1. Holomorphic functions

Before we start, let us very briefly recall the complex numbers. We define the *complex numbers*, denoted $\mathbb{C}$, to be the set of pairs (x, y) of real numbers together with the following two binary operations, called the "addition" and the "multiplication" of $\mathbb{C}$, defined by

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

and

$$(x_1, y_1) \cdot (x_2, y_2) := (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$$

for all real numbers x_i, y_i $(i = 1, 2)$. Using the laws of distributivity and commutativity in $\mathbb{R}$, one easily verifies that the operations $+, \cdot$ introduced above are also distributive and commutative. Furthermore, one checks that $(\mathbb{C}, +)$ and $(\mathbb{C} - \{(0, 0)\}, \cdot)$ form abelian groups with neutral elements $(0, 0)$ and $(1, 0)$ respectively. F.e. let $z = (x, y) \neq 0$, then the multiplicative inverse of (x, y) is given by

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right).$$

So $\mathbb{C}$ is in fact a field. Moreover, the map

$$\phi : \mathbb{R} \ni x \mapsto (x, 0) \in \mathbb{C}$$

is an injection of $\mathbb{R}$ into $\mathbb{C}$, i.e. it is an injective map which is "structure preserving" in the sense that

$$\phi(x_1 + x_2) = \phi(x_1) + \phi(x_2) \text{ and } \phi(x_1 x_2) = \phi(x_1)\phi(x_2)$$

for all x_1, x_2 in $\mathbb{R}$. So we may identify $\mathbb{R}$ as a subfield of $\mathbb{C}$, which we will do in what follows. Now, if we introduce the vector $i = (0, 1)$, then $\mathbb{C}$ as a real vector space of dimension two will have $1 = (1, 0)$ and i as a basis so that every z in $\mathbb{C}$ can be uniquely written as $z = x + yi$ where x, y are real numbers. Also, we introduce the *complex conjugate* $\bar{z}$ of the complex number $z = x + yi$ as the number $x - yi$. Note that $z\bar{z} = x^2 + y^2$. We let $|z| := \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$ and get the Euclidean distance of the vector $z = (x, y)$ from the origin. This is an absolute value. In particular, we have

$$|z_1 z_2| = |z_1||z_2|$$

for all z_1, z_2 in $\mathbb{C}$. Finally, we will denote ∂_x, ∂_y the partial derivatives with respect to x, y.

Let us now come to the central object of study.

Definition 1. Given a function

$$f : U \subset \mathbb{C} \to \mathbb{C}$$

where $U \subset \mathbb{C}$ is open, we say that f is *holomorphic* iff at every point z_0 in U the function f is *complex-differentiable*, i.e. if there is a point in $\mathbb{C}$, written $f'(z_0)$, such that

$$\frac{f(z_0+h)-f(z_0)}{h} \to f'(z_0)$$

whenever $h \to 0$ ($h \in \mathbb{C} - \{0\}$).

Obviously, *polynomials* over $\mathbb{C}$, i.e. functions of the form

$$p(z) = a_0 + a_1 z + ... + a_n z^n$$

with *coefficients* a_i in $\mathbb{C}$, are holomorphic over the whole complex plane $\mathbb{C}$. Note that complex differentiability is a quite strong notion of differentiability, since we are considering limits $h = h_1 + ih_2 \to 0$ (h_1, h_2 real) where we have $h_1 \to 0$ and $h_2 \to 0$ independently. Letting in particular $h = h_1 + i \cdot 0 \to 0$, we see that $\partial_x f(z_0) = f'(z_0)$ and letting $h = 0 + ih_2 \to 0$, we get $-i\partial_y f(z_0) = f'(z_0)$. From this it follows that

$$\partial_{\bar{z}} f(z_0) = 0 \tag{0.1}$$

where we let $\partial_{\bar{z}} := \partial_x + i\partial_y$. Writing $f = u + iv$ where u, v are functions $U \to \mathbb{R}$, we see that (0.1) is equivalent to the *Cauchy-Riemann-equations*

$$\partial_x u(z_0) = \partial_y v(z_0) \text{ and } \partial_x v(z_0) = -\partial_y u(z_0)$$

being fulfilled.

Let us now recall that a function $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at a point v_0 in U iff there is a real 2×2 matrix A and a norm $\|\cdot\|$ on $\mathbb{R}^2$ such that

$$\frac{\|F(v_0+h) - F(v_0) - Ah\|}{\|h\|} \to 0$$

as $h \to 0$ ($h \neq 0$). Since all norms in $\mathbb{R}^2$ are equivalent, the above notion of differentiability is independent of the choice of $\|\cdot\|$. Also, as one verifies, A is the Jacobian matrix given by $(\partial_j F_i(v_0))_{1 \le i,j \le 2}$.

Using our absolute value $|\cdot|$ on $\mathbb{C}$ defined above, which is of course a norm on $\mathbb{C}$ viewed , and considering again our holomorphic function $f : U \subset \mathbb{C} \to \mathbb{C}$, we have for every z_0 in U that

$$|(f(z_0+h) - f(z_0))/h - f'(z_0)| = |f(z_0+h) - f(z_0) - f'(z_0)h|/|h| \to 0$$

as $h \to 0$. So we see in particular that complex-differentiability implies differentiability. If we write down the matrix of our linear map $h \to f'(z_0)h = \partial_x f(x_0)h$ with respect to our (ordered) basis $1, i$ of $\mathbb{C}$, we get

$$\begin{pmatrix} \partial_x u(z_0) & -\partial_x v(z_0) \\ \partial_x v(z_0) & \partial_x u(z_0) \end{pmatrix}. \tag{0.2}$$

Lemma 2. Given a function $f : U \subset \mathbb{C} \to \mathbb{C}$, where U is an open subset of $\mathbb{C}$, then following are equivalent:

1. f is complex-differentiable.

2. f is differentiable and $\partial_{\bar{z}} f(z) = 0$ for all z in U.

Proof. We have already seen that 1. implies 2. Assume therefore that 2. holds and choose z_0 in U. Writing down the Jacobian of f with respect to our basis $1, i$ of $\mathbb{C}$, we get

$$\begin{pmatrix} \partial_x u(z_0) & \partial_y u(z_0) \\ \partial_x v(z_0) & \partial_y v(z_0) \end{pmatrix}.$$

Because of $\partial_{\bar{z}} f(z) = 0$, we may use the Cauchy-Riemann-equations on the second column of our matrix and get the matrix (0.2). From this 1. follows easily.

□

If we have reviewed the facts about complex analysis we need, we will use them to look more closely at some properties of the Riemann zeta function.