Bachelor Thesis

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Abstract

This thesis acts as a companion essay to an online Tool developed to enable people to explore how some major theories of choice under risk evaluate (real-life) decisions with uncertain outcomes. While the Tool provides a rich interface to compare the different predictions and experiment with their different functional forms and parameters, this essay contains additional documentation concerning the implementation of the Tool and the underlying theories. After reading the essay and the case-studies in the end, everybody should be able to explore choice situations with these theories and contrast their different outcomes to inform the final decision.

1 Introduction

Almost everybody faces the need to make decisions over actions with uncertain outcomes. From low stakes choices about which restaurant will serve the best food to high stakes decisions over future careers or making an investment in one's home, we often lack sufficient information to determine with certainty how one option will play out against the others. Classical economics offers some answers when it comes to rationally deciding between different uncertain outcomes but lacks the flexibility to account for human emotions such as regret or the fear of losses. This is why a growing body of behavioral theories has developed to model these emotions and give better insights into how human decision makers (DMs) make judgements and evaluate the outcome of their decisions ex post.

The purpose of my Bachelor Thesis is two-fold. First and foremost, it aims to make some of the insights of these behavioral theories accessible to non-specialist readers and provide them with a handy tool to see how those theories judge some of their real-life decisions. The second aim is to provide readers already involved with economics with a quick introduction to the theories and offer a quick and flexible way to compare the different theories' outcomes. I am also confident, that the Tool and the backend code will enable faster exploratory analysis for their own work.

To my knowledge, there is no comparable tool available anywhere on the internet. So, the Tool and its open-sourced code as well as this essay represent the core contribution of my Bachelor Thesis.

Section I will clarify some distinction and conventions followed in behavioral economics and proceed to present all of the theories implemented in the Tool in detail. Section II will focus on the basic layout of the Tool, as well as some of the issues concerning the theories implementation. Afterwards, it will give further information on some of the advanced features of the Tool. Lastly, Section III will go through three case-studies as an example of how non-academic choice situations might be translated to problems that can be input into the Tool and how the different theories might be applied to them.

2 Theoretical Basis

This section is concerned with giving the non-specialist reader an introduction into the field of decision-making under risk and stating for the specialist reader, which version of a given theory I implemented. This means looking at some conventions and decisions and then systematically explaining and motivating all the major theories I chose to include in the Tool.

2.1 Concepts and conventions

A decision in this context must be the choice between two ore more actions, which have one or more possible outcomes. Traditionally, decision-science distinguishes at least three kinds of decisions in relation to the information available to the DM at the point of decision.

- Decisions under certainty occur, when the DM has knowledge of all available actions and can accurately predict the exact outcome of each action. A basic example is deciding between action A: eating a banana and action B: not eating a banana. In this case, the DM is aware of all available actions and knows exactly what happens when he decides for either.

- Decisions under risk occur, when the DM has reasonable knowledge of all available actions and can accurately predict the probabilities with which each outcome will occur and the payoffs which any outcome will yield. Imagine that you have to decide between action A: eating a banana and action B: eating a banana, when a coinflip comes up heads and not eating a banana, when it comes up tails. Now you know all the possible actions and can assign the outcome probabilities and payoffs accurately. But at least one action : eating / not eating of the banana conditional on the outcome of a coin flip leads to a probabilistic outcome (we assume a fair coin flip with each side being equally likely to be on top). These are the kinds of decisions we will be looking at in this thesis.

- Lastly, there are decision under uncertainty. They occur, when we lack some fundamental knowledge about the actions we face such as the probabilities and payoffs of each outcome or do not know all the possible actions we could take. Imagine that instead of basing action B: eating / not eating a banana based on the outcome of a coinflip, you base it on whether a penguin egg is laid in your local zoo on that day. Chances are this is a fact that is nearly impossible to predict for you and you can therefore not construct a reasonable representation of this decision.

As said above, any decision under risk is a decision among different actions at least one of which has a probabilistic instead of a sure outcome. The actions with probabilistic outcomes are commonly referred to as lotteries, because facing them is like participating in a lottery where the odds of winning a certain amount are known. Lotteries are often depicted graphically in addition to mathematical and text descriptions. Imagine you face the following decision: Either eat a banana or eat a banana when a coin flip comes up heads and not eating a banana when it comes up tails. A more formal notation of this is A = (1: eat banana) and B = (0.5: eat banana; 0.5: do not eat banana). A represents the action with the sure outcome and B is the alternative action with probabilistic outcomes. Notice that for each action, the first number in parentheses multiplied by 100 gives the probability in percent. The text after the colon describes the payoff. Lastly, we can depict this decision as decision trees as follows:

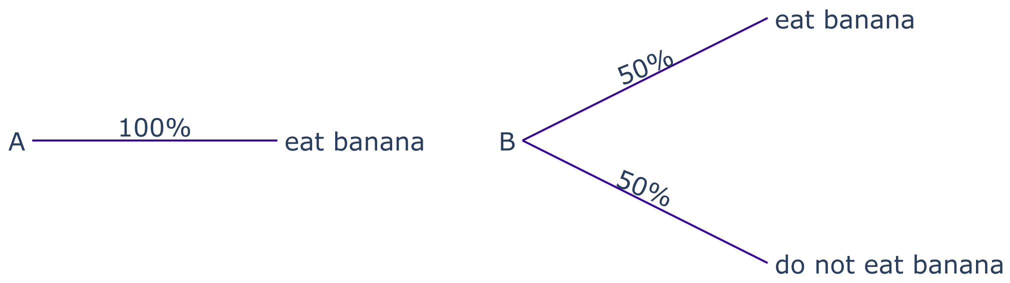

Figure 1: A graphical depiction of the set of actions available in this decision.

In Figure 1 you see the two available actions from before. Instead of giving the probability and the outcome in brackets, we now depict them in a kind of decision tree, where probabilities are given as percent.

One last necessary convention is how to measure the outcomes of any action. In the decisions above we state the outcomes of deciding for either A or B in words. You either eat a banana or you do not. But calculating with these outcomes is impossible because they are no numbers. To solve this problem, decision theorists often equate any outcome with a monetary payoff that either directly follows from the outcome or is considered equivalent to the outcome by the DM. The above outcomes are obviously not directly linked to a monetary payoff as decisions about choosing a suitable career path might be. So, assume that the DM equates the outcome of eating a banana in the above setup with receiving two euros and the outcome of not eating a banana with receiving only one euro. Then we could redraw the above graphs like this.

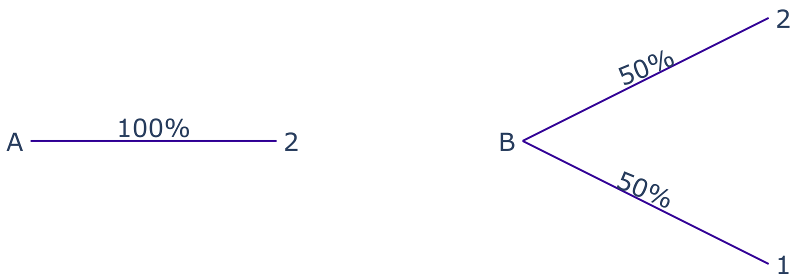

Figure 2: The lottery from Figure 1 rewritten in monetary terms.

In Figure 2 you see the two available actions from before. The only difference is that now, instead of showing a description of the outcome we reduce it to an equivalent monetary payoff.

Now that we know which kinds of decisions this thesis is concerned with and how we may depict them, how does one go about analyzing these cases with the theories presented in the Tool?

2.2 Expected Utility Theory

Most readers will have an intuitive idea of how to easily solve a dilemma like the one above. It seems easy enough to just calculate the weighted average of all possible outcomes for each action and choose the action with the higher average payoff. In this case, this would come down to average payoff$(A) = 1 \cdot 2 = 2$ and average payoff$(B) = 0.5 \cdot 2 + 0.5 \cdot 1 = 1.5$ meaning one should choose action A eating the banana for sure. Mathematically, this concept can be expressed like this:

$$\text{Expected payoff}(L) = \sum_{i=1}^{I} p_i \cdot x_i \tag{1}$$

where p_i is the probability of the i^{th} outcome and x_i its payoff.[1]

While this seems like a sensible solution to easy cases like this, attitudes might change for other kinds of problems. To illustrate this, please consider the following two decisions and their respective actions.

In this case it seems sensible to apply the rule from above and calculate the expected payoff. Expected payoff$(A) = 1 \cdot 0.5 = 0.5$ and Expected payoff$(B) = 0.1 \cdot 5 + 0.9 \cdot 5 = 0.5$, meaning that the DM should value the two actions about the same and not feel the strong urge to take one over the other.

If we calculate the expected payoff of the actions in Figure 4, we get the following results: Expected payoff$(A') = 1 \cdot 5,000.0 = 5,000.0$ and Expected payoff$(B') = 0.1 \cdot 50,000.0 + 0.9 \cdot 0.0 = 5,000.0$. While the DM should be indifferent between the two

[1]Since the aim of this essay is to be reasonably accessible to everybody, I will explain some notation and other background information in footnotes:

Expected payoff(L) is a way of declaring that we are looking at a mathematical function of a lottery L. In the above examples we had the lotteries A and B describing different actions concerning eating a banana. A function in this sense is simply a mapping of the input (in this case the lottery) to an output (the expected value or average payoff of this lottery). The theories discussed in this essay attempt to reduce the complexity of lotteries of different dimensions to single values via the application of different functions. While the lotteries might look complicated, the outputs of the theories can be easily compared to facilitate decisions or explain how actual decisions are made.

The Sum-Notation $\sum$ above will come up repeatedly in this essay. The right-hand side of the function describing expected payoffs should be read like this: "Sum the product of the probability and the payoff for all possible outcomes of an action which are numbers from i to n". In easy cases this is no different than what we did for action B in Figure 2.

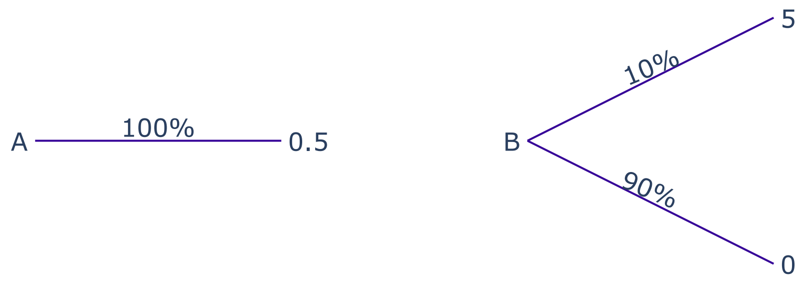

Figure 3: A low stakes gamble.

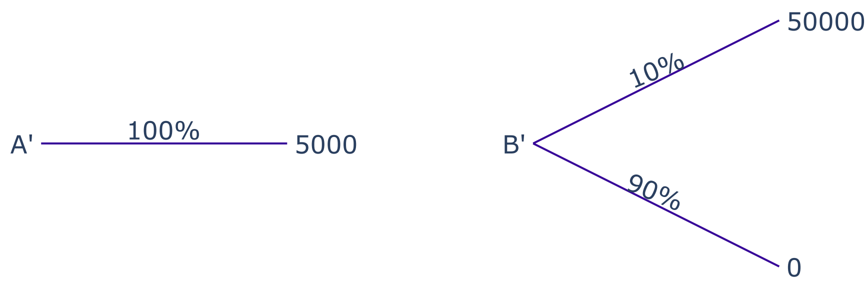

Figure 4: A high stakes gamble.

actions, many will exhibit a significant preference for taking the sure action A' over the risky action B'. For human DMs, this tendency becomes more pronounced, the higher the respective payoffs are, given that the basic structure of the choice (a sure payoff and a risky action providing the same expected value) remains intact.

This is where one of the first major theories of decision under risk comes into play. A version of Expected Utility (EU) was first proposed in 1738 by Bernoulli (Bernoulli 1954). I will focus on the later formulation by Neumann and Morgenstern 1944 which formalized many of the implicit assumptions and represents the basis for many of the later theories presented in this thesis. EU remains to this day the most important normative theory of decision under risk.

The basic structure of EU theory is quite similar to the calculation of expected values in equation 1:

$$\mathrm{EU}(L) = \sum_{i=1}^{I} p_i \cdot u(x_i) \tag{2}$$

Again, the outcome is a function of the lottery. What changed in contrast to equation 1 is that now we have a second function inside the calculation of each outcome's individual value. $u(x)$ commonly called the "utility function" and is where Expected utility theory gets its name. $u(x)$ converts any (monetary) outcome x_i to the utility gained from it. Remember the decision in Figure 4 involving high stakes from above and how preferences change when stakes rise. This can be easily modeled by choosing a suitable utility function.

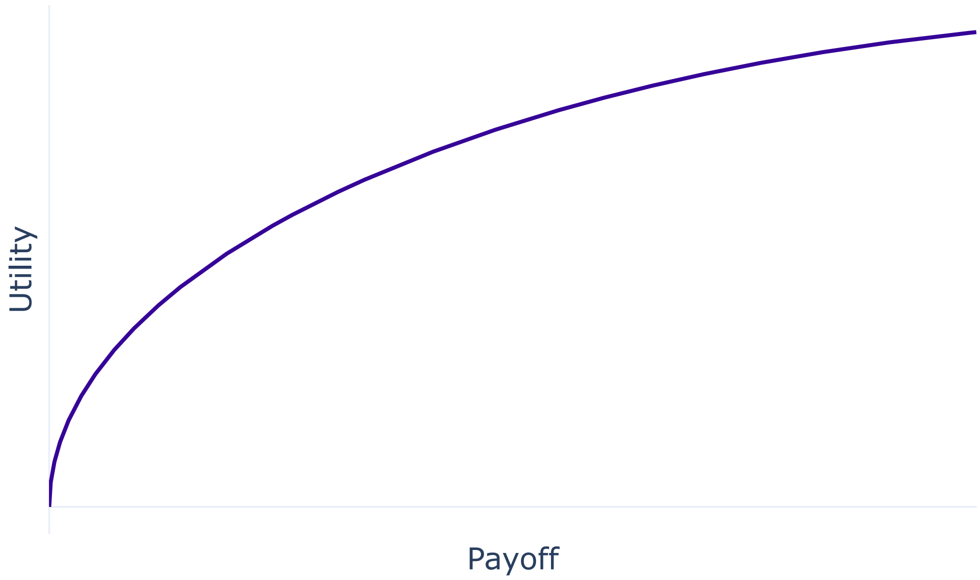

Figure 5: This figure shows the utility the DM feels dependent on his final wealth. The final wealth is depicted on the x-axis and the associated utility on the y-axis.

The function depicted in Figure 5 is a typical example of a utility function. It belongs to a family of function already proposed by Bernoulli for their property of being concave for positive values of x. This concavity ensures two things. Firstly, it encodes diminishing sensitivity to lottery payoffs with rising wealth levels and to increasing payoffs. This means that if somebody with concave utility wins €11.0 instead of winning €10.0 the marginal utility gained through that additional €1.0 is bigger than if he were to win €10,001.0 compared to winning €10,000.0, even though the difference between the two outcomes is the same. Secondly, concave utility functions model a behavior called risk aversion. Risk aversion is what we saw in Figure 4. A risk averse DM prefers the sure amount over a lottery yielding an expected value equal to the sure amount.

Figure 6 illustrates how a concave utility function models diminishing sensitivity. Two pairs (€10.0, €11.0 and €10,000.0, €10,001.0) of initial and final monetary wealths and their associated utilities are highlighted with green, dashed lines, showing the different evaluation of incremental payoffs as wealth increases. If your current wealth is €10.0, then winning another €1.0 and ending up with a final wealth of €11.0 leads to a big increase in utility. At the same time, a wealth increase from €10,000.0 to €10,001.0 leads to a much smaller increase in utility according to this function.

Figure 7 shows the second major property of classic utility functions – risk aversion. It illustrates the utilities resulting from the decision A = (1: 55) and B = (0.5: 10; 0.5: 100). While the payoff (55) of the sure action A is equal to the weighted average of $B = (0.5 \cdot 10 + 0.5 \cdot 100)$, the utility of getting 55 for sure is obviously higher than the weighted average of the utilities of lottery $B = (0.5 \cdot u(10) + 0.5 \cdot u(100))$. This means

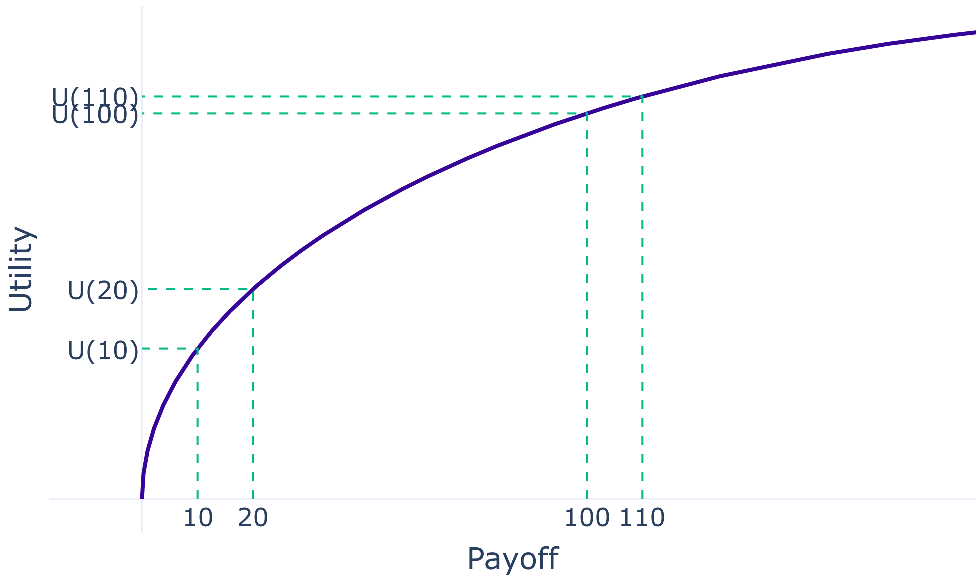

Figure 6: The function displayed in this figure is the same as in Figure 5 with wealth on the x-axis and the utility resulting from a certain wealth-level on the y-axis. The green, dashed lines mark two pairs of initial and final wealths with the same distance on the x-axis and their utilities. It becomes apparent that the marginal utility of the additional wealth at the higher wealth-level is much smaller than that of the smaller wealth-level.

that a DM conforming to EU with a concave utility function as the one above will always prefer sure amounts over mean-preserving lotteries with probabilistic outcomes.

2.3 Axioms of rational choice

In addition to their formulation of EU, von Neumann and Morgenstern proposed a system of axioms for rational choice which will become useful in order to understand the differences between EU and some of the other theories I will present. In the description of the axioms, $\preceq$ will signify a weak preference. This means that A $\preceq$ B can be read as "A is not preferred to B" and "B is at least as preferable as A". This allows us to define strict preferences with the sign $\prec$ as A$\prec$B if and only if A$\preceq$B and not B$\preceq$A – "B is strictly preferred to A if B is weakly preferred to A and at the same time A is not weakly preferred to B".

- *Completeness:* For any two lotteries A and B, their preference relation (A$\preceq$B or B$\preceq$A) must be defined. This means that for us to be able to choose rationally between two lotteries, we need to be able to rank them in terms of which one we prefer.

- *Transitivity:* For any three lotteries A, B and C, if B is (weakly) preferred to A and C is (weakly) preferred to B, then C must be (weakly) preferred to A (if A$\preceq$B and B$\preceq$C, then A$\preceq$C). The property of transitivity ensures that we can produce a finite ranking of all available lotteries without getting stuck in infinite loops during our

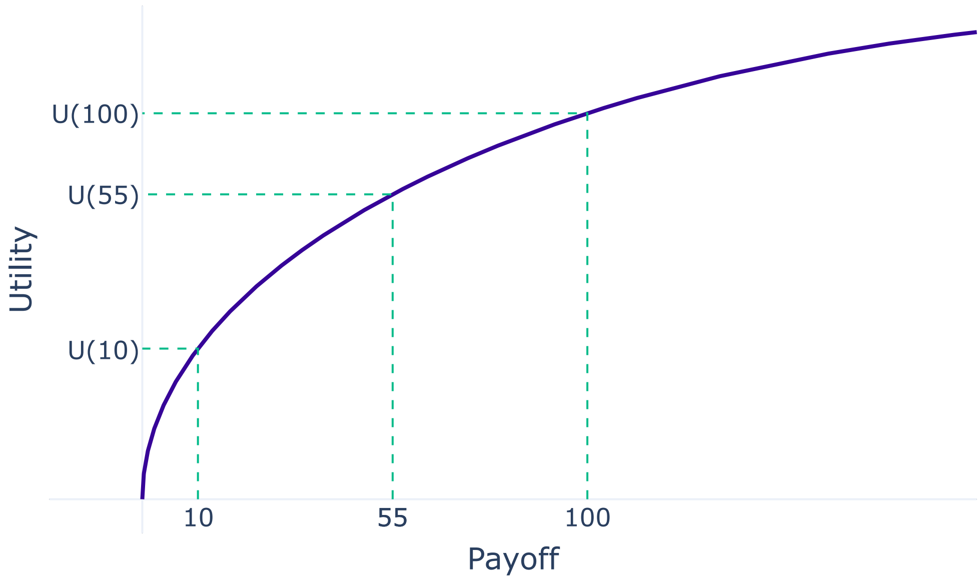

Figure 7: This figure shows a utility function associating final wealth on the x-axis with its respective utility on the y-axis. The green, dashed lines mark three equally spaced wealths and their utility. Visual inspection makes clear, that the utility of the middle value, which is also the average of the outer two values is significantly higher than the average of the utilities of the outer two wealth-levels, thus illustrating risk aversion.

decision. Imagine after a long day of work you prefer watching a movie to reading a book and you prefer reading a book to replying to all the emails you missed, then you should prefer watching the movie to answering all those emails.

- *Continuity:* Assume B is (weakly) preferred to A and C is (weakly) preferred to B. Then there must be a probability p between 0 and 1 such that a lottery $L_1 = (A \cdot p; c \cdot (1-p)$ is exactly as preferable as the sure payoff $L_2 = (B \cdot 1)$. Continuity means that no single outcome can be infinitely bad or good by stating that we must be able to construct an equally preferable lottery out of other outcomes when using an appropriate probability distribution. It also ensures that our evaluation is appropriately sensitive to the probabilities with which they are expected to occur. In practice, this means that if you prefer spending you holidays in Prague to spending them at home and if you prefer spending them in Paris to spending them in Prague, then there must be a lottery between staying at home and going to Paris with a probability such that you are indifferent between this lottery and going to Prague.

- *Independence:* Suppose A$\preceq$B. Then for any additional lottery C, and probability p between 0 and 1 the following holds $L_1(A \cdot p; C \cdot (1-p)) \preceq L_2 = (B \cdot p; C \cdot (1-p))$. This means that adding identical outcomes to two lotteries must not influence our ranking of them. Suppose you prefer eating Pizza to eating Noodles, then adding to both these options the chance of a penguin laying an egg in your local zoo should not change your preferences.

Some of these axioms have been called into question throughout the years, but there is a wide consensus, that they represent a good starting point to explore how to make rational decisions.

2.4 Cumulative Prospect Theory

Amos Tversky and Daniel Kahneman presented Cumulative Prospect Theory (CPU) a modification of their earlier Prospect theory in 1992 (Tversky and Kahneman 1992). CPU is similar to EU in that it transforms payoffs according to a function $u(x)$ take account of systematic deviations from rational behavior they observed in studies. In addition, CPU also transforms the probabilities associated with the payoffs, allowing it to explain more of these irrational tendencies than EU could before. Some of the most important among the observed irrational tendencies were:

- *Framing:* The way in which a problem is presented can influence the decision the DM settles on. Describing a problem in terms of losing out on higher payoffs instead of avoiding the risk of losing sure payoffs and similar techniques can significantly impact the chosen outcomes. Similarly, people seem to represent extreme outcomes (highest or lowest payoffs) differently than outcomes with payoffs of medium size.

- *Nonlinear preferences:* Subjects seem to overestimate very small and very big probabilities while underestimating probabilities of medium size. People value getting from a probability of 99% of winning a certain amount to 100% more highly than getting from 52% to 53% even though the difference in both cases is exactly 1%. They will also pay more to reduce the risk of losing from 1% to 0% than they would to reduce it from 61% to 60%.

- *Risk seeking:* We saw before that concave utility functions in EU model risk. At the same time, there are two contexts in which risk seeking behavior can be observed. Firstly, people are attracted to the chance of winning large amounts over their respective expected value. An example of this is engaging in state-run lotteries which on average yield negative payoffs but promise instantaneous riches to a few winners. The second area of risk seeking behavior is observed when people can avert small losses; most will accept more risk when trying to avoid small losses than they would to attain small gains.

- *Loss aversion:* Lastly, people attach more weight to losses than they do to gains of equivalent size.

CPU was designed to model as many of these irrationalities as possible. The basic composition of CPU is as follows:

$$\text{CPU}(L) = \sum_{i=1}^{I} \pi_i \cdot u(x_i) \tag{3}$$

Where π_i represents the transformation of the probabilities based on the value of $u(x_i)$, which Kahneman and Tversky call the "value function". The value function's purpose is very similar to the utility function from EU. Kahneman and Tversky propose the following parametrization:

$$\text{u}(x, r) = \begin{cases} (x - r)^{\alpha}, & \text{if } x \geq r \\ -\lambda) - (x - r))^{\alpha}, & \text{if } x < r \end{cases} \tag{4}$$

In addition to the payoff x, this function also takes a reference point r into account.[2] This means that in contrast to EU and the theories discussed later on, CPU does not evaluate final wealth but payoffs in relation to a reference point. This is a much bigger difference than it might seem in the examples in this essay, because they are already stated in terms of gains and losses as is not uncommon in economic papers.

Formally, I would have to make the distinction clearer and always state, that the DM is (incorrectly) assumed to have a wealth of 0 in my examples, but to make the text more readable, I will always mean payoff in relation to a wealth of 0. In the case study using CPU this will be achieved by setting r to 0, thus making the economic variable of interest comparable between the different theories.

Kahneman and Tversky's parametrization features two additional parameters, λ and α, which represent the degree of loss aversion (λ) and of diminishing sensitivity (α) respectively. The graph in Figure 8 displays the value function. It is easy to see that the part to the right of r looks similar to the utility function before. It is concave thus modeling the same risk aversion and diminishing sensitivity as in EU. The part to the left of r is steeper but otherwise equivalent to the part on the right, thus modeling the loss aversion that Kahneman and Tversky found in their observations.

As mentioned above, CPU transforms the payoff of the lotteries but also the probability to account for the way people seem to perceive extreme payoffs and very big and very small probabilities. Since Quiggin 1982 and Luce and Fishburn 1991, this is done in two steps and referred to as cumulative probability weighting in contrast to the marginal

[2] While the assignment of the function $\text{u}(x, r)$ is similar to before, it is defined in two pieces. The upper case for $x \geq r$ defines a function for all values bigger than or equal to r. The bottom case assigns values to all payoffs smaller than r.

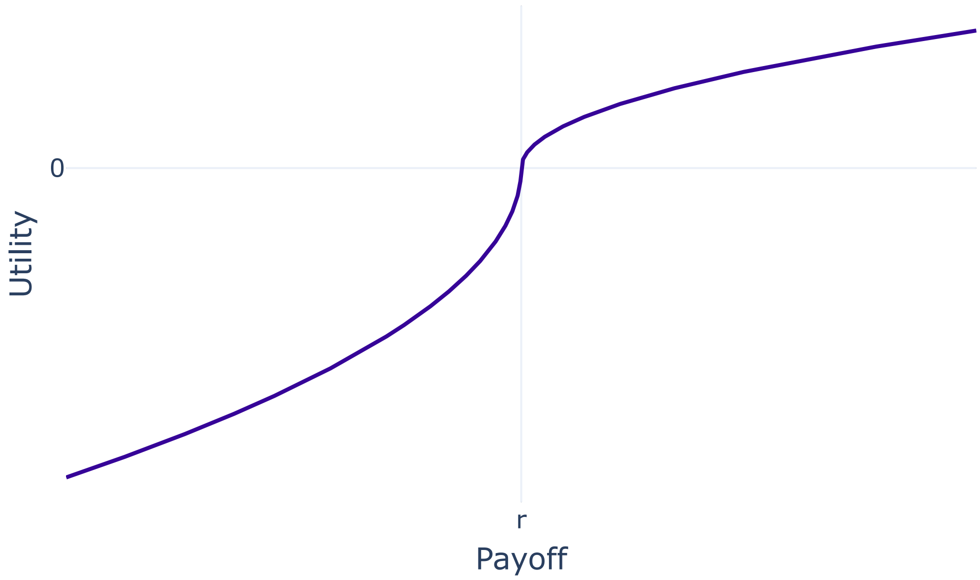

Figure 8: This figure displays the value function proposed by Kahneman and Tversky with standard parameters of $\lambda = 2.25$ and $\alpha = 0.88$. In contrast to earlier graphs, the x-axis shows payoffs in relation to the reference point r. As before, the y-axis displays the utility associated with x-values.

utility weighting practiced before. First, individual outcomes are ordered by the size of their payoffs and afterwards their probabilities are transformed in that order according to a probability weighting function.

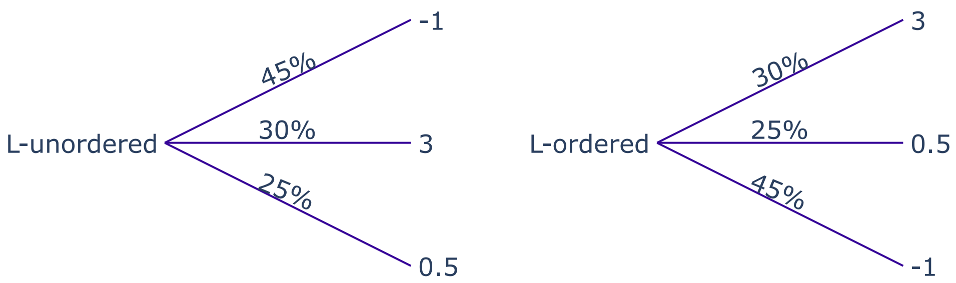

Figure 9: This figure illustrates the ordering step of applying CPU to a lottery. While the left-hand lottery is unordered, the right-hand lottery is ordered by the size of the payoff.

Figure 9 shows the ordering step; on the left you see the unordered outcomes and on the right the ordered outcomes. Now, the probabilities associated with positive and negative payoffs are respectively transformed according to the following formula:

$$\begin{aligned} \pi_i^+ &= w(p_1^+ + ... + p_i^+) - w(p_1^+ + ... + p_{i-1}^+), \qquad 1, ..., k \\ \pi_i^- &= w(p_1^- + ... + p_i^+) - w(p_1^- + ... + p_{i-1}^-), \qquad 1, ..., k^3 \end{aligned} \tag{5}$$

[3]This can be read as follows: “The probability weight for outcome i equals the weighted probability of the sum of all probabilities with outcomes of the same sign up to the outcome i minus the weighted probability of the sum of all probabilities with outcomes of the same sign up to but excluding the outcome

π_i the weighted (transformed) probability of the probability p_i and $w(p)$ is the weighting function. The fact that we input not only the probability of the outcome we are currently evaluating p_i but also the probabilities of the outcomes with smaller payoffs is due to Quiggin 1982.[4] Kahneman and Tversky proposed the following weighting function:

$$w(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{\frac{1}{\delta}})} \tag{6}$$

Here, p is simply the probability to be evaluated and δ can roughly be interpreted as the degree to which very small and very big probabilities (for example 1% and 99%) are overestimated compared to probabilities of medium size.

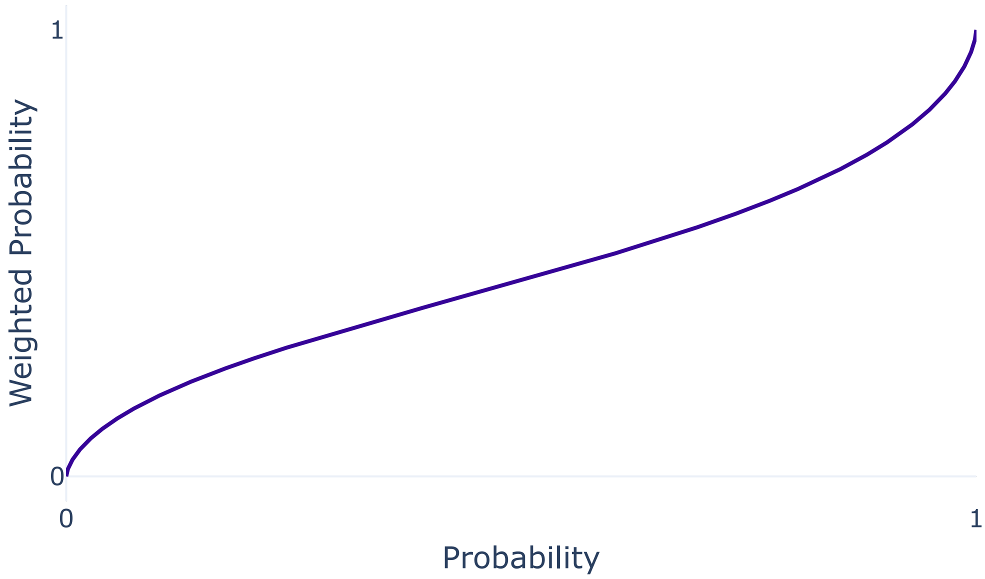

Figure 10: This figure shows the probability weighting function by Kahneman and Tversky for a standard estimate of $\delta = 0.65$. The x-axis displays the unmodified, objective probabilities and the y-axis the transformed, weighted probabilities.

When I introduced the Axioms of rational choice proposed by von Neuman and Morgenstern, I pointed out that the later theories would not be in line with all of these axioms. CPU is not in line with the Independence axiom, which stated that if we add the same additional outcome to two lotteries, our preferences over them should not change. While Kahneman and Tversky's value functions is functionally equivalent to EU's utility function, probability weighting can mean that the added outcomes may be evaluated differently depending on the context of the two lotteries we are looking at. It may for example be the case that the added outcome represents an extreme outcome in the first lottery and is thus overweighted and represents a medium outcome in the other lottery

i".

[4]You can find an example of how to calculate the weighted probabilities for the ordered lottery in Figure 9 in the appendix

and therefore has a much lower impact on its perceptions. This is especially likely when we compare lotteries with sure payoffs, because if we add another outcome to sure payoffs, it is by definition an extreme outcome. This is why many decision theorists including Kahneman and Tversky see CPU not as a normative but rather as a descriptive theory of choice.

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2.5 Savoring and Disappointment Theory

Gollier and Muermann 2010 introduce a new theory of decision making under risk based on the intuition that DMs derive savoring and disappointment from realizing higher/lower than anticipated payoffs. They call it Optimal Choice and Beliefs with Ex Ante Savoring and Ex Post Disappointment (Optimal Anticipation with Savoring and Disappointment or OSAD). Unlike EU and CPT, where a lottery always yields the same utility, without depending on the DM's environment, OSAD takes some context information into account. Building on Bell 1985's Disappointment theory, Gollier and Muermann suggest that expectations about payoffs matter. After all, winning €10.0 when you expected to win €5.0 is much more satisfying than when you expected to win €15.0. In addition, Gollier's and Muermann's model allows the DM to influence his own expectations to maximize his utility.

Unlike the other theories presented in this essay, OSAD is originally focused on optimizing the DM's expectations in order for him to obtain maximum utility from a given lottery. This means that inter-lottery comparisons are not the main goal. Still, the implementation in the Tool could be used to test different sets of expectations for a given lottery or to compare two lotteries given optimal expectations. Having said this, the Tool cannot optimize your inputs, instead this must be done manually.[5]

[5]An example of how this might be achieved via a kind of sensitivity analysis is provided in the second case-study

Unlike the earlier theories, OSAD compares a lottery of objective probabilities with subjective beliefs. More concretely, the DM has to enter an objective lottery (for example: $L = (0.4 : 3; 0.4 : 6; 0.2 : 10)$ with probabilities $p_i, ..., p_n$ and payoffs $x_i, ..., x_n$ and then add a set of subjective probabilities $(q_i, ...q_n)$ to it (for example subjective probs= $(0.3, 0.3, 0.4)$). Figure 11 shows exactly this comparison. The left part shows the lottery with the objective probabilities. The right side shows the same lottery and adds the subjective probabilities behind the objective ones as a reference.[6]

[6]Both this way of displaying the lottery and the subjective probabilities and the input in the Tool encourage continuity of the subjective probabilities with the objective probabilities in line with Jouini, Karehnke, and Napp 2013. Still, they do not enforce it in a strict way.

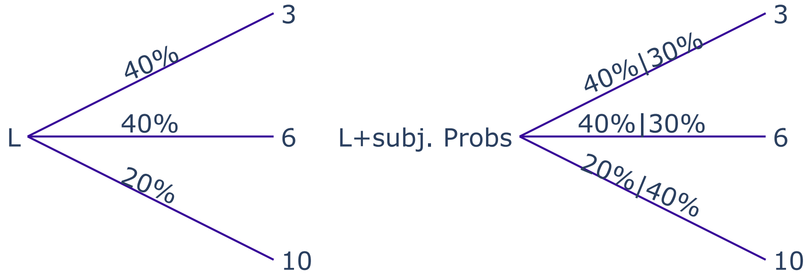

Figure 11: This figure introduces the depiction of lotteries with additional context information. Lottery L on the left side shows only the basic outcomes with their payoffs and respective probabilities. The right-hand lotterydisplays the same information as on the left but adds the context information required by OSAD. In thiscase it is the subjective probabilities added behind the objective probabilities.

The subjective probabilities serve to calculate a reference point endogenously which can then be used to calculate the effects of anticipation on the overall utility. These effects are two-fold. On one hand DMs derive utility from anticipating high payoffs, but on the other hand they are disappointed, when the anticipated payoffs are higher than realized payoffs. Now that the intuitions are clear, this is OSAD's basic formula:

$$OSAD(L) = k \cdot u(y) + \sum_{i=1}^{I} p_i \cdot BU(x_i, y) \tag{7}$$

This formula can be split into two terms. $k \cdot u(y)$ measures the utility derived from anticipatory feelings, where k is a multiplier to calibrate the relative impact of positive anticipation on overall utility. $u()$ is a simple utility function similar to those we have seen before and y is the certainty equivalent of the subjective lottery $\sum_{i=1}^{I} q_i \cdot u(x_i)$. To a certain degree, k can be interpreted as the time-difference between forming the subjective reference point and resolving the actual risk with the objective probabilities. The longer this time is, the higher one can assume is the share of anticipatory utility in relation to the outcome of the objective lottery. Meanwhile, $\sum_{i=1}^{I} p_i \cdot BU(x_i, y)$ calculates the utility of the objective lottery and the impact of disappointment by using the bivariate utility function BU. BU takes as its first argument a (realized) payoff x_i and as its second an anticipated payoff y, which is equal to the certainty equivalent of the subjective lottery. Like a simple utility function, BU is increasing and concave in its first argument which means that DMs derive higher utility from higher realized payoff subject to diminished sensitivity and loss aversion. Secondly, BU is negative in its second argument, meaning that higher anticipated payoffs will lead to lower utility for any realized payoff. Gollier and Muermann state the additional requirement, that BU should be positive for simultaneous increases in realized and anticipated payoffs. This means that DMs prefer high realized payoffs with high anticipated payoffs to low realized payoffs with low anticipated payoffs.

One example of such a BU is the additive habits formulation of utility:

$$BU(x, y) = u(x - \eta \cdot y) \tag{8}$$

Where $u()$ is a simple utility function as before and η is a measure of the impact of disappointment on utility.

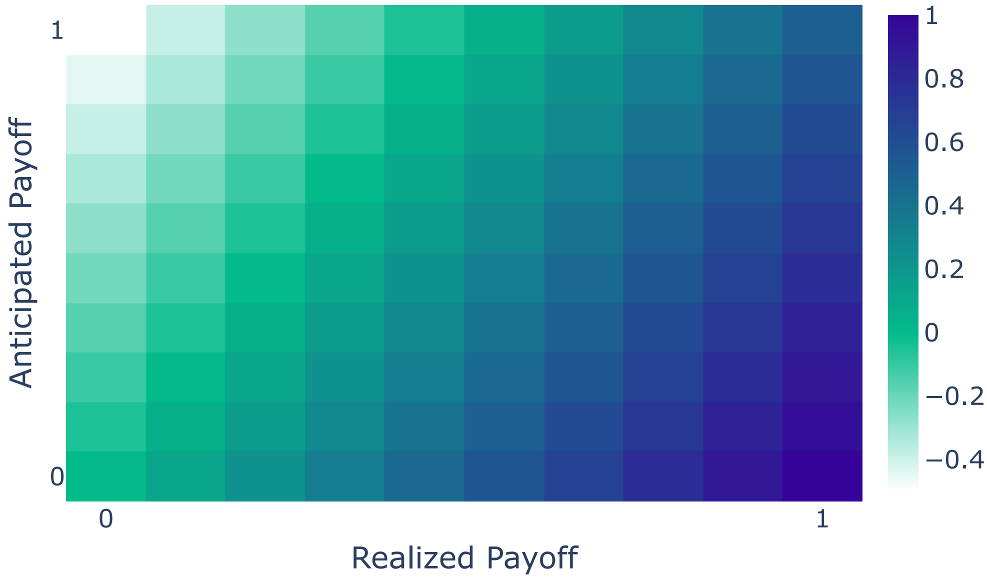

Figure 12: This figure illustrates the utility values gained by a combination of realized payoffs on the x-axis and anticipated payoffs on the y-axis. It assumes the additive habit, bivariate utility function from equation 9 and assumes a linear utility function for simplicity. Lighter values indicate lower utility gained from the combination and darker values higher utility.

Figure 12 shows a depiction of the values, the additive habits bivariate utility function assumes for different combinations of realized payoffs on the x-axis and anticipated payoffs on the y-axis. As can be seen in the equation of the formula, it uses a simple univariate utility function inside. Figure 12 assumes linear utility for simplicity. The figure depicts these values for combinations of realized and anticipated payoffs between 0 and 1 on the x-axis and y-axis, respectively. Of course, most decisions will be between actions with higher realized and anticipated payoffs than 1, but the displayed section gives a fair picture of the underlying characteristics of the bivariate utility function.

2.6 Reference Dependent Risk Attitudes

Like OSAD, Reference dependent Risk Attitudes (RDRA) as proposed by Kőszegi and Rabin 2006, 2007 aims to introduce an endogenous reference point. While This figure illustrates the utility values gained by a combination of realized payoffs on the x-axis and anticipated payoffs on the y-axis. It assumes the additive habit, bivariate utility function from equation 9 and assumes a linear utility function for simplicity. Lighter values indicate lower utility gained from the combination and darker values higher utility. CPU depends on a reference point and has been tested with various specification like status quo beliefs, lagged status quo beliefs or the mean of the target lottery, these different specifications produce mutually exclusive predictions about the behavior of DMs. Building a model that can explain these divergences and when which predictions are appropriate in a systematic way was a major motivation for RDRA.

Köszegi and Rabin take recent (lagged) expectations as the reference point compared to which different actions are evaluated, because preferences do not immediately adjust to new expectations, but rather take some time during which DMs still use old beliefs to construct their preferences. In contrast to earlier attempts at using lagged beliefs as reference-points for example in the disappointment theory of Bell 1985, they allow for probabilistic beliefs and do not condense these to simple certainty equivalents or similar measures in order to retain the initial structure and full information of the recent beliefs as a reference.

An important mechanism of RDRA is encoded in its bivariate utility function, which depends on the actual payoff x and the reference-payoff y:

$$BU(x, y) = u(x) + m(u(x) - u(y)) \tag{9}$$

Take as an example the case that we have the target lottery $T = (0.3 : 100; 0.3 : 50; 0.4)$ and the context lottery $C = (0.5 : 25; 0.5 : 75)$ as the reference point. To assign a value to the target reference, we first need to go through all realizable payoffs and calculate their utility conditional on the reference lottery and then take their weighted average:[7]

$$RDRA(T, C) = \sum_{i=1}^{I} \sum_{j=1}^{J} p_{T_i} \cdot p_{C_j} \cdot BU(x_{T_i}, x_{C_j}) \tag{10}$$

In addition to this way of calculating the utility of a target lottery in relation to a reference lottery, Köszegi and Rabin propose two (meta) concepts to compare several target lotteries with each other. Both of these concepts are based on the fact that in real life, DMs often face a time-delay between deciding and receiving the realized payoff. This reaches from decisions about whether to buy insurance to what plans to make for the weekend. Depending on how long this time delay is, Köszegi and Rabin propose that the reference point based on recent expectations may change or remain the same as at the time of decision. When the reference point remains stable, they propose the concept of Unacclimated personal equilibrium (UPE). UPE requires that DMs only plan actions which they know they will go through with. Then, any plan a DM knows he will go

[7] You can find an example-calculation for the example in Figure 16 in the Appendix.

through with in line with their preferences based on recent beliefs at the time of decision is a UPE. This means that a plan-decision combination for which $RDRA(T_{\text{plan}}, C) \succeq RDRA(T_{\text{all other plans}}, C)$ where $T_{\text{plan}} = C$ is a UPE. The DM's preferred UPE is the one yielding the highest overall utility of all UPEs and is called his Preferred personal equilibrium (PPE).

When there is a bigger delay between making the plan and deciding, the reference point can change in the meantime and become equal to the plan made. Köszegi and Rabin propose that the plan which, when the reference-point becomes equal to that plan, yields the highest utility according to RDRA, be called the Choice-acclimating personal equilibrium (CPE). CPE should be considered the optimal plan-decision combination for these cases and it can be stated as $RDRA(T_{\text{plan}}, C_{\text{plan}}) \succeq RDRA(T_{\text{all other plans}}, C_{\text{all other plans}})$.[8]

2.7 Regret Theory

In many ways, CPU has been the most influential theory of decision making under risk when it comes to describing the irrational tendencies most human DMs show. Its conscious breaking of the independence axiom led many later theories to follow a similar route. Regret theory (RT) is not one of those theories. Rather than breaking the independence axiom, it breaks with the transitivity axiom as you will see later on. While both Bell 1982 and Loomes and Sugden 1982 proposed early versions of this theory, the version implemented in the Tool and focused on in this essay, is based on Loomes and Sugden 1982.

The basic motivation for RT is that DMs rarely make decisions in isolation. That means they compare the possible outcomes of one lottery with the outcomes of comparable lotteries instead of evaluating every lottery solely based on its own merits. In contrast to the more recent OSAD, RT assumes that the context cannot be influenced by the DM but is instead given exogenously. Consider a target lottery $T = (0.4 : -1;\ 0.6 : 3)$, which the DM can choose to enter. RT proposes that actually, DMs are unlikely to look at the simple lottery in isolation. Rather, the DMs will compare the simple lottery to context information producing the following lottery:

The right side of Figure 13 assumes correlated state-spaces between the target lottery and context-information. This means that we compare two lotteries of equal dimensions (same number of outcomes with distinctive payoffs and probabilities) such that every outcome of the target lottery has exactly one corresponding outcome in the context lottery (i.e. if the first outcome of the target lottery occurs we know that the first outcome of the context lottery would have occurred had we chosen it instead of the target lottery).

[8] A short example of how these concepts feature in actual calculations can be found in

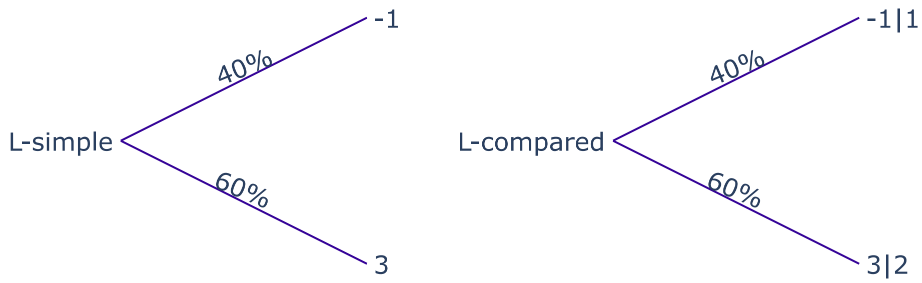

Figure 13: A depiction of the target on the left and the target with the context information on the right. Similarly to Figure 11, the right-hand side shows the target lottery and the context payoffs.

Loomes and Sugden 1982 explain, why this simplifying assumption is compatible with the way Kahneman and Tversky and other presented information to participants during experiments. Then, the DM can predict his regret, when he chooses to enter the target lottery and gets a payoff of -1 because he could have gotten a higher payoff of 1 had he chosen the context lottery. Similarly, he predicts rejoicing in case his payoff is 3 because the associated payoff of the context lottery is only 2. RT works through two mechanisms. Like the theories discussed earlier, it features a utility/value function that is used to transform the payoffs of the lotteries directly. As stated above, its second feature is that it compares the (target) lottery with a context lottery leading to the following basic equation:

$$RT(L) = \sum_{i=1}^{I} p_i \cdot Q(x_{T_i}, x_{C_i}) \tag{11}$$

Where $Q(x_{T_i}, x_{C_i}$ is a bivariate function of the correlated payoffs of the target lottery and the context-information. This "regret function" is where the prediction of regret or pleasure described before happens. A simplified version of the parametrization by Loomes and Sugden 1982 looks as follows:

$$Q(x_T, x_C) = u(x_T) + w \cdot (u(x_T) - u(x_C)) \tag{12}$$

Here, $u()$ performs a similar function to the value/utility function in EU and CPU and is called a choiceless-utility function. Q() as stated above is made up of the utility derived from the realized payoff and by the utility derived from comparing the realized payoff and the context-payoff. The second term models the context-payoff and is weighted with the variable w to be able to calibrate the importance of the regret/pleasure portion of the final payoff.

Figure 14 plots the values of the regret function (Q()) for target payoffs and context payoffs between 0 and 1, respectively. Intuitively it becomes clear that the final, "choice-adjusted" value becomes higher, the higher the target-payoff is and the lower the context lottery is. The parameter w (weight of the regret term) is set to 0.5 and choiceless-utility

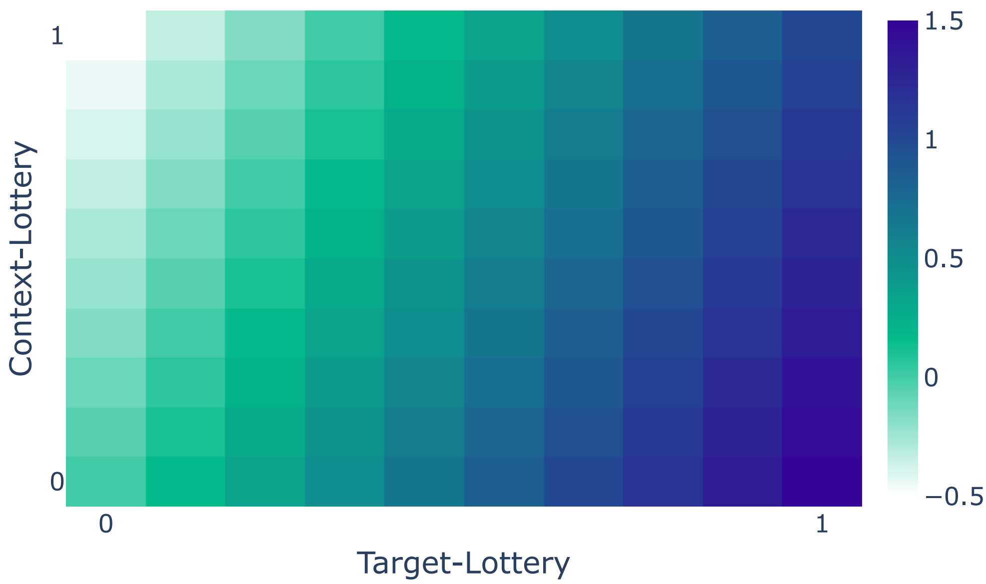

Figure 14: This figure displays the context-dependent utility indicated by the regret function in Equation 12 for target-payoff and context-payoff pairs ranging from 0 to 1 with higher utility values indicated by darker tiles and lower utility by lighter ones.

is kept linear to make the graph easier to interpret. the overall pattern of the diagram does not depend on the chosen ranges of target and context payoffs.

While RT's initial ambition was largely focused on evaluating pairwise lotteries (one target-lottery and one context-lottery) Herweg and Mueller 2019, Loomes and Sugden 1982 already point out that RT can break the transitivity axiom (A$\preceq$B and B$\preceq$C, then A$\preceq$C) if applied to situations where more than two actions are available to the DM. Lotteries are evaluated in relation to a context lottery and so, it may be that such intransitive preferences are formed.

2.8 Salience Theory

Salience Theory (ST) as proposed by Bordalo, Gennaioli, and Shleifer 2012 is similar to RT in its motivation and mechanisms. It also accepts intransitive preferences rather than breaking the independence axiom. In fact, Herweg and Mueller 2019 show that ST is contained by generalized RT. Like RT, ST evaluates its target-lottery in comparison to a context-lottery, but instead of modelling regret and pleasure terms added to the basic utility, ST calculates decision weights which are supposed to capture the salience of all outcomes. The basic idea is that payoffs in the target-lottery are more salient the more different they are from their corresponding payoffs in the context-lottery.[9] The basic setup

[9]Bordalo, Gennaioli, and Shleifer 2012 present two versions of ST, rank based salience theory and smooth salience theory, arguing that rank-based salience theory allows for easy analysis theoretical problems, smooth salience theory is more suited for actual computation. Because the focus of the Tool and this essay

of ST is as follows:

$$ST(L) = \sum_{i=1}^{I} p_i \cdot \omega_i \cdot u(x_i) \tag{13}$$

Where p_i is the basic probability and ω_i its associated decision weight. As in earlier theories, $u()$ is a transformation of the payoff similar to a utility/value function.

For smooth salience theory, ω is defined as:

$$\omega = \frac{\delta^{-\sigma(x_T, x_C)}}{\sum_{i=1}^{I} p_i \cdot \delta^{\sigma(x_{Ti}, x_{Ci})}} \tag{14}$$

Where δ between 0 and 1 can be interpreted as the inverse coefficient of distortion induced by salience (i.e., how big is the impact of salience on the perceived probability). When $\delta = 1$, the DM is acts as if he were a purely rational agent. The smaller δ becomes, the more the DM will overweigh salient states in comparison to less salient ones. If The numerator in this definition represents the salience of the current outcome and the denominator the average salience of all outcomes in a lottery so that δ is regularized to account for the average salience. Lastly, $\sigma(x_T, x_C)$ is called the salience function. Bordalo, Gennaioli, and Shleifer 2012 suggest that this function should have three properties to constrain ST to a tractable set of predictions:

- *Ordering:* If the difference between the target-payoff and context-payoff of one outcome is bigger than that of another outcome, the salience assigned to the first should always be bigger than that of the second.
- *Diminishing sensitivity:* If two outcomes feature the same difference between the target- and context-payoff, the outcome with the lower target-payoff should be assigned the larger salience.
- *Reflection:* If an outcome with positive target- and context-payoff is assigned higher salience than another payoff with positive target- and context-payoff, the same must be true if the signs of all considered payoffs are switched (i.e., the outcomes are reflected to negative signs).

As an example of a salience function with all the above characteristics, Bordalo, Gennaioli, and Shleifer 2012 propose the following function:

$$\sigma(x_T, x_C) = \frac{|x_T - x_C|}{|x_T| + |x_C| + \theta} \tag{15}$$

is on calculations and a short introduction to the theories rather than the analysis of complex problems, only smooth salience theory will be described.

Where Ordering is ensured in the numerator and Diminishing Sensitivity in the denominator, while taking the absolute of all values ensures strong (exact) reflection. θ serves as another degree of freedom to trade of the relative strength of ordering and diminishing sensitivity.

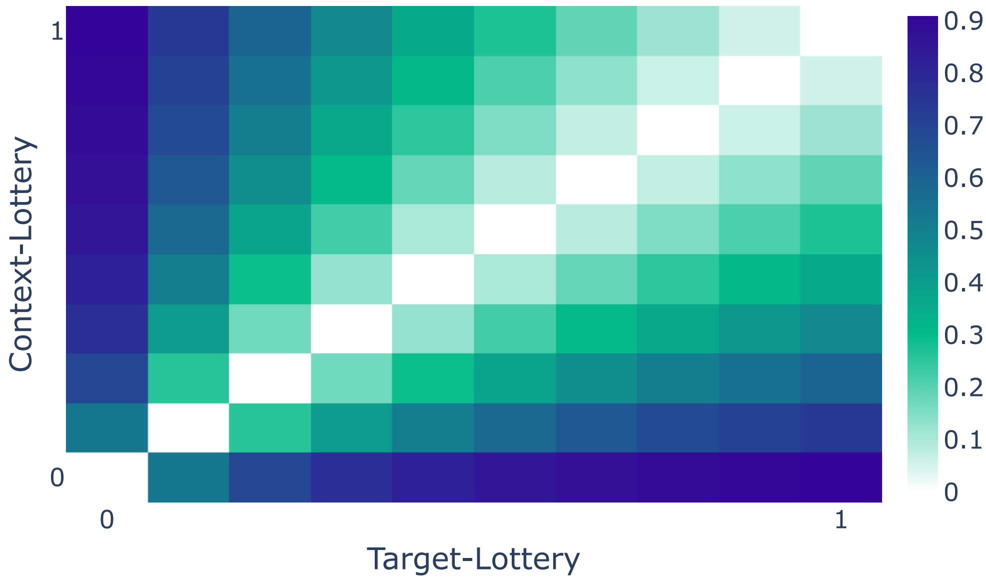

Figure 15: This figure displays the salience, assigned by the salience function in Equation 15 to pairs of target and context payoffs ranging between O and 1. Higher salience is indicated by darker tiles and lower salience by lighter ones. θ is set to 0.1

Figure 15 shows the values, the salience function proposed by Bordalo, Gennaioli, and Shleifer 2012 assumes for different combinations of target-payoffs and context-payoffs showing all the characteristics discussed above. It becomes apparent that whenever the target and context payoffs are equal, the salience of that state is minimal, while higher differences lead to higher salience. Again the chosen range of payoffs from 0 to 1 is representativ of the broader picture.

3 Tool

The Tool is the core of this project and contains an implementation of all the theories described before in the programming language Python.[10]

Instead of publishing the resulting code as a library, I chose to wrap the functionalities in a Graphical, web-based interface created with the (Python) Dash library and hosted on external servers. This is in line with the general aim of this thesis to give a short introduction to these theories for both specialist and non-specialist readers.

[10]While Python is generally often used in an object-oriented fashion, this implementation is in the widest sense functional.

The complete code of the Tool and all theories is open-sourced and freely available. It can be cloned from there and reused for other projects.

3.1 Implementation of the theories

The Tool is as flexible as possible when it comes to the choice of different utility functions or other important sub-functions of the theories while retaining the characteristic properties of each theory. This enables the analysis of different classes of problems (think lotteries with payoffs of very different sizes) and the quick testing of the result of different parametrizations of the functions on the results. Let me illustrate this point with one example:

Remember the simplest theory from the beginning of this essay, Expected utility theory and its formal representation:

$$\mathrm{EU}(L) = \sum_{i=1}^{I} p_i \cdot u(x_i) \tag{2 revisited}$$

It should be obvious from our earlier discussion, that the lottery L is the key input or argument which we map to a singular value, but the utility function u() is another implicit argument, which we have to determine before we can get any result. Remember that u() was supposed to be concave over all positive values to encode diminishing sensitivity and risk aversion in EU. In Python, this is achieved by passing u() as a generic argument to the function calculating the result of applying EU to a lottery:

```
1   def expected_utility(
2       pays: List[float],
3       probs: List[float],
4       um_function=um.bern_utility,
5       um_kwargs={},
6       ce_function=um.bern_ce,
7   ) -> List[float]:
8       """Implementation of Expected Utility Theory and its
           Certainty
9
10      Args:
11          pays (List[float]): Vector of the payoffs for all
               outcomes
12          probs (List[float]): Vector of the probabilities for
                all outcomes (must be the same length as pays)
13          um_function (function, optional): The utility
               function used to transform payoffs to utilities.
               Defaults to um.bern_utility.
14          um_kwargs (dict, optional): Generic keyword-
               arguments supplied to the utility function .
               Defaults to {}.
15          ce_function (function, optional): The 'inverse' of
               the utility function used to calculate the
               certainty equivalent. Defaults to um.bern_ce.
16
17      Returns:
18          List[float]: The utility assigned to the lottery by
               EU and the associated certainty equivalent
19      """
20      pays_ch, probs_ch = he.list_cleaning(pays, probs)
21      pays_ch_ut = [um_function(i, **um_kwargs) for i in
           pays_ch]
22      ind_vals = [pays_ch_ut[i] * probs_ch[i] for i in range(
           len(pays_ch))]
23      utility = sum(ind_vals)
24      try:
25          ce = ce_function(utility, **um_kwargs)
26      except:
27          ce = nan
28      return utility, ce
```

Figure 16: The back-end implementation of EU

The code in Figure 16 shows the basic structure of the calculation of EU values. Without going into too much detail, the first 7 lines serve as the setup of the function where the arguments needed for calculation are defined. Line 4 requires a utility function as an argument, which can then be used in the body of this EU calculation. Lines 8 through 19 document the necessary arguments and outputs of the function. This might seem exaggerated in this example but is very useful for the more complex, other theories.

Line 20 does some type and input checking using an external function and line 21 takes every payoff of the input lottery and calculates the value the utility function assigns to it. Lines 22/23 calculate the weighted average of these payoffs according to the provided probabilities. Afterwards, Lines 24 through 27 try to calculate the certainty equivalent. In case a custom input function was supplied without a suitable "inverse" function to calculate the certainty equivalent this check keeps the program from crashing and helps provide helpful notifications in the Tool.

```
1  def lin_utility(x: float) -> float:
2      """ A linear utility function; the utility of a value x equals x """
3      return x
4
5
6  def lin_ce(x: float) -> float:
7      """ Inverse of lin utility """
8      return x
9
10
11 def root_utility(x: float, exp: float = 2.0, mult: float = 3) -> float:
12     """
13     A simple root utility function with u(x) = x**1/exp;
14     by default the quadratic root is used and loss aversion means
15     that losses are evaluated as 3 times as high as wins.
16     """
17     return x ** (1 / exp) if x > 0 else -mult * (-x) ** (1 / exp)
18
19
20 def root_ce(x: float, exp: float = 2.0, mult: float = 3) -> float:
21     """ inverse of root utility """
22     return x ** exp if x > 0 else -((x / (-mult)) ** exp)
```

Figure 17: Two utility functions and their respective certainty equivalent functions

Figure 17 shows two of the utility functions already implemented in the Tool and their respective "certainty equivalent functions". These "certainty equivalent functions are simply the utility functions solved for their primary input, x. The first utility function is a linear utility function simply returning the payoff supplied to it in line 3. Its counterpart from line 6 to 8 is just as simple. The root utility function starting in line 11 is more complicated assigning the (by default square) root of the payoff when it is positive and the scaled (square) root of the negative of the payoff when it is negative in line 17. This is both to model loss aversion and to extend the domain of this function to negative numbers. This is achieved using a simple, inline if statement. Its certainty equivalent is calculated similarly.

Because the utility functions are supplied as generic arguments to EU, they are easily changeable in the Tool. While EU takes only the utility functions as an argument, we saw in the first section, that Cumulative Prospect Theory (CPT) also takes the probability weighting function as an argument. Similarly, Regret theory (RT) and Salience theory (ST) contain Regret and Salience functions. All of these can be freely changed and alternatives can be added to the code when needed.

3.2 Layout of the Tool

The Tool is based on the Dash Framework which made it possible to build the web-based graphical interface. This framework manages the server and provides the different components to the user's browser. In addition, it is also closely integrated with the graphing library (Plotly) used to produce the figures in this thesis and the Tool itself.

The actual Tool is divided into three main segments. There is the main input segment on top, the middle segment where different utility functions and potentially other functions (probability weighting function for CPT, . . .) can be chosen and their parameters changed and lastly, the output section, where you will find a short summary of the inputs and the utility, certainty equivalent and risk premium assigned to the input-lottery.

The input segment shows a table on the left, where you can input the lottery in. This table can be extended or shortened by deleting a row or clicking the "Add Row" button, respectively. Above that table, there is a dropdown menu in which the theory used can be chosen. The right side of this segment shows a more classical representation of the lottery as well as a graphic depicting the probability density function and the cumulative density function of the lottery. The first displays the probability of each payoff as a bar-chart and the second shows how likely a payoff of the size x or smaller is. Beneath those figures, there is a table with some standard statistical characteristics of the lottery.

If you choose a theory that requires some context information like Regret theory, a new column will be added to the existing input table in which you can enter the context in relation to which the target lottery should be evaluated. Note that if you enter the context information in that table, the Tool automatically assumes correlated state spaces meaning that both outcomes do not only have the same probability but are intrinsically linked to the same state of the world.Alternatively, you may choose a sure amount in comparison to which your target lottery should be evaluated by clicking on the "Use single input" button and entering that certain value in the additional input table appearing beneath the initial input table. In addition to the changes of the input table, the figures on the right will display the additional data where appropriate to make the comparison of the target and context-lottery easier.

The second major section includes the utility function and additional changeable functions for each theory. Like in the input segment, you can choose among some predefined functions in the dropdown menu on the top left of that section. The predefined functions are some of the most prominent parametrizations in the case of the utility function and otherwise the ones proposed by the initial authors of the theories.

Choosing among them will display their formula beneath the dropdown-menu and input fields for their respective parameters. On the right of these segments, there is a plot of the functions chosen. These plots should be familiar to you as they are based on the same data discussed in section one. Beneath that plot, the values to be displayed can be chosen. Note that some utility function may not be able to process negative values or 0 in which case a notification will appear on the top right of the window. To reset all values in a middle segment, click the "Reset all values" button.

Something that is new about these segments is that you will see the possibility to define a custom (utility, probability weighting, regret function, etc.. . .) below all the pre-defined functions. Choosing this option will open a text input below the dropdown-menu in which you can enter the body of your own function definition. On a technical level, this input accepts a subset of the standard Python-Syntax and functions and evaluates them. The input is parsed and "sanitized" with Simpleeval Python library and the following rules apply:

- Only single-line inputs are accepted
- Spaces are ignored
- In the case of univariate functions (utility function and probability weighting function), the independent variable is always called "x". In the case of bivariate functions (Regret and Salience functions, etc.) the two independent variables are called "x_1" and "x_2". The input formula must follow this convention.
- No other variables are allowed
- Floating point values have to be entered using a point and not a comma as a separator (i.e., 34 can be entered as 0.75 but not 0,75)
- Only the right-handside of the equations is entered. $u(x) = 3{*}x$ becomes "$3{*}x$"
- Every operation must be explicitly declared (i.e., $3x$ must be written as $3{*}x$)
- Piecewise definitions of arbitrary complexity can be defined by using a shortened if statement (do something if condition else do something else). The Utility function proposed by Tversky and Kahneman in Equation 4 with parameters of $\lambda = 2.25$, $\alpha = 0.88$ and a reference point of $r = 0$ can be entered as "$(x - 0){*}{*}0.88$ if $x >= 0$ else $-2.25{*}(-(x - 0)){*}{*}0.88$". Nesting if statements is possible.
- Simpleeval (the parser) imposes some (generous) restrictions on the computational complexity of evaluated expression to keep the server from crashing. This includes restrictions on the size of power-operations (exponents may not exceed a certain size) and similar measures which are not likely to impact normal usage of the Tool.

Allowed operators		Allowed mathematical functions	
+	"plus"	abs()	"the absolute vale of"
-	"minus"	exp()	"e to the power of"
/	"divided by"	log()	"the natural logarithm of"
*	"multiplied by"	log10()	"the base 10 logarithm of"
**	"to the power of"	u()	"the univariate utility function"[11]
<	"smaller than"	sqrt()	"the square root of"
>	"greater than"	pi	"π – Pi"
<=	"smaller equal"	e	"e – Euler's Number"
>=	"greater equal"	sin()	"the sine of"
==	"equal to"	cos()	"the cosine of"

From a technical standpoint, evaluating user input code is risky, which is why Simpleeval is used to prevent most malevolent inputs. Unfortunately, this means that user input utility functions cannot be evaluated in a way that allows the construction of an appropriate certainty equivalent function. In this case only the utility resulting from the lottery is shown in the output section.

The output section is conceptually the simplest of the three major sections. It displays a short summary of the lottery from the input section, the theory chosen and the parametrized functions. Below that, it displays the final utility resulting from participating in a lottery given the chosen theory of decision under risk and in most cases its certainty equivalent and the risk premium. The risk premium is the difference between the certainty equivalent and the expected value of a lottery and indicates how much a DM would be willing to pay to insure against the lottery or participate in it.

4 Case-studies

This last section will go through three examples of how the Tool could be used to evaluate real-life decision problems. The first example will be a choice between a sure payoff and a simple, binary lottery evaluated with EU and CPT. The second is an analysis of a planned project and optimal expectations with OSAD and RDRA, while the third will look at a big lottery in comparison to a benchmark using RT and ST.

While Reading this section, please keep in mind, that most of the theories presented in this essay do not make any normative claims. While EU is the most important normative

[11] This is only provided when appropriate, such as in the case of the bivariate utility function of OSAD and the regret function of RT.

theory of decision making under risk, the other theories were originally designed to be descriptive of how human DMs decide instead of how a perfectly rational DM should decide. Humans often do not act perfectly rationally and so all of these descriptive theories either break the independence or the transitivity axiom. The one exception to this is RT, (Loomes and Sugden 1982, p. 820) argue that the restrictions imposed by the axioms of rational choice according to (Neumann and Morgenstern 1944) are too restrictive and claim RT as a normative theory as well. Even without this claim, using the theories in the Tool to analyze decisions can be justified, when we just assume that the theories predict utility somewhat accurately and that human DMs are utility maximizers.

4.1 Deciding on a career – EU and CPT

Consider the following scenario: After finishing his study in Economics at a prestigious, German university, Tom is approached by a friend with an idea for a risky business venture. Using his newly acquired knowledge in modeling, he analyzes the situation and condenses all information to a simple decision. If the two decide to pursue the idea, they can expect to achieve their goals with a probability of 50% and to fail completely again with a probability of 50%. If they do not pursue the idea Tom is confident that he will be able to earn the average wage in Germany for sure (i.e., 100% probability). The average wage works out to about €48,000 a year. The nature of the business idea only allows for complete success or failure which will render Tom basically unemployable because of reputational or health reasons, resulting in an upside of €86,400 a year in case of success and about €9,600 in case of failure:

Figure 18

Starting with EU, we can analyze this decision quite easily because all the data has already been condensed to an appropriate format. In a first step we need to come up with a basic parametrization we would like to use in line with the way I presented EU in the first section.

In order to make the comparison to the analysis with CPT more interesting, I chose to use a basic Root utility function as already supplied in the Tool with the following equation:

$$
\mathrm{u}(x) = \begin{cases} \sqrt[exp]{x}, & \text{if } x > 0 \\ -lm \cdot \sqrt[exp]{-x}, & \text{if } x \leq 0 \end{cases} \tag{16}
$$

Where exp is an the degree of the root and lm is a multiplier to model loss aversion. The piecewise definition supplied in the Tool is fine for this because all payoffs in this problem are positive. This means I will leave the parameters at 2 for the exponent and 1 for the loss-multiplier. This is all the setup we need for EU.

Inputting the sure payoff into the table expectedly shows a mean of €48,000.0, a standard deviation of €0.0 and skewness of 1.0. Using the root utility function described before, this option yields a utility of 219.1 (no unit), and a certainty equivalent of €48,000.0. As expected the risk premium (how much would you pay to insure against this risk) is €0, after all this is a sure payoff without any risk.

Entering the venture option, shows a mean of €48,000.0 indicating that this is a mean-preserving lottery. The standard deviation is €38,400.0 and the skewness 0.0. Using the same utility function as above, this option yields a utility of 196.0, a certainty equivalent of €38,400.0 and a risk premium of €9,600.0. Clearly, according to EU, taking the sure payoffs should be preferred over engaging in the risky venture.

How does CPT judge this? Again, the first thing we need to do when setting up the analysis with CPT is to decide on the appropriate functional forms of the utility function and in this case also the probability weighting function. To keep things simple and because these are examples meant to showcase the capabilities of the Tool and some differences between similar seeming theories, I will stick with Kahneman and Tversky's proposals in both cases and use the default values for all parameters but one.

When we enter the sure payoff this time with the CPT analysis set up, the input section will show the same information as with EU. When looking at the output of this setup, the utility is seemingly much higher than in the EU case (13,167.2). Because we only have one input, probability weighting does not play any role in this. The whole difference stems from the different utility functions. This is an important result of this comparison of theories because the outcomes of different utility functions cannot be compared, which means that the utility from lotteries which were evaluated with different functions cannot be compared. The functions may be designed to handle values of different sized and differ in their sensitivity to changing inputs. The only information that can be transferred between the usage of different utility functions is how the decision between two options ends.

When we enter the risky venture in the Tool, this yields a utility of 11,483.9, a certainty equivalent of 41,090.0 and a risk premium of 6,910.0, again lower than the sure payoff indicating we should not enter in the risky venture.

So far I acted as if the outcomes of CPT should be comparable to those of EU. As I explained in the first section, one of CPT's major contributions was the introduction of a reference point r in the utility function and focusing on payoffs instead of final wealth levels. This comparison can therefore only be made if we set r to 0 as we did before, because then the payoffs are equivalent to the final wealth and we can analyze the same economic variable as we did with EU.

4.2 Making a business decision – OSAD and RDRA

Tom learned a lot of great management and entrepreneurial theories during his studies. Ironically, he is still rather risk averse and acts the way, EU and CPT indicated would lead to higher utility and took an offer at a small consulting firm. He has been working there for three years and is now the leader of a small team evaluating an investment case for a client.

The client identified the opportunity to start a new product line in a market segment very similar to his current operations. They would need to invest €1,000,000.0 to do so and asked Tom's team to figure out whether this is a good investment. After a few weeks of research into the broader market and the client's current position, Tom's team identifies three possible scenarios. They identify a base case in which the project returns 14% on the invested capital and which they estimate will occur with 50% probability. The best-case scenario is a return of 18% with a probability of 30% and the worst case a return of 10% with a probability of 20%. Usually, the client does not invest in projects returning less than 12% and sees anything below that as a loss.

Multiplying the investment with the returns minus the hurdle rate leads to the following lottery:

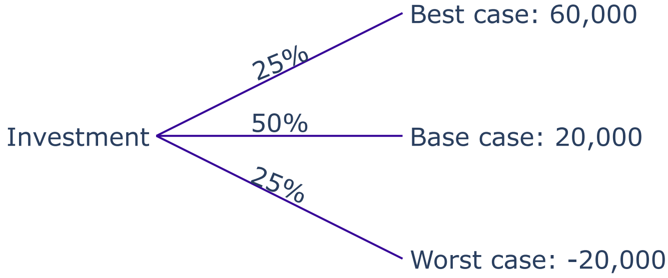

Figure 19: A depiction of the investment decision faced by Tom's client

Being a consultant, Tom has learned that it is much better to maximize the utility of his clients than to actually maximize their financial outcomes. After all, the better they feel, the more likely he is to be hired again. So, he decides not to present them with the actual lottery but to manage their expectations according to OSAD and RDRA.[12]

As you might remember, the implementation of OSAD in the Tool allows you to enter an alternative set of probabilities, which determine the amount of anticipatory utility you gain, while the actual probabilities (those in Figure 19) are used to resolve the risk and determine the objective utility and the disappointment term. Tom comes up with three sets of subjective probabilities one of which he wants to present to the client.

[12]Usually, applying OSAD or similar theories aimed at optimizing one's utility by adjusting one's expectations requires a certain amount of cognitive dissonance (Gollier and Muermann 2010, p. 1275). This is not the case for Tom since he is actually managing the client's expectations.

First, he wants to test an overly optimistic distribution assigning the best case a higher probability than objective probabilities:

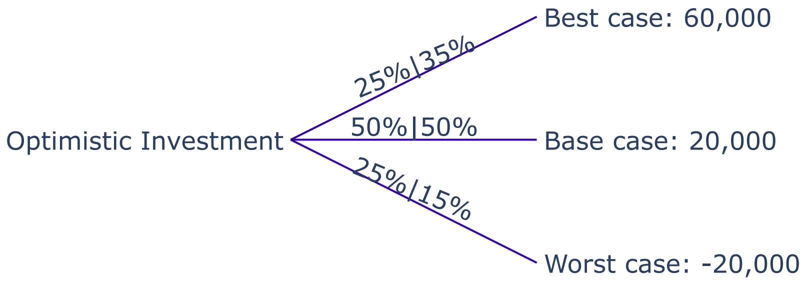

Figure 20: An overly positive set of subjective probabilities

Next, Tom wants to know what utility the clients would get from a truthful report of his findings:

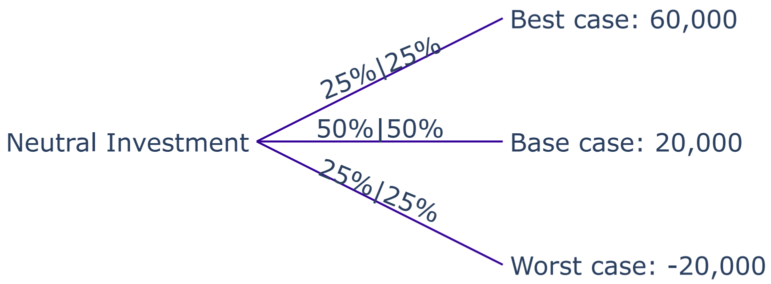

Figure 21: An neutral set of subjective probabilities

Lastly, Tom wants to know whether it might be worth it to undersell the chances of success in order to surprise his clients with the more positive return of the project afterwards:

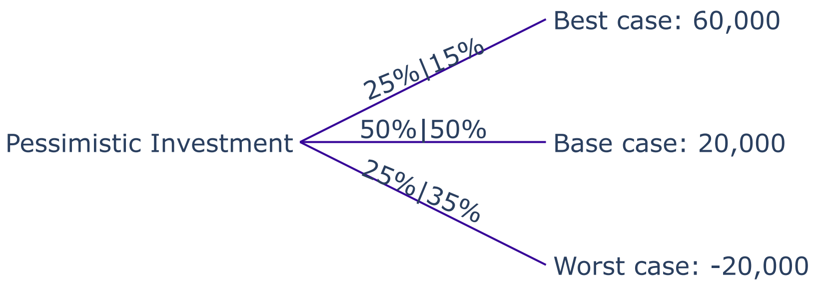

Figure 22: An overly pessimistic set of subjective probabilities

For his model, Tom decides to use Kahneman and Tversky's utility function with standard parameters and the Additive Habits bivariate utility function, again with a standard parameter of $\eta = 0.1$. As a last variable, he decides to set the savoring coefficient (in the input section) lower than 0.5, as the returns are expected soon, and he therefore expects the savoring part of the utility to have less of an impact. He decides to calculate the

Subjective Probabilities

		pessimistic	neutral	optimistic
	0.35	3,8	4,5	5,2
Savoring	0.25	3,8	4,2	4,6
Coefficient	0.15	3,7	3,8	3,9
	0.025	3,6	3,4	3,2

Table 1: This table displays the utility indicated by OSAD for all three sets of subjective probabilities and a range of different values of the savoring coefficient.

utility calculated for savoring coefficients of 0.35, 0.25, 0.15 and 0.05. This combination of objective lotteries and savoring coefficient produces the following utilities.

Table 1 depicts the utilities assigned by OSAD to a given combination of a set of subjective probabilities in the columns and a savoring coefficient in the rows. The largest utility in any row is marked and can be read as an indication of which set of subjective probabilities should communicated to the client as objective probabilities to maximize their utility for that particular savoring coefficient.[13] Communicating truthfully is disadvantageous for any of the savoring coefficients Tom looked at. The smaller the utility gained from savoring, the less attractive it becomes to report overly optimistic probabilities as clients will be disappointed when the actual outcome underperforms their expectations.

Let us now look at how Tom might look at the decision using RDRA. Tom and his team find out that the current operations of the client on average yield a return which is slightly above the 12% hurdle rate imposed on new projects:

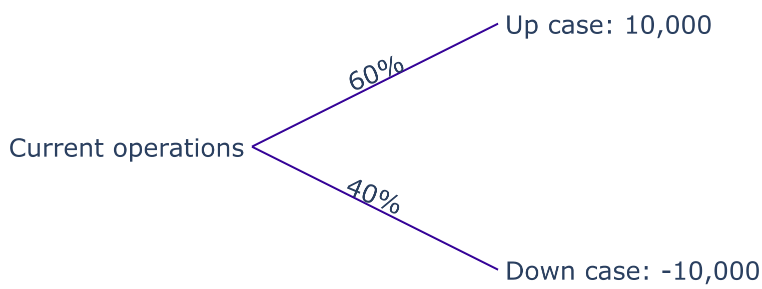

Figure 23: A lottery describing the profit profile of the clients current operations, that can be used as a context lottery for RDRA.

The lottery in Figure 23 represents the distribution of marginal returns if the client were to invest the additional €1,000,000 into his normal business instead of the new project. Entering this lottery in the Tool shows a mean return of €2,000.0 on top of

[13] A more granular analysis with a richer set of subjective probabilities could optimize the set of subjective probabilities even more, but the basic message would remain the same.

		Planned Action	
		Investment	Current Operations
Pursued Action	Investment	19,9	20,0
	Current Operations	1,7	19,3

Table 2: This table displays the utility indicated by RDRA All possible combinations of the target and context lottery as planned and realized actions. The columns indicate, which action was planned and the rows, which action was realized.

the required 12%, a standard deviation of €9,798.0 and a skewness of the distribution of - 0.4082 for the current operations. To keep things simple, Tom decides, to model the simple outcome-based utility with a linear utility. In line with Köszegi and Rabin's suggestion, he decides to model risk aversion and diminishing sensitivity in the bivariate utility function. To achieve this, he uses the piecewise defined root function from Equation 16 with a loss-multiplier of 3.0, indicating that losses are weighed three times as heavy as wins. Evaluating the new investment in regard to the risk profile of the current operations using RDRA in the above parametrization indicates a utility of 20,016.7. In the case of RDRA, the Tool calculates the certainty equivalent in relation to a sure payoff of 0.0.

Table 2 depicts the outcomes of all possible combinations of the two Investment opportunity and Current operations lotteries and the utility indicated by RDRA based on the above specifications. It is apparent that the combination of expecting to invest into the extension of current operations and then investing into the new product line ("Investment") yields the highest utility. This is however not one of the classic solution concepts proposed by Köszegi and Rabin discussed in the first section.

If we plan to Invest in the new project, going through with this plan yields higher utility than extending current operations, meaning that this is a UPE. Meanwhile, planning to extend current operations and then actually doing so yields lower utility than diverging from the plan. This means that planning to invest in the new project and actually doing so is the UPE with the highest resulting utility and therefore also the PPE.

The Köszegi and Rabin's notion of a CPE demands that a followed through plan yield higher utility than any other possible followed through plan. Planning to invest into the new product line and following through yields higher utility than planning to extend current operations and later doing so making the first option the CPE as well.

4.3 Investing a bonus – RT and ST

After his analysis, Tom decides to recommend the investment in the new product line. His supreme expectation management leads to a lasting relationship with the client and at the end of the year, he receives a big bonus. He decides to invest €5,000 of his bonus into the stock-market for at least 5 years and finds that American financial institutions might be interesting.

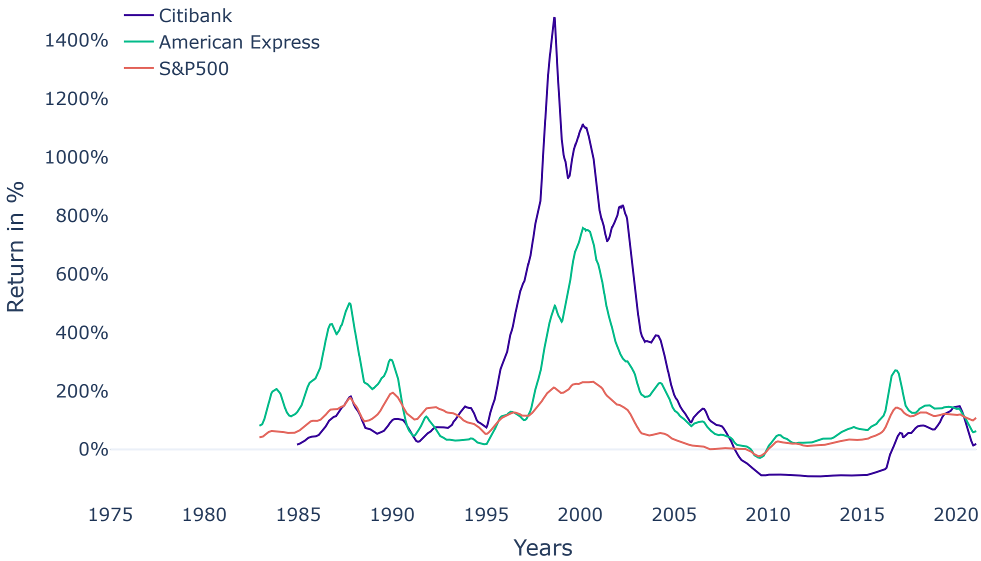

Figure 24: This figure displays the 180-day rolling average returns gained from holding a stock or ETF for a five year period in percent.

Figure 24 shows the percentage return achieved by holding the stock for five years for Citibank, American Express and ETF covering the S&P500. The graph depicts the 180-day rolling average of these values. It is very obvious, that both stocks correlate strongly with the broader market and have Betas above 1. This means that during the past, when the market went up, the individual stocks of Citibank and American Express went up as well and when the market went down, they went down as well. A second thing, that becomes clear from looking at Figure 30 is that Citibank "reacts" more strongly to shifts in the overall market than American Express.

For his decision, Tom assumes that the past performance of the stocks is predictive of their future returns. At the meantime he decides to evaluate both individual stocks in relation to the broader market using RT and ST. To get the past data into a form that can be entered in the Tool he calculates all percentiles that are a multiple of five, yielding him 21 individual values per title:

Figure 25 shows the predicted returns (percentile-values times the invested amount of €5,000). The quantiles give a similar picture like the percentage returns in Figure 24.

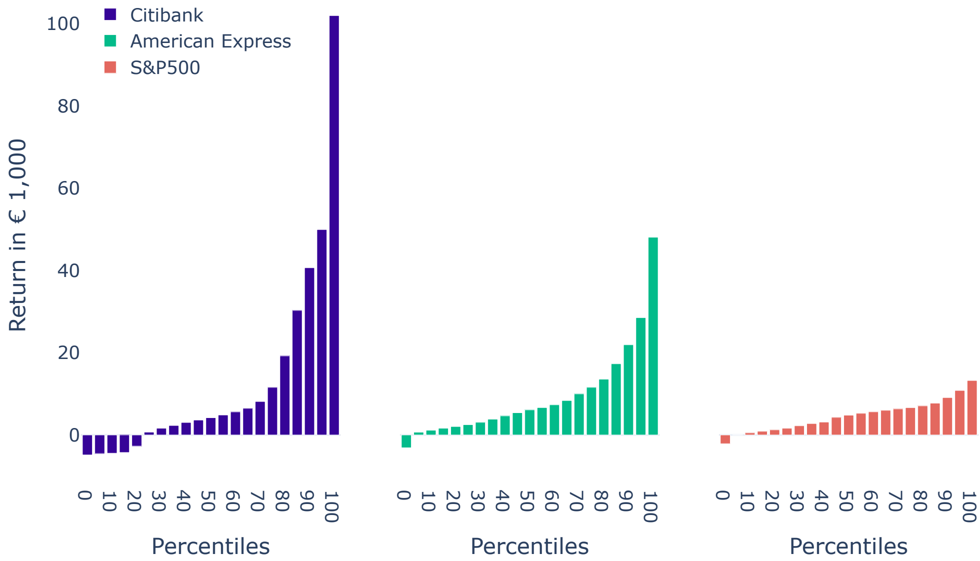

Figure 25: The above charts illustrate the returns in €1,000 relative to an investment €5,000. For this purpose they plot the returns representing equally spaced quantiles for all three titles described in Figure 24

Citibank is more volatile than American express, which is in turn more volatile than the overall market. In Addition, Figure 25 illustrates, that Citibank has a higher mean and standard deviation than American, which in turn is higher than the S&P500. So, it seems plausible that the final decision will be a tradeoff of higher returns and higher risks. For the purpose of this case study, I will enter all percentiles that are bigger than 0 and a multiple of 5 (i.e., 5, 10, 15, ...) and their corresponding values as if they were a discrete distribution.[14]

Because all of these values are equally likely with a probability of 5% and their great number, the charts in the input section might not look as nice as they do with smaller calculations. This should not be a big problem and the calculations should run without any problem.

If we want to use RT or ST for the analysis, we can enter the target lottery and context lottery in the input-table. Because of the way the Tool implements RT and ST, the state-spaces of the two lotteries must be correlated. This means that payoffs of the two lotteries in the same row do not just occur with the same probability but are inherently linked. If you choose the target lottery and receive a certain payoff, you know that in the case of the context lottery you would have received the payoff in the same row. Usually,

[14]Of course, the approximation of the underlying distribution could be made more exact by using a bigger number of percentiles. While the Tool is in principle able to accommodate arbitrarily granular lotteries, there comes a point where it is probably easier to use the underlying functions. You can find an example of how to do this in the appendix.

obtaining examples which have these characteristics is hard, but in the case of the stocks Tom is looking at, a simple visual inspection of the data in Figure 30 reveals that all three titles are strongly (linearly) correlated with positive Betas. While not a formal proof, this should be sufficient to assume that the distribution of the percentiles in Figure 31 indeed represent correlated state-spaces. In the case of weaker correlations or negative Betas, justifying this assumption would be much harder.

Entering Citibank as the target lottery reveals a mean payoff for an investment of €5,000 over a period of 5 years of €13,911 with a standard deviation of €24,873 and a skewness of 2.3. The corresponding measures for the S&P500 are a mean of €4,960, a standard deviation of €3,467 and a skewness of 0.6. Using a root utility function with a loss multiplier of 3 and an exponent of 2 (implying the square root) and the Regret function proposed by Loomes and Sugden 1982 with a weight of the regret term of 1 yields a utility of 37.3, a Certainty Equivalent of €1,391.2 and a Risk Premium of €12,520.2. This low utility value is due to the extreme diminishing sensitivity implied by the root utility function. Experiments find much lower diminishing sensitivity, but as long as the parametrization is not changed between investing into Citibank and investing into American Express, this should be fine for the sake of this example.

The investment into American Express has a mean of €10,211.0, a standard deviation of €11,247.0 and a skewness of 2.0. So, it is less profitable on average than the investment into Citibank, but the smaller standard deviation also implies less risk. Using the same parametrization as before, American Express as the target with S&P500 as the context yields a utility of 113.4, a Certainty Equivalent of €12,855.0 and a Risk Premium of €-2,643.9. In this parametrization of RT, it is possible for a target lottery to become more attractive by adding the context, than it would be standing alone. This is what happened in the case of American express, where the regret term increased the utility so much that the risk premium is actually negative, meaning that Tom should want to pay to enter in this lottery. As stated before, the weighting of the regret term in the regret function is quite high with a value of 1.0. A value of 1 implies, that Tom receives just as much utility from the comparison of his target lottery to the context as he does from the final payoff he receives.

Setting up ST, Tom first needs to decide on the appropriate value of δ, the degree to which salient states are overweighed compared to less salient ones. He estimates that 0.5 might be a good fit but decides to check whether the outcomes are very sensitive to changes in this parameter. He decides to use the same root utility function as for RT and use the Salience function proposed by Bordalo, Gennaioli, and Shleifer 2012 and the default parameters.

Target Lottery

		Citibank			American Express		
		Utility	Certainty Equivalent	Risk Premium	Utility	Certainty Equivalent	Risk Premium
$\delta -$ local thinking	0.6	38.7	1,5	12	90	8	2
	0.55	36.3	1,3	13	90	8	2
	0.5	33.6	1,3	13	90	8	2
	0.45	30.4	9	13	90	8	2
	0.4	26.8	7	13	90	8	2

Table 3: This table displays the utiltiy, certainty equivalent and risk premium indicated by ST All possible combinations of the target and context lottery as planned and realized actions. The columns indicate, which action was planned and the rows, which action was realized.

Table 3 shows the outcomes for the Citibank and American Express stocks with the SP500 as the context conditional on different values for δ. American Express yields the higher utility, certainty equivalent and risk premium for any of the tested values of δ. At the same time, the outcomes of American Express are much less sensitive to changes in δ than Citibank's. All in all, both RT and ST imply that investing Tom's €5,000 should be invested into American Express to gain the higher utility.

5 Conclusion

This essay provided a short introduction to some of the most influential theories of choice under risk. It mostly kept to the initial parametrizations without trying to follow all variations and developments of the theories, opting for conceptual clarity. The Tool at the same time offers many degrees of freedom to test different parametrizations and run quick evaluations of different examples.

The case-studies in the last section showed some ways to look at real-world examples.

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