Homework 2.

Theory question

Question 1. Derive the normal equations and the $\hat{\beta}$ from the multivariate version of the multiple regression likelihood

$$\frac{\partial \ell(\beta, \sigma^2)}{\partial \beta} = \frac{\partial}{\partial \beta}\left[-\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(Y - X\beta)'(Y - X\beta)\right]$$

Since $-\frac{n}{2}\log(2\pi\sigma^2)$ is constant with respect to β:

$$\frac{\partial \ell}{\partial \beta} = 0 - \frac{1}{2\sigma^2}\frac{\partial}{\partial \beta}\left[(Y - X\beta)'(Y - X\beta)\right]$$

Let $Q = (Y - X\beta)'(Y - X\beta) = Y'Y - 2\beta'X'Y + \beta'X'X\beta$. Then:

$$\frac{\partial Q}{\partial \beta} = -2X'Y + 2X'X\beta$$

Set the derivative to zero:

$$\begin{aligned} \frac{1}{\sigma^2}(2X'Y - 2X'X\beta) &= 0 \\ X'X\beta &= X'Y \\ \hat{\beta}_{MLE} &= (X'X)^{-1}X'Y \end{aligned}$$

Question 2. Demonstrate that $\hat{\beta}$ is an unbiased estimator of β

$$
\begin{aligned}
Y &= X\beta + \epsilon \\
\hat{\beta} &= (X'X)^{-1}X'Y \\
\hat{\beta} &= (X'X)^{-1}X'(X\beta + \epsilon) \\
\hat{\beta} &= (X'X)^{-1}X'(X\beta) + (X'X)^{-1}X'(\epsilon) \\
\hat{\beta} &= (X'X)^{-1}(X'X)\beta + (X'X)^{-1}X'(\epsilon) \\
\hat{\beta} &= \beta + (X'X)^{-1}X'(\epsilon) \\
\mathrm{E}(X) &= \beta + \mathrm{E}((X'X)^{-1}X'(\epsilon)) \\
\mathrm{E}(X) &= \beta + (X'X)^{-1}X'\,\mathrm{E}(\epsilon) \\
\mathrm{E}(\epsilon) &= 0 \qquad\qquad \text{(by assumption)}
\end{aligned}
$$

$$\boxed{\mathrm{E}(X) = \beta}$$

Question 3. Demonstrate that $\hat{\beta}$ follows a multivariate normal distribution

$$
\begin{aligned}
\mathrm{Cov}(\hat{\beta}) &= \mathrm{Cov}((X'X)^{-1}X'\varepsilon) \\
&= (X'X)^{-1}X' \cdot \mathrm{Cov}(\varepsilon) \cdot [(X'X)^{-1}X']' \\
&= (X'X)^{-1}X' \cdot (\sigma^2 I_n) \cdot X(X'X)^{-1} \\
&= (\sigma^2 I_n)(X'X)^{-1}X'X(X'X)^{-1} \\
&= (\sigma^2 I_n)(X'X)^{-1}
\end{aligned}
$$

$$\mathrm{E}(X) = \beta \qquad \mathrm{Cov}(\hat{\beta}) = (\sigma^2 I_n)(X'X)^{-1}$$

A **linear transformation** of a multivariate normal is also a **multivariate normal**, and

$$\hat{\beta} = A + C(\epsilon)$$

$$\hat{\beta} = \beta + (X'X)^{-1}X'\epsilon$$

Therefore

Since $\epsilon \sim N(0, \sigma^2 I_n)$ and $\hat{\beta}$ is a linear transformation of ϵ, it follows that:

$$\hat{\beta} \sim N(\beta, \sigma^2(X'X)^{-1})$$