[Marcus: Let's try to have one title per planet, so we keep track of how many planets we're planning here. I've marked topics and questions that I consider particular good or essential with a + or even a ++.]

Bijections between finite sets (++)

Boss level (theorem 2.1.4 from Fischer's Linear Algebra):
Sind X und Y endliche Mengen mit gleich vielen Elementen, so sind für eine Abbildung $f : X \to Y$ folgende Bedingungen äquivalent:
i) f ist injektiv,
ii) f ist surjektiv,
iii) f ist bijektiv.

Remark: This theorem admits different proofs (e.g. proof by contradiction using the pigeonhole principle, proof by induction, etc). These different proofs use different APIs of finite sets.

Remark: use the tactic `TFAE` to prove the equivalence of the conditions. (if it's nice to use)

Isomorphisms between finite-dimensional vector spaces (++)

Follow-up planet to the planet "bijections between finite stes".

The lattice of subspaces 1 (++)

- Introduce lattices.
- Introduce the lattice of subspaces of a vector space under the inclusion order.

Boss level, option 1a: Show that the abstract definition of `U₁ ⊔ U₂` agrees with the more concrete description as `span (U₁ \cup U₂)`.
Boss level, option 1b: Prove that the union of two subspaces of a vector space is a subspace if and only if one of the
subspaces is contained in the other.

The lattice of subspaces 2 (+)

Boss level, option 2a: Show that the lattice of subspaces of a vector space satisfies the modular law.
Boss level, option 2b: When does a collection of subspaces form a decomposition of the entire space? (3 subspaces or general version with n subspaces)

Marcus: I would be happy with any one of the Boss levels in these lattice planets. I imagine that if you pick, say, 1b as a Boss level, you could still include 1a as a walk-through exercise that leads up to that Boss level. Similarly for 2a/2b.

Vector spaces of infinite fields

Follow-up to planets on the lattice of subspaces.

Boss level: Show that a vector space over an infinite field cannot be a finite union of proper subspaces.

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example {V : Type} [Module k V] (U W : submodule k V) : U ⊔ W = ⊤ ↔ U ≤ W ∨ W ≤ U :=
```

The reals over the rationals -- abstract version (+)

Walk-through: Show that $\mathbb{Q}^n$ is a finite dimensional vector space over $\mathbb{Q}$.
Boss level: Show that $\mathbb{R}$ with its standard addition is an infinite dimensional vector space over $\mathbb{Q}$.

Remark: Once of proof of the last part requires uncountability of $\mathbb{R}$: all finite linear combinations of rational is not going to span all of $\mathbb{R}$ because $\mathbb{R}$ is uncountable (we have this). This is written in the game

Unique factorization (++)

Walk-through: Show that there is no integer n such that $n^2 = 2$.
Boss level: Show that there is no rational r such that $r^2 = 2$.
Boss level: Show that $\sqrt{p}$ is not rational for any prime p.

The reals over the rationals -- concrete version (++)

Follow-up to planets on unique factorization (and reals of rationals -- abstract version)

Boss level Show that $\log p_i$ are linearly independent over $\mathbb{Q}$ where p_i are prime numbers.

This uses cardinality argument (cardinal_eq_of_finite_basis) and the unique factorization of integers into primes.

Matrices 1

Boss level every matrix can be written as a sum of a symmetric (A^+ = A) and a skew-symmetric matrix. (this might be in `mathlib` already, we should figure this out first.)

Matrices 2 (++)

~~**Boss level** Show that the space of $n \times n$ matrices with real entries is a vector space over $\mathbb{R}$. (`Fin n \to Fin n \to ℝ`) (`E_ij`) - Note: this is already in mathlib and can be done by inferInstance.~~

Matrices 3 (+)

Boss level Suppose A is an $n \times n$ matrix with real entries. Show that the space generated by the powers of A, i.e. the set $\{I, A, A^2, A^3, \ldots\}$ is a proper subspace of the space of $n \times n$ matrices with real entries. ($n \geq 2$)

proof-sketch: any two matrix in the span of these vectors commute: S T = T S.

Quotients (++)

Boss level Any function $f \colon A \to B$ is a function can be factored into three functions $f = i \circ g \circ q$ where q is a surjection, h is a bijection, and i is an injection.

Alternative questions for quotients

1. Construction of the field $\mathbb{F}_p$ via quotient construction of its underlying cyclic group for a prime number p.
2. (Advanced) (Gaussian Coefficients) Let V be an n-dimensional vector space over $\mathbb{F}_p$. Show that the number of subspaces of V is given by the Gaussian Coefficients.

TODO: I will break down the question 2 into four separate parts (Sina.)

- Could use tensor product of vector spaces?
- (1st) Isomorphism theorem. Two questions about this: set, vector spaces

Trace (++)

Boss level Suppose f is a linear transformation over the space of $n \times n$ matrices such that $f(AB) = f(BA)$ for all A, B. Show that there exists a scalar c such that $f(A) = c\operatorname{tr}(A)$ for all A.

Boss Level every linear map on the space of $n \times n$ matrices is of the form $\operatorname{tr}(A \bullet)$ for some matrix A.

Boss Level Show that the trace of a matrix is the sum of its eigenvalues.

Walk through:

- ~~Show that $\operatorname{tr}(AA^T) \geq 0$ and the equality holds if and only if $A = 0$. (Proof in the repo, but not part of any levels yet.)~~
- Show that for any matrix A the map $\operatorname{tr}(A \bullet) : X \mapsto \operatorname{tr}(AX)$ is a linear map on the space of $n \times n$ matrices.
- Show that the map above is a zero map if and only if $A = 0$.
- Show that the map $A \mapsto \operatorname{tr}(A \bullet)$ is an isomorphism.
  ~~- Suppose A is an $m \times n$ and B is an $n \times m$ matrix. Show that the trace of AB is the same as the trace of BA.~~

Marcus: Good source of questions, just pick one or two of these.

Determinantes (+)

Van der Monde Matrix (???)

Fix 100 distinct points $t_0, \ldots, t_{99}$ in the interval $I = [-1, 1]$. Consider the map $L\colon \mathbb{R}^{200} \to \mathbb{R}^{100}$ defined by the assignment

$$c = (c_0, ..., c_{199}) \mapsto (p_c(t_0), ..., p_c(t_{99}))$$

where $p_c = \sum_{i=0}^{i=199} c_i x^i$, i.e. from vectors of coefficients of polynomials of degree ≤ 199 to the vectors $(p_c(t_i))_{i=0}^{99}$ of values of such polynomials at nodes t_i.

1. Show that L is linear.
2. Show that this map is represented, upon choosing the standard basis in $\mathbb{R}^{200}$ and $\mathbb{R}^{100}$, by the 100×200 *Vendermonde* matrix.

$$\begin{bmatrix} 1 & t_0 & t_0^2 & \ldots & t_0^{199} \\ 1 & t_1 & t_1^2 & \ldots & t_1^{199} \\ 1 & t_2 & t_2^2 & \ldots & t_2^{199} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_{99} & x_{99}^2 & \ldots & t_{99}^{199} \end{bmatrix}$$

3. Show that this map is never invertible.