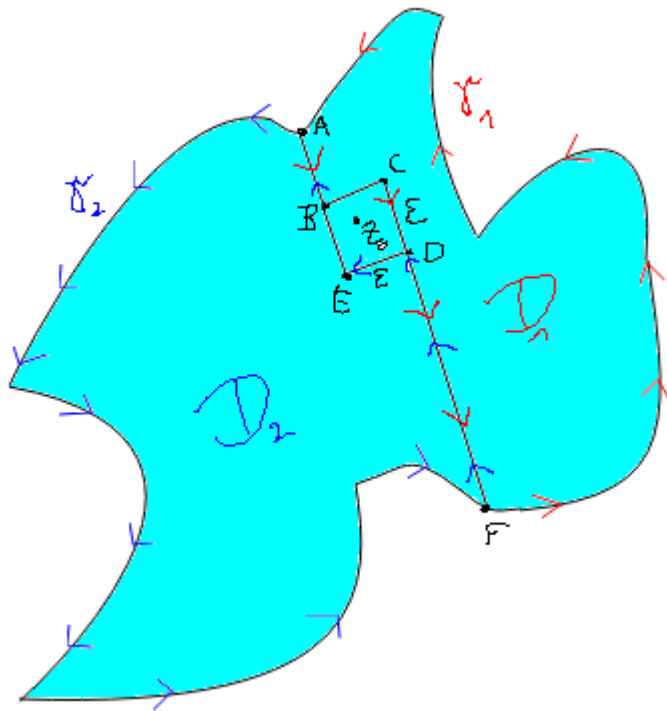


3. Cauchy's integral theorem II and Cauchy's integral formula

In this post, we extend Cauchy's theorem from the last post a little bit and give a corollary of it (another theorem of Cauchy).

In what follows, we let $f : U \subset \mathbb{C} \rightarrow \mathbb{C}$ with U open be a continuous function. Within U we assume lies a piecewise C^1 curve γ of Jordan type (in the picture below, this is represented as the boundary of the green colored region) together with its interior $\text{int}(\gamma)$. We will further assume f to be holomorphic on U except for one point z_0 in the interior of γ .

Since z_0 is an interior point, we may draw a small (closed) square $BCDE$ (see the following picture) of side length $\epsilon > 0$ around z_0 which lies in the interior of γ . Extending the lines EB and CD to the boundary, we get two piecewise C^1 curves of Jordan type, namely $\gamma_1 = \gamma_{1,\epsilon}$, following the red arrows and traveling along A, B, C, D, F, A , and $\gamma_2 = \gamma_{2,\epsilon}$, following the blue arrows through A, F, D, E, B, A , with interiors D_1 and D_2 :



Now we let $\gamma_0 = \gamma_{0,\epsilon}$ be the closed curve we get when we travel along B, C, D, E, B . With the help of Theorem 1 from the last post applied to f and

the curves $\gamma_{1,\epsilon}$ and $\gamma_{2,\epsilon}$ respectively, it follows that

$$0 = \int_{\gamma_{1,\epsilon}} f dz + \int_{\gamma_{2,\epsilon}} f dz = \int_{\gamma_{0,\epsilon}} f dz + \int_{\gamma} f dz,$$

where we use that in the first sum the integrals over the lines AB and DF cancel each other out since we integrate in different directions. So, since f is continuous and therefore bounded on the closure of $\text{int}(\gamma)$ (which is compact), letting $c := \sup_{z \in \text{int}(\gamma)} |f(z)| < \infty$ gives

$$\left| \int_{\gamma_{0,\epsilon}} f dz \right| \leq \text{length}(\gamma_{0,\epsilon}) \cdot c = 4\epsilon c \rightarrow 0$$

as $\epsilon \searrow 0$. So we have the following extension of our theorem of Cauchy we presented in the last post (which is also due to Cauchy):

Theorem 1. (*Cauchy's integral theorem*) *We let γ be a positively oriented piecewise C^1 curve of Jordan type. We let further $f : U \subset \mathbb{C} \rightarrow \mathbb{C}$ be a continuous function where U is open and contains the closure of the interior $\text{int}(\gamma)$ of γ . Then if f is holomorphic on U except for possibly one point z_0 in $\text{int}(\gamma)$, we have*

$$\int_{\gamma} f(z) dz = 0.$$

Denoting $D(r, z_0) \subset \mathbb{C}$ the (open) disc of radius r and center z_0 and $\partial D(r, z_0)$ its positively oriented boundary, we may easily deduce:

Theorem 2. (*Cauchy's integral formula*) *Let $f : U \subset \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic function. Let z_0 be in U and $r > 0$ for which the closure of $D := D(r, z_0)$ lies in U , then we have for every z in D*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} dw.$$

Proof. Let z be in D and let

$$g(w) = \begin{cases} \frac{f(w) - f(z)}{w - z} & \text{for } w \neq z \\ f'(z) & \text{for } w = z \end{cases}.$$

Since f is holomorphic on U , g is both holomorphic on $U - \{z\}$ and continuous on $w = z$ and therefore on U . Applying g and ∂D to Theorem 1 above and recalling that in the last post we found that $\int_{\partial D} 1/(w - z) dw = 2\pi i$, we get the assertion of our theorem:

$$\int_{\partial D} \frac{f(w)}{w - z} dw = \int_{\partial D} g(w) dw + \int_{\partial D} \frac{f(z)}{w - z} dw = 2\pi i f(z).$$

□

Note that Theorem 2 is rather remarkable. By differentiating under the integral sign with respect to z (we leave it to the reader to check that we are allowed to do this) and therefore essentially differentiating the function $1/(w-z)$ with respect to z , we find that *a holomorphic function is in fact infinitely many often complex differentiable (!)*:

Corollary 3. *Let $f : U \subset \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic function, then f is infinitely often complex differentiable with*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{n+1}} dw,$$

where $D = D(r, z_0)$ is as in Theorem 2.