Representations of Rotations

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November 14, 2017

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Representations

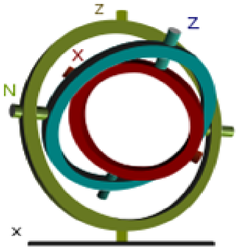
- Rotation Representations:
  - Rotation Matrix

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \rightarrow \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \\ r_{12} \\ r_{22} \\ r_{32} \\ r_{13} \\ r_{23} \\ r_{33} \end{bmatrix}$$

- Where are the redundancies? $|r_i| = 1, r_i \cdot r_j = 0, i \neq j$
- Not applicable for motion interpolation.

Rotational Representations

Euler angles[1]



[1] images source: Wikipedia

Three Angle Representations

Euler angles

- 24 different representations involving x,y,z
- Distinction between intrinsic (relative or Euler axis) vs. extrinsic (fix axis) rotations
- 12 possibilities for each flavor (intrinsic, extrinsic)
- in the following 6 combinations, the third axis is the same as first x-y-x, x-z-x, y-x-y, y-z-y, z-x-z, z-y-z
- for the next 6 combinations, rotation about all three axis: x-y-z, x-z-y, y-x-z, y-z-x, z-y-x, z-x-y

Euler Angles vs. Fixed Angles

Z-Y-X Euler Angles:

$$ {}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) $$

X-Y-Z Fixed Angles:

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) $$

In general: Three rotations taken about fixed axes yield the same final orientation as the same three rotations taken in opposite order about the axes of a moving frame.

$$ {}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) \quad = \quad {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) $$

Inverse Problem

How to find the Euler angles for a given matrix? Given ${}^A_BR$, find (α, β, γ).

$$
\begin{aligned}
{}^A_BR &= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \\
&= \begin{bmatrix} c\alpha \cdot c\beta & c\alpha \cdot s\beta \cdot s\gamma - s\alpha \cdot c\gamma & c\alpha \cdot s\beta \cdot c\gamma + s\alpha \cdot s\gamma \\ s\alpha \cdot c\beta & s\alpha \cdot s\beta \cdot s\gamma + c\alpha \cdot c\gamma & s\alpha \cdot s\beta \cdot c\gamma - c\alpha \cdot s\gamma \\ -s\beta & c\beta \cdot s\gamma & c\beta \cdot c\gamma \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\sin(\beta) &= s\beta = -r_{31} \\
\text{with: } \cos^2(\alpha) + \sin^2(\alpha) = 1 \rightarrow \cos(\beta) &= c\beta = \sqrt{r_{11}^2 + r_{21}^2}
\end{aligned}
$$

Inverse Problem

How to find the Euler angles for a given matrix?

$$\begin{bmatrix} c\alpha \cdot c\beta & X & X \\ s\alpha \cdot c\beta & X & X \\ -s\beta & c\beta \cdot s\gamma & c\beta \cdot c\gamma \end{bmatrix} \rightarrow$$

$$\begin{aligned} \sin(\beta) &= s\beta = -r_{31} \\ \cos(\beta) &= c\beta = \sqrt{r_{11}^2 + r_{21}^2} \\ \beta &= \arctan \frac{\sin\beta}{\cos\beta} \end{aligned}$$

if $\beta = 90^o$, $\cos\beta = 0 \rightarrow$ Singularity of representation!

Inverse Problem

What happens during a singularity? Example:

$$\beta = 90^o, \cos\beta = 0, \sin\beta = -1$$

What does it mean for the rotation matrix?

$$^A_BR = \begin{bmatrix} 0 & -s(\alpha-\gamma) & c(\alpha-\gamma) \\ 0 & c(\alpha-\gamma) & s(\alpha-\gamma) \\ -1 & 0 & 0 \end{bmatrix}$$

Singularity

Using Euler angles, when choosing one angle poorly, instead of having two remaining degrees of freedom, it is possible to lose another d.o.f. This configuration is often referred to as **Gimbal lock**.

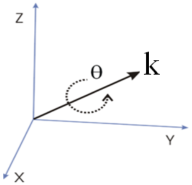
Euler Angles

We know now how to calculate rotation matrices from Euler angles and vice versa. But the inverse problem is not always solvable.

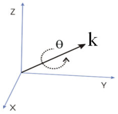
Pro and contra Euler angles

Pro: Intuitive, require a minimum number of parameters.
Contra: Singularities can occur, regardles of which Euler angle representation we chose.

Angle-Axis Representation



Angle-Axis Representation



- Contains three parameters.
- Defined through a rotation vector $\Theta = \theta k$
- Length of the vector Θ defines the amount of the rotation agle
- Its unit vector k defines the rotation axis

$$< \text{normalized axis, angle} >=< k, \theta >= \left(\begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}, \theta \right)$$

Angle-Axis Representation

Calculate axis unit vector k and angle θ from rotation matrix R:

$$\theta = \arccos\left(\frac{trace(R) - 1}{2}\right) = \arccos\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$k = \frac{1}{2\sin(\theta)} \begin{bmatrix} R(3,2) - R(2,3) \\ R(1,3) - R(3,1) \\ R(2,1) - R(1,2) \end{bmatrix}$$

Singularity for small angles θ, the rotation axis for very small vectors becomes ill-defined. For $\theta = 0$ the axis becomes undefined.

Angle-Axis Representation

How to get from angle-axis representation to rotation matrix?
Remember k is a unit vector, θ represents the rotation angle.

$$\begin{aligned} k &\in \mathfrak{so}(3) = \mathbb{R}^3 \\ R &\in SO(3) \subset \mathbb{R}^{3\times 3} \end{aligned}$$

Exponential Map

The exponential map effects a transformation from the axis-angle representation of rotations to rotation matrices:

$$\exp : \mathfrak{so}(3) \rightarrow SO(3)$$

Angle-Axis Representation

- $R = \exp(\theta K)$, exponential map of (θK),
- where $K \in \mathbb{R}^{3\times 3}$ is the cross product skew matrix - with the property of $K \cdot v = k \times v$ for all vectors $v \in \mathbb{R}^3$
- $K = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$

Example:

$$k = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, v = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \rightarrow K = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Angle-Axis Representation

Taylor expansion helps us to get a closed form solution.

$$\begin{aligned} R = \exp(\theta K) &= \sum_{k=0}^{\infty} \frac{(\theta K)^k}{k!} \\ &= \mathbf{I} + \theta K + \frac{(\theta K)^2}{2!} + \frac{(\theta K)^3}{3!} + \ldots \end{aligned}$$

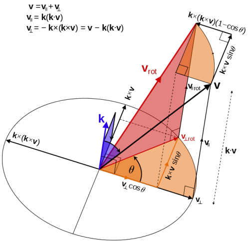
and because: $K^3 = -K, K^4 = -K^2, K^5 = K, K^6 = K^2, K^7 = -K$ and so on

We get:

$$\begin{aligned} R &= \mathbf{I} + (\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots)K + (\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots)K^2 \\ R &= \mathbf{I} + \sin(\theta)K + (1 - \cos(\theta))K^2 \end{aligned}$$

Rotating a Vector, Rodrigues' Rotation Formula[2]

Rodrigues' Formula to rotate a vector v around a unit vector k by angle θ:



$$v_{rot} = v + (1 - \cos\theta) k \times (k \times v) + (\sin\theta)(k \times v)$$

[2]image source: Wikipedia

Summary

Rotations and Number of Parameters Parameters

There are no three parametric rotations without singularities.

Let's try four parameters.

Quaternions

- a number system extending complex numbers
- first described by William Rowan Hamilton in 1843
- can be interpreted as the quotient of two 3d vectors
- can be thought of complex numbers with one real and three imaginary components: $a1 + bi + cj + dk$, e.g. $4 + 3i + 2j + 5k$
- algebra of quaternions is often denoted by $\mathbb{H}$

Some basic properties:

$$i^2 = j^2 = k^2 = ijk = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$$

Quaternions are non-commutative.

Quaternions

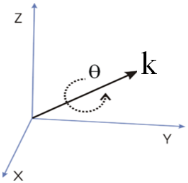
- can be written as four dimensional vectors $\varepsilon \in \mathbb{R}^4$ using the basis $1, i, j, k$ of $\mathbb{H}$ then the basis elements are:

$$\begin{aligned} 1 &= (1,0,0,0); \\ i &= (0,1,0,0); \\ j &= (0,0,1,0); \\ k &= (0,0,0,1) \end{aligned}$$

- if we require quaternions to be normalized, they can be applied to rotations in 3d space - sometimes referred to as Euler parameters.

Quaternions / Euler Parameters

Revisit angle-axis rotation. Euler parameters are a quaternion in scalar vector representation. They are defined as:



$$\begin{aligned}\varepsilon \in \mathbb{R}^4, \text{with } \varepsilon_0 &= \cos(\frac{\theta}{2}) \\ \varepsilon_1 &= k_X \sin(\frac{\theta}{2}) \\ \varepsilon_2 &= k_Y \sin(\frac{\theta}{2}) \\ \varepsilon_3 &= k_Z \sin(\frac{\theta}{2})\end{aligned}$$

Quaternion - Derivation from Rotation Matrix

$$
\begin{aligned}
{}^{A}_{B}R &= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \\
&= \begin{bmatrix} 1-2\varepsilon_2^2-2\varepsilon_3^2 & 2(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_0) & 2(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_0) \\ 2(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_0) & 1-2\varepsilon_1^2-2\varepsilon_3^2 & 2(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_0) \\ 2(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_0) & 2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_0) & 1-2\varepsilon_1^2-2\varepsilon_2^2 \end{bmatrix}
\end{aligned}
$$

$$
r_{11}+r_{22}+r_{33} = 3-4(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2)
$$

$$
\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2 \triangleq 1-\varepsilon_0^2
$$

Quaternion - Derivation from Rotation Matrix

$$\varepsilon_0 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$

$$\varepsilon_1 = \frac{r_{32} - r_{23}}{4\varepsilon_0}$$
$$\varepsilon_2 = \frac{r_{13} - r_{31}}{4\varepsilon_0}$$
$$\varepsilon_3 = \frac{r_{21} - r_{12}}{4\varepsilon_0}$$

What if: $\varepsilon_0 = 0$?

Lemma: Always at least one parameter is larger than $\frac{1}{2}$. Apply this one for the computation $\rightarrow$ no Singularity.

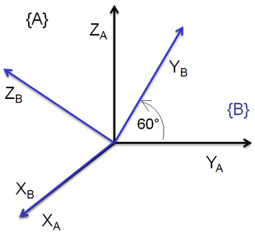
Derive Angle and Axis from Quaternion

Pretty straightforward: $\theta_{1,2} = \pm 2 \arccos \varepsilon_0$
if $\theta \in \{n \cdot 2\pi | n \in \mathbb{Z}\} \rightarrow k_{X_{1,2}}, k_{Y_{1,2}}, k_{Z_{1,2}}$ not defined; otherwise :

$$k_{X_{1,2}} = \frac{\varepsilon_1}{\sin(\frac{\theta_{1,2}}{2})}$$
$$k_{Y_{1,2}} = \frac{\varepsilon_2}{\sin(\frac{\theta_{1,2}}{2})}$$
$$k_{Z_{1,2}} = \frac{\varepsilon_3}{\sin(\frac{\theta_{1,2}}{2})}$$

Two solutions, θ_1 and θ_2 and corresponding $k_2 = -k_1$.

Example



Euler parameters:

$$\varepsilon = (\sqrt{(3/4)}, 1/2, 0, 0)$$

Rotation matrix:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3/4} \\ 0 & \sqrt{3/4} & 1/2 \end{bmatrix}$$

Hamilton Product

Let $q_1, q_2 \in \mathbb{H}$:

$$q_1 = a_1 + b_1 i + c_1 j + d_1 k$$
$$q_2 = a_2 + b_2 i + c_2 j + d_2 k$$

the result of the Hamilton product $q_1 q_2$ is

$$\begin{aligned} q_1 q_2 &= (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2) \\ &+ (a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2) i \\ &+ (a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2) j \\ &+ (a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2) k \end{aligned}$$

Hamilton Product

Hamilton product is useful for multiplication of two quaternions.

- useful for vector rotation,
- associative but not commutative,
- Application: The result of the rotation of $p \in \mathbb{H}$ with quarternion $q_A \in \mathbb{H}$ followed by a rotation with a second quaternion $q_B \in \mathbb{H}$ leads to the same quaternion $p' \in \mathbb{H}$ as when rotating a vector p via the Hamilton product of both quaternions $q_1 q_2$, see further below.

Vector Rotation Using Quaternions

Rotating a vector p with a quaternion q, using the Hamilton product:

$$\hat{p}' = q\hat{p}q^{-1}$$

, where q^{-1} is the conjugation of q, i.e., all imaginary components of q^{-1} take the negative value of q.

Vector Quaternion Rotation - Example

Rotating a vector $p = (1,0,0)^T \in \mathbb{R}$, i.e., it needs to be mapped into quaternion space, resulting in $\hat{p}$, with all components of p becoming imaginary in $\hat{p}$. p shall be rotated by 90^o or $\frac{\pi}{2}$ around the y-axis (unit vector $k = (0,1,0)^T$). We have:

$$\hat{p}' = q\hat{p}q^{-1}$$

$$\hat{p} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, q = \begin{bmatrix} \cos\frac{\theta}{2} \\ k_X \sin\frac{\theta}{2} \\ k_Y \sin\frac{\theta}{2} \\ k_Z \sin\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} \\ 0\sin\frac{\pi}{4} \\ 1\sin\frac{\pi}{4} \\ 0\sin\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} \\ 0 \\ \sin\frac{\pi}{4} \\ 0 \end{bmatrix}$$

, where q^{-1} is the conjugation of q, i.e., all imaginary components of q^{-1} are negated w.r.t. q.

Vector Quaternion Rotation - Example

$$\begin{aligned}
\hat{p}' &= q\hat{p}q^{-1} \\
&= \begin{bmatrix} \cos\frac{\pi}{4} \\ 0 \\ \sin\frac{\pi}{4} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \cos\frac{\pi}{4} \\ 0 \\ -\sin\frac{\pi}{4} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \cos\frac{\pi}{4} \\ 0 \\ -\sin\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} \cos\frac{\pi}{4} \\ 0 \\ -\sin\frac{\pi}{4} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \rightarrow p' = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}
\end{aligned}$$

Multiple Rotation Using Quaternions

Rotating a vector p for n times by $q_1, q_2, \cdots, q_n$, results in p'':

$$\begin{aligned} \hat{p}'' &= q_n(\cdots(q_2(q_1\hat{p}q_1^{-1})q_2^{-1})\cdots)q_n^{-1} \\ &= (q_1q_2\cdots q_n)\hat{p}(q_1q_2\cdots q_n)^{-1} \end{aligned}$$

A quaternion and its negation lead to the same rotation.

Summary

- Joint space and operational space
- Homogenious Transformations for mapping between coordinate frames and for operations
- Rotations
  - Rotations with three angles are minimal but have singularities.
  - Quaternions: no singularities, easy to interpolate.

Literature

[1] John J. Craig, Introduction to Robotics - Mechanics and Control, 3rd edition, Pearson Prentice Hall, 2005.