5. Zeros and poles

Let us now talk about zeros and poles. Suppose we have a holomorphic function $f : U \subset \mathbb{C} \to \mathbb{C}$. A *zero* of f is any point z_0 in U such that $f(z_0) = 0$. By the analytic continuation theorem from the last post, it follows that on any region $G \subset U$ we must have that f is either completely zero or it has at most *isolated zeros,* i.e. the zeros of f have no limit point in G. Note that the latter means that for every isolated zero z_0 in U, we have $f(z_0) = 0$ and $f(z) \neq 0$ for all $z \neq z_0$ in a small non-empty open disc with center z_0. Given an isolated zero z_0 of f in U, then by writing f around z_0 as a complex power series (see Theorem 5 of our last post)

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

we see that there is a smallest integer $n \geq 1$ such that $a_n \neq 0$ and $a_k = 0$ for $k < n$. Note that by the uniqueness of the coefficients a_k (see Proposition 4 (2.) of the last post), we have that n is also unique. The number n is called the *order* (or the *multiplicity*) of our zero z_0. If $n = 1$ then z_0 is called a *simple zero.* So we have

$$f(z) = (z - z_0)^n [a_n + a_{n+1}(z - z_0) + ...] = (z - z_0)^n g(z),$$

where g is a holomorphic function which is non-vanishing in a small neighbourhood of z_0 and for which $g(z_0) = a_n \neq 0$.

To define what a pole is, let us recall that the *punctured disc* of radius R and center z_0, written $D'(z_0, R)$, is the open set

$$D(z_0, R) - \{z_0\} = \{z : 0 < |z - z_0| < R\}.$$

Suppose that the function f is holomorphic on a punctured disc $D' = D'(z_0, R)$ of positive radius R with center z_0. By the analytic continuation theorem, we may assume that f does not vanish on D'. We say that f has a *pole* at z_0 iff the function $1/f$ defined on D' can be extended holomorphically to a function of the full disc $D = D(z_0, R)$ with value 0 at z_0. By what we have reasoned above in the case of zeros, we have that there is a unique positive integer n with

$$1/f(z) = (z - z_0)^n g(z) \text{ or } f(z) = (z - z_0)^{-n} h(z),$$

where $h = 1/g$ is holomorphic and non-vanishing on the full disc $D = D(z_0, R)$. We call n the *order* of the pole at z_0 and call the pole a *simple pole* iff $n = 1$. Expanding h around z_0 as a complex power series (of positive convergence radius), renaming its unique coefficients appropriately and dividing the summands by $(z - z_0)^n$, we see that for every z in D' we have

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + ... + \frac{a_{-1}}{(z - z_0)} + r(z),$$

where $a_{-n} \neq 0$, the coefficients a_k are unique and where $r(z)$ is holomorphic on D. The sum above to the left of $r(z)$ is called the *principal part* of f at z_0 and a_{-1} is called the *residue* of f at z_0, written $\mathrm{Res}_{z_0} f$.

Now, let C be a positively oriented circle in D, where z_0 is in the interior of C. Recall that by Cauchys integral formula from the last post, we have for every constant function $F(z) = a$

$$F^{(k)}(z_0) = \frac{k!}{2\pi i} \int_C \frac{a}{(z - z_0)^{k+1}} dz,$$

where $F^{(0)}(z_0) = a$ and $F^{(k)}(z_0) = 0$ for $k \geq 1$. Using this on the principal part of f at z_0 and using Cauchys integral theorem on r, we see that

$$\int_C f(z)dz = \sum_{k=1}^{n} \int_C \frac{a_{-k}}{(z - z_0)^k} dz + \int_C r(z)dz = 2\pi i a_{-1} = 2\pi i \mathrm{Res}_{z_0} f. \qquad (0.1)$$

Using what we have observed above, we can prove the famous *residue theorem*:

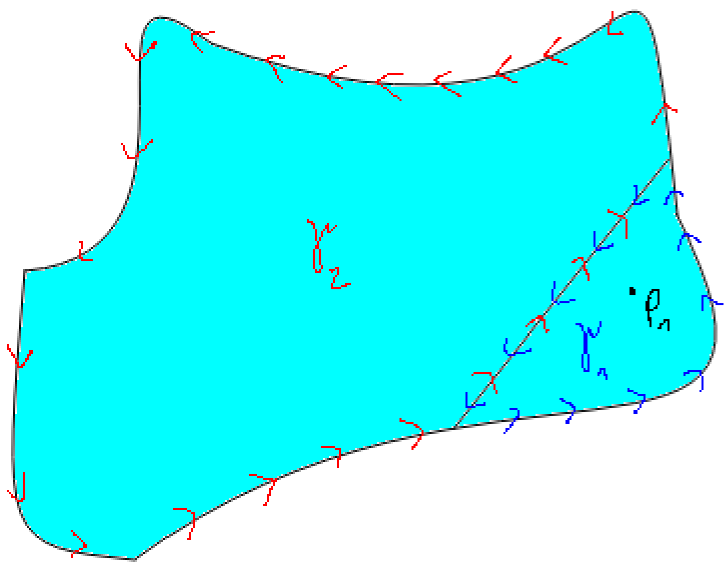
Theorem 1. *(Residue theorem) Let γ be a piecewise C^1 curve of Jordan type in the complex plane. Suppose further that we have a function f which is holomorphic on an open set containing γ and its interior, except for some poles at points $z_1, z_2, ..., z_n$ in the interior of γ. Then*

$$\int_\gamma f(z) = 2\pi i \sum_{k=1}^{n} Res_{z_i} f.$$

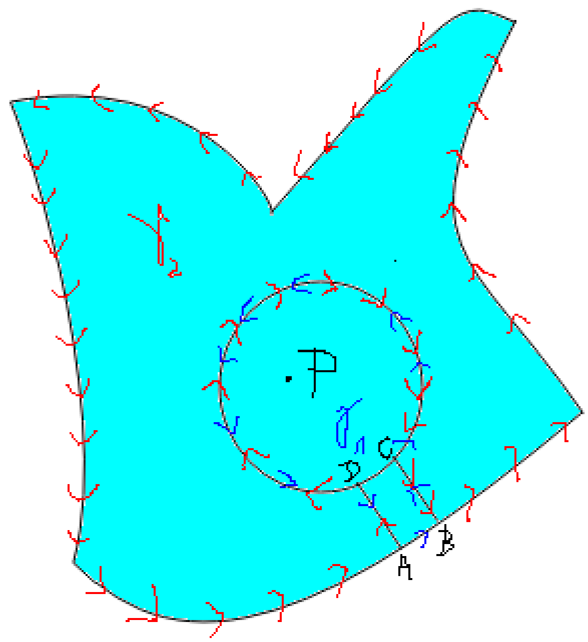
Proof. Writing

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$$

where γ_1 and γ_2 are the piecewise C^1 curves of Jordan type drawn in the following picture (for γ_1 we follow the blue arrows and for γ_2 we follow the red arrows, the boundary of the green area shows γ and during integration the additional path we introduced and which separates the interiors of γ_1 and γ_2 cancles out)



and where the interior of γ_1 contains exactly one pole P_1 of f, we see that we are by induction reduced to the case $n = 1$. For the proof of the case $n = 1$ we consider the following picture:



The circle around the pole P, which we will denote c, is assumed to be small enough such that the integral formula (0.1) holds. Now as in the picture indicated, we introduce, appart from γ which as always is depicted as the positively oriented boundary of the green region, two additional piecewise C^1 curves of Joran type, namely γ_1 and γ_2, where for γ_1 we travel along the blue arrows beginning say at D and going through A, B, C, around the circle to D again and where for γ_2 we travel along the red arrows beginning say at D going along the circle to C, then to B and along γ to A and to D again. Using Cauchys integral theorem on γ_2, we certainly have

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Introducing the piecewise C^1 curve of Jordan type γ_3 where we start say at C travel along c from C to D, then to A, to B and to C again, we see again by Cauchys integral theorem and the fact that during integration we have to pass the line between the points D, C shared by γ_3 and c in opposite directions,

$$\int_c f(z)dz = \int_c f(z)dz + \int_{\gamma_3} f(z) = \int_{\gamma_1} f(z)dz.$$

So putting everything together and using formula (0.1), we see that

$$\int_{\gamma} f(z)dz = \int_{c} f(z)dz = 2\pi i \mathrm{Res}_P f.$$

This concludes our proof. $\square$