In this post, we first discuss two methods of how we can get new holomorphic functions out of old ones. The first method is via a parameter integral:

**Proposition 1.** *Suppose that we have a piecewise $C^1$ curve $\gamma : [a,b] \to \mathbb{C}$ and a continuous function $F : im(\gamma) \times U \to \mathbb{C}$ with $U \subset \mathbb{C}$ open. Suppose further that for every $w_0 \in im(\gamma)$ the function $z \mapsto F(w_0, z)$ on $U$ is complex differentiable and that the function $(w,z) \mapsto \partial_z F(w,z)$ is continuous. Then the function*

$$G : U \to \mathbb{C}, z \mapsto \int_\gamma F(w,z)dw$$

*is also complex differentiable with $G'(z) = \int_\gamma \partial_z F(w,z)dw$.*

*Proof.* Let $z$ be in $U$. Since $F$ is continuous, we have that for every disc $D = D(0,r)$ with center zero for which $D + z \subset U$ is relatively compact, the function

$$\psi(w,h) := \begin{cases} F(w, z+h) - F(w,z)/h & \text{for } h \neq 0 \\ \partial_z F(w,z) & \text{for } h = 0 \end{cases}$$

is uniformly continuous on $\mathrm{im}(\gamma) \times D$. So integrating $\psi(w,h)$ along $\gamma$ and letting $h \to 0$, we get the assertion of our proposition. □

As a second method we now show that we may get holomorphic functions out of locally uniformly convergent series of holomorphic functions:

**Proposition 2.** *Suppose we have a sequence $(f_n)_{n\geq 0}$ of holomorphic functions $U \subset \mathbb{C} \to \mathbb{C}$. Then, if the infinite series $\sum_{n=0}^\infty f_n$ converges locally uniformly on $U$ to the function $f : U \to \mathbb{C}$, then $f$ itself is holomorphic and we have that $f^{(n)}$ is given by the infinite series $\sum_{k=0}^\infty f_k^{(n)}$, which is also locally uniformly convergent.*

*Proof.* First note that since $f = \sum f_k$ is locally uniformly convergent and the $f_k$ are in particular continuous, $f$ must be continuous (proof is left to the reader). Let $D \subset U$ be a relatively compact disc of radius $> 0$ with positively oriented boundary $\partial D$, then $(f_k)$ converges on $D$ uniformly. By definition this means that given $\epsilon > 0$, there is an integer $N$ such that

$$|f(z) - \sum_{k\leq m} f_k(z)| < \epsilon$$

for all $m \geq N$ and $z \in D$. Choosing $n \geq 0$ and letting

$$F_n(z) := \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{n+1}} dw,$$

which is of course holomorphic on $D$ by Proposition 1, then for every compact subset $K$ of $D$ and every $z$ in $K$

$$\left|F_n(z) - \sum_{k\leq m} f^{(n)}(z)\right| \leq \frac{n!}{2\pi}\int_{\partial D}\left|\frac{f(w) - \sum_{k\leq m} f_k(w)}{(w-z)^{n+1}}\right| dw \leq C_n \cdot \epsilon,$$

where $C_n = n!\mathrm{length}(\partial D)/2\pi\mathrm{dist}(K, \partial D)^{n+1} > 0$ with

$$\mathrm{dist}(K, \partial D) := \inf\{|z_1 - z_2| : z_1 \in K, z_2 \in \partial D\} > 0$$

and where we have used Cauchy's integral formula for the $n-$th derivative of a holomorphic function. Now, since $D, \epsilon$ and $K$ were chosen arbirary, we see for all $n$ that $F_n = \sum f_k^{(n)}$ locally uniformly, but since $F_0 = f$ by assumption, we have that $f$ is holomorphic and that $F_n = f^{(n)}$ for all $n \geq 0$. □

We will now look at complex power series. Given a point $z_0$ in the complex plane, then a *complex power series in $z$ around $z_0$* is an infinite series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

where the numbers $a_n$ are in $\mathbb{C}$. Recall that for a sequence $(b_n)_{n\geq 0}$ of real numbers one defines the *limes superior*, written $\overline{\lim}_{n\geq 0} b_n$, to be

$$\lim_{n\to\infty}\sup_k\{b_k : k \geq n\}$$

which, being the limes of a monotonically decreasing sequence, is a unique element in $[-\infty, \infty]$. To $f$ above we may associate a unique element $R$ out of $[0, \infty]$ given by $R = 1/L$, where $L = \overline{\lim}_{n\geq 0} \sqrt[n]{|a_n|}$. $R$ is called the *convergence radius* of $f$. Furthermore, the open disc $D = D(z_0, R)$ around $z_0$ of radius $R$ will be called the *(open) disc of convergence* of $f$. The names we gave $R$ and $D$ are justified by the following well known result on power series:

**Theorem 3.** *Let $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a complex power series with convergence radius $R$. Let further $D = D(z_0, R)$ be the disc of convergence of $f$. Then $f$ converges absolutely and locally uniformly on $D$. It may converge at points on the boundary $\partial D$ of $D$ and it diverges outside the closure $\overline{D}$ of $D$.*

*Proof.* (omitted) □

Let us give two more facts about complex power series which we will use below:

**Proposition 4.** *Let $f(z) = \sum a_n(z - z_0)^n$ be a complex power series. Then the following hold:*

1. *$f$ restricted to its disc of convergence gives a holomorphic function.*

2. *Let $g(z) = \sum b_n(z - z_0)^n$ be a second complex power series, and assume that both $f$ and $g$ have a positive radius of convergence. Suppose further that for a sequence $(w_k)_{k \geq 0}$ of points in $\mathbb{C} - \{z_0\}$ which converges to $z_0$ we have $f(w_k) = g(w_k)$ for almost all $k \geq 0$, then $a_n = b_n$ for all $n \geq 0$.*

*Proof.* Assertion 1. follows directly from Proposition 2 where we let $f_n(z) = a_n(z - z_0)^n$. For assertion 2., note first that $\lim f(w_k) = f(z_0) = g(z_0) = \lim g(w_k)$, i.e. we have $a_0 = b_0$. Assuming now that $a_k = b_k$ for $k = 0, 1, 2, ..., n$ for some $n \geq 0$, then the function

$$F(z) := \frac{f(z) - g(z)}{(z - z_0)^{n+1}}$$

is locally continuous around $z_0$ and we have $F(w_k) = 0$ for almost all $k$. So we have $\lim F(w_k) = 0$, i.e. $a_{n+1} = b_{n+1}$. By induction 2. follows. ☐

Having introduced complex power series, we may now formulate another characterization of a function being holomorphic.

**Theorem 5.** *A function $f : U \subset \mathbb{C} \to \mathbb{C}$ is holomorphic iff at every point $z_0$ in $U$ and for every $R > 0$ such that the closure of $D(z_0, R)$ lies in $U$, we have that $f$ restricted to $D(z_0, R)$ is a complex power series around $z_0$ with a convergence radius larger than $R$.*

*Proof.* "$\Rightarrow$": Suppose $f$ is holomorphic, then by Corollary 3 of our last post, $f$ is infinitely many times complex differentiable. So given $z_0$ in $U$, we may consider the (formal) power series

$$F(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Using the integral representation of $f^{(n)}(z_0)/n!$ from Corollary 3, then for every $D = D(z_0, R)$ whose closure lies in $U$, we have

$$\left|\frac{f^{(n)}(z_0)}{n!}\right| = \left|\frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{(z - z_0)^{n+1}} dz\right| \leq \frac{1}{2\pi} \cdot 2\pi R \cdot C \cdot R^{-(n+1)} = C/R^n.$$

where $C = \sup_{z \in D}|f(z)|$. From this we get $\overline{\lim}_{n \geq 0} \sqrt[n]{\left|\frac{f^{(n)}(z_0)}{n!}\right|} \leq 1/R$, using that for a positive constant $c$ we have $\sqrt[n]{c} \to 1$ as $n \to \infty$. Since we may make

$D$ a little larger by slightly increasing $R$ and have that the closure still stays in $U$, we see that $f$ restricted to $D$ is a complex power series around $z_0$ with convergence radius greater than $R$.

"$\Leftarrow$": Since at every point $z_0$ the function $f$ a complex power series with positive convergence radius, it is holomorphic there. So $f$ itself is holomorphic. $\square$

Let us now end this post with a remarkable fact about holomorphic functions. We say that an open subset $G$ of $\mathbb{C}$ is a *region* (*"Gebiet"* in german) iff it cannot be written as a union of two non-empty and disjoint open sets. Then the following holds:

**Theorem 6.** *(Analytic continuation) Given two holomorphic functions $f, g : G \subset \mathbb{C} \to \mathbb{C}$, where $G$ is a region in $\mathbb{C}$. Suppose that $f(z) = g(z)$ on a sequence of distinct points with limit point in $G$. Then $f(z) = g(z)$ throughout $G$.*

*Proof.* Combining Theorem 5 and Proposition 4. (2.), we see that $f$ and $g$ agree on a small open disc $D \subset G$ of positive radius.
We let $U_1 := D$ and for $n \geq 1$ build the set $U_{n+1}$ out of $U_n$ as the union of all relatively compact open discs in $G$ whose centers are in $U_n$. Then obviously for all $n \geq 1$ we have $U_n \subset U_{n+1} \subset G$, $U_n$ is open and we have $f(z) = g(z)$ on $U_n$ by using Theorem 5 and Proposition 4. (2.). We let $V$ be the open subset $\cup_{n \geq 1} U_n$ of $G$ and let $V' := G - V$. Suppose now there was a $z_0$ in $V'$ which was a boundary point of $V$. Then we would have that very close to $z_0$ there was a point $w_0$ in one of the sets $U_n$ for which a relatively compact disc in $G$ with center $w_0$ existed which contained $z_0$ , i.e. then $z_0 \in U_{n+1} \subset V$, a contradiction. So $V'$ contains no boundary point of $V$ and must therefore be open since $G$ is open. But since $G = V \cup V'$ is a region and $V \neq \emptyset$, we must have $V' = \emptyset$, which concludes our proof. $\square$