Homogenious Transformations

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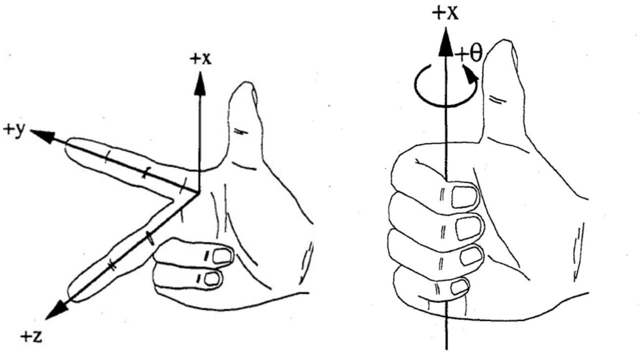
Right-Handed Coordinate Frames

We use right-handed coordinate frames[1]:



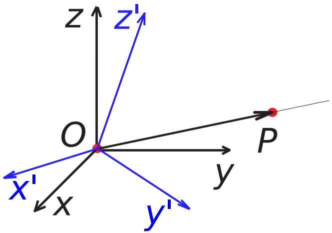
[1] image source: SAE J1733 [2]

Homogenious Transformations

Coordinate transformations

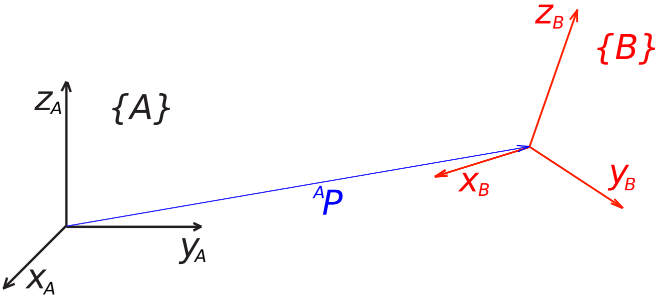
At first we want to focus on how to **transform coordinates** from one frame to another.

Rigid Body Configuration - Cartesian Frames



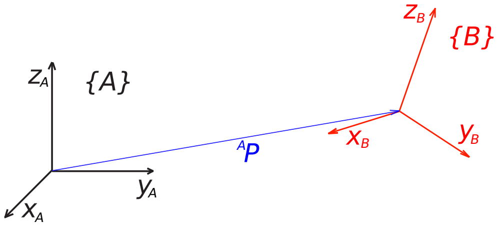
- if the orientation of the coordinate frame changes (and the origin stays the same), the vector is the same, but components of the vector will change

Rigid Body Configuration



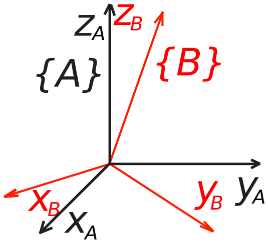
- how to describe frame $\{B\}$ with respect to fixed frame $\{A\}$?
- ^AP defines the origin of frame $\{B\}$ represented in frame $\{A\}$

Rigid Body Configuration



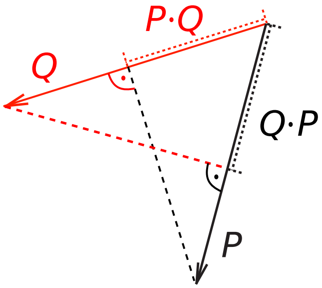
- the orientation of frame $\{B\}$ relative to frame $\{A\}$ is defined as the rotations of base vectors of frame B with respect to frame A
- one idea is, to have a mathematical structure, a **rotation matrix**, representing that orientation of frame $\{B\}$ with respect to frame $\{A\}$

Rotation Matrix



- we want a matrix $R \in \mathbb{R}^{\mathrm{nxn}}$ which maps a vector $P_B \in \mathbb{R}^{\mathrm{n}}$ in frame $\{B\}$ to a vector P_A in frame $\{A\} \in \mathbb{R}^{\mathrm{n}}$
- this rotation matrix is denoted as ${}^A_B R$
- the new vector is calculated as $P_A = {}^A_B R \ P_B$
- How can ${}^A_B R$ be inferred from $\{A\}$ and $\{B\}$?

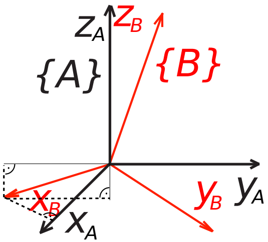
Let's recall: Scalar Product (Dot Product)



- the scalar product, when applied on unit vectors, can be interpreted as the projected distance of a vector P on another vector Q, or vice versa
- equals $\cos(\alpha)$, where α is the enclosed angle

$$\begin{aligned} P \cdot Q &= P_1 Q_1 + \cdots + P_n Q_n \\ &= \sum_{i=1}^{n} P_i Q_i \end{aligned}$$

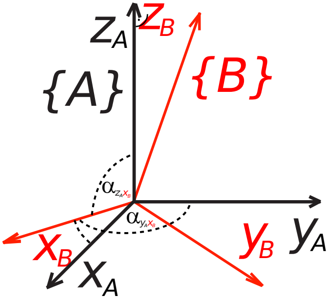
Mapping Coordinates



- to map a vector from one frame $\{B\}$ to another frame $\{A\}$ we have to map its components X_B, Y_B, Z_B into frame $\{A\}$
- e.g., and assuming there is a reference frame $\{O\}$:

$$^{A}X_B = \begin{bmatrix} ^{O}X_A \cdot ^{O}X_B \\ ^{O}Y_A \cdot ^{O}X_B \\ ^{O}Z_A \cdot ^{O}X_B \end{bmatrix}$$

Properties of the Rotation Matrix



$$
\begin{aligned}
{}^A_BR &= ({}^AX_B, {}^AY_B, {}^AZ_B) \\
{}^A_BR &= \begin{bmatrix} X_A \cdot X_B & X_A \cdot Y_B & X_A \cdot Z_B \\ Y_A \cdot X_B & Y_A \cdot Y_B & Y_A \cdot Z_B \\ Z_A \cdot X_B & Z_A \cdot Y_B & Z_A \cdot Z_B \end{bmatrix}
\end{aligned}
$$

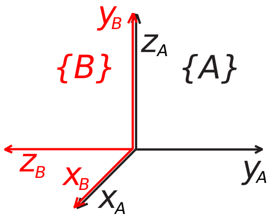
- elements of R are cosines of angles enclosed by coordinate frame axes
- thus, **rotation matrices** are often referred to as **direction cosines**

Properties of the Rotation Matrix

- the rotation matrix is an orthonormal matrix, its transpose is its inverse:

$$\begin{aligned}
{}^A_B R &= \begin{bmatrix} {}^A X_B & {}^A Y_B & {}^A Z_B \end{bmatrix} = \begin{bmatrix} {}^B X_A^T \\ {}^B Y_A^T \\ {}^B Z_A^T \end{bmatrix} \\
&= \begin{bmatrix} {}^B X_A & {}^B Y_A & {}^B Z_A \end{bmatrix}^T = {}^B_A R^T \\
{}^A_B R &= {}^B_A R^T \\
{}^A_B R^{-1} &= {}^B_A R = {}^A_B R^T
\end{aligned}$$

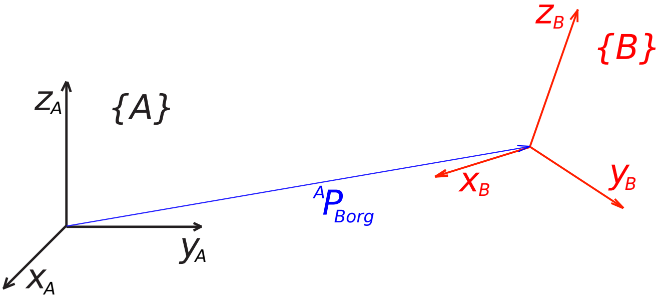
Example



$$
{}^A_BR = \begin{bmatrix} X_A \cdot X_B & X_A \cdot Y_B & X_A \cdot Z_B \\ Y_A \cdot X_B & Y_A \cdot Y_B & Y_A \cdot Z_B \\ Z_A \cdot X_B & Z_A \cdot Y_B & Z_A \cdot Z_B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} {}^BX_A^T \\ {}^BY_A^T \\ {}^BZ_A^T \end{bmatrix}
$$

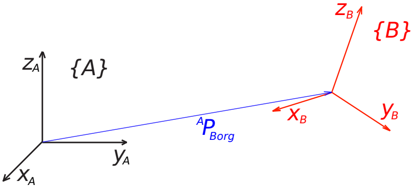
$$
= ({}^AX_B, {}^AY_B, {}^AZ_B)
$$

Full Description of a Frame



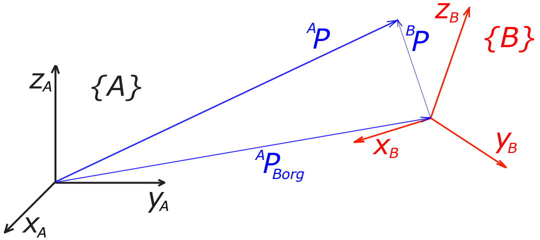
$$\{B\} = \begin{bmatrix} {}^A_BR & {}^AP_{Borg} \end{bmatrix}$$

Example



$$
\begin{aligned}
\{B\} &= \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{Borg} \end{bmatrix} \\
&= \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Translations



$$^AP_{O_A} =^A P_{O_B} +^A P_{Borg}$$

$$^AP =_B^A R \cdot^B P +^A P_{Borg}$$

Homogenious Transformation

$$^{A}P =^{A}_{B} R \cdot^{B} P +^{A} P_{Borg}$$

$$\begin{bmatrix} ^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} & ^{A}_{B}R & & ^{A}P_{Borg} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^{B}P \\ 1 \end{bmatrix}$$

Example

How does a homogenious transformation matrix T look, which does not perform any rotation, only translation by vector?

$$T = \begin{bmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

How does a homogenious transformation matrix T look, which does not perform any translation, only rotation with angle α around the z-axis?

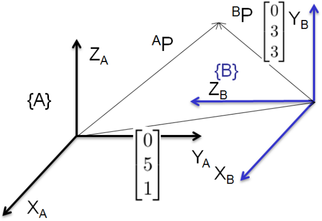
$$T = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

How does a homogenious transformation matrix T look, which does not perform any rotation or translation?

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example



$$^A_BR = \begin{bmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix}$$

$$^AP = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & -1 & 5\\0 & 1 & 0 & 1\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\3\\3\\1\end{bmatrix} = \begin{bmatrix}0\\2\\4\\1\end{bmatrix}$$

Literature

[1] John J. Craig, Introduction to Robotics - Mechanics and Control, 3rd edition, Pearson Prentice Hall, 2005.

[2] SAE J1733: Sign Convention for Vehicle Crash Testing- https://law.resource.org/pub/us/cfr/ibr/005/sae.j1733.1994.pdf