Homework 1.

Question 1.

1. Show that β_0 is unbiased

$$
\begin{aligned}
y_i &= \beta_0 + \beta_1 x_i + \epsilon_i \\
\mathrm{E}(\beta_0) &= \mathrm{E}(y_i - \beta_1 x_i - \epsilon_i) \\
\mathrm{E}(\beta_0) &= \mathrm{E}(y_i) - \beta_1 \mathrm{E}(x_i) - \mathrm{E}(\epsilon_i) \\
\mathrm{E}(\beta_0) &= \bar{y} - \beta_1 \bar{x} - 0 \\
\mathrm{E}(\beta_0) &= \bar{y} - \beta_1 \bar{x} \\
\mathrm{E}(\beta_0) &= \beta_0 \\
\beta_0 &= \bar{y} - \beta_1 \bar{x}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} \\
\mathrm{E}(\hat{\beta}_0) &= \mathrm{E}(\bar{y} - \hat{\beta}_1 \bar{x}) \\
\mathrm{E}(\hat{\beta}_0) &= \mathrm{E}(\bar{y}) - \mathrm{E}(\hat{\beta}_1 \bar{x}) \\
\mathrm{E}(\hat{\beta}_0) &= \bar{y} - \bar{x}\,\mathrm{E}(\hat{\beta}_1) \\
\mathrm{E}(\hat{\beta}_1) &= \beta_1 && \text{(we showed this in class)} \\
\mathrm{E}(\hat{\beta}_0) &= \bar{y} - \beta_1 \bar{x} \\
\mathrm{E}(\hat{\beta}_0) &= \beta_0
\end{aligned}
$$

2. Determine the variance of β_0

$$\operatorname{Var}(X+Y) = \operatorname{Var}(X) + \operatorname{Var}(Y) + 2\operatorname{Cov}(X,Y)$$

$$\begin{aligned}
\hat{\beta}_0 &= \overline{y} - \hat{\beta}_1\overline{x} \qquad\qquad\qquad \text{(from class)}\\
\operatorname{Var}(\hat{\beta}_0) &= \operatorname{Var}(\overline{y} - \hat{\beta}_1\overline{x})\\
\operatorname{Var}(\hat{\beta}_0) &= \operatorname{Var}(\overline{y}) + \operatorname{Var}(\hat{\beta}_1\overline{x}) + 2\operatorname{Cov}(\overline{y}, -\hat{\beta}_1\overline{x})
\end{aligned}$$

$$\begin{aligned}
\overline{y} &= \frac{1}{n}\sum_{i=1}^{n} y_i\\
\operatorname{Var}(\overline{y}) &= \operatorname{Var}(\frac{1}{n}\sum_{i=1}^{n} y_i)\\
\operatorname{Var}(\overline{y}) &= (\frac{1}{n})^2\sum_{i=1}^{n} \operatorname{Var}(y_i)\\
\operatorname{Var}(\overline{y}) &= (\frac{1}{n})^2\sum_{i=1}^{n} \operatorname{Var}(\beta_0 + \beta_1 x_i + \epsilon_i)\\
\operatorname{Var}(\overline{y}) &= (\frac{1}{n})^2\sum_{i=1}^{n} \operatorname{Var}(\beta_0) + \operatorname{Var}(\beta_1 x_i) + \operatorname{Var}(\epsilon_i)\\
\operatorname{Var}(\overline{y}) &= (\frac{1}{n})^2\sum_{i=1}^{n} 0 + 0 + \sigma^2\\
\operatorname{Var}(\overline{y}) &= (\frac{1}{n})^2 . (n\sigma^2)\\
\operatorname{Var}(\overline{y}) &= \frac{\sigma^2}{n}
\end{aligned}$$

$$\begin{aligned}
\operatorname{Var}(-\beta_1\overline{x}) &= (-\overline{x})^2\operatorname{Var}(\hat{\beta}_1)\\
\operatorname{Var}(-\beta_1\overline{x}) &= (-\overline{x})^2\operatorname{Var}(\hat{\beta}_1)\\
\operatorname{Var}(-\beta_1\overline{x}) &= (-\overline{x})^2\frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \overline{x})} \qquad\qquad \text{(we showed this in class)}
\end{aligned}$$

$$\operatorname{Var}(\hat{\beta}_0) = \operatorname{Var}(\overline{y}) + \operatorname{Var}(\hat{\beta}_1\overline{x})$$

$$\boxed{\operatorname{Var}(\hat{\beta}_0) = \frac{\sigma^2}{n} + \overline{x}^2\frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \overline{x})}}$$