7. Meromorphic functions and products

In this post, we look at products. Given an open subset U of $\mathbb{C}$, then a *meromorphic function* on U is a any holomorphic function $f : U - N_f \to \mathbb{C}$, where N_f is a discrete subset of U where f has poles. F.e., given a non-zero polynomial $p(z)$, then $f(z) = 1/p(z)$ is a meromorphic function on $\mathbb{C}$ where N_f is the set of zeros of $p(z)$. N_f in this case is not only discrete but also finite. In fact Gauss proved the *fundamental theorem of algebra* which says that for every non-zero complex polynomial $p(z)$ we have

$$p(z) = c \prod_{i=1}^{n} (z - \lambda_i)$$

for some complex numbers λ_i, c where $c \neq 0$ and where the unique number $n \geq 0$ is called the *degree* of $p(z)$. Using induction on the degree of $p(z)$, this is equivalent to saying that every complex polynomial of positive degree has a zero. Since for every non-constant polynomial $p(z)$ we have that $p(z) \to \infty$ as $z \to \infty$, we see that the fundamental theorem of algebra is then a consequence of

Theorem 1. *(Liouville) Given a holomorphic function f on the complex plane which has no zeros and such that $1/f$ is bounded, then f must be constant.*

Proof. By assumption, there is a real number $c > 0$ such that $|1/f(z)| \leq c$ for all complex numbers z. Given a complex number z_0, then we have for every circle C_r of radius r and center z_0 that

$$|(1/f)'(z_0)| = \left| \frac{1}{2\pi i} \int_{C_r} \frac{1}{f(z)(z - z_0)^2} dz \right| \leq \frac{1}{2\pi} 2\pi r c r^{-2} = \frac{c}{r}$$

for all $r > 0$. Letting $r \to \infty$, we see that $(1/f)'(z_0) = 0$. And since c was arbitrary, we see that $(1/f)'(z) = 0$ for all z in the plane, i.e. $1/f$ and therefore also f must be constant. $\square$

Obviously, given two meromorphic functions f_1, f_2 on an open subset U of the complex plane, then also their sum $f_1 + f_2$ and their (pointweise) product $f_1 f_2$ is meromorphic. Moreover, if f_1, f_2 are meromorphic with f_2 being non-trivial, i.e. not completely zero, in any region within U, then f_1/f_2 is meromorphic since by analytic continuation f_2 can only have a discrete set of zeros.

We will call a function $D : U \to \mathbb{Z}$ a *divisor* on U iff the set $\mathrm{supp}(D)$ of points where D is non-zero is discrete. We say that a divisor D is *non-negative* iff $D(z) \geq 0$ for all z in U. Furthermore, D is called *finite* iff $\mathrm{supp}(D)$ is a finite set and *infinite* otherwise. If f is a meromorphic function which is non-trivial in every region of U, then letting $D(z) := n$ if f has a zero of order n at z, $D(z) := -n$ if f has a pole of order n at z and $D(z) = 0$ everywhere else in U, gives a divisor on U which we will denote D_f. If f is holomorphic, then D_f is non-negative. Also

$$D_{fg} = D_f + D_g \text{ and especially } D_{1/f} = -D_f. \tag{0.1}$$

Divisors can be added and subtracted and clearly form an abelian group which will be denoted $\mathrm{Div}(U)$. Sometimes it is convenient to write D as the well defined sum

$$D = \sum_{w \in U} D_w,$$

where D_w is the divisor with $D_w(w) = D(w)$ and $D_w(z) = 0$ for $z \neq w$. Consider the map

$$\left\{ \begin{array}{c} f \text{ meromorphic and non-trivial} \\ \text{on every region of } U \end{array} \right\} \to \mathrm{Div}(U), f \mapsto D_f \tag{0.2}$$

A divisor in the image of this map will be called a *principal divisor*. Is every divisor a principal divisor? I.e. given a divisor D on U, is there always a meromorphic function f on U where $D = D_f$? The well known answer to this question is ***yes***! We will see this shortly in the case $U = \mathbb{C}$.

Recall that the infinite product $\prod_{n \geq 0}(1 - a_n)$ converges absolutely iff $\sum a_n$ converges absolutely. Moreover, a product $\prod(1 - f_n(z))$ converges absolutely and uniformly on a subset U of the complex plane iff $\sum f_n(z)$ does.

Theorem 2. *Every divisor in Div($\mathbb{C}$) is a principal divisor.*

Proof. Let $D = \sum_{w \in \mathbb{C}} D_w$ be a divisor on $\mathbb{C}$. Because of (0.1) and the fact that for every finite divisor D on $\mathbb{C}$, there is obviously a polynomial p with $D = D_p$, we may assume that D is positive and infinite. Suppose that for every w in $\mathrm{supp}(D)$ we have a holomorphic function $f_w(z)$ on $\mathbb{C}$ such that $D_w = D_{f_w}$ and such that $f(z) = \prod(1 + f_w(z))$ is absolutely and uniformly convergent, then $f(z)$ satisfies $D = D_f$. $\square$