Finite Sets

Perhaps a good boss-level exercise would be the theorem 2.1.4 from Fischer's Linear Algebra.

Theorem:
Sind X und Y endliche Mengen mit gleich vielen Elementen, so sind für eine Abbildung $f : X \to Y$ folgende Bedingungen äquivalent:
i) f ist injektiv,
ii) f ist surjektiv,
iii) f ist bijektiv.

Remark: This theorem admits different proofs (e.g. proof by contradiction using the pigeonhole principle, proof by induction, etc). These different proofs use different APIs of finite sets.

Union of Subspaces

Prove that the union of two subspaces of a vector space is a subspace if and only if one of the
subspaces is contained in the other.

Remark: This may not be suitable for a boss level exercise but it is a good exercise to understand the definition of a subspace.

Infinite Dimensional Vector Space

1. Show that $\mathbb{Q}^n$ is a finite dimensional vector space over $\mathbb{Q}$.
2. Show that if $\mathbb{R}$ with its standard addition is a vector space over $\mathbb{Q}$ then the scalar multiplication is given by the standard multiplication of real numbers.
3. Show that $\mathbb{R}$ with its standard addition is an infinite dimensional vector space over $\mathbb{Q}$.

Remark: Once of proof of the last part requires uncountability of $\mathbb{R}$. Another proof uses the fact that $\log p_i$ are linearly independent over $\mathbb{Q}$ where p_i are prime numbers.

Proper Subspaces of $n \times n$ Matrices

1. Show that the space of $n \times n$ matrices with real entries is a vector space over $\mathbb{R}$.
2. Suppose A is an $n \times n$ matrix with real entries. Show that the space generated by the powers of A, i.e. the set $\{I, A, A^2, A^3, \ldots\}$ is a proper subspace of the space of $n \times n$ matrices with real entries.
3. Show that a vector space over an infinite field cannot be a finite union of proper subspaces.

The lattice of subspaces of a vector space and the modular law

1. Show that the subspaces of a vector space form a lattice under the inclusion order.
2. Show that the lattice of subspaces of a vector space satisfies the modular law.
3.

Van der Monde Matrix

Fix 100 distinct points $t_0, \ldots, t_{99}$ in the interval $I = [-1, 1]$. Consider the map $L \colon \mathbb{R}^{200} \to \mathbb{R}^{100}$ defined by the assignment

$$c = (c_0, ..., c_{199}) \mapsto (p_c(t_0), ..., p_c(t_{99}))$$

where $p_c = \sum_{i=0}^{i=199} c_i x^i$, i.e. from vectors of coefficients of polynomials of degree ≤ 199 to the vectors $\left(p_c(t_i)\right)_{i=0}^{99}$ of values of such polynomials at nodes t_i.

1. Show that L is linear.
2. Show that this map is represented, upon choosing the standard basis in $\mathbb{R}^{200}$ and $\mathbb{R}^{100}$, by the 100×200 *Vendermonde* matrix.

$$\begin{bmatrix} 1 & t_0 & t_0^2 & \dots & t_0^{199} \\ 1 & t_1 & t_1^2 & \dots & t_1^{199} \\ 1 & t_2 & t_2^2 & \dots & t_2^{199} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_{99} & x_{99}^2 & \dots & t_{99}^{199} \end{bmatrix}$$

3. Show that this map is never invertible.

Quotients

1. Construction of the field $\mathbb{F}_p$ via quotient construction of its underlying cyclic group for a prime number p.
2. (Advanced) (Gaussian Coefficients) Let $\mathbb{F}$ be a finite field of size q and let V be an n-dimensional vector space over $\mathbb{F}$. Show that the number of subspaces of V is given by the Gaussian Coefficients.

TODO: I will break down the question 2 into four separate parts (Sina.)

Trace

1. Show that the trace of a matrix, the sum of its diagonal entries, is a linear map from the space $n \times n$ matrices to the field of scalars.
2. Show that $\operatorname{tr}(AA^T) \geq 0$ and the equality holds if and only if $A = 0$.
3. Suppose f is a linear transformation over the space of $n \times n$ matrices such that $f(AB) = f(BA)$ for all A, B. Show that there exists a scalar c such that $f(A) = c\operatorname{tr}(A)$ for all A.
4. Show that for any matrix A the map $\operatorname{tr}(A\,\bullet) : X \mapsto \operatorname{tr}(AX)$ is a linear map on the space of $n \times n$ matrices.
5. Show that the map above is a zero map if and only if $A = 0$.
6. Show that the map $A \mapsto \operatorname{tr}(A\,\bullet)$ is an isomorphism.
7. Use (4) to prove that every linear map on the space of $n \times n$ matrices is of the form $\operatorname{tr}(A\,\bullet)$ for some matrix A.
8. Suppose A is an $m \times n$ and B is an $n \times m$ matrix. Show that the trace of AB is the same as the trace of BA.
9. Show that the trace of a matrix is the sum of its eigenvalues.

Singular Value Decomposition (?)