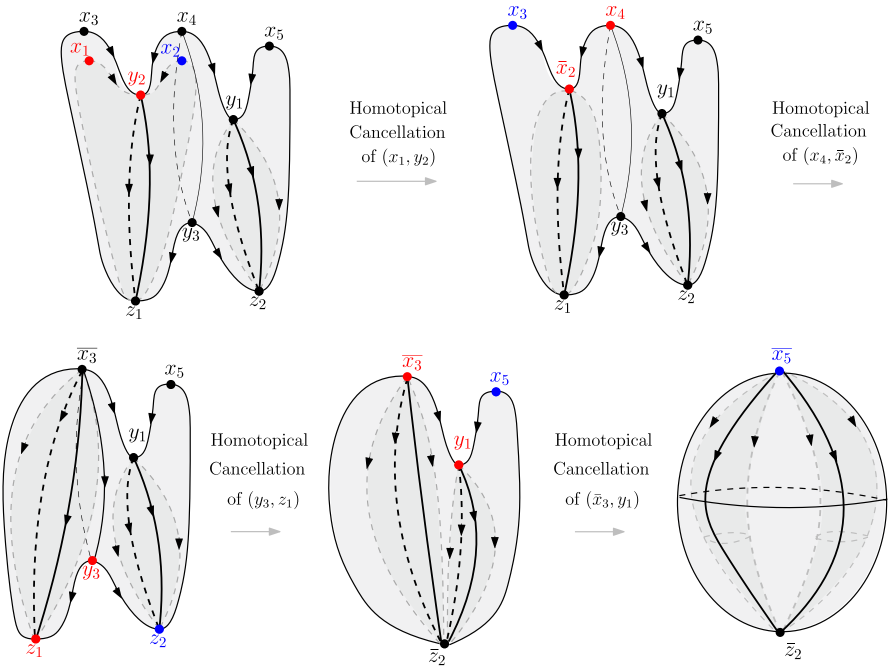


Fig. 38 Homotopical cancellation of the pairs (y_2^e, x_1), (y_2^i, x_4), (z_2^e, y_3) and $(y_1^e, \bar{x}_3)$, sucessively

The algebraic cancellations are associated to dynamical homotopical cancellations as follows: the algebraic cancellation of E_9^3 and E_6^3 determines the dynamical cancellation of (y_2^e, x_1); the algebraic cancellation of E_{12}^5 and E_7^5 determines the dynamical homotopical cancellation of (y_2^i, x_4); the algebraic cancellation of E_8^6 and E_2^6 determines the dynamical homotopical cancellation of (z_2^e, y_3); the algebraic cancellation of E_{10}^6 and E_4^6 determines the dynamical homotopical cancellation of $(y_1^e, \bar{x}_3)$ (Fig. 37).

Figure 38 shows the dynamical cancellation of the pair of singularities (y_2^e, x_1), (y_2^i, x_4), (z_2^e, y_3) and $(y_1^e, \bar{x}_3)$, respectively.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no Conflict of interest.

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