Advanced Computer Graphics Proseminar

Univ.-Prof. Dr. Matthias Harders

Winter semester 2015



Monte-Carlo Integration

- Used to approximate integral in rendering equation
- Requires generation of random samples
- Image quality and convergence dependent on sampling strategy
- Often requires mapping of canonical uniform random variable $\xi \in [0,1]$ to another distribution

$$I = \iint f(x,y)\,dxdy \qquad I \approx \hat{I} = \frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i,y_i)}{p(x_i,y_i)}$$

Sampling Triangular Patches

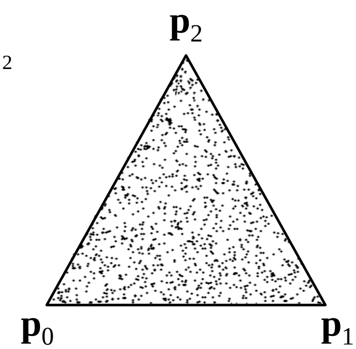
- Triangle in 3D given by vertices $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$
- Arbitrary vertex $\mathbf{q}$ given by barycentric coordinates
- Correct sampling approach

$$\mathbf{q} = \lambda_0 \mathbf{p}_0 + \lambda_1 \mathbf{p}_1 + (1 - \lambda_0 - \lambda_1)\mathbf{p}_2$$

$$\lambda_0 = 1 - \sqrt{\xi_0}$$

$$\lambda_1 = \xi_1 \sqrt{\xi_0}$$

$$\lambda_2 = 1 - \lambda_0 - \lambda_1 = \sqrt{\xi_0}(1 - \xi_1)$$



Generating Random Numbers

- Computer-based approaches **pseudo-random** (i.e. following algorithm)
- Example: `drand48()`

$$x_{n+1} = (a \cdot x_n + c) \bmod m$$

- 48-bit parameters given as

$$m = 2^{48} \qquad a = (5\mathrm{DEECE66D})_{16} \qquad c = (\mathrm{B})_{16}$$

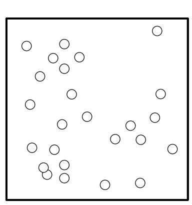
- Generates non-negative, double-precision, floating-point values, uniformly distributed over $[0.0, 1.0]$

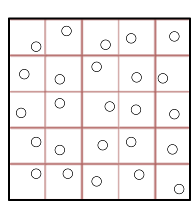
Generating Random Numbers

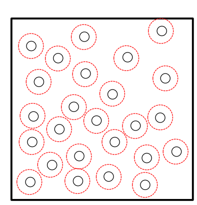
- Algorithm provides deterministic, pseudo-random, periodic numbers
- Allows setting (random) seed value x_0 via `srand48()`
- Same seed always results in same sequence
- Initialization with system time may not be optimal
- Higher-quality random number generators available, e.g. RANMAR, RANLUX (available in CERN-library)

Sampling Strategy

- Fully randomized sampling possibly suboptimal due to clumping of samples
- Stratified or Poisson disk sampling reduces clumping and non-uniformity


Random samples


Stratified


Poisson disk

Quasi-Random Sampling

- Deterministic sequence of numbers that only appear random, and regularly cover domain
- Values are said to be of low discrepancy
- Example: **Halton sequence**
- Representation of positive integer n with base b

$$n = \sum_{i=1}^{\infty} d_i b^{i-1} \qquad 0 \le d_i < b$$

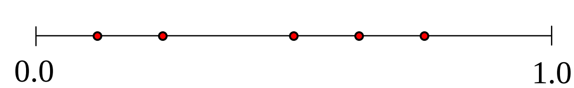
- Associated **radical inverse function**

$$\Phi_b(n) = 0.d_1 d_2 \ldots d_m \qquad d_{m+i} = 0 \quad i > 0$$

Halton Sequence

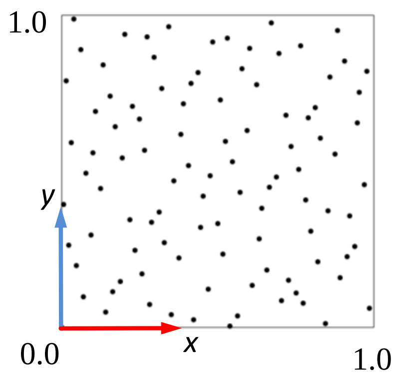
- Example for base 2

n (Base 10)	Base 2 number	$\Phi_2(n)$
1	$(1)_2$	$(0.1)_2 = 1/2$
2	$(10)_2$	$(0.01)_2 = 1/4$
3	$(11)_2$	$(0.11)_2 = 3/4$
4	$(100)_2$	$(0.001)_2 = 1/8$
5	$(101)_2$	$(0.101)_2 = 5/8$
…	…	…



Halton Sequence in 2D

- In higher dimensional cases, choose different prime number as base for each dimension

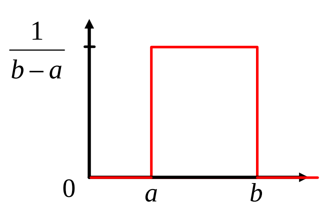


$$(x_i, y_i) = (\Phi_2(i), \Phi_3(i))$$

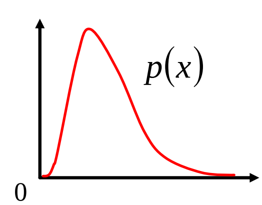
(100 samples)

Sampling Arbitrary Probability Density Function

- Evaluating integrals via Monte-Carlo method requires random samples according to arbitrary PDF
- Option: **Inversion method**



Uniform distribution



Arbitrary PDF

Inversion Method

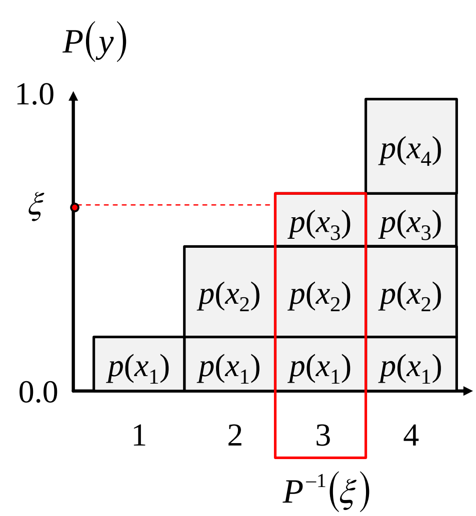
- Mapping uniform random variable to distribution
- Discrete example: assume random variable X with four possible outcomes of non-uniform probability

$$0 \le p(x_1), p(x_2), p(x_3), p(x_4) \le 1 \qquad \sum_i p(x_i) = 1$$

- Discrete cumulative distribution function (CDF)

$$P(y) = \Pr(X \le y) = \sum_{x_i \le y} p(x_i)$$

Inversion Method Graphically



Inversion Method – Continuous PDF

1) Compute cumulative distribution function for PDF

$$P(y) = \int_{-\infty}^{y} p(x)dx$$

2) Compute inverse of CDF (not always feasible)
3) Sample uniformly distributed random number ξ
4) Obtain new random variable, adhering to probability density function $p(x)$

$$X = P^{-1}(\xi)$$

Conditional Probability

- Probability of outcome given another event occurred

$$P(X|Y) = \frac{P(X,Y)}{P(Y)}$$

- Example: rolling two fair dice
- **Joint probability** of sum being 6 and 2nd die showing 2

$$P(X,Y) = 1/36$$

- **Conditional probability** that sum of two-dice throw equals 6, given that first roll shows 2

$$P(X|Y) = \frac{P(X,Y)}{P(Y)} = \frac{1/36}{1/6} = 1/6$$

2D Joint Distribution Random Sampling

1) Compute **marginal probability density function**

$$p(x) = \int p(x, y)\,dy$$

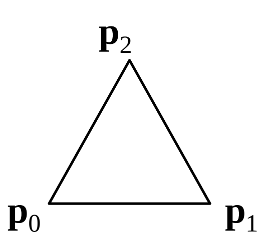
2) Obtain conditional probability density function

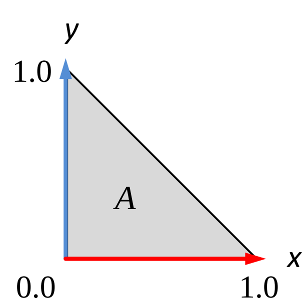
$$p(y|x) = \frac{p(x, y)}{p(x)}$$

3) Sample marginal probability function using $p(x)$
4) Sample conditional probability function using $p(y|x)$

Uniformly Sampling Triangle Area

- Without loss of generality, assume isosceles triangle with area $A = ½$
- Extends to general case via barycentric coordinates





Uniformly Sampling Triangle Area

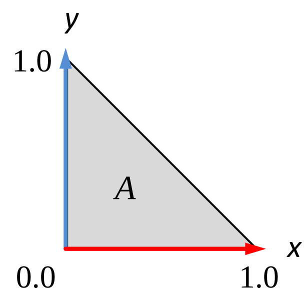
- Determine joint probability density function $p(x,y)$ (constant for uniform probability over area)
- According to definition of PDF

$$\int_A p(x,y)\,dA = 1$$

$$p(x,y)\int_A dA = 1$$

$$p(x,y)\frac{1}{2} = 1$$

$$p(x,y) = 2$$



Uniformly Sampling Triangle Area

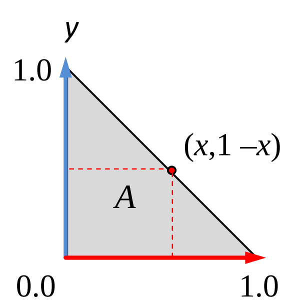
- Compute marginal probability density function

$$p(x) = \int_0^{1-x} p(x,y)\,dy$$

$$p(x) = \int_0^{1-x} 2 \cdot dy$$

$$p(x) = 2y\Big|_0^{1-x}$$

$$p(x) = 2 - 2x$$



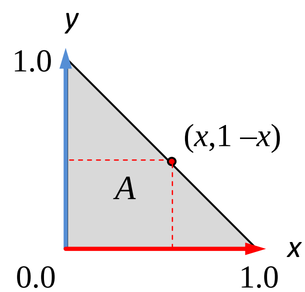
Uniformly Sampling Triangle Area

- Compute conditional probability density function

$$p(y|x) = \frac{p(x,y)}{p(x)}$$

$$p(y|x) = \frac{2}{2-2x}$$

$$p(y|x) = \frac{1}{1-x}$$



Uniformly Sampling Triangle Area

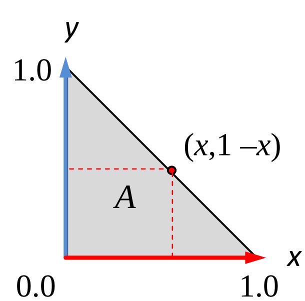
- Determine cumulative distribution functions for applying inversion method

$$P(x) = \int_0^x p(\hat{x})d\hat{x}$$

$$P(x) = \int_0^x 2 - 2\hat{x} \cdot d\hat{x}$$

$$P(x) = 2\hat{x} - \hat{x}^2 \Big|_0^x$$

$$P(x) = 2x - x^2$$



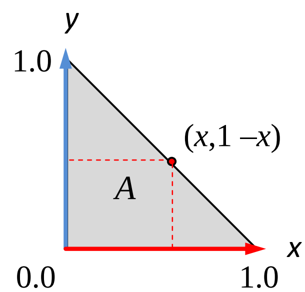
Uniformly Sampling Triangle Area

- Determine cumulative distribution functions for applying inversion method

$$P(y) = \int_0^y p(\hat{y}|x)\, d\hat{y}$$

$$P(y) = \int_0^y \frac{1}{1-x}\, d\hat{y}$$

$$P(y) = \frac{y}{1-x}$$



Uniformly Sampling Triangle Area

- Invert CDF for sampling with canonical uniformly distributed variable

$$\hat{\xi}_0 = 2x - x^2$$

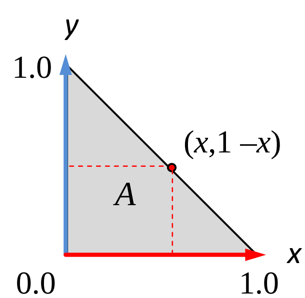
$$1 - \xi_0 = 2x - x^2$$

$$\xi_0 = 1 - 2x + x^2$$

$$\xi_0 = (1-x)^2$$

$$\sqrt{\xi_0} = 1 - x$$

$$x = 1 - \sqrt{\xi_0}$$



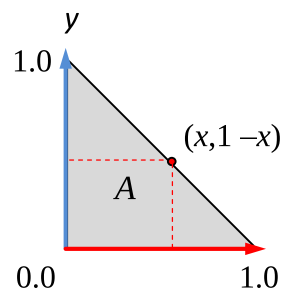
Uniformly Sampling Triangle Area

- Invert CDF for sampling with canonical uniformly distributed variable

$$\xi_1 = \frac{y}{1-x}$$

$$\xi_1 = \frac{y}{\sqrt{\xi_0}}$$

$$y = \xi_1 \sqrt{\xi_0}$$



Sampling Triangular Patches

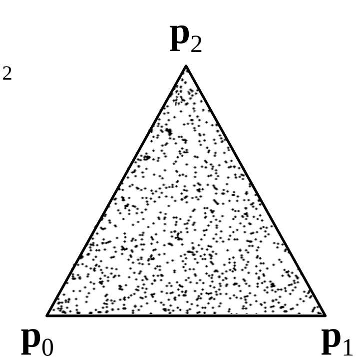
- Triangle in 3D given by vertices $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$
- Arbitrary vertex $\mathbf{q}$ given by barycentric coordinates
- Correct sampling approach

$$\mathbf{q} = \lambda_0 \mathbf{p}_0 + \lambda_1 \mathbf{p}_1 + (1 - \lambda_0 - \lambda_1)\mathbf{p}_2$$

$$\lambda_0 = 1 - \sqrt{\xi_0}$$

$$\lambda_1 = \xi_1 \sqrt{\xi_0}$$

$$\lambda_2 = 1 - \lambda_0 - \lambda_1 = \sqrt{\xi_0}(1 - \xi_1)$$



Tasks for Next Time

- Download path tracing example from OLAT
- Compile, run program, view output
- Examine source code (detailed explanations next proseminar)

Proseminar Schedule

Date	Topic	Remark
12.10.	Introduction	
19.10.	Theory – Radiometry	Radiosity example code
26.10.	*(no proseminar - Nationalfeiertag)*	
2.11.	*(no proseminar - Allerseelen)*	
9.11.	Discussion of Radiosity code	Programming assignment 1
16.11.	Programming support and advice	
23.11.	Theory – Random sampling	Path Tracer example
30.11.	Discussion of Path Tracer code	Programming assignment 2, *Hand-in PA1*
7.12.	Presentation of solutions	
14.12.	Programming support and advice	*Project proposal (21.12. Hand-in PA2)*
	Christmas break	
14.1.	Presentation of solutions	Presentation of solutions
21.1.	Geometric Modelling	
28.1.	Programming support and advice	
4.2.	Project presentation	*Submission final project*