2. Line integrals, Jordan curves and Cauchy's integral theorem I

The goal of this post is to prove a very famous and important theorem of Cauchy (with the help of Greens theorem).

We will first need some terminology. A *piecewise C^1 curve* in the complex plane $\mathbb{C}$ is any continuous map

$$\gamma : [a,b] \to \mathbb{C}$$

$(a < b)$ such that there is a decomposition of $[a,b]$

$$a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

for which the restriction of γ to each piece $[x_{i-1}, x_i]$ $(i = 1, 2, ..., n)$ is one times continuously differentiable. We let $\mathrm{im}(\gamma) := \gamma([a,b])$ and inspired by real analysis define *the length of* γ, written $\mathrm{length}(\gamma)$, to be the integral

$$\int_a^b \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2} dt = \int_a^b |\gamma'(t)| dt\,,$$

where $\gamma = \gamma_1 + i\gamma_2$ with real-valued functions γ_1, γ_2.

Recall that a complex 1-form on an open subset U of $\mathbb{C}$ is an expression

$$\alpha(z) = f_1(z)dx + f_2(z)dy$$

where f_1, f_2 are maps $U \subset \mathbb{C} \to \mathbb{C}$. These forms obviously can be added to each other and multiplied by functions $f : U \subset \mathbb{C} \to \mathbb{C}$ and form a module of rank two over the ring of all complex valued functions defined on U.

Given such a form α, where we assume f_1, f_2 to be continuous and given a piecewise C^1 curve $\gamma = \gamma_1 + i\gamma_2 : [a,b] \to \mathbb{C}$ $(a < b)$ such that $\gamma([a,b]) \subset U$, we define the *line integral of* α *with respect to* γ as

$$\int_\gamma \alpha := \int_a^b f_1(\gamma(t))\gamma_1'(t) + f_2(\gamma(t))\gamma_2'(t)dt.$$

We call any piecewise C^1 map $\psi : [c,d] \to [a,b]$ which is surjective and for which $\psi' > 0$ a *parameter transformation from* $[c,d]$ *to* $[a,b]$. Note that this notion of parameter transform respects the direction we travel on the curve. Now for γ as above also $\gamma \circ \psi$ will be a piecewise C^1curve and using the substitution rule of integration, we see that

$$\int_\gamma \alpha = \int_{\gamma\circ\psi} \alpha,$$

i.e. the integral is invariant of how we parametrize γ. Note that also the quantities $\mathrm{im}(\gamma)$ and $\mathrm{length}(\gamma)$ do not depend on how we parametrize γ. Given in particular a continuous function $f : U \subset \mathbb{C} \to \mathbb{C}$ and defining the 1-form

$$dz := dx + idy,$$

we may integrate the form

$$\alpha(z) = f(z)dz = f(z)dx + if(z)dy$$

and get

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Let us now come to an important example. For some $r > 0$, let

$$\gamma : [0, 2\pi] \to \mathbb{C}, t \mapsto z_0 + re^{it},$$

then $\gamma([0, 2\pi])$ is nothing but a circle of radius r around the point z_0. It is the boundary of the open disc $D = D(z_0, r)$ of radius r around z_0. Letting z be in D and $q := z - z_0/r$ and noting that $|q| < 1$ and $\gamma'(t) = ire^{it}$, we see that

$$\int_\gamma \frac{1}{w-z}dw = \int_0^{2\pi} \frac{ire^{it}}{z_0 - z + re^{it}}dt = i\int_0^{2\pi} \frac{1}{1 - qe^{-it}}dt.$$

Using the famous geometric series $1/(1-u) = \sum u^k$ for $|u| < 1$, we have

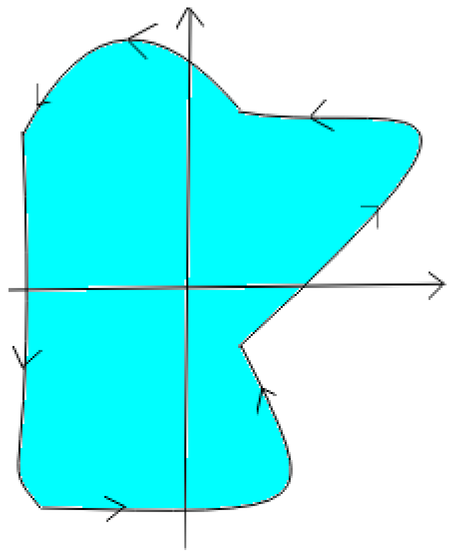
$$i\int_0^{2\pi} \sum q^k e^{-itk}dt = i\sum q^k \int_0^{2\pi} e^{-itk}dt = 2\pi i,$$

where we are allowed to interchange integration with summation (!). Putting everything together, we see that

$$\int_{\partial D} \frac{1}{w-z}dw = 2\pi i$$

for every $z \in D$. Note that our integral doesn't depend on the radius r we chose. Before we now get to the afore mentioned theorem of Cauchy, let us introduce more terminology:

A piecewise C^1 curve γ is said to be *of Jordan type* iff we have $\gamma(a) = \gamma(b)$, γ restricted to $[a, b[$ is injective and lastly $\mathbb{C} - \mathrm{im}(\gamma)$ consists of exactly two connected components of which exactly one is relatively compact, we call the relatively compact one *the interior* and denote $\mathrm{int}(\gamma)$. F.e. the curve we used in our example above, the circle arount a point z_0 of some radius r, is of Jordan type. Given a piecewise C^1 curve of Jordan type, we say that γ is *positively oriented*, iff "traveling" on γ, the interior should be on the left:



Finally, let us recall *Greens theorem*: Let $D \subset \mathbb{C}$ be the interior of a positively oriented piecewise C^1 curve γ of Jordan type, then given two C^1 functions $u, v : U \subset \mathbb{C} \to \mathbb{R}$ where U is open and contains the closure of D, then

$$\int_\gamma u dx + v dy = \int_D \partial_x v - \partial_y u dx dy.$$

Theorem 1. *(Cauchy's integral theorem) We let γ be a positively oriented piecewise C^1 curve of Jordan type. Then, if $f : U \subset \mathbb{C} \to \mathbb{C}$ is a holomorphic function which contains the closure of the interior of γ, then*

$$\int_\gamma f(z) dz = 0.$$

Proof. Let D be the interior of γ and write $f = u + iv$, where u, v are realvalued. Then by Greens theorem

$$\begin{aligned} \int_\gamma f(z) dz &= \int_\gamma u dx - v dy + i \int_\gamma v dx + u dy \\ &= \int_D -\partial_x v - \partial_y u dx dy + i \int_D \partial_x u - \partial_y v dx dy \\ &= i \int_D \partial_{\bar{z}} f(z) dx dy = 0 \end{aligned}$$

by the charactarization of holomorphic functions given in Lemma 2 of our previous post. $\square$