MLHW

Q1: (a) Symmetric matrix $M \in \mathbb{R}^n$ is a positive semi-definite if $\forall x \in \mathbb{R}^n$ $x^T M x \geq 0$, given matrix $A \in \mathbb{R}^{n \times n}$. Show that AA^T is a positive semedefinite

$\Rightarrow$ Assume a matrix $v \in \mathbb{R}^n$

$$v^T A A^T v \geq 0$$

$$= (A^T v)^T (A^T v) = \|A^T v\|^2$$

Because of the power of the matrix, $\|A^T v\|$ is alway ≥ 0 $\forall v$

Then AA^T is positive semi-definite.

(b) If $f(x_1, x_2) = x_1 \sin(x_2) e^{-x_1 x_2}$, $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = ?$

$$\frac{\partial f}{\partial x_1} = \sin(x_2)\left[e^{-x_1 x_2} + x_1 \cdot e^{-x_1 x_2} \cdot (-x_2)\right]$$

$$= \boxed{e^{-x_1 x_2} \cdot \sin(x_2) \cdot (1 - x_1 x_2)} \; *$$

$$\frac{\partial f}{\partial x_2} = x_1\left[\cos(x_2) e^{-x_1 x_2} + \sin(x_2) e^{-x_1 x_2} \cdot (-x_1)\right]$$

$$= \boxed{x_1 e^{-x_1 x_2}\left[\cos(x_2) - x_1 \sin(x_2)\right]} \; *$$

(c) Given $f(x;p) = p^x (1-p)^{1-x}$ for $x \in \{0,1\}$

The MLE of p is:

(1) construct $L(p) = f(x_1;p) f(x_2;p) \cdots f(x_n;p) = p^{x_1}(1-p)^{1-x_1} p^{x_2}(1-p)^{1-x_2} \cdots p^{x_n}(1-p)^{1-x_n}$

$$= p^{\sum_{i=1}^n x_i} (1-p)^{n - \sum_{i=1}^n x_i}$$

(2) $\ln(L(p)) = \ln(p^{\sum_{i=1}^n x_i}) + \ln((1-p)^{n - \sum_{i=1}^n x_i}) = \sum_{i=1}^n x_i \ln(p) + (n - \sum_{i=1}^n x_i) \ln(1-p)$

given $\sum_{i=1}^n x_i = n\bar{x}$ where $\bar{x}$ is average of all sample

(3) $\frac{d}{dp} \ln(L(p)) = n\bar{x} \cdot \frac{1}{p} + (n - n\bar{x}) \cdot \frac{-1}{1-p}$, suppose $\frac{d}{dp} \ln(L(p)) = 0$

$$\Rightarrow \frac{n\bar{x}}{p} = \frac{n - n\bar{x}}{1-p} \Rightarrow \boxed{p = \bar{x}} \; *$$

make this correct

examine the 2nd derivative (4) $\frac{d^2}{dp^2} \ln(L(p)) = \frac{-n\bar{x}}{p^2} + \frac{-(n - n\bar{x})}{(1-p)^2}$, substitude $p = \bar{x} \Rightarrow \frac{-n}{\bar{x}} - \frac{n(1-\bar{x})}{(1-\bar{x})^2} = \frac{-n}{\bar{x}} - \frac{n}{1-\bar{x}}$

because of $x \in \{0,1\}$, $0 \leq \bar{x} \leq 1$ and $1 - \bar{x} \geq 0 \Rightarrow \frac{d^2}{dp^2} \ln(L(p)) < 0$