Advanced Statistics

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Lecture 2

Back to our dice problem

- IN THEORY, each of our dice roll outcomes has the same probability:

$$
\begin{aligned}
p(1) = \frac{n_1}{n_{total}} = \frac{1}{6}, p(2) = \frac{n_2}{n_{total}} &= \frac{1}{6}\\
p(3) = \frac{n_3}{n_{total}} = \frac{1}{6}, p(4) = \frac{n_4}{n_{total}} &= \frac{1}{6}\\
p(5) = \frac{n_5}{n_{total}} = \frac{1}{6}, p(6) = \frac{n_6}{n_{total}} &= \frac{1}{6}
\end{aligned}
$$

- But how can we test whether that is actually true?
- We need some way to compare the data to the theoretical probability distribution

Aggregating our dice results

- We obviously need to take more than one dice roll into account. But how can we aggregate all our results in a convenient number?
- As luck would have it, descriptive statistics provides us with a number of standard measures to characterise the properties of a *sample* (e.g. rolling the dice 10 times):
- Measures of *central tendency*:
- The mean: $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$
- The median: The number separating the higher half of a sample from the lower half
- The mode: The most frequent observation
- Measures of dispersion:
- The variance: $s_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$

Statistics basics: Random variables

- We need to consider our sample as a random variable
- What is a random variable?
- A random variable is a function that assigns a number to each possible outcome of our experiment (the dice roll)
- The outcome of a single dice roll can be described by a very obvious function: just assign the numbers from 1 to 6
- The outcome of *multiple* dice rolls is little trickier
- Regardless of which one we choose, we can then come up with a *theoretical* probability distribution for the random variable.
- The opposite of a random variable is a *constant*, a value that's the same for every sample.

Statistics basics: Random variables (formal definition!)

- Warning: Some mathematical notation follows.

- A random variable X is a function $X : O \to \mathbb{R}$ that associates to each outcome $\omega \in O$ exactly one number $X(\omega) = x$.
- Note: $\mathbb{R}$ = real numbers
- O_X is all the x's (all the possible values of X, the support of X). i.e., $x \in O_X$.
- Good example: number of coin tosses till you get Heads (H) for the first time
- $X : \omega \to x$
  - ω: H, TH, TTH,... (infinite)
  - $x = 0, 1, 2, \ldots; x \in O_X$

Random variables (2)

Every discrete random variable X has associated with it a **probability mass function**, also called *distribution function*.

$$p_X : S_X \to [0, 1]$$

defined by

$$p_X(x) = P(X(\omega) = x), x \in S_X$$

- Back to the example: number of coin tosses till H

- $X : \omega \to x$
- ω: H, TH, TTH,... (infinite)
  - $x = 0, 1, 2, \ldots; x \in S_X$
- $p_X = .5, .25, .125, \ldots$

What is a probability distribution?

From Wikipedia: In probability and statistics, a probability distribution assigns a probability to each measurable subset of the possible outcomes of a random experiment, survey, or procedure of statistical inference. Here's an example of a discrete probability distribution, the distribution of the *sum of two dice rolls*:

Discrete probability distribution

Every possible outcome (sum of the numbers rolled on two dice) is assigned a corresponding probability. This is called a *probability mass function*.

Important: all values sum to 1.

A quick clarification: Sample and population

- With the introduction of *theoretical* probability distributions, we need to be very careful to not confuse properties of the theoretical distribution with properties of an individual sample.
- Standard practice is to use
  - roman letters (e.g. m or $\bar{x}$ for the mean, s for the standard deviation) for properties of the sample
  - greek letters (e.g. μ "mu" for the mean and σ "sigma" for the standard deviation) for properties of the distribution (or the population that is represented by the distribution).

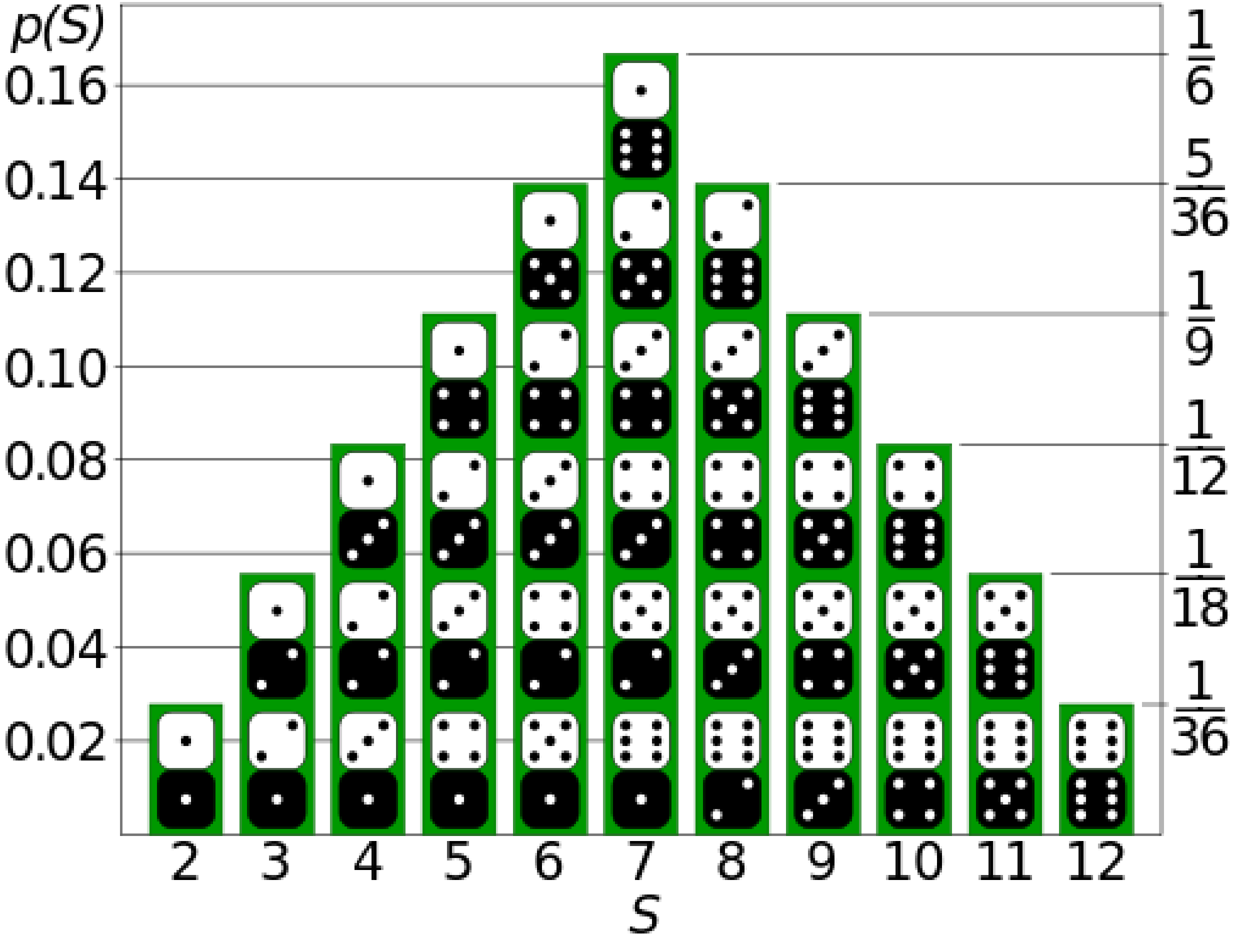
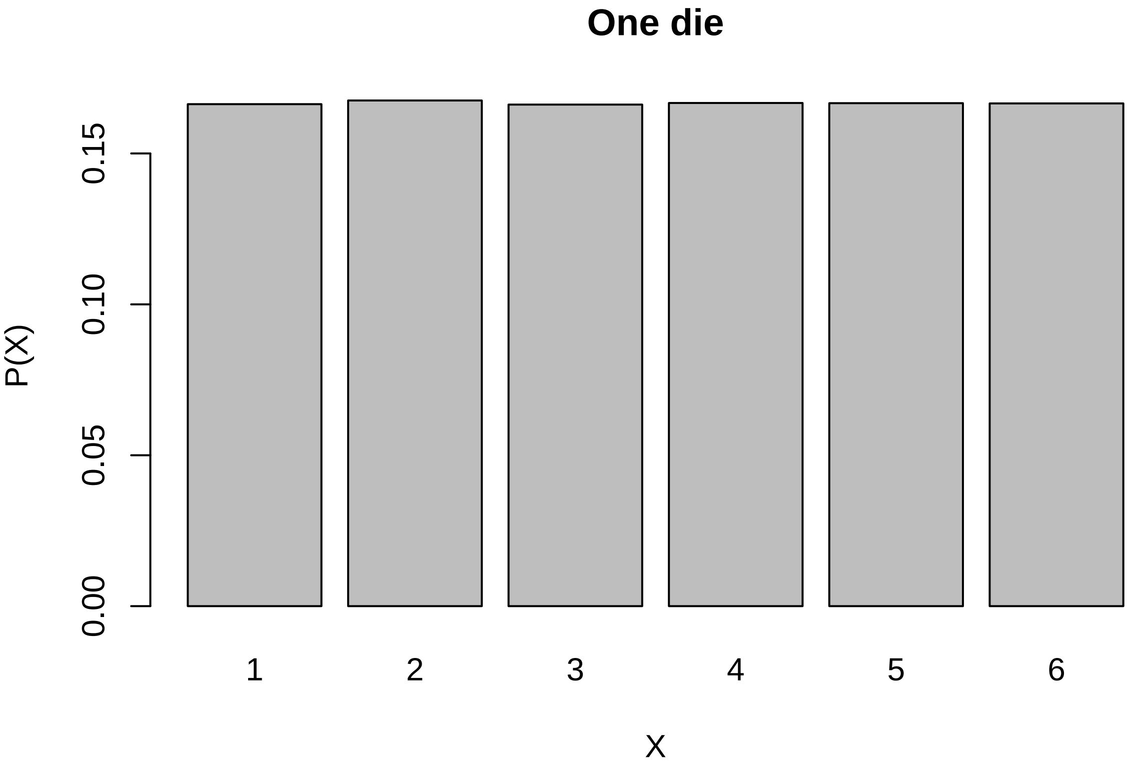


Figure 1: Dice Distribution (from Wikipedia)

From discrete to continuous

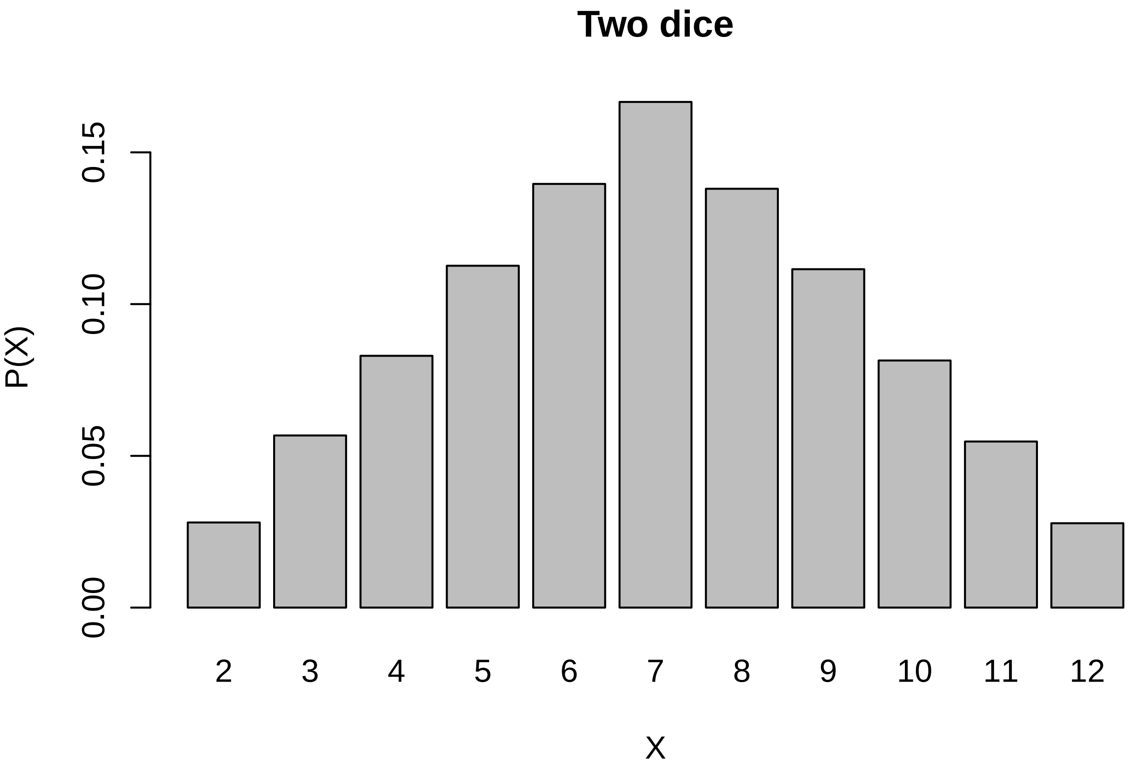
- Let's look at the probability distributions we get from rolling one, two, three etc. dice and summing up the results.
- We'll start with rolling one die (note that the bars may be a tiny bit uneven since I used a simulation to produce this graph).

From discrete to continuous

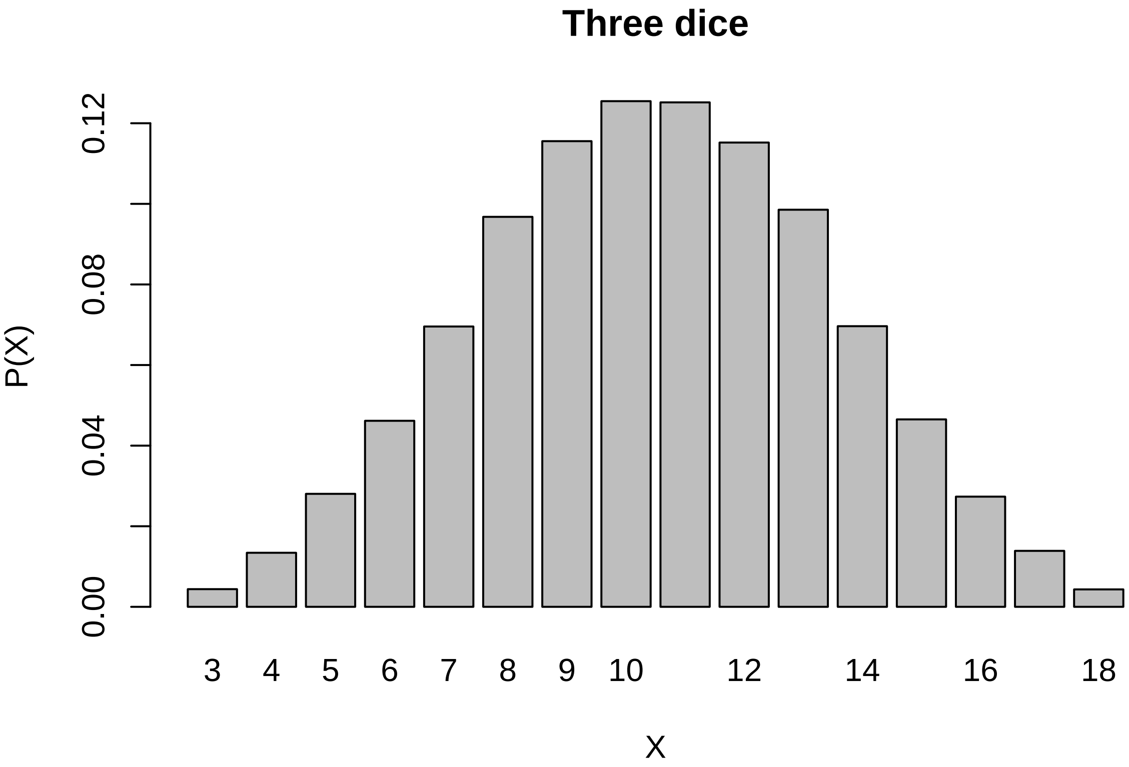


Two dice

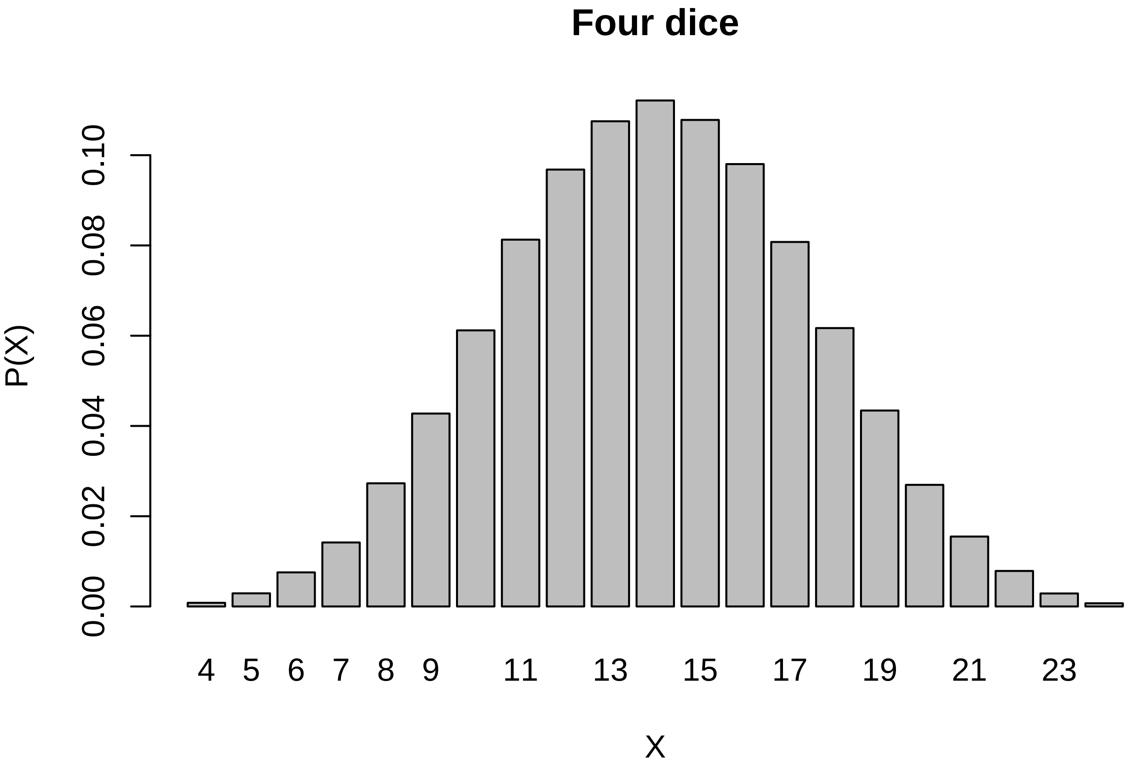
- This is the same as the image from Wikipedia.


Two dice
P(X)
0.00
0.05
0.10
0.15
2
3
4
5
6
7
8
9
10
11
12
X

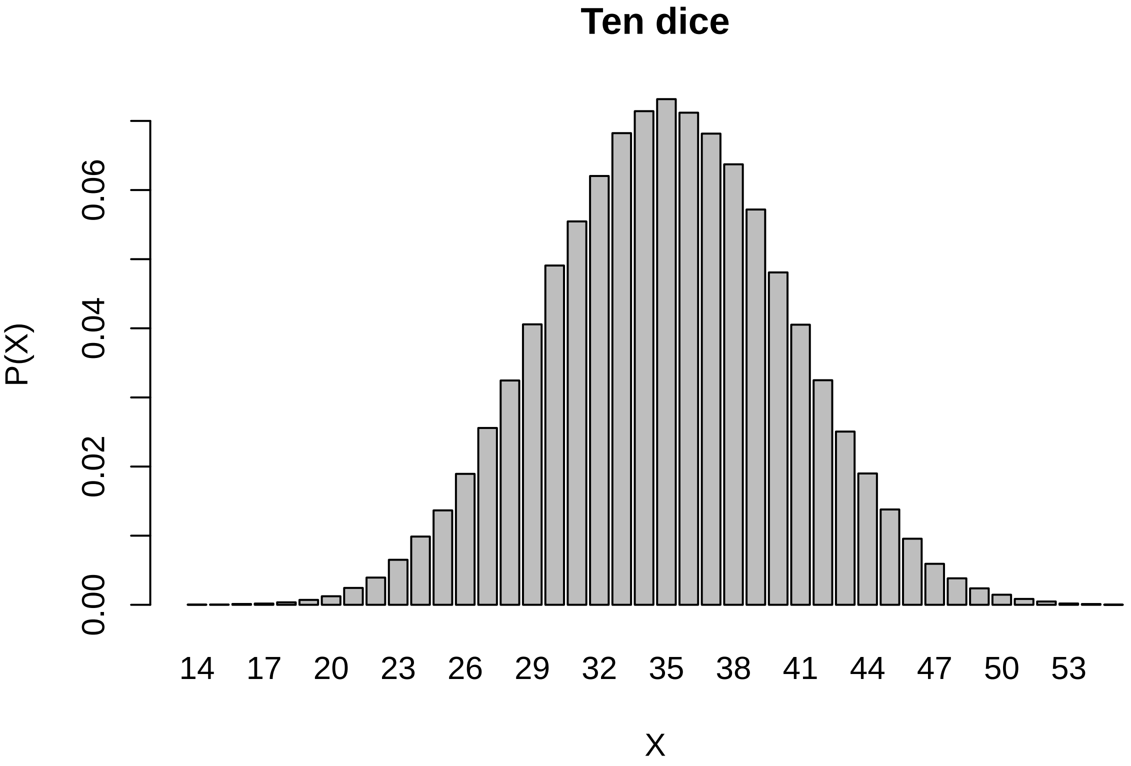
Three dice


Three dice
P(X)
0.00
0.04
0.08
0.12
3
4
5
6
7
8
9
10
12
14
16
18
X

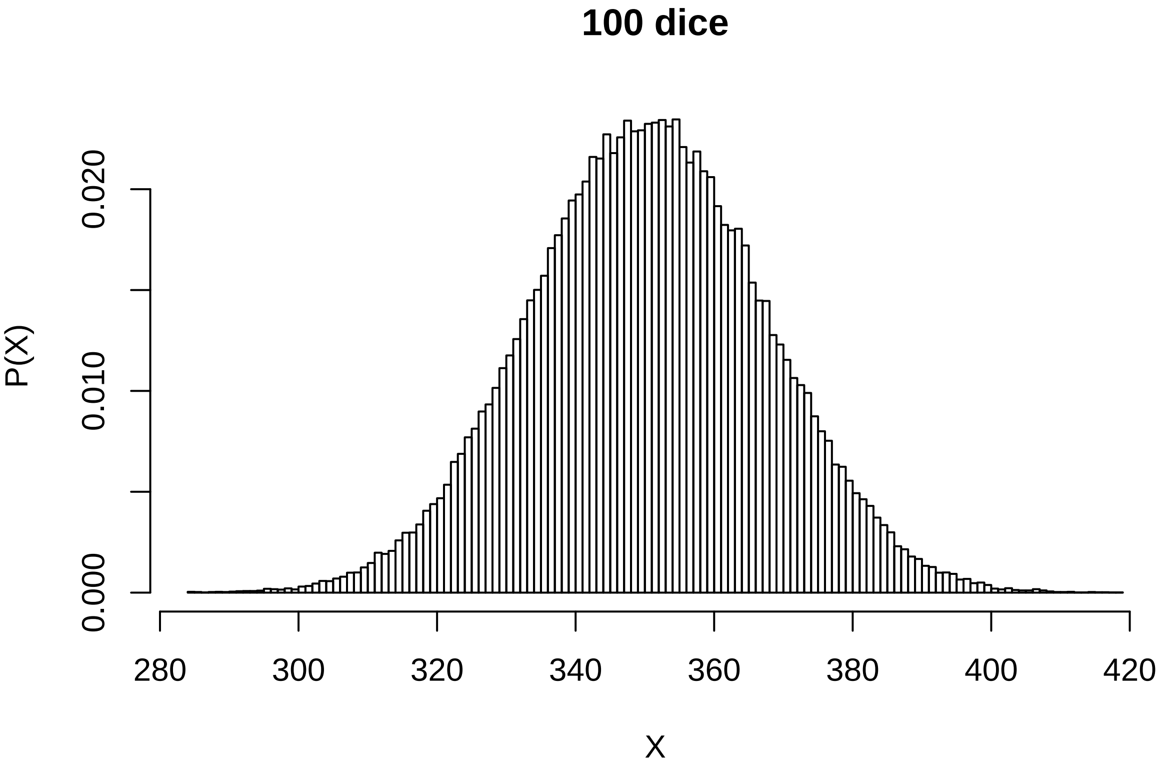
Four dice



Ten dice



100 dice



Do you see a pattern?

- Central limit theorem (CLT)

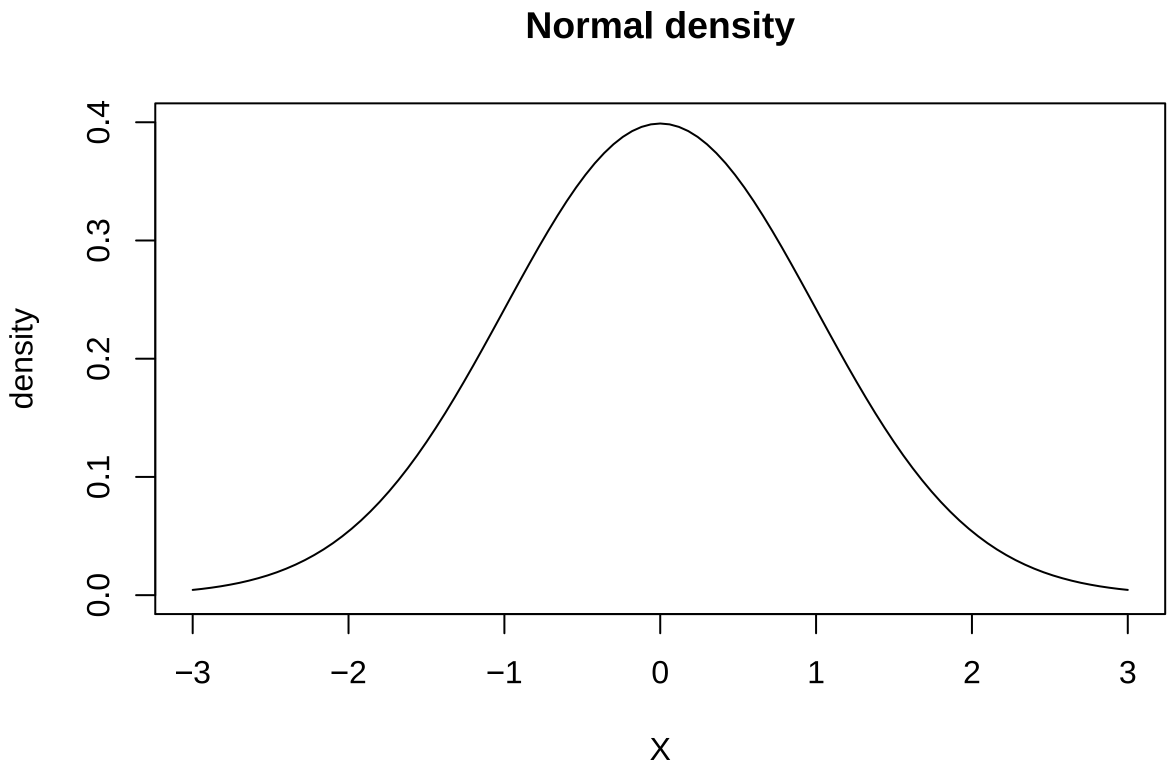
  When sampling from a population that has a mean, provided the sample size is large enough, the sampling distribution of the sample mean will be close to normal regardless of the shape of the population distribution

- (Technically, we were sampling the sum of X rather than the mean, but the mean of X is simply the sum divided by the number of observations. Do you care about this distintion? Didn't think so. It makes me feel better, though.)

What does this mean?

- For our dice problem, it means that we can compute the means of our samples (e.g. the mean of the 5, or 10, or 100 samples)
- Remember, the *sample mean* is a random variable as well, since it is different every time we take a sample
- We can then use a *continuous* probability distribution – the **normal distribution** as the theoretical probability distribution for our random variable (i.e. the sample mean).
- This makes our life easy, because the normal distribution is very simple to handle mathematically (really!).

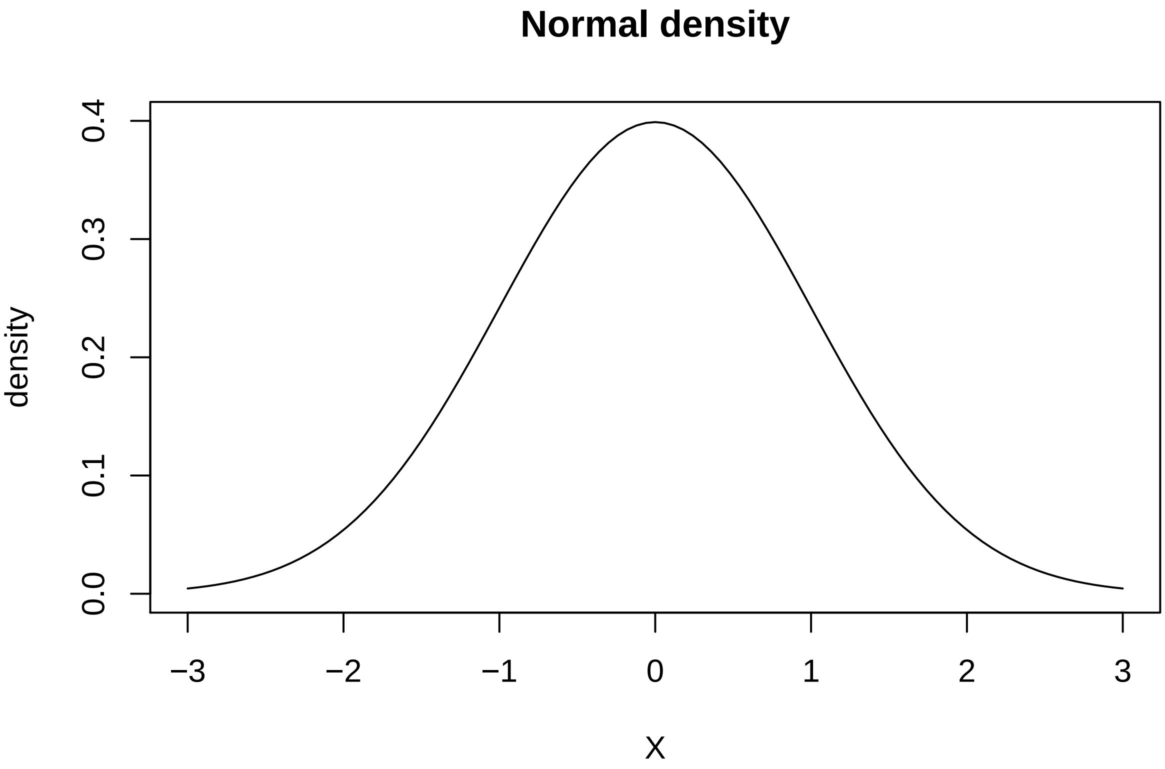
Continuous probability distributions



Here, the outcomes are continuous, so it doesn't make sense to ask about the probability of any point on the x-axis. - What is the probability of x = 1? - What do you mean by "1"? The function is continuous, so does 1.00001 still qualify as 1? - It makes more sense to ask these questions about intervals. The probability is then the area under the curve for the interval. - Important: the total area under the curve is 1.

Normal probability density function (PDF)

$$f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-((x-\mu)^2/2\sigma^2)}$$



Normal probability density function (PDF)

$$f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-((x-\mu)^2/2\sigma^2)}$$

- This looks scary, but it really isn't. This is simply a mathematical function that happens to describe the distribution of a lot of random variables in nature. - If you look closely, you can see that the function has three parameters, x, μ, and σ (π and e are constants). - The first parameter, x is the random variable. The function gives you the probability density at each value of x - The second parameter, μ, is called the **expected value** or the **mean** of the distribution. - The third parameter, σ, is called the standard deviation.

Standard normal distribution

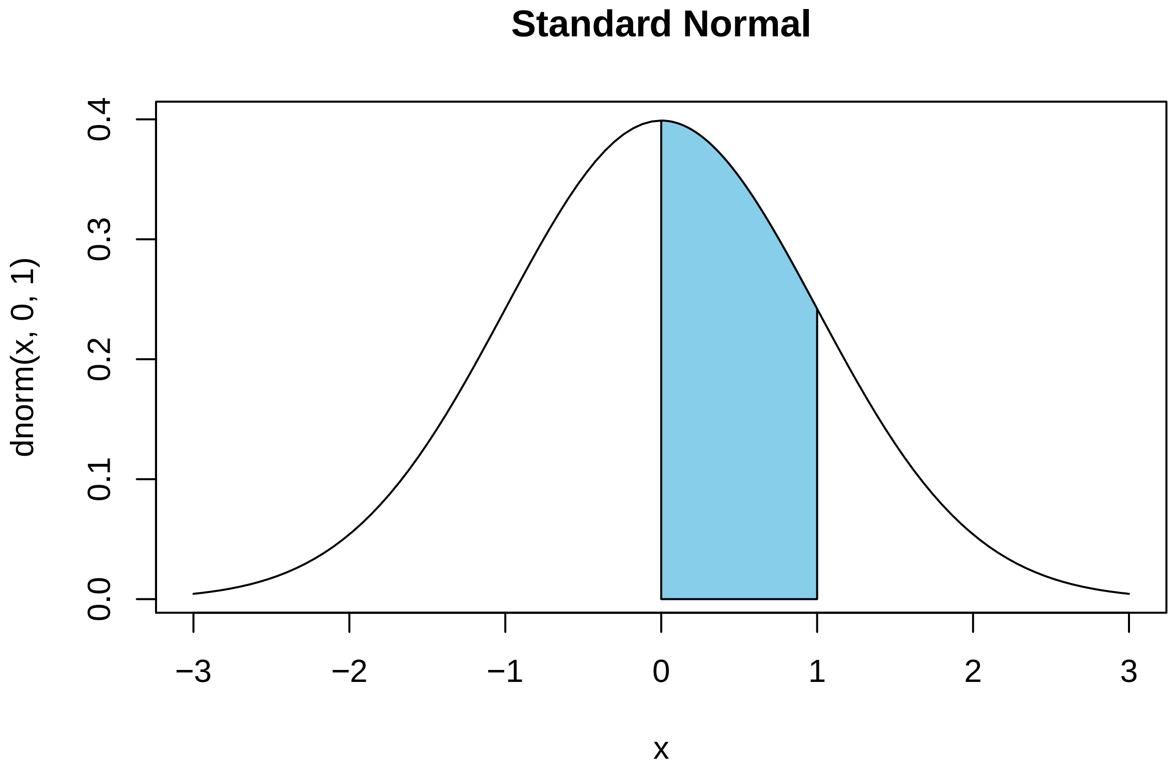
- There is an infinite number of normal distributions with different parameters μ and σ. The one with $\mu = 0$ and $\sigma = 1$ is particularly useful and is called the *standard* normal distribution.
- Look at how simple and nice the normal distribution appears when we plug in those values:

$$f(z, \mu, \sigma) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

- You can *transform* values from any normal distribution to the normal distribution.
- This is known as a *z-transformation*: $z = \frac{x-\mu}{\sigma}$
- By transforming all our observations to z-values and then looking up their probability in the standard normal distribution, this is the only distribution we'll ever need (. . . mostly).

The probability of outcomes in the standard normal distribution

- Remember, we can't really get the probability of a *point* event in a continuous distribution, since there are no "points" in a continuous variable
- But we can ask questions about *intervals*:
- What's the probability of x being between 0 and 1?



Getting the area under the curve

- Since we know exactly what the function is, we can get the area under the curve.
- Remember how to do that from maths class? Your best friend, integration :

$$p(0 < z < 1) = \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

- Don't want to do integration? Well, you're in luck, because most statistical software (and Excel!) can do this for you.
- `=NORM.S.DIST(0,TRUE)` will give you the area under the curve to the left of 0 (i.e. the probability that $z < 0$)
- `=NORM.S.DIST(1,TRUE)` will give you the area under the curve to the left of 1 (i.e. the probability that $z < 1$)

Getting the area under the curve

- Remember, Excel gives us the *lower* tail (the area under the curve to the left of the z value)

- So we can calculate our interval: $p(0 < z < 1) = p(z < 1) - p(z < 0)$
- In Excel:
  - `=NORM.S.DIST(1,TRUE)-NORM.S.DIST(0, TRUE)`
  - Result: 0.3413447
- Success!

Things you can do with this knowledge

- Say I'm looking at random numbers from a standard normal distribution, and I see that one of them is 4.
- That seems very unusual
- Just how unusual?
  - What's the probability of getting a value of 4 when sampling from a standard normal distribution (mean = 0, sd = 1)?

Just how unusual is a value of 4?

- Remember, when you have a continuous distribution, you can't think about point values (e.g. 5). Rather, what you want to know is:
  - What is the probability of getting a value of 4 *or greater* (or $p(z > 4) = 1 - p(z < 4)$)?
- Let's ask Excel: `=1-NORM.S.DIST(4,TRUE)`
  - Result: 0.0001338302
- So it's very unusual.
- Can we come up with a similar test for our dice sample mean?
- We'll have to figure out how the dice sample means are distributed.
  - Then we can take our sample mean and see how likely (or unlikely) it is that it comes from the theoretical distribution.

The theoretical distribution of dice sample means

- We've seen that we can approximate our theoretical distribution (which is actually discrete) using a continuous distribution function, namely the normal distribution, which makes our lives very easy (yes, really!).
- We have to figure out the μ and the $sigma$ parameters for our theoretical normal distribution of sample means, though.
- Note for those of you who care (probably no one): In the case that we actually know exactly what the probabilities for our discrete probability distribution should look like, we could also use a different distribution, the χ^2 (chi square) distibution. More about that next week, otherwise our heads may explode.

Random variables: Expected value

- Random variables have expected values

- For discrete random variables, the expected value is the outcome value multiplied by the probability of the outcome:

$$E(X) = \mu = \sum_{i=1}^{k} p(x_i) \cdot x_i$$

- where $E(X)$ is the expected value of a discrete random variable X with the outcomes $(x_1 \dots x_k)$ and the associated probabilities $(p(x_1) \dots p(x_k))$
- The equivalent for continuous random variables:

$$\mu = \int\limits_{-\infty}^{\infty} x f(x) dx$$

Random variables: Variance

- Random variables also have variances
- For discrete random variables, the variance is the difference between the outcome value and the mean multiplied by the probability of the outcome:

$$\sigma^2 = \sum_{i=1}^{k} p(x_i) \cdot (x_i - \mu)^2$$

- where σ^2 is the variance of a discrete random variable X with the outcomes $(x_1 \dots x_k)$ and the associated probabilities $(p(x_1) \dots p(x_k))$
- The equivalent for continuous random variables:

$$\mu = \int\limits_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

Maths basics: Expected values

- For example, the expected value μ of rolling a six-sided die is:

$$\begin{aligned} E(X) = & \sum_{i=1}^{6} p(x_i) \cdot x_i \\ & = x_1 \cdot p(x_1) + x_2 \cdot p(x_2) + x_3 \cdot p(x_3) + x_4 \cdot p(x_4) \\ & + x_5 \cdot p(x_5) + x_6 \cdot p(x_6) \\ & = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} \\ & + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} \\ & = \frac{21}{6} = 3.5 \end{aligned}$$

Maths basics: Variance

- For example, the variance σ^2 of rolling a six-sided die is:

$$\begin{aligned}\sigma^2 &= \sum_{i=1}^{6} p(x_i) \cdot (x_i - \mu)^2 \\ &= (x_1 - \mu)^2 \cdot p(x_1) + (x_2 - \mu)^2 \cdot p(x_2) + (x_3 - \mu)^2 \cdot p(x_3) \\ &+ (x_4 - \mu)^2 \cdot p(x_4) + (x_5 - \mu)^2 \cdot p(x_5) + (x_6 - \mu)^2 \cdot p(x_6) \\ &= \frac{1}{6} \cdot \Big((1 - 3.5)^2 + (2 - 3.5)^2 + (3 - 3.5)^2 \\ &+ (4 - 3.5)^2 + (5 - 3.5)^2 + (6 - 3.5)^2\Big) \\ &= \frac{17.5}{6} = 2.9167\end{aligned}$$

Back to the dice example again

- So, we know that, if our dice are fair, our dice rolls come from a discrete theoretical distribution with $\mu = 3.5$ and $\sigma^2 = 2.9167$.
- But remember, we don't want to evaluate single dice rolls, but rather the mean of a sample of dice rolls, since that will enable us to use the nice and easy normal distribution to calculate the probabilities.
- So, what is the mean $\mu_{\bar{x}}$ and what is the variance $\sigma_{\bar{x}^2}$ for the **distribution of sample means**?

Maths basics: Expected values (3)

- What is the expected value of rolling two dice and adding the spots?
- What is the expected value of rolling two dice and multiplying the number of spots?
- What is the expected value of an IQ test result?
- Imagine you and your friend both take IQ tests. What is the expected value of the differences between your scores (assuming that you both come from the general population)?
    - Don't know? Well, stay tuned. This will require some maths, though.

Maths basics: Computing expected values

- The expected value of a random variable is often also called μ:

$$E(X) = \mu$$

- μ is also called the distribution *mean*
- What if the expected value is constant across all the possible outcomes?
- e.g what is the expected value of a die with 1 on all sides?
    - 1, of course!
- More general:
- if the value is the same across all outcomes, we can call it a constant
- e.g. if $x_1 = x_2 = x_3 = \cdots = x_i = 1$
    - then $E(X) = E(1) = 1$

Maths basics: Computing expected values (2)

Even more general: If a is a constant, then $E(a) = a$ - If X is a random variable and a is a constant, what is the expected value of $a \cdot X$?

$$E(a \cdot X) = a \cdot E(X)$$

- For example, if the expected value of rolling a 6-sided die is 3.5, what is the expected value of rolling a 6-sided die and then multiplying the number of spots by 3?

$$E(3 \cdot X) = 3 \cdot E(X) = 3 \cdot 3.5 = 10.5$$

- Try it if you don't believe me!

Maths basics: Computing expected values (3)

- If X is a random variable and a is a constant, what is the expected value of $a + X$?

$$E(a + X) = a + E(X)$$

- For example, if the expected value of rolling a 6-sided die is 3.5, what is the expected value of rolling a 6-sided die and then adding 3 to the number of spots?

$$E(3 + X) = 3 + E(X) = 3 + 3.5 = 6.5$$

- Try it if you still don't believe me!

Maths basics: Computing expected values (4)

- If X is a random variable and Y is a random variable, what is the expected value of $X + Y$?

$$E(X + Y) = E(X) + E(Y)$$

- For example, if the expected value of rolling two 6-sided dice and adding the two results?

$$E(X + Y) = E(X) + E(Y) = 3.5 + 3.5 = 7$$

- Try it if you still don't believe me!

Maths basics: Computing expected values (5)

- If X is a random variable and Y is a random variable, what is the expected value of $X \cdot Y$?

$$E(X \cdot Y) = E(X) \cdot E(Y)$$

- But *ONLY* if X and Y are *INDEPENDENT*
- For example, if the expected value of rolling two 6-sided dice and multiplying the two results?

$$E(X + Y) = E(X) \cdot E(Y) = 3.5 \cdot 3.5 = 12.25$$

- Try it if you still don't believe me!

The expected value of the sample mean

- If X is a random variable, and we take a sample of size n from X, what is the expected value of the mean of that sample?
- Remember, this is how you compute the sample mean:

$$\bar{x} = \frac{\sum\limits_{i=1}^{n}}{n}$$

- The expected value of the sample mean is:

$$\begin{aligned} E(\bar{X}) &= E\left(\frac{\sum\limits_{i=1}^{n}}{n}\right) \\ &= \frac{1}{n} \cdot \big(E\sum_{i=1}^{n} X_i\big) \end{aligned}$$

The expected value of the sample mean (2)

$$\begin{aligned} &= \frac{1}{n} \cdot \big(E\sum_{i=1}^{n} X_i\big) \text{ (since } E(a \cdot X) = a \cdot E(X)) \\ &= \frac{1}{n} \cdot \sum_{i=1}^{n} E(X_i) \text{ (since } E(X+Y) = E(X) + E(Y)) \\ &= \frac{1}{n} \cdot \sum_{i=1}^{n} \mu_x \text{ (since } E(X) = \mu) \\ &= \frac{1}{n} \cdot n \cdot \mu_x = \mu_x \end{aligned}$$

The expected value of the sample mean (3)

- We just found that the expected value of the sample mean $E(\bar{X})$ is identical to the expected value (the mean) of the population μ_x, *regardless of the sample size.*
- We can say that the sample mean $\overline{X}$ is an *unbiased estimator* of the population mean μ
- No matter what we do and what crazy population we're taking samples of, the sample means will always be distributed around the true population mean.
  - *Isn't that cool?*
  - I know what you're thinking right now, but this is *actually* cool. Just think about it: If you want to know the true population mean of any population, all you have to do is take enough samples.

Our dice example

- Remember, we want to know how the means of our dice rolls should be distributed if the dice are fair. So, now we know that they are normally distributed (because of the Central Limit Theorem) with an expected value of $\mu_{\bar{x}} = 3.5$.
- What about the *variance* of the *distribution of sample means*?

  - Well, this will take some work. Sorry, people. It's maths time.
  - First, we need to know how the sample variance is related to the population variance.
  - In other words, we need to know the expected value of the sample variance.

The expected value of the sample variance

- We could now go through the same process for the sample variance s^2, but this would take a lot of time and most of you may not care about the details.
- Instead, I will just tell you what we would find if we did this:
- It turns out that the sample variance

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}$$

  is **not** an unbiased estimator of the population variance. In fact, it *underestimates* the population variance.
- I have posted a link to a video that very nicely and intuitively demonstrates this point on myBU in case you don't believe me.

The expected value of the sample variance (2)

- We can correct our sample variance if we divide by $n - 1$ instead of n:

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n-1}$$

- This definition of the sample variance *does* give us an unbiased estimator of the population variance σ^2.
- I have posted a link to another video that gives you the proof of this in case you don't believe me.

The expected value of the variance of sample means

- Remember, now we are no longer talking about the population variance σ^2 now, but the **variance of the sample means** $\sigma^2_{\bar{x}}$.
- Also remember that the population does not have to be distributed normally, but the means of samples from this distribution will be if the sample size is large enough (that's the whole reason we're concerning ourselves with the means rather than individual samples here).
- We'll start by rewriting $E(\sigma^2_{\bar{x}})$ as $Var(\bar{X})$, that is, the expected variance of the sample mean. Same thing, but makes the following steps more obvious.

The expected value of the variance of sample means (2)

- Now we replace $\bar{X}$ with the definition of the sample mean:

$$\begin{aligned} Var(\bar{X}) &= Var(\frac{X_1 + X_2 + \cdots + X_n}{n}) \\ &= \frac{1}{n^2} \cdot Var(X_1 + X_2 + \cdots + X_n) \end{aligned}$$

- What have we done now? We just moved the constant out of the expected variance
  - Tiny not-that-intuitive detail: The variance is a square (s^2), so as we move $1/n$ out of the term, we need to square it too.

The expected value of the variance of sample means (3)

- We can treat expected variances just like expected values.
  - If (and only if!) $X_1, X_2, \ldots, X_n$ are independent (i.e. the value of X_1 doesn't depend on the value of X_2, or X_3, etc.), we can rewrite the expected variance of a sum as the sum of its expected variances

$$\begin{aligned} Var(\bar{X}) &= \frac{1}{n^2} \cdot Var(X_1 + X_2 + \cdots + X_n) \\ &= \frac{1}{n^2} \cdot (Var(X_1) + Var(X_2) + \cdots + Var(X_n)) \end{aligned}$$

The expected value of the variance of sample means (4)

- We just established that the expected value of the variance of each individual sample (if we apply the correction and divide by $n-1$) is the population variance σ^2. So let's replace each term $Var(X_i)$ with σ^2:

$$\begin{aligned} Var(\bar{X}) &= \frac{1}{n^2} \cdot (Var(X_1) + Var(X_2) + \cdots + Var(X_n)) \\ &= \frac{1}{n^2} \cdot (\sigma^2 + \sigma^2 + \cdots + \sigma^2) \\ &= \frac{1}{n^2} \cdot n \cdot \sigma^2 \\ &= \frac{\sigma^2}{n} \end{aligned}$$

- So, the expected value of the variance of sample means $\sigma^2_{\bar{x}}$ is $\frac{\sigma^2}{n}$ and the expected value of the standard deviation of sample means (also called **standard error**) is $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$.

Please, let's FINALLY finish the dice example

- OK, OK. We now have everything we need to determine whether our dice roll sample mean is unusual assuming fair dice.
- More formally, we call this a **Hypothesis Test**
- We establish a *Null Hypothesis* H_0 (e.g. the dice are fair),
  - determine a theoretical probability distribution of the random variable (our dice roll means) given that the H_0 is true:
    - a normal distribution with $\mu_{\bar{x}} = 3.5$ and $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{\sqrt{2.9167}}{\sqrt{n}}$, where n is the number of dice rolls in our sample,
  - and finally we can calculate the probability that you would observe the sample mean you observed given the H_0.

Final steps

- So, let's assume you did 10 dice rolls for this example, and that your mean was 4.
- Since we know that the sample means should be normally distributed, we can transform your mean into a z-value:
- Since $\mu_{\bar{x}} = 3.5$ and $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{\sqrt{2.9167}}{\sqrt{10}} = 0.5400648$:

$$z(4) = \frac{4 - 3.5}{0.5400648} = 0.9258148$$

- Let's ask Excel what the probability of observing a sample mean this far away (or farther) from the population mean is for the standard normal distribution: `1-NORM.S.DIST(0.9258,TRUE)`
  - Result: 0.177274964

Final steps (2)

- Fisher (who popularised this sort of hypothesis testing) suggested that we should consider data with a probability of less than 5% (or .05) given the null hypothesis as **significant** evidence for rejecting the null hypothesis.
- In our case, we are far away from a probability (or short, p-value) of .05. So, we can't reject the null hypothesis. Try it for yourselves, though.
- More on this next week.

Technical note for those who really care

- We really don't care about the direction of the effect here, just the absolute distance from the mean (i.e. this is a *two-tailed* test).
- So, to be absolutely correct, we should ask Excel to give us the probability of z being at least this far away from the mean on either side: $p(z < -.9258 \cup z > .9258) = p(z < -.9258) + (1 - p(z < .9258))$.
- When we ask Excel for the p-value `=NORM.S.DIST(-0.9258,TRUE)+(1-NORM.S.DIST(0.9258,TRUE))` we get the actual, correct result of 0.3545499.
- Good times. Until next week!