Dynamics, PID Control

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November 20, 2017

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Outline

- So far we saw how the motion of joints is related to motions of the rigid bodies of a robot.
- We assumed we could command arbitrary joint level trajectories, which would be faithfully executed by the real-world robot.
- Most robots are driven by electrical, pneumatic or hydraulic actuators, which apply torques (or for linear actuators forces)
- The dynamics of a robot manipulator describes how the robot moves in response to these actuator forces.

Dynamics [3]

- The dynamics of a system describes how the controls u_t influence the change-of-state of the system

$$x_{t+1} = f(x_t, u_t)$$

  - The notation x_t refers to the dynamic state of the system: e.g., joint positions and velocities $x_t = (q_t, \dot{q}_t)$
  - f is an arbitrary function
- We define a **nonholonomic system** as one with differential constraints:

$$\dim(u_t) < dim(x_t)$$

$\rightarrow$ Not all degrees of freedom are **directly** controllable

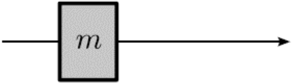
Dynamics

- Examples:
  - An air plane flying: You cannot command it to hold still in the air or to move straight up
  - A car: you cannot command it to move sideways
  - One can set controls u_t (air planes control stick, car's steering wheel, your muscles activations, send torque/voltage/current to a robot's motors)
  - But these controls only indirectly influence the dynamics of a state, $x_{t+1} = f(x_t, u_t)$

The Math Behind

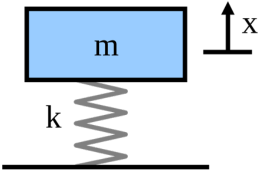
- In general the calculation can be done by summing up all the forces acting on the coupled rigid bodies of the robot
- We shall rely on the Lagrangian to derive the system dynamics, requiring only the potential and kinematic energies of the system to be computed.

Proportional Control



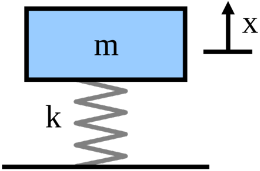
- The simplest possible example:
- Task: Control a force $u(t)$ at time t to move a 1-D point mass m from a given position x towards a certain position x^*
- Proportional Control: the bigger the error $(x^* - x)$, the bigger should be u(t)
- Physical analogy: Mass-Spring System

Natural Systems



- Mass-Spring System
- Conservative System
- Assume no gravity
- Position and velocity $x, \dot{x}$
- System has kinetic energy (K) and potential energy (V)

Natural Systems

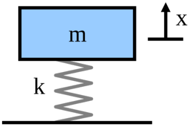


- System has kinetic energy (K) and potential energy (V)
- $L = K - V$

$$\frac{d}{dt}\left(\frac{\partial(K-V)}{\partial \dot{x}}\right) - \frac{\partial(K-V)}{\partial x} = 0$$

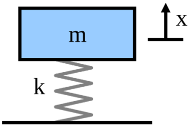
Mass-Spring Systems

- Pull the spring and let it go: oscillation starts.
- While oscillating, kinetic energy is being transformed to potential energy and back.
- m is the mass, $\ddot{x}$ the acceleration, x is position, F the force
- k is spring constant (F/x)
  - Kinetic energy $K = \frac{1}{2}m\dot{x}^2$
  - Potential spring energy $V = \text{Work} = \int_x^0 (-kx)\text{d}x = \frac{1}{2}kx^2$



Mass-Spring Systems

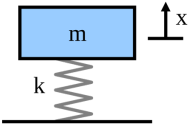
$$
\begin{aligned}
K &= \frac{1}{2}m\dot{x}^2 \\
V &= \frac{1}{2}kx^2 \\
\frac{d}{dt}\left(\frac{\partial(K-V)}{\partial\dot{x}}\right) - \frac{\partial(K-V)}{\partial x} &= 0 \\
\frac{d}{dt}\left(\frac{\partial(K)}{\partial\dot{x}}\right) - \frac{\partial(K)}{\partial x} &= -\frac{\partial(V)}{\partial x} \\
m\ddot{x} &= ma = F \\
&= -kx \\
m\ddot{x} + kx &= 0
\end{aligned}
$$



Mass-Spring Systems

- Conservative Systems:

$$m\ddot{x} + kx = 0 \Longleftrightarrow \ddot{x} + \frac{k}{m}x = 0$$



- What's the frequency, given k and m

Assume:

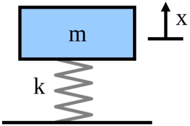
$$x = a + be^{\omega t} \text{ , with } \omega \text{ is complex}$$

$$\begin{aligned}
\left[a + \ddot{b}e^{\omega t}\right] + \left[a + be^{\omega t}\right]\frac{k}{m} &= 0 \\
\omega^2 be^{\omega t} + \frac{k}{m}be^{\omega t} + \frac{k}{m}a &= 0, \text{assume: } a = 0
\end{aligned}$$

Mass-Spring Systems

$$\omega^2 + \frac{k}{m} = 0$$

$$\omega^2 = -\frac{k}{m} \implies \omega = i\sqrt{\frac{k}{m}}$$



Natural Frequency:

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}}$$

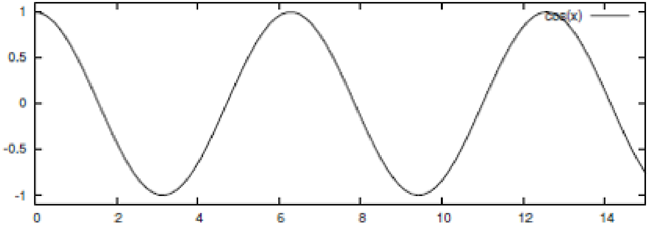
Natural Frequency increases with stiffness and inverse mass

$$\begin{aligned} x(t) &= a + be^{i\sqrt{\frac{k}{m}}t} \\ \text{real part:} \quad x(t) &= a + b\cos(\omega_n t) \end{aligned}$$

Mass-Spring Systems

$$x(t) = a + b\cos(\omega_n t)$$

- Oscillation around a with amplitude b and natural frequency: $\omega_n = \sqrt{\frac{k}{m}}$



P-Controller

$$
\begin{aligned}
V(x) &= \frac{1}{2} K_P (x_t^* - x_t)^2 \\
f &= -\frac{\partial V}{\partial x} \\
0 &= m\ddot{x} + K_P (x_t^* - x_t) \\
u(t) &= K_P (x_t^* - x_t)
\end{aligned}
$$

with:

$$
\begin{aligned}
x^* &= a \text{ target position} \\
K_P &= \text{Spring parameter} \\
u(t) &= \text{resulting output (e.g. force / torque)}
\end{aligned}
$$

Natural System

- System is stable (Lyapunov stable) but oscillates.
- We need a system that converges asymptotically.
- Dissipative system, energy is taken out (in contrast to conservative system)
- External friction force which increases linearly with velocity
- Idea: control the position AND velocity, e.g. $\dot{x}^* = 0$
- Physical analogy: Mass-Spring-Damper System

Mass-Spring-Damper

Dissipative systems:

$$\frac{d}{dt}\left(\frac{\partial(K-V)}{\partial \dot{x}}\right) - \frac{\partial(K-V)}{\partial x} = f_{friction}$$

Viscous friction: $f_{friction} = -c\dot{x}$

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\boxed{\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0}$$

Mass-Spring-Damper

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \omega_n^2 x = 0$$

$$\frac{\frac{c}{m}}{2\omega_n}2\omega_n \ , \ \omega_n = \sqrt{\frac{k}{m}} \quad \text{Natural frequency}$$

$$\xi_n = \frac{c}{2\omega_n m} = \frac{c}{2\sqrt{km}} \quad \text{Natural damping ratio}$$

$$\ddot{x} + 2\xi_n\omega_n\dot{x} + \omega_n^2 x = 0$$

Critically damped system: $\xi_n = 1 \ , \ c = 2\sqrt{km}$

Time Response

- Overdamped: $c^2 > 4mk \iff \xi > 1$
- Underdamped: $c^2 < 4mk \iff \xi < 1$
- Critically damped: $c^2 = 4mk \iff \xi = 1$

For non-overdamped systems ($\xi \leq 1$):

$$x(t) = a + be^{-\xi_n \omega_n t} \cos(\omega_n \sqrt{1 - \xi_n^2}\; t)$$

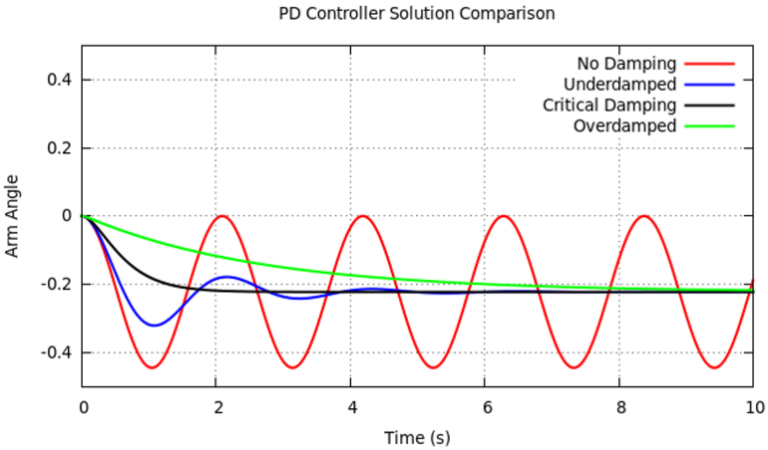
$$\boxed{\omega_n' = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \omega_n \sqrt{1 - \xi_n^2}}$$ Damped natural frequency

Time Response

If there is a small damping, the oscillation frequency decreases, critically damped and overdamped systems do not oscillate (frequency becomes imaginary).

Damping Ratio Examples

Mass-Spring-Damper



Example

- $m = 2.0 \quad c = 4.8 \quad k = 8.0$
- What is the damped Natural frequency?

$$
\begin{aligned}
\omega_n &= \sqrt{\frac{k}{m}} = 2 \\
\omega_n^{'} &= \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \omega_n\sqrt{1-\xi^2} \\
\xi_n &= \frac{c}{2\sqrt{km}} = 0.6
\end{aligned}
$$

Asymptotic Stability

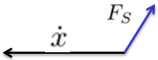
A system

$$\frac{d}{dt}\left(\frac{\partial(K)}{\partial \dot{x}}\right) - \frac{\partial(K - V_{goal})}{\partial x} = F_s$$

is asymptotically stable if a force counteracts the velocity

$$F_S^T \dot{x} < 0; \quad \text{for} \quad \dot{x} \neq 0$$

$$F_S = -k_v \dot{x} \rightarrow k_v > 0$$



scalar product negative

PD-Contol:

$$F = u(t) = K_p(x_t^* - x_t) + K_d(-\dot{x}_t)$$

PD-Control

If we are following a trajectory, we apply a desired velocity $\dot{x}_t^*$ to our control rule, in order to account for a changing desired position x_t*

$$u(t) = K_p(x_t^* - x_t) + K_d(\dot{x}_t^* - \dot{x}_t)$$

Selecting Parameters

- Start with a mass,
- Select spring and damper, k and c
- Masses can be modeled
- Compensate for gravity, friction, centrifugal and coreolis force
- Compensating for friction is dangerous, usually setting to zero is better than to make a bad guess.

Selecting Parameters

- How to select the weights?
- Large ω or small ω desired?

Controlling a 1D-Mass [3]

- Proportional and derivative feedback (PD control) are like adding a spring and damper to the point mass
- PD is a linear control law

$$\pi : (x, \dot{x}) \rightarrow K_p(x^* - x) + K_d(\dot{x}^* - \dot{x})$$

, linear in the dynamic system state $(x, \dot{x})$

- With these linear control laws it is possible to design approach trajectories by tuning the gains, but no optimality principle behind such motions yet

Controlling a 1D-Mass-Integral Feedback

- What happens if a steady force acts against the desired state

$$u = K_p(x^* - x) + K_d(\dot{x}^* - \dot{x}) + K_i \int_{s=0}^{t} (x^*(s) - x(s))ds$$

- This is not a linear control law (not linear w.r.t $(x, \dot{x})$, it is hard to solve the equation analytically

1D-Point Mass, PID-Control

$$u = K_p(x^* - x) + K_d(\dot{x}^* - \dot{x}) + K_i \int_{s=0}^{t} (x^*(s) - x(s))ds$$

PID control

- Proportional Control (”Position control”)
  $f \propto K_p(x^* - x)$
- Derivate Control (”Damping”)
  $f \propto K_d(\dot{x}^* - \dot{x}) \qquad (\text{if } \dot{x}^* = 0 \rightarrow \text{damping})$
- Integral Control (”Steady state error compensation”)
  $f \propto K_i \int_{s=0}^{t} (x^*(s) - x(s))ds$

Literature

- [1] John J. Craig, Introduction to Robotics - Mechanics and Control, 3rd edition, Pearson Prentice Hall, 2005.
- [2] Oussama Khatib: Introduction to Robotics, Online Lecture, 2008.
- [3] Marc Toussaint: Lecture Notes on "Robotics", 2010