

# 1 Holomorphic functions

Before we start, let us very briefly recall the complex numbers. We define the *complex numbers*, denoted  $\mathbb{C}$ , to be the set of pairs  $(x, y)$  of real numbers together with the following two binary operations, called the “addition” and the “multiplication” of  $\mathbb{C}$ , defined by

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

and

$$(x_1, y_1) \cdot (x_2, y_2) := (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$$

for all real numbers  $x_i, y_i$  ( $i = 1, 2$ ). Using the laws of distributivity and commutativity in  $\mathbb{R}$ , one easily verifies that the operations  $+, \cdot$  introduced above are also distributive and commutative. Furthermore, one checks that  $(\mathbb{C}, +)$  and  $(\mathbb{C} - \{(0, 0)\}, \cdot)$  form abelian groups with neutral elements  $(0, 0)$  and  $(1, 0)$  respectively. F.e. let  $z = (x, y) \neq 0$ , then the multiplicative inverse of  $(x, y)$  is given by

$$z^{-1} = \left( \frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right).$$

So  $\mathbb{C}$  is in fact a field. Moreover, the map

$$\phi : \mathbb{R} \ni x \mapsto (x, 0) \in \mathbb{C}$$

is an injection of  $\mathbb{R}$  into  $\mathbb{C}$ , i.e. it is an injective map which is “structure preserving” in the sense that

$$\phi(x_1 + x_2) = \phi(x_1) + \phi(x_2) \text{ and } \phi(x_1x_2) = \phi(x_1)\phi(x_2)$$

for all  $x_1, x_2$  in  $\mathbb{R}$ . So we may identify  $\mathbb{R}$  as a subfield of  $\mathbb{C}$ , which we will do in what follows. Now, if we introduce the vector  $i = (0, 1)$ , then  $\mathbb{C}$  as a real vector space of dimension two will have  $1 = (1, 0)$  and  $i$  as a basis so that every  $z$  in  $\mathbb{C}$  can be uniquely written as  $z = x + yi$  where  $x, y$  are real numbers. Also, we introduce the *complex conjugate*  $\bar{z}$  of the complex number  $z = x + yi$  as the number  $x - yi$ . Note that  $z\bar{z} = x^2 + y^2$ . We let  $|z| := \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$  and get the Euclidean distance of the vector  $z = (x, y)$  from the origin. This is an absolute value. In particular, we have

$$|z_1z_2| = |z_1||z_2|$$

for all  $z_1, z_2$  in  $\mathbb{C}$ . Finally, we will denote  $\partial_x, \partial_y$  the partial derivatives with respect to  $x, y$ .

Let us now come to the central object of study.

**Definition 1.1.** Given a function

$$f : U \subset \mathbb{C} \rightarrow \mathbb{C}$$

where  $U \subset \mathbb{C}$  is open, we say that  $f$  is *holomorphic* iff at every point  $z_0$  in  $U$  the function  $f$  is *complex-differentiable*, i.e. if there is a point in  $\mathbb{C}$ , written  $f'(z_0)$ , such that

$$\frac{f(z_0 + h) - f(z_0)}{h} \rightarrow f'(z_0)$$

whenever  $h \rightarrow 0$  ( $h \in \mathbb{C} - \{0\}$ ).

Obviously, *polynomials* over  $\mathbb{C}$ , i.e. functions of the form

$$p(z) = a_0 + a_1 z + \dots + a_n z^n$$

with *coefficients*  $a_i$  in  $\mathbb{C}$ , are holomorphic over the whole complex plane  $\mathbb{C}$ . Note that complex differentiability is a quite strong notion of differentiability, since we are considering limits  $h = h_1 + ih_2 \rightarrow 0$  ( $h_1, h_2$  real) where we have  $h_1 \rightarrow 0$  and  $h_2 \rightarrow 0$  independently. Letting in particular  $h = h_1 + i \cdot 0 \rightarrow 0$ , we see that  $\partial_x f(z_0) = f'(z_0)$  and letting  $h = 0 + ih_2 \rightarrow 0$ , we get  $-i\partial_y f(z_0) = f'(z_0)$ . From this it follows that

$$\partial_{\bar{z}} f(z_0) = 0 \quad (1.1)$$

where we let  $\partial_{\bar{z}} := \partial_x + i\partial_y$ . Writing  $f = u + iv$  where  $u, v$  are functions  $U \rightarrow \mathbb{R}$ , we see that (1.1) is equivalent to the *Cauchy-Riemann-equations*

$$\partial_x u(z_0) = \partial_y v(z_0) \text{ and } \partial_x v(z_0) = -\partial_y u(z_0)$$

being fulfilled.

Let us now recall that a function  $F : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is differentiable at a point  $v_0$  in  $U$  iff there is a real  $2 \times 2$  matrix  $A$  and a norm  $\|\cdot\|$  on  $\mathbb{R}^2$  such that

$$\frac{\|F(v_0 + h) - F(v_0) - Ah\|}{\|h\|} \rightarrow 0$$

as  $h \rightarrow 0$  ( $h \neq 0$ ). Since all norms in  $\mathbb{R}^2$  are equivalent, the above notion of differentiability is independent of the choice of  $\|\cdot\|$ . Also, as one verifies,  $A$  is the Jacobian matrix given by  $(\partial_j F_i(v_0))_{1 \leq i, j \leq 2}$ .

Using our absolute value  $|\cdot|$  on  $\mathbb{C}$  defined above, which is of course a norm on  $\mathbb{C}$  viewed as  $\mathbb{R}^2$ , and considering again our holomorphic function  $f : U \subset \mathbb{C} \rightarrow \mathbb{C}$ , we have for every  $z_0$  in  $U$  that

$$|(f(z_0 + h) - f(z_0))/h - f'(z_0)| = |f(z_0 + h) - f(z_0) - f'(z_0)h|/|h| \rightarrow 0$$

as  $h \rightarrow 0$ . So we see in particular that complex-differentiability implies differentiability. If we write down the matrix of our linear map  $h \rightarrow f'(z_0)h = \partial_x f(z_0)h$  with respect to our (ordered) basis  $1, i$  of  $\mathbb{C}$ , we get

$$\begin{pmatrix} \partial_x u(z_0) & -\partial_x v(z_0) \\ \partial_x v(z_0) & \partial_x u(z_0) \end{pmatrix}. \quad (1.2)$$

**Lemma.** Given a function  $f : U \subset \mathbb{C} \rightarrow \mathbb{C}$ , where  $U$  is an open subset of  $\mathbb{C}$ , then following are equivalent:

1.  $f$  is complex-differentiable.

2.  $f$  is differentiable and  $\partial_{\bar{z}}f(z) = 0$  for all  $z$  in  $U$ .

*Proof.* We have already seen that 1. implies 2. Assume therefore that 2. holds and choose  $z_0$  in  $U$ . Writing down the Jacobian of  $f$  with respect to our basis  $1, i$  of  $\mathbb{C}$ , we get

$$\begin{pmatrix} \partial_x u(z_0) & \partial_y u(z_0) \\ \partial_x v(z_0) & \partial_y v(z_0) \end{pmatrix}.$$

Because of  $\partial_{\bar{z}}f(z) = 0$ , we may use the Cauchy-Riemann-equations on the second column of our matrix and get the matrix (1.2). From this 1. follows easily.

□

If we have reviewed the facts about complex analysis we need, we will use them to look more closely at some properties of the Riemann zeta function.