# 6. Logarithms

In this post we we will talk about logarithms. To do that, let first us recall/reintroduce the exponential. The *exponential function* is defined as the complex power series

$$\exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + z^2/2 + z^3/6 + \ldots .$$

Since $R = 1/\lim\,\sup_{k\geq 0} \sqrt[k]{1/k!} = \infty$, exp converges everywhere absolutely and locally uniformly. It therefore gives an example of a so called *analytic function*, i.e. a function which is holomorphic on the whole complex plane. Using the famous binomial theorem, we get

$$\frac{(z+w)^n}{n!} = \frac{1}{n!} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) z^k w^{n-k} = \sum_{k+l=n} \frac{z^k}{k!} \frac{w^l}{l!}.$$

Summing this over all $n \geq 0$, we see that for all complex numbers $z, w$ we have

$$\exp(z+w) = \exp(z)\exp(w).$$

This is called the *functional equation* for the exponential function. Since in particular

$$1 = \exp(z)\exp(-z),$$

we see that exp vanishes nowhere on $\mathbb{C}$ and that it is positive on $\mathbb{R}$, since it is positive for every non-negative number and therefore has to be positive on the negative numbers. Also, since for $x$ in $\mathbb{R}$ we have

$$\overline{\exp(ix)} = \exp(\overline{ix}) = \exp(-ix),$$

it follows also that $|\exp(ix)| = 1$, i.e. that $\exp(ix)$ lies on the unit circle. More precisely, introducing the complex power series

$$\cos(z) := \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \text{ and } \sin(z) := \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!},$$

which are, as one easily checks, both analytic, then obviously

$$\exp(iz) = \cos(z) + i\sin(z).$$

So we have in particular for real numbers $x$ that

$$\exp(ix) = \cos(x) + i\sin(x) \text{ and } 1 = \cos(x)^2 + \sin(x)^2.$$

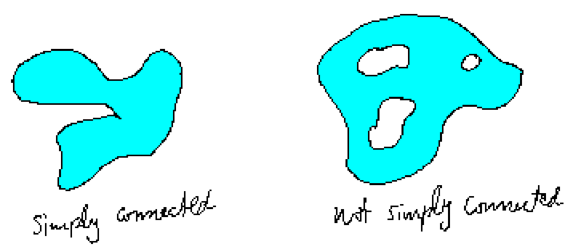
Finally, the reader may verify that using $\exp(ix)$, we hit every point on the unit circle. So every point in the complex plane, except zero, can be expressed as

$$\exp(x+iy) = r \cdot (\cos(\alpha) + i\sin(\alpha))$$

where we let $y = \alpha$ and $r = \exp(x) > 0$.

Having considered (briefly) the exponential function as a function on the complex plane, let us now come to logarithm functions. First some notation:

Given two points $z_1, z_2$ in the complex plane, then an *integration path from* $z_1$ *to* $z_2$ is any piecewise $C^1$ curve $\gamma : [a, b] \to \mathbb{C}$ $(a < b)$ which is injective on $[a, b[$ and for which $\gamma_z(a) = z_1, \gamma_z(b) = z_2$. The point $z_1$ is then called the *start point* and $z_2$ is called the *end point* of $\gamma$. As defined earlier, $\gamma$ is called *closed* iff start and end point are equal and it is called *of Jordan type* iff it is closed and positively oriented. Let $U$ be an open subset of $\mathbb{C}$. We call $U$ *path-connected* iff for every pair $z_1, z_2$ of points in $U$ there is an integration path from $z_1$ to $z_2$. Furthermore, we call $U$ *simply connected*, iff $U$ is path-connected and contains the interiors of all the closed integration paths within $U$:



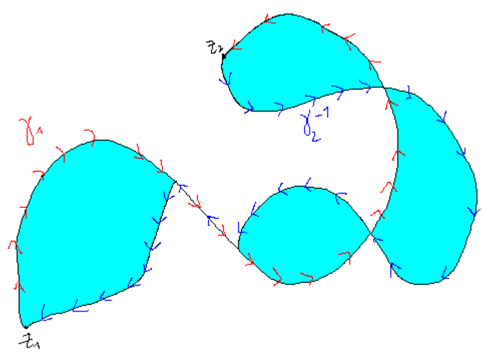

Finally, we call a subset $U$ of $\mathbb{C}$ a *simply connected domain* or a *domain without holes* iff $U$ is open and simply connected.

Using Cauchy's integral theorem, we first prove the following proposition:

**Proposition 1.** *Let $\gamma_1, \gamma_2$ be two integration paths which have the same start and end points and which lie within a simply connected domain $U \subset \mathbb{C}$. Then we have for every holomorphic function $f : U \subset \mathbb{C} \to \mathbb{C}$ that*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

*Proof.* Let us denote the start point of our curves $z_1$ and their end point $z_2$. Denoting $\gamma_1\gamma_2^{-1}$ the closed curve where we travel first along $\gamma_1$ from $z_1$ to $z_2$ and then in reversed direction along $\gamma_2$ from $z_2$ back to $z_1$ ($\gamma_2$ reversed is usually denoted $\gamma_2^{-1}$), we get a closed curve which consists of finitely many closed integration paths that are connected by lines which are travelled in both directions exactly once:




Obviously, we have

$$\int_{\gamma_1\gamma_2^{-1}} f(z)dz = \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz,$$

where we use the fact that integration over a curve in the opposite direction changes the sign of the integral. Since the integral on the left vanishes on closed integration paths by Cauchy's integral theorem and since the integration of a line in both directions exactly once gives zero, we have that our integral on the left is zero and the proposition follows. □

In light of the previous proposition, we may make the following definition:

**Definition 2.** Let $U \subset \mathbb{C}$ be a simply connected domain and $f : U \to \mathbb{C}$ a holomorphic function. Then for every integration path $\gamma$ from $z_1$ to $z_2$ in $U$, we write

$$\int_{z_1}^{z_2} f(z)dz := \int_{\gamma} f(z)dz.$$

As a consequence of our Proposition 1, we have the following corollary:

**Corollary 3.** *Let $U \subset \mathbb{C}$ be a simply connected domain and let $z_0$ be a point in $U$. Then for every holomorphic function $f : U \to \mathbb{C}$, the function*

$$F : U \to \mathbb{C}, z \mapsto \int_{z_0}^{z} f(w)dw$$

*is holomorphic with $F' = f$. If $G' = f$ for another holomorphic function $G$ on $U$, then $F = G + c$ for some constant $c$ in $\mathbb{C}$.*

*Proof.* Let $z$ be in $U$. Then for all $h$ with $z+h$ in $U$, we have by our proposition above

$$(F(z+h) - F(z))/h = \frac{1}{h}\int_{z}^{z+h} f(w)dw.$$

For $h$ small enough, the line sement $\gamma(t) = z + th$ with $t \in [0,1]$ and where $\gamma'(t) = h$ lies within $U$ and it follows

$$\frac{1}{h}\int_z^{z+h} f(w)dw = \frac{1}{h}\int_\gamma f(w)dw = \int_0^1 f(z+th)dt \to f(z)$$

as $h \to 0$. So the the first assertion of our collary follows. For the second assertion, note that $(F-G)' = 0$ on $U$ and that therefore $F-G$, written locally as a powerseries, must be constant because of the uniqueness of powerseries coefficients. ☐

**Definition 4.** A *logarithm* of a holormorhic function $F : U \to \mathbb{C}$ is any holomorphic function $f : U \to \mathbb{C}$ such that

$$F = e^f$$

on $U$.

**Theorem 5.** *If $U \subset \mathbb{C}$ is a simply connected domain, then any holomorphic function $F : U \to \mathbb{C}$ which is non-vanishing on $U$ possesses a logarithm. In fact, every logarithm of $F$ can be written as*

$$f(z) = \int_{z_0}^{z} \frac{F'(w)}{F(w)}dw + C$$

*for some fixed $z_0$ in $U$ and some constant $C$ in $\mathbb{C}$. In particular, two logarithms of $F$ differ only by a constant.*

*Proof.* First note that if $f$ is a logarithm of $F$, then $F' = f'e^f = f'F$, or $f' = F'/F$, so by Corollary 3 above, $f$ has an integral-representation as claimed in our theorem.

Let now $G(z) := e^{g(z)}$ where

$$g(z) = \int_{z_0}^{z} \frac{F'(w)}{F(w)}dw$$

with $z \in U$. We show that $cG = F$ for some non-zero constant $c$ in $\mathbb{C}$. Then for every $C$ with $c = e^C$, the function $f = g + C$ will be a logarithm of $F(z)$. We have

$$G'(z) = g'(z)G(z) = \frac{F'(z)}{F(z)}G(z)$$

or

$$G'(z)F(z) = F'(z)G(z) \tag{0.1}$$

on $U$. Fixing a $w$ in $U$ and expanding $F, G$ around $w$ as power series

$$F(z) = \sum_{k=0}^{\infty} a_k(z-w)^k \text{ and } G(z) = \sum_{k=0}^{\infty} b_k(z-w)^k,$$

where $a_0, b_0 \neq 0$ because $F, G$ do not vanish everywhere on $U$, we will show by induction that $a_n = cb_n$ where $c = a_0/b_0$ and $n \geq 0$:

Obviously, we have $a_0 = cb_0$. Suppose now that for some $n \geq 0$, we have $a_k = cb_k$ when $n \geq k \geq 0$. Using equation (0.1) and comparing the $n-$th coefficient of both sides, we have

$$\sum_{k=0}^{n}(k+1)b_{k+1}a_{n-k} = \sum_{k=0}^{n}(k+1)a_{k+1}b_{n-k}$$

or

$$(n+1)b_{n+1}a_0 = (n+1)a_{n+1}b_0 + \sum_{k=0}^{n-1}(k+1)(a_{k+1}b_{n-k} - b_{k+1}a_{n-k}).$$

So since for $k = 0, .., n-1$ we have by induction

$$a_{k+1}b_{n-k} - b_{k+1}a_{n-k} = cb_{k+1}b_{n-k} - cb_{k+1}b_{n-k} = 0,$$

it follows $a_{n+1} = cb_{n+1}$. So since $U$ is in particular a region, we have $F = cG$ by analytic continuation. This concludes our proof. $\square$