1: Fundamentals of Null-Hypothesis Significance Testing

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Why are we doing this?

Hello everyone and welcome to our first “real” lecture on Advanced Statistics. In our introductory session, I have already explained to you a bit about my idea of what “advanced” means in this context. In our undergraduate statistics units, we try to give students the absolute minimum in statistical education acceptable. This may surprise you, but given the resistance that many undergraduate students have to statistics (I’m not saying that this is the students’ fault – far too many of them – and you –have internalised the misconception that you can’t do maths), the lack of mathematical skills (compounded by the resistance to anything mathematical), and the limited time, this is the best we can do. We give you a small toolkit of step-by-step instructions to perform basic statistical tests without making catastrophic mistakes. If we are really good, we give you some degree of understanding of the basic ideas behind null-hypothesis significance testing (NHST). This is the kind of statistical education I have received myself and that you probably received as well (if you didn’t receive any statistical education at all, don’t worry: you were spared all the downsides of this type of

Karl Popper: Falsification and severe tests

Philosophers have been thinking about the question of what we can know for certain, for a very long time. There are some philosophers, like Hume, who thought that really nothing we think we know is certain. Everything could be an illusion. Most philosophers don't go this far. In the early 20th century, the "Vienna Circle" of logical positivists (e.g. Kurt Gödel, Rudolf Carnap, and Carl Hempel) proposed a distinction between statements that are definitions (like "a triangle has three sides"), which are necessarily true, statements that are verifiable through observation of the world ("my desk is three foot tall"), and statements that are unverifiable ("The world is an illusion"). The logical positivists thought that unverifiable statements were essentially meaningless metaphysics, and we should focus on definitions and verifiable statements (i.e. logic, mathematics, and science). How do you verify a statement based on empirical observations? For "my desk is three foot tall", that is easy, you just get a ruler and measure it, but what about generalisations like "all swans are white"? For this, the logical positivists proposed a process called induction: inferring universal rules given particular

Falsifiable and non-falsifiable theories

What is a falsifiable theory? Consider psychoanalysis: A psychoanalyst proposes that a patient's fear of flying is caused by an Oedipus complex (i.e. sexual attraction to his mother in his infancy). If the patient confirms this, the psychoanalyst will take this as evidence for his theory. If the patient strongly denies this (which is probably more likely), the psychoanalyst will conclude that the Oedipus complex must be repressed by the patient. The patient's denial is then also evidence for the psychoanalyst's theory. Essentially, psychoanalytic theories are not falsifiable, therefore Popper would say that they are not a scientific.

As Psychologists and scientists, we must ensure that our theories actually yield testable, that is, falsifiable hypotheses.

Dienes (2008) gives a good example (Box 1.5, p. 9):

> *Consider the following two factor theory of liking: Factor 1: We are preprogrammed to like familiar things (e.g. foods, people, animals, tools, etc.) because our knowledge and skills are likely to apply to them. They are not dagerous,*

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Neyman and Pearson: Null-hypothesis significance testing

In the 1920s and 30s, Jerzy Neyman and Egon Pearson established a consistent logical procedure for significance testing (although the term “significance” itself comes from Fisher, as does the suggestion of using .05 as the cutoff).

Maths basics: Probability

- Basic rules:
  - All probabilities are between 0 and 1: $P(A) \in [0, 1]$
  - The complementary probability of an event (i.e. the probability that an event will NOT happen) is 1-the probability of the event: $P(A^c) = 1 - P(A)$
  - A probability can be interpreted as the number of outcomes that form an event (e.g. the outcome “Heads” when flipping a coin) over the total number of outcomes (e.g. “Heads” and “Tails”)

$$p(A) = \frac{n_A}{n}$$

  - But note that a probability of .5 (e.g. for getting “Heads” on a coin flip) doesn’t mean that you will get “Heads” on exactly 50% of coin flips.

Maths basics: Probability (2)

- Basic rules:
  - What is the probability that Event A and Event B will happen together?

$$\begin{aligned} P(A \cap B) &= P(A|B)P(B) = P(B|A)P(A) \\ P(A \cap B) &= P(A)P(B) \end{aligned}$$

if A and B are independent

- What is the probability that either Event A OR Event B will happen?

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ P(A \cup B) &= P(A) + P(B) \end{aligned}$$

if A and B are mutually exclusive

Maths basics: Conditional probability

- What is the probability of Event A *GIVEN THAT* Event B happened?
  - Divide the number of outcomes where A and B happen together by the number of all outcomes where B happens (regardless of whether A happened, too).
  - If we divide both nominator and denominator by n, we can convert this into probabilities:

$$P(A \mid B) = \frac{n_{AB}}{n_B} = \frac{n_{AB}/n}{n_B/n} = \frac{P(A \cap B)}{P(B)}$$

- Now plug in our definition of joint probability (see last slide):

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

- This is known as *Bayes' theorem*. Keep it in mind for later!

Back to our dice problem

- IN THEORY, each of our dice roll outcomes has the same probability:

$$\begin{aligned}
p(1) &= \frac{n_1}{n_{total}} = \frac{1}{6}, p(2) = \frac{n_2}{n_{total}} &= \frac{1}{6} \\
p(3) &= \frac{n_3}{n_{total}} = \frac{1}{6}, p(4) = \frac{n_4}{n_{total}} &= \frac{1}{6} \\
p(5) &= \frac{n_5}{n_{total}} = \frac{1}{6}, p(6) = \frac{n_6}{n_{total}} &= \frac{1}{6}
\end{aligned}$$

- But how can we test whether that is actually true?
- We need some way to compare the data to the theoretical probability distribution

Aggregating our dice results

- We obviously need to take more than one dice roll into account. But how can we aggregate all our results in a convenient number?
- As luck would have it, descriptive statistics provides us with a number of standard measures to characterise the properties of a *sample* (e.g. rolling the dice 10 times):
- Measures of *central tendency*:
- The mean: $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$
- The median: The number separating the higher half of a sample from the lower half
- The mode: The most frequent observation
- Measures of dispersion:
- The standard deviation: $s_x^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$

Statistics basics: Random variables

- We need to consider our sample as a random variable
- What is a random variable?
- A random variable is a function that assigns a number to each possible outcome of our experiment (the dice roll)
- The outcome of a single dice roll can be described by a very obvious function: just assign the numbers from 1 to 6
- The outcome of *multiple* dice rolls is little trickier
- Regardless of which one we choose, we can then come up with a *theoretical* probability distribution for the random variable.
- The opposite of a random variable is a *constant*, a value that's the same for every sample.

Statistics basics: Random variables (formal definition!)

- Warning: Some mathematical notation follows.
- A random variable X is a function $X : O \rightarrow \mathbb{R}$ that associates to each outcome $\omega \in O$ exactly one number $X(\omega) = x$.
- Note: $\mathbb{R}$ = real numbers
- O_X is all the x's (all the possible values of X, the support of X). i.e., $x \in O_X$.
- Good example: number of coin tosses till you get Heads (H) for the first time
- $X : \omega \rightarrow x$
  - ω: H, TH, TTH,... (infinite)
  - $x = 0, 1, 2, \ldots ; x \in O_X$

Random variables (2)

Every discrete random variable X has associated with it a **probability mass function**, also called *distribution function*.

$$p_X : S_X \rightarrow [0, 1]$$

defined by

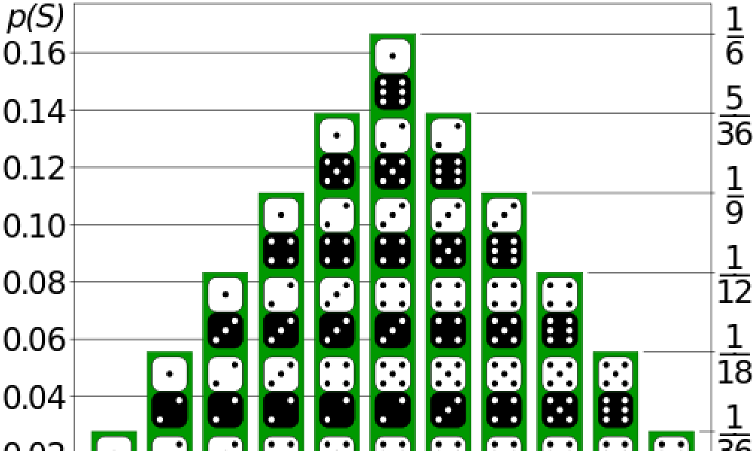
$$p_X(x) = P(X(\omega) = x), x \in S_X$$

- Back to the example: number of coin tosses till H

- $X : \omega \rightarrow x$
- ω: H, TH, TTH,... (infinite)
    - $x = 0, 1, 2, \ldots ; x \in S_X$
- $p_X = .5, .25, .125, \ldots$

What is a probability distribution?

From Wikipedia: In probability and statistics, a probability distribution assigns a probability to each measurable subset of the possible outcomes of a random experiment, survey, or procedure of statistical inference. Here's an example of a discrete probability distribution, the distribution of the *sum of two dice rolls*:



Discrete probability distribution

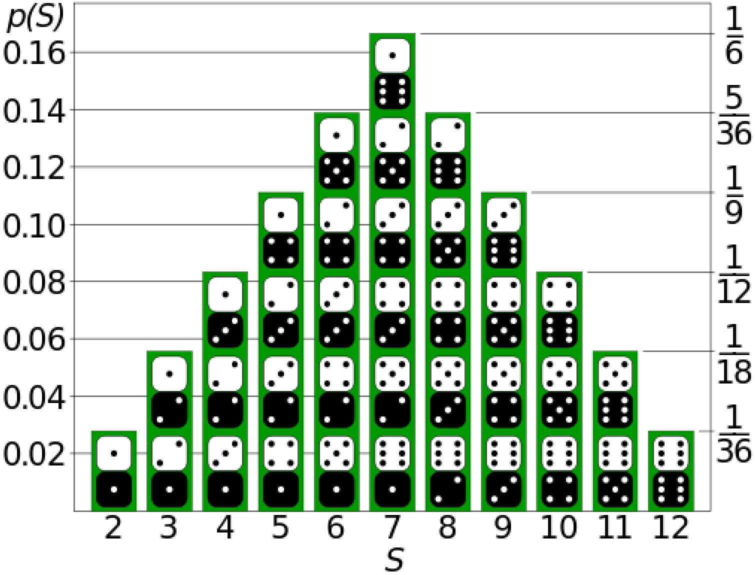


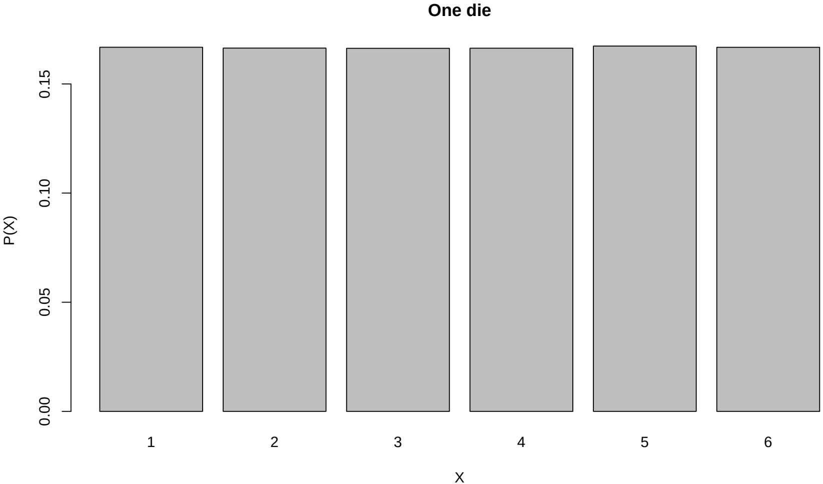
Figure 2: Dice Distribution (from Wikipedia)

A quick clarification: Sample and population

- With the introduction of *theoretical* probability distributions, we need to be very careful to not confuse properties of the theoretical distribution with properties of an individual sample.
- Standard practice is to use
  - roman letters (e.g. m or $\bar{x}$ for the mean, s for the standard deviation) for properties of the sample
  - greek letters (e.g. μ "mu" for the mean and σ "sigma" for the standard deviation) for properties of the distribution (or the population that is represented by the distribution).

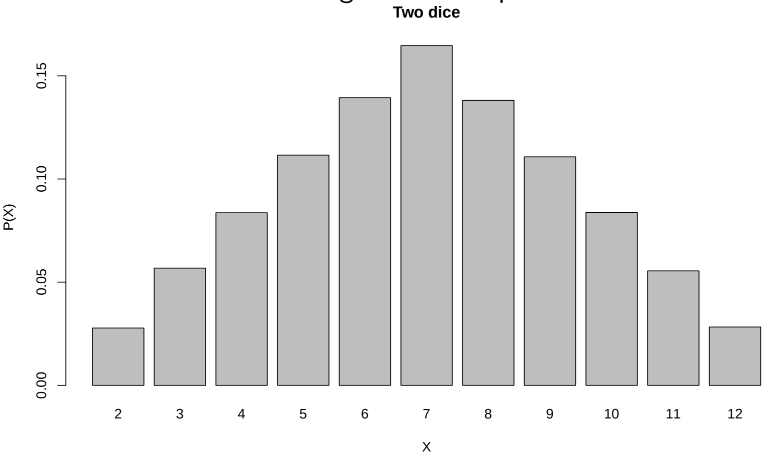
From discrete to continuous

- Let's look at the probability distributions we get from rolling one, two, three etc. dice and summing up the results.
- We'll start with rolling one die (note that the bars may be a tiny bit uneven since I used a simulation to produce this graph).

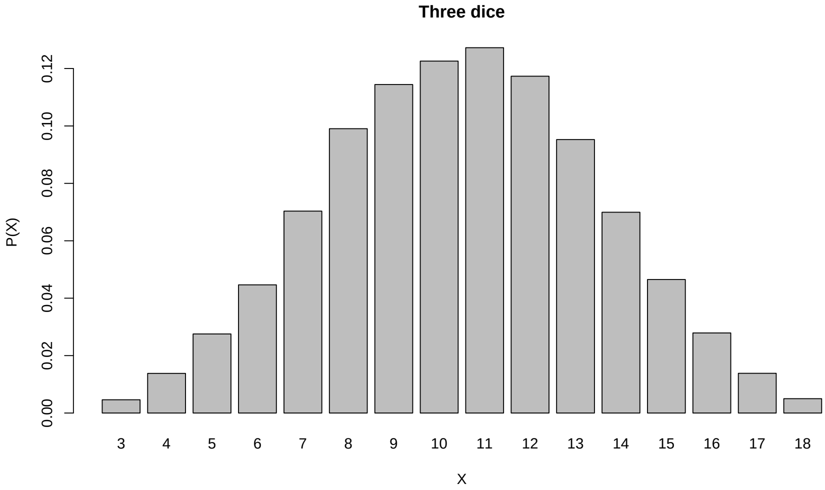


Two dice

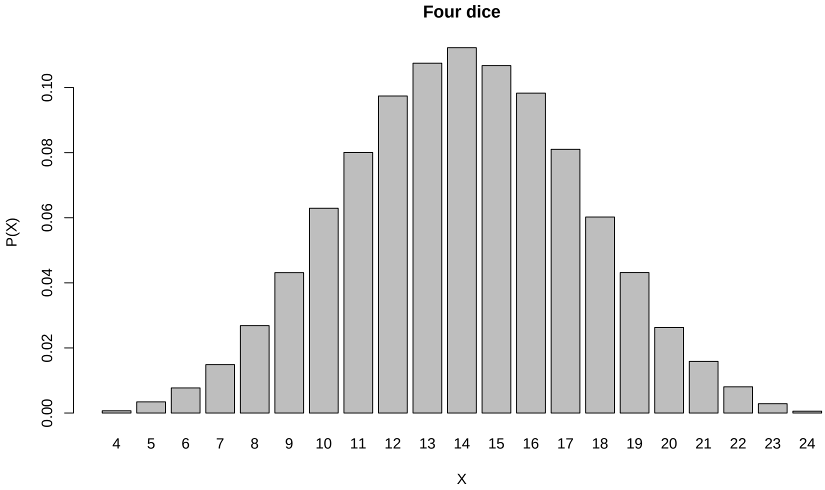
- This is the same as the image from Wikipedia.



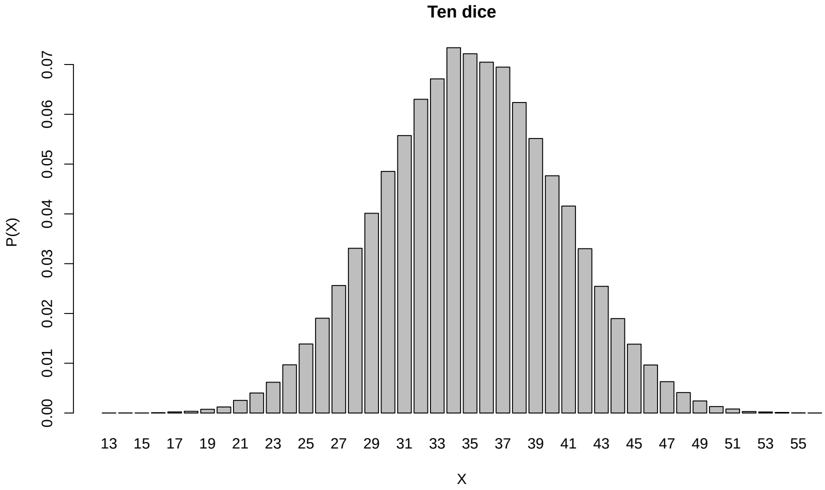
Three dice


Three dice
P(X)
0.00
0.02
0.04
0.06
0.08
0.10
0.12
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
X

Four dice

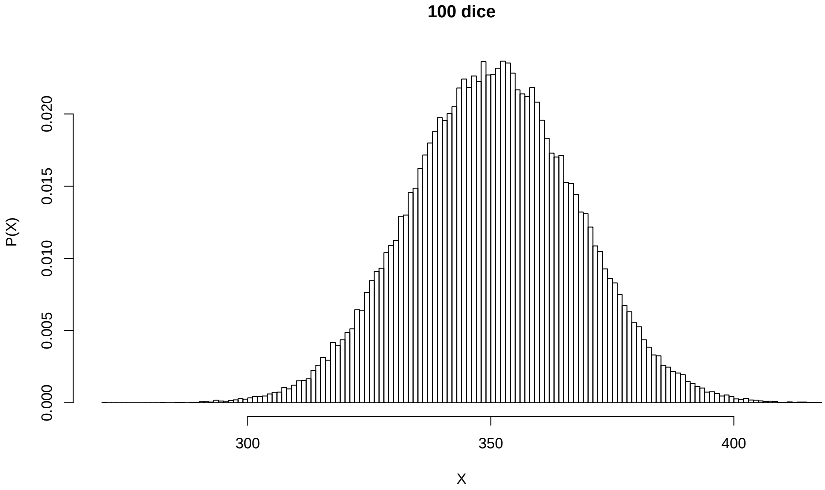

Four dice
P(X)
0.00
0.02
0.04
0.06
0.08
0.10
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
X

Ten dice



Ten dice
P(X)
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
13
15
17
19
21
23
25
27
29
31
33
35
37
39
41
43
45
47
49
51
53
55
X

100 dice



Do you see a pattern?

- Central limit theorem (CLT)

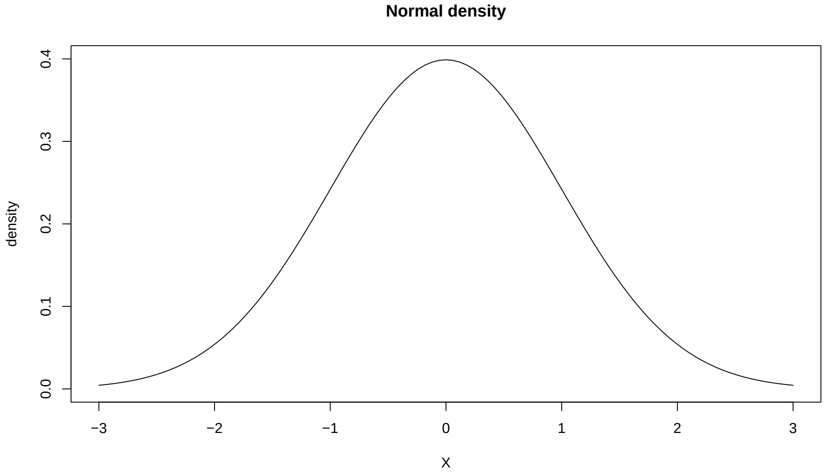
  When sampling from a population that has a mean, provided the sample size is large enough, the sampling distribution of the sample mean will be close to normal regardless of the shape of the population distribution

- (Technically, we were sampling the sum of X rather than the mean, but the mean of X is simply the sum divided by the number of observations. Do you care about this distintion? Didn't think so. It makes me feel better, though.)

What does this mean?

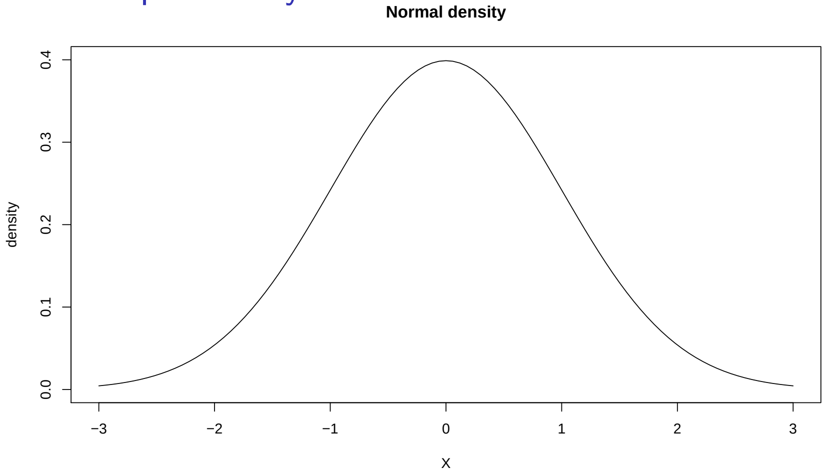
- For our dice problem, it means that we can compute the means of our samples (e.g. the mean of the 5, or 10, or 100 samples)
- Remember, the *sample mean* is a random variable as well, since it is different every time we take a sample
- We can then use a *continuous* probability distribution – the **normal distribution** as the theoretical probability distribution for our random variable (i.e. the sample mean).
- This makes our life easy, because the normal distribution is very simple to handle mathematically (really!).

Continuous probability distributions



- Here, the outcomes are continuous, so it doesn't make sense to ask about the probability of any point on the x-axis.
- What is the probability of $x = 1$?

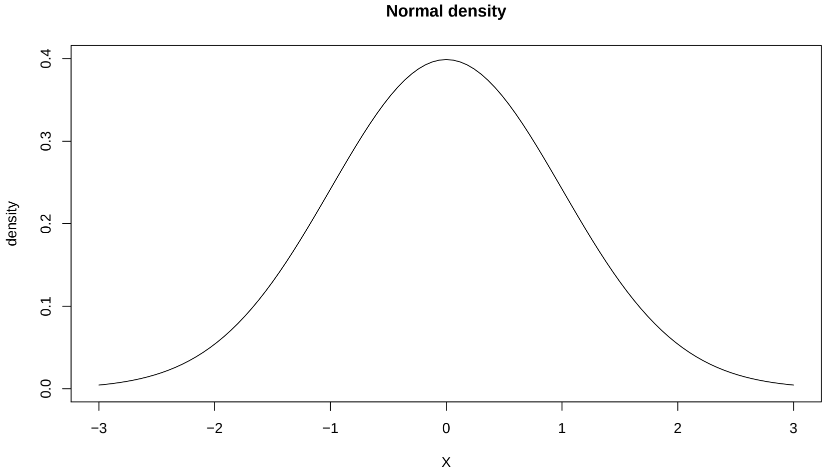
Continuous probability distributions



- What do you mean by “1”? The function is continuous, so does 1.00001 still qualify as 1?
- It makes more sense to ask these questions about intervals. The probability is then the area under the curve for the interval.
- Important: the total area under the curve is 1.

Normal probability density function (PDF)

$$f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-((x-\mu)^2/2\sigma^2)}$$



Normal probability density function (PDF)

$$f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-((x-\mu)^2/2\sigma^2)}$$

- This looks scary, but it really isn't. This is simply a mathematical function that happens to describe the distribution of a lot of random variables in nature. - If you look closely, you can see that the function has three parameters, x, μ, and σ (π and e are constants). - The first parameter, x is the random variable. The function gives you the probability density at each value of x - The second parameter, μ, is called the **expected value** or the **mean** of the distribution. - The third parameter, σ, is called the standard deviation.

Standard normal distribution

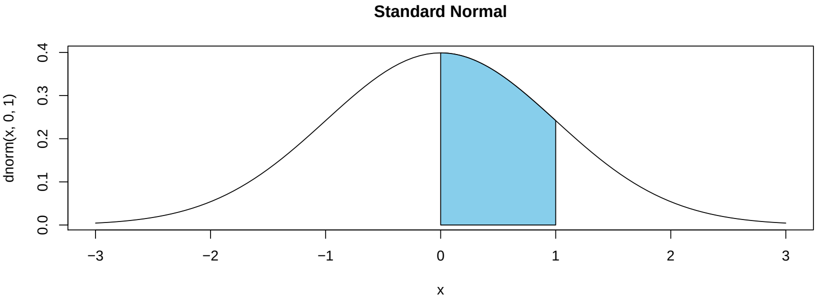
- There is an infinite number of normal distributions with different parameters μ and σ. The one with $\mu = 0$ and $\sigma = 1$ is particularly useful and is called the *standard* normal distribution.
- Look at how simple and nice the normal distribution appears when we plug in those values:

$$f(z, \mu, \sigma) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

- You can *transform* values from any normal distribution to the normal distribution.
- This is known as a *z-transformation*: $z = \frac{x-\mu}{\sigma}$
- By transforming all our observations to z-values and then looking up their probability in the standard normal distribution, this is the only distribution we'll ever need (... mostly).

The probability of outcomes in the standard normal distribution

- Remember, we can't really get the probability of a *point* event in a continuous distribution, since there are no "points" in a continuous variable
- But we can ask questions about *intervals*:
- What's the probability of x being between 0 and 1?



Getting the area under the curve

- Since we know exactly what the function is, we can get the area under the curve.
- Remember how to do that from maths class? Your best friend, integration :

$$p(0 < z < 1) = \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

- Don't want to do integration? Well, you're in luck, because most statistical software (and Excel!) can do this for you.
- `=NORM.S.DIST(0,TRUE)` will give you the area under the curve to the left of 0 (i.e. the probability that $z < 0$)
- `=NORM.S.DIST(1,TRUE)` will give you the area under the curve to the left of 1 (i.e. the probability that $z < 1$)

Getting the area under the curve

- So, for our interval: $p(0 < z < 1) = p(z < 1) - p(z < 0)$
- Remember, Excel gives us the upper tail (the area under the curve to the right of the z value)
- So we rewrite our interval: since $p(z < 1) = 1 - p(z > 1)$ and $p(z < 0) = 1 - p(z > 0)$, $p(0 < z < 1) = 1 - p(z > 1) - (1 - p(z > 0))$
- In Excel:
  - `=NORM.S.DIST(1,TRUE)-NORM.S.DIST(0, TRUE)`
  - Result: 0.3413447
- Success!

Things you can do with this knowledge

- Say I'm looking at random numbers from a standard normal distribution, and I see that one of them is 4.
- That seems very unusual
- Just how unusual?
  - What's the probability of getting a value of 4 when sampling from a standard normal distribution (mean $=$ 0, sd $=$ 1)?

Just how “unusual” is a value of 4?

- Remember, when you have a continuous distribution, you can’t think about point values (e.g. 5). Rather, what you want to know is:
  - What is the probability of getting a value of 4 *or greater* (or $p(z > 4) = 1 - p(z < 4)$)?
- Let’s ask Excel: `=1-NORM.S.DIST(4,TRUE)`
  - Result: 0.0001338302
- So it’s very unusual.
- Can we come up with a similar test for our dice sample mean?
- We’ll have to figure out how the dice sample means are distributed.
  - Then we can take our sample mean and see how likely (or unlikely) it is that it comes from the theoretical distribution.

The theoretical distribution of dice sample means

- We've seen that we can approximate our theoretical distribution (which is actually discrete) using a continuous distribution function, namely the normal distribution, which makes our lives very easy (yes, really!).
- We have to figure out the μ and the *sigma* parameters for our theoretical normal distribution of sample means, though.
- Note (in case anyone very critical reads this): In the case that we actually know exactly what the probabilities for our discrete probability distribution should look like, we could also use a different distribution, the χ^2 (chi square) distibution. We will talk more about that next week.

Random variables: Expected value

- Random variables have expected values
- For discrete random variables, the expected value is the outcome value multiplied by the probability of the outcome:

$$E(X) = \mu = \sum_{i=1}^{k} p(x_i) \cdot x_i$$

- where $E(X)$ is the expected value of a discrete random variable X with the outcomes $(x_1 \ldots x_k)$ and the associated probabilities $(p(x_1) \ldots p(x_k))$
- The equivalent for continuous random variables:

$$\mu = \int\limits_{-\infty}^{\infty} xf(x)dx$$

Random variables: Variance

- Random variables also have variances
- For discrete random variables, the variance is the difference between the outcome value and the mean multiplied by the probability of the outcome:

$$\sigma^2 = \sum_{i=1}^{k} p(x_i) \cdot (x_i - \mu)^2$$

- where σ^2 is the variance of a discrete random variable X with the outcomes $(x_1 \ldots x_k)$ and the associated probabilities $(p(x_1) \ldots p(x_k))$
- The equivalent for continuous random variables:

$$\mu = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

Maths basics: Expected values

- For example, the expected value μ of rolling a six-sided die is:

$$\begin{aligned} E(X) &= \sum_{i=1}^{6} p(x_i) \cdot x_i \\ &= x_1 \cdot p(x_1) + x_2 \cdot p(x_2) + x_3 \cdot p(x_3) + x_4 \cdot p(x_4) \\ &+ x_5 \cdot p(x_5) + x_6 \cdot p(x_6) \\ &= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} \\ &+ 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} \\ &= \frac{21}{6} = 3.5 \end{aligned}$$

Maths basics: Variance

- For example, the variance σ^2 of rolling a six-sided die is:

$$\begin{aligned}
\sigma^2 &= \sum_{i=1}^{6} p(x_i) \cdot (x_i - \mu)^2 \\
&= (x_1 - \mu)^2 \cdot p(x_1) + (x_2 - \mu)^2 \cdot p(x_2) + (x_3 - \mu)^2 \cdot p(x_3) \\
&+ (x_4 - \mu)^2 \cdot p(x_4) + (x_5 - \mu)^2 \cdot p(x_5) + (x_6 - \mu)^2 \cdot p(x_6) \\
&= \frac{1}{6} \cdot \Big((1 - 3.5)^2 + (2 - 3.5)^2 + (3 - 3.5)^2 \\
&+ (4 - 3.5)^2 + (5 - 3.5)^2 + (6 - 3.5)^2 \Big) \\
&= \frac{17.5}{6} = 2.9167
\end{aligned}$$

Back to the dice example again

- So, we know that, if our dice are fair, our dice rolls come from a discrete theoretical distribution with $\mu = 3.5$ and $\sigma^2 = 2.9167$.
- But remember, we don't want to evaluate single dice rolls, but rather the mean of a sample of dice rolls, since that will enable us to use the nice and easy normal distribution to calculate the probabilities.
- So, what is the mean $\mu_{\bar{x}}$ and what is the variance $\sigma_{\bar{x}^2}$ for the **distribution of sample means**?

Maths basics: Expected values (3)

- What is the expected value of rolling two dice and adding the spots?
- What is the expected value of rolling two dice and multiplying the number of spots?
- What is the expected value of an IQ test result?
- Imagine you and your friend both take IQ tests. What is the expected value of the differences between your scores (assuming that you both come from the general population)?
  - Don't know? Well, stay tuned. This will require some maths, though.

Maths basics: Computing expected values

- The expected value of a random variable is often also called μ:

$$E(X) = \mu$$

- μ is also called the distribution *mean*
- What if the expected value is constant across all the possible outcomes?
- e.g what is the expected value of a die with 1 on all sides?
    - 1, of course!
- More general:
- if the value is the same across all outcomes, we can call it a constant
- e.g. if $x_1 = x_2 = x_3 = \cdots = x_i = 1$
    - then $E(X) = E(1) = 1$

Maths basics: Computing expected values (2)

Even more general: If a is a constant, then $E(a) = a$ - If X is a random variable and a is a constant, what is the expected value of $a \cdot X$?

$$E(a \cdot X) = a \cdot E(X)$$

- For example, if the expected value of rolling a 6-sided die is 3.5, what is the expected value of rolling a 6-sided die and then multiplying the number of spots by 3?

$$E(3 \cdot X) = 3 \cdot E(X) = 3 \cdot 3.5 = 10.5$$

- Try it if you don't believe me!

Maths basics: Computing expected values (3)

- If X is a random variable and a is a constant, what is the expected value of $a + X$?

$$E(a + X) = a + E(X)$$

- For example, if the expected value of rolling a 6-sided die is 3.5, what is the expected value of rolling a 6-sided die and then adding 3 to the number of spots?

$$E(3 + X) = 3 + E(X) = 3 + 3.5 = 6.5$$

- Try it if you still don't believe me!

Maths basics: Computing expected values (4)

- If X is a random variable and Y is a random variable, what is the expected value of $X + Y$?

$$E(X + Y) = E(X) + E(Y)$$

- For example, if the expected value of rolling two 6-sided dice and adding the two results?

$$E(X + Y) = E(X) + E(Y) = 3.5 + 3.5 = 7$$

- Try it if you still don't believe me!

Maths basics: Computing expected values (5)

- If X is a random variable and Y is a random variable, what is the expected value of $X \cdot Y$?

$$E(X \cdot Y) = E(X) \cdot E(Y)$$

- But *ONLY* if X and Y are *INDEPENDENT*
- For example, if the expected value of rolling two 6-sided dice and multiplying the two results?

$$E(X + Y) = E(X) \cdot E(Y) = 3.5 \cdot 3.5 = 12.25$$

- Try it if you still don't believe me!

The expected value of the sample mean

- If X is a random variable, and we take a sample of size n from X, what is the expected value of the mean of that sample?
- Remember, this is how you compute the sample mean:

$$\bar{x} = \frac{\sum\limits_{i=1}^{n}}{n}$$

- The expected value of the sample mean is:

$$\begin{aligned} E(\bar{X}) &= E\left(\frac{\sum\limits_{i=1}^{n}}{n}\right) \\ &= \frac{1}{n} \cdot (E \sum_{i=1}^{n} X_i) \end{aligned}$$

The expected value of the sample mean (2)

$$
\begin{aligned}
&= \frac{1}{n} \cdot (E \sum_{i=1}^{n} X_i) \text{ (since } E(a \cdot X) = a \cdot E(X)) \\
&= \frac{1}{n} \cdot \sum_{i=1}^{n} E(X_i) \text{ (since } E(X + Y) = E(X) + E(Y)) \\
&= \frac{1}{n} \cdot \sum_{i=1}^{n} \mu_x \text{ (since } E(X) = \mu) \\
&= \frac{1}{n} \cdot n \cdot \mu_x = \mu_x
\end{aligned}
$$

The expected value of the sample mean (3)

- We just found that the expected value of the sample mean $E(\bar{X})$ is identical to the expected value (the mean) of the population μ_x, *regardless of the sample size*.
- We can say that the sample mean $\overline{X}$ is an *unbiased estimator* of the population mean μ
- No matter what we do and what crazy population we're taking samples of, the sample means will always be distributed around the true population mean.
  - *Isn't that cool?*
  - I know what you're thinking right now, but this is *actually* cool. Just think about it: If you want to know the true population mean of any population, all you have to do is take enough samples.

Our dice example

- Remember, we want to know how the means of our dice rolls should be distributed if the dice are fair. So, now we know that they are normally distributed (because of the Central Limit Theorem) with an expected value of $\mu_{\bar{x}} = 3.5$.
- What about the *variance* of the *distribution of sample means*?
  - Well, this will take some work. Sorry, people. It's maths time.
  - First, we need to know how the sample variance is related to the population variance.
  - In other words, we need to know the expected value of the sample variance.

The expected value of the sample variance

- OK, first we want to know how the variance of your samples (s^2) is related to the population variance σ^2.
- Remember that the variance of a sample is

$$s^2 = \frac{\sum\limits_{i=1}^{n}(x_i - \bar{x})^2}{n}$$

- We can rewrite this as:

$$s^2 = \frac{\sum\limits_{i=1}^{n}(x_i - \bar{x})^2}{n} = \frac{\sum\limits_{i=1}^{n}(x_i^2 - 2 \cdot x_i + \bar{x})^2}{n}$$

$$= \frac{\sum\limits_{i=1}^{n} x_i^2 - 2 \cdot \bar{x} \cdot \sum\limits_{i=1}^{n} x_i + \bar{x}^2}{n}$$

The expected value of the sample variance (2)

- Further rewriting: Since $\sum_{i=1}^{n} x_i = n \cdot \bar{x}$:

$$
\begin{aligned}
s^2 &= \frac{\sum_{i=1}^{n} x_i^2 - 2 \cdot \bar{x} \cdot \sum_{i=1}^{n} x_i + \bar{x}^2}{n} \\
&= \frac{\sum_{i=1}^{n} x_i^2 - 2 \cdot \bar{x} \cdot n \cdot \bar{x} + \bar{x}^2}{n} \\
&= \frac{\sum_{i=1}^{n} x_i^2 - n \cdot \bar{x}^2}{n} = \frac{\sum_{i=1}^{n} x_i^2}{n} - \bar{x}^2
\end{aligned}
$$

The expected value of the sample variance (3)

- Now we can calculate the expected value of s^2:

$$
\begin{aligned}
E(S^2) &= E\left(\frac{\sum\limits_{i=1}^{n} X_i^2}{n} - \bar{X}^2\right)\\
&= E\left(\frac{\sum\limits_{i=1}^{n} X_i^2}{n}\right) - E(\bar{X}^2)\\
&= \frac{\sum\limits_{i=1}^{n} E(X_i^2)}{n} - E(\bar{X}^2) = \frac{n \cdot E(X_i^2)}{n} - E(\bar{X}^2)
\end{aligned}
$$

The expected value of the sample variance (4)

- And $\frac{n \cdot E(X_i^2)}{n} - E(\bar{X}^2)$ of course simplifies to $E(X_i^2) - E(\bar{X}^2)$
- So, now we have to figure out what $E(X_i^2)$ and $E(\bar{X}^2)$ are.
- The "easiest" (I know, right?) way to do this is to start with the population variance σ^2 and the variance of the sample means $\sigma_{\bar{x}}^2$

The expected value of the sample variance (5)

- We can define the population variance as $\sigma^2 = E(X_i - \mu)^2$, the expected value of the squared deviations of X from the population mean μ
- Let's rewrite this:

$$\begin{aligned}\sigma^2 = E(X_i - \mu)^2 &= E(X_i^2 - 2X_i\mu + \mu^2)\\ &= E(X_i^2) - E(2X_i\mu) + E(\mu^2)\\ &= E(X_i^2) - 2\mu E(X_i) + \mu^2\end{aligned}$$

since μ is a constant (and μ^2 is too, of course).

The expected value of the sample variance (6)

Continuing from previous slide: - We already determined that $\mu = E(X)$, so:

$$\begin{aligned}\sigma^2 &= E(X_i^2) - 2\mu E(X_i) + \mu^2 \\ &= E(X_i^2) - 2\mu^2 + \mu^2 = E(X_i^2) - \mu^2\end{aligned}$$

- Solving for $E(X_i^2)$:

$$\begin{aligned}\sigma^2 &= E(X_i^2) - \mu^2 \\ &\Leftrightarrow E(X_i^2) = \sigma^2 + \mu^2\end{aligned}$$

- OK, so now we know that the expected value of a squared random variable is equal to the sum of the population variance σ^2 and the square of the population mean μ^2.

The expected value of the sample variance (7)

- Next up: the variance of sample means $\sigma_{\bar{x}}^2$
  - This is the square of the *standard error* of the mean $\sigma_{\bar{x}}$
- We can define the variance of sample means as $\sigma_{\bar{x}}^2 = E(\bar{X} - \mu)^2$, i.e. the expected value of the squared deviations of the sample means from the true population mean
- We can rewrite this just like we did for the sample variance (this works exactly the same as before; if you are bored, you can skip the next two slides).

The expected value of the sample variance (7a)

- We can define the variance of the sample means as $\sigma^2_{\bar{x}} = E(\bar{X} - \mu)^2$
- Let's rewrite this:

$$\begin{aligned}\sigma^2 &= E(\bar{X} - \mu)^2 = E(\bar{X}^2 - 2\bar{X}\mu + \mu^2) \\ &= E(\bar{X}^2) - E(2\bar{X}\mu) + E(\mu^2) \\ &= E(\bar{X}^2) - 2\mu E(\bar{X}) + \mu^2\end{aligned}$$

since μ is a constant (and μ^2 is too, of course).

The expected value of the sample variance (7b)

Continuing from previous slide: - We already determined that $\mu = E(\bar{X})$, so:

$$\begin{aligned}\sigma_{\bar{x}}^2 &= E(\bar{X}^2) - 2\mu E(\bar{X}) + \mu^2 \\ &= E(\bar{X}^2) - 2\mu^2 + \mu^2 = E(\bar{X}^2) - \mu^2\end{aligned}$$

- Solving for $E(\bar{X}^2)$:

$$\begin{aligned}\sigma_{\bar{x}}^2 &= E(\bar{X}^2) - \mu^2 \\ &\Leftrightarrow E(\bar{X}^2) = \sigma_{\bar{x}}^2 + \mu^2\end{aligned}$$

- OK, so now we know that the expected value of the squared mean of a random variable is equal to the sum of the variance of the sample means $\sigma_{\bar{x}}^2$ and the square of the population mean μ^2.

The expected value of the sample variance (8)

- Plugging $E(X_i^2) = \sigma^2 + \mu^2$ and $E(\bar{X}^2) = \sigma_{\bar{x}}^2 + \mu^2$ into our term for the expected value of the sample variance:

$$\begin{aligned} E(S^2) &= E(X_i^2) - E(\bar{X}^2) = \sigma^2 + \mu^2 - (\sigma_{\bar{x}}^2 + \mu^2) \\ &= \sigma^2 - \sigma_{\bar{x}}^2 \end{aligned}$$

- In words: the expected value of the sample variance is equal to the population variance minus the variance of the sample means.
  - This means that the sample variance *systematically* underestimates the population variance
  - The sample variance is *NOT* an unbiased estimator of the population variance.

The expected value of the variance of sample means

- We start with the relationship we just figured out: $E(\sigma_{\bar{x}}^2) = \sigma_{\bar{x}}^2 = E(\bar{X}^2) - \mu^2$ (since $E(\bar{X}^2)$ and μ^2 are both constants).
- We can rewrite $\bar{X}^2$ as:

$$\begin{aligned}
\bar{X}^2 &= \frac{(X_1 + X_2 + \cdots + X_n)^2}{n^2} \\
&= \frac{1}{n^2} \cdot \left(X_1^2 + X_2^2 + \cdots + X_n^2 \right. \\
&\quad \left. + 2 \sum_{i=1} \sum_{j=i+1} X_i \cdot X_j \right)
\end{aligned}$$

The expected value of the variance of sample means (2)

- If (and only if!) $X_1, X_2, \ldots, X_n$ are independent (i.e. the value of X_1 doesn't depend on the value of X_2, or X_3, etc.), we can write the expected value of the final term of this expression as:

$$\begin{aligned} E\left(2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} X_i \cdot X_j\right) &= n \cdot *(n-1) \cdot E(X_i) \cdot E(X_j) \\ &= n \cdot (n-1) \cdot \mu^2 \end{aligned}$$

- Since $E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i)$ and $E(X_i) = \mu$

The expected value of the variance of sample means (3)

- With that, we can rewrite $E(\bar{X^2})$ as

$$E(\bar{X^2}) = \frac{1}{n^2} \cdot \Big(E(X_1)^2 + E(X_2)^2 + \dots \\ + E(X_n)^2) + n \cdot (n-1) \cdot \mu^2 \Big)$$

- But we know already (through our hard work earlier) that the expected value of the square of X_i is $E(X_i^2) = \sigma^2 + \mu^2$.
- So we can replace $E(X_1)^2 + E(X_2)^2 + \dots + E(X_n)^2$ with $n \cdot (\sigma^2 + \mu^2) = n \cdot \sigma^2 + n \cdot \mu^2$.

The expected value of the variance of sample means (4)

- Let's do that now:

$$E(\bar{X}^2) = \frac{1}{n^2} \cdot \left(n \cdot \sigma^2 + n \cdot \mu^2 + n \cdot (n-1) \cdot \mu^2\right)$$
$$= \frac{\sigma^2}{n} + \frac{n \cdot \mu^2 + n * 2 \cdot \mu^2 - n \cdot \mu^2}{n^2} = \frac{\sigma^2}{n} + \mu^2$$

- Plugging this into our previous equation $\sigma_{\bar{x}}^2 = E(\bar{X}^2) - \mu^2$ we get:

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} + \mu^2 - \mu^2 = \frac{\sigma^2}{n}$$

- If we take the square root of this, we *FINALLY* get the **standard error of the mean**:

$$\sigma_{\bar{x}} = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$$

Correcting the bias in the expected value of the sample variance

- Before we actually use our hard-earned $\mu_{\bar{x}}$ and $\sigma_{\bar{x}}$, a quick detour:
- Remember that the expected value of the sample variance was biased by the variance of the sample mean, i.e. $E(S^2) = \sigma^2 - \sigma_{\bar{x}}^2$?
- Now we know what the variance of the sample mean is, so let's plug it in:

$$\begin{aligned} E(S^2) &= \sigma^2 - \sigma_{\bar{x}}^2 = \sigma^2 - \frac{\sigma^2}{n} = \frac{n \cdot \sigma^2 - \sigma^2}{n} \\ &= \sigma^2 \cdot \frac{n-1}{n} \end{aligned}$$

Correcting the bias in the expected value of the sample variance (2)

- We just found out that the expected value of the sample variance $E(s^2)$ underestimates the true population variance σ^2 by a factor of $\frac{n-1}{n}$.
- That means we can apply a correction factor to the sample variance so that it becomes an *unbiased* estimator of the population variance:

$$
\begin{aligned}
s^2_{n-1} &= s^2 / \frac{n-1}{n} = s^2 \cdot \frac{n}{n-1} = \frac{\sum\limits_{i=1}^{n} (x_i - \bar{x})^2}{n} \cdot \frac{n}{n-1} \\
&= \frac{\sum\limits_{i=1}^{n} (x_i - \bar{x})^2}{n-1}
\end{aligned}
$$

Correcting the bias in the expected value of the sample variance (3)

- Most statistical software will use this corrected formula for computing the sample variance: $s^2_{n-1} = \frac{\sum\limits_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$
- If you want a more intuitive explanation of what is going on here, watch the videos at EasyStats: `http://easystats.org/`

Please, let's FINALLY finish the dice example

- OK, OK. We now have everything we need to determine whether our dice roll sample mean is unusual assuming fair dice.
- More formally, we call this a **Hypothesis Test**
- We establish a *Null Hypothesis* H_0 (e.g. the dice are fair),
  - determine a theoretical probability distribution of the random variable (our dice roll means) given that the H_0 is true:
    - a normal distribution with $\mu_{\bar{x}} = 3.5$ and $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{\sqrt{2.9167}}{\sqrt{n}}$, where n is the number of dice rolls in our sample,
  - and finally we can calculate the probability that you would observe the sample mean you observed given the H_0.

Final steps

- So, let's assume you did 10 dice rolls for this example, and that your mean was 4.
- Since we know that the sample means should be normally distributed, we can transform your mean into a *z*-value:
- Since $\mu_{\bar{x}} = 3.5$ and $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{\sqrt{2.9167}}{\sqrt{10}} = 0.5400648$:

$$z(4) = \frac{4 - 3.5}{0.5400648} = 0.9258148$$

- Let's ask Excel what the probability of observing a sample mean this far away (or farther) from the population mean is for the standard normal distribution: `1-NORM.S.DIST(0.9258,TRUE)`
  - Result: 0.177274964

Final steps (2)

- Fisher suggested that we should consider data with a probability of less than 5% (or .05) given the null hypothesis as **significant** evidence for rejecting the null hypothesis.
- In our case, we are far away from a probability (or short, p-value) of .05. So, we can't reject the null hypothesis. Try it for yourselves, though.
- More on this next week.

Technical note for those who really care

- We really don't care about the direction of the effect here, just the absolute distance from the mean (i.e. this is a *two-tailed* test).
- So, to be absolutely correct, we should ask Excel to give us the probability of z being at least this far away from the mean on either side:

$$p(z < -.9258 \cup z > .9258) = p(z < -.9258) + (1 - p(z < .9258))$$

- When we ask Excel for the p-value `=NORM.S.DIST(-0.9258,TRUE)+(1-NORM.S.DIST(0.9258,TRUE))` we get the actual, correct result of 0.3545499.