ML HW

Q.3 : Logistic Sigmoid Function and Hyperbolic Tangent Function

Given: $\sigma(a) = \frac{1}{1+e^{-a}}$, $\tanh(a) = \frac{e^{a}-e^{-a}}{e^{a}+e^{-a}}$

(1) Show $\tanh(a) = 2\sigma(2a) - 1$

$$\Rightarrow \tanh(a) = \frac{e^{a}-e^{-a}}{e^{a}+e^{-a}} = \frac{1-e^{-2a}}{1+e^{-2a}} = \frac{1}{1+e^{-2a}} - \frac{e^{-2a}}{1+e^{-2a}}$$

$$= \frac{1}{1+e^{-2a}} - \frac{1}{e^{2a}+1} = \sigma(2a) - \sigma(-2a)$$

$\because \sigma(z) = 1-\sigma(-z)$ $\quad \therefore \tanh(a) = \sigma(2a) - 1 + \sigma(2a)$

$$= 2\sigma(2a) - 1$$

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(2) Given
$$\begin{cases} y(x, \vec{w}) = w_0 + \sum_{j=1}^{M} w_j \sigma\left(\frac{x-\mu_j}{s}\right) \\ y(x, \vec{u}) = u_0 + \sum_{j=1}^{M} u_j \tanh\left(\frac{x-\mu_j}{2s}\right) \end{cases}$$

Assume $a = \frac{x-\mu_j}{2s}$

$$\Rightarrow w_j \sigma(a) = u_j \tanh(a) = u_j\left[2\sigma(2a) - 1\right] = u_j\left[2\frac{1}{1+e^{-2a}} - 1\right]$$

$$= w_j \frac{1}{1+e^{-a}} \Rightarrow \frac{w_j}{u_j} = 2\frac{(1+e^{-a})}{1+e^{-2a}} - (1+e^{-a}) = \frac{2(1+e^{-a}) - (1+e^{-a})(1+e^{-2a})}{1+e^{-2a}}$$

$$= \frac{\cancel{2} + 2\cancel{e^{-a}} - \cancel{1} - \cancel{e^{-a}} - e^{-2a} - e^{-3a}}{1+e^{-2a}} = \frac{1+e^{-a}-e^{-2a}-e^{-3a}}{1+e^{-2a}}$$

$$= \frac{1-e^{-2a}}{1+e^{-2a}} + \frac{e^{-a}-e^{-3a}}{1+e^{-2a}} = \frac{e^{a}-e^{-a}}{e^{a}+e^{-a}} + e^{-a}\left(\frac{1-e^{-2a}}{1+e^{-2a}}\right)$$

$$= \tanh(a) + e^{-a}\left(\frac{e^{a}-e^{-a}}{e^{a}+e^{-a}}\right) = \tanh(a) + e^{-a}\tanh(a) = \boxed{(1+e^{-a})\tanh(a)}$$

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$$w_j \sigma(2a) = u_j \tanh(a) = u_j\left[2\sigma(2a) - 1\right] \Rightarrow \frac{w_j}{u_j} = \frac{2\sigma(2a)-1}{\sigma(2a)} = 2 - \frac{1}{\sigma(2a)}$$

$$= \boxed{2 - \frac{1}{\sigma\left(\frac{x-\mu_j}{s}\right)}}$$

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