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A. Deriving Linear Constrained Minimum Variance Dectector (LCMV)

Let $\mathbf{w}^* = \underset{\mathbf{w}}{\operatorname{argmin}}\{\mathbf{w}^T\mathbf{R}\mathbf{w}\}$ such that $\mathbf{D}^T\mathbf{w} = \mathbf{c}$

where $\mathbf{w} \in \mathfrak{R}^{Lx1}$, $\mathbf{R} \in \mathfrak{R}^{LxL}$, $\mathbf{D} = [\mathbf{d}_1|\mathbf{d}_2|\ldots|\mathbf{d}_p] \in \mathfrak{R}^{Lxp}$ and $\mathbf{c} \in \mathfrak{R}^{px1}$

B. Lagrange Multipler Form

$J(\mathbf{w}) = \mathbf{w}^T\mathbf{R}\mathbf{w} + \boldsymbol{\lambda}^T(\mathbf{D}^T\mathbf{w} - \mathbf{c})$ such that $\boldsymbol{\lambda} \in \mathfrak{R}^{px1}$

C. Derivatives of Cost Equation

$\left.\frac{\partial J(\mathbf{w})}{\partial \mathbf{w}}\right|_{\mathbf{w}^*} = \mathbf{0}$ and $\left.\frac{\partial J(\mathbf{w})}{\partial \lambda}\right|_{\lambda^*} = \mathbf{0}$

$2\mathbf{w}^T\mathbf{R} + \boldsymbol{\lambda}^T\mathbf{D}^T = \mathbf{0}$

$\mathbf{D}^T\mathbf{w} - \mathbf{c} = \mathbf{0}$

D. Solve for $\mathbf{w}$

$\mathbf{w}^T\mathbf{R} = -\frac{1}{2}\boldsymbol{\lambda}^T\mathbf{D}^T$

$\mathbf{R}\mathbf{w} = -\frac{1}{2}\mathbf{D}\boldsymbol{\lambda}$, note $\mathbf{R} = \mathbf{R}^T$ is symmetric and invertible

$\mathbf{w} = -\frac{1}{2}\mathbf{R}^{-1}\mathbf{D}\boldsymbol{\lambda}$

E. Solve for $\boldsymbol{\lambda}$ by substituting in $\mathbf{w}$

$\mathbf{D}^T\mathbf{w} - \mathbf{c} = \mathbf{0}$

$-\frac{1}{2}\mathbf{D}^T\mathbf{R}^{-1}\mathbf{D}\boldsymbol{\lambda} = \mathbf{c}$, note that $\mathbf{D}^T\mathbf{R}^{-1}\mathbf{D}$ is square and full rank and invertible

$\boldsymbol{\lambda} = -2(\mathbf{D}^T\mathbf{R}^{-1}\mathbf{D})^{-1}\mathbf{c}$

F. Solve for $\mathbf{w}$ by substituting in expression for λ

$\mathbf{w} = -\frac{1}{2}\mathbf{R}^{-1}\mathbf{D}\boldsymbol{\lambda}$

$\mathbf{w}^{LCMV} = \mathbf{R}^{-1}\mathbf{D}(\mathbf{D}^T\mathbf{R}^{-1}\mathbf{D})^{-1}\mathbf{c}$

A. Deriving Linear Constrained Minimum Variance Classifier (LCMVC)

Let $\mathbf{w}_j^* = \underset{\mathbf{w}_j}{\operatorname{argmin}}\{\mathbf{w}_j^T \mathbf{R} \mathbf{w}_j\}$ such that $\mathbf{M}^T \mathbf{w}_j = \mathbf{c}_j$, where $1 \le j \le p$

where $\mathbf{w}_j \in \Re^{Lx1}$, $\mathbf{R} \in \Re^{LxL}$, $\mathbf{M} = [\mathbf{m}_1 | \mathbf{m}_2 | \ldots | \mathbf{m}_p] \in \Re^{Lxp}$ and $\mathbf{c} = (0, \ldots, c_j, \ldots, 0)^T \in \Re^{px1}$

B. Lagrange Multipler Form

$J(\mathbf{w}_j) = \mathbf{w}_j^T \mathbf{R} \mathbf{w}_j + \boldsymbol{\lambda}^T (\mathbf{M}^T \mathbf{w}_j - \mathbf{c}_j)$ such that $\boldsymbol{\lambda} \in \Re^{px1}$

C. Derivatives of Cost Equation

$$\left.\frac{\partial J(\mathbf{w}_j)}{\partial \mathbf{w}_j}\right|_{\mathbf{w}_j^*} = \mathbf{0} \quad \text{and} \quad \left.\frac{\partial J(\mathbf{w}_j)}{\partial \boldsymbol{\lambda}}\right|_{\boldsymbol{\lambda}^*} = \mathbf{0}$$

$$2\mathbf{w}_j^T \mathbf{R} + \boldsymbol{\lambda}^T \mathbf{M}^T = \mathbf{0}$$

$$\mathbf{M}^T \mathbf{w}_j - \mathbf{c}_j = \mathbf{0}$$

D. Solve for w

$$\mathbf{w}_j^T \mathbf{R} = -\tfrac{1}{2} \boldsymbol{\lambda}^T \mathbf{M}^T$$

$\mathbf{R}\mathbf{w}_j = -\tfrac{1}{2} \mathbf{M} \boldsymbol{\lambda}$, note $\mathbf{R} = \mathbf{R}^T$ is symmetric and invertible

$$\mathbf{w}_j = -\tfrac{1}{2} \mathbf{R}^{-1} \mathbf{M} \boldsymbol{\lambda}$$

E. Solve for λ by substituting in w

$$\mathbf{M}^T \mathbf{w}_j - \mathbf{c}_j = \mathbf{0}$$

$-\tfrac{1}{2} \mathbf{M}^T \mathbf{R}^{-1} \mathbf{M} \boldsymbol{\lambda} = \mathbf{c}_j$, note that $\mathbf{M}^T \mathbf{R}^{-1} \mathbf{M}$ is square and full rank and invertible

$$\boldsymbol{\lambda} = -2(\mathbf{M}^T \mathbf{R}^{-1} \mathbf{M})^{-1} \mathbf{c}_j$$

F. Solve for w by substituting in expression for λ

$$\mathbf{w}_j = -\tfrac{1}{2} \mathbf{R}^{-1} \mathbf{M} \boldsymbol{\lambda}$$

$$\mathbf{w}_j^{LCMVC} = \mathbf{R}^{-1} \mathbf{M} (\mathbf{M}^T \mathbf{R}^{-1} \mathbf{M})^{-1} \mathbf{c}_j$$

A. DERIVING TARGET-CONSTRAINED INTERFERENCE-MINIMIZED FILTER (TCIMF)

Let $\mathbf{w}^* = \underset{\mathbf{w}}{\operatorname{argmin}} \{\mathbf{w}^T \mathbf{R} \mathbf{w}\}$ such that $[\mathbf{DU}]^T]\mathbf{w} = [\mathbf{1}_{px1}^T \mathbf{0}_{qx1}^T]^T$

where $\mathbf{w} \in \mathfrak{R}^{Lx1}$, $\mathbf{R} \in \mathfrak{R}^{LxL}$, $\mathbf{D} = [\mathbf{d}_1|\mathbf{d}_2|...|\mathbf{d}_p]$ and $\mathbf{U} = [\mathbf{u}_1|\mathbf{u}_2|...|\mathbf{u}_q]$

B. LAGRANGE MULTIPLIER FORM

$J(\mathbf{w}) = \mathbf{w}^T \mathbf{R} \mathbf{w} + \boldsymbol{\lambda}^T ([\mathbf{DU}]^T \mathbf{w} - [\mathbf{1}_{px1}^T \mathbf{0}_{qx1}^T]^T)$ such that $\boldsymbol{\lambda} \in \mathfrak{R}^{(p+q)x1}$

C. DERIVATIVES OF COST EQUATION

$\left.\frac{\partial J(\mathbf{w})}{\partial \mathbf{w}}\right|_{\mathbf{w}^*} = \mathbf{0}$ and $\left.\frac{\partial J(\mathbf{w})}{\partial \boldsymbol{\lambda}}\right|_{\boldsymbol{\lambda}^*} = \mathbf{0}$

$2\mathbf{w}^T \mathbf{R} + \boldsymbol{\lambda}^T [\mathbf{DU}]^T = \mathbf{0}$

$[\mathbf{DU}]^T \mathbf{w} - [\mathbf{1}_{px1}^T \mathbf{0}_{qx1}^T]^T = \mathbf{0}$

D. SOLVE FOR $\mathbf{w}$

$\mathbf{w}^T \mathbf{R} = -\frac{1}{2} \boldsymbol{\lambda}^T [\mathbf{DU}]^T$

$\mathbf{R}\mathbf{w} = -\frac{1}{2} [\mathbf{DU}] \boldsymbol{\lambda}$, note $\mathbf{R} = \mathbf{R}^T$ is symmetric and invertible

$\mathbf{w} = -\frac{1}{2} \mathbf{R}^{-1} [\mathbf{DU}] \boldsymbol{\lambda}$

E. SOLVE FOR $\boldsymbol{\lambda}$ BY SUBSTITUTING IN $\mathbf{w}$

$[\mathbf{DU}]^T \mathbf{w} = [\mathbf{1}_{px1}^T \mathbf{0}_{qx1}^T]^T$

$-\frac{1}{2} [\mathbf{DU}]^T \mathbf{R}^{-1} [\mathbf{DU}] \boldsymbol{\lambda} = [\mathbf{1}_{px1}^T \mathbf{0}_{qx1}^T]^T$, note that $[\mathbf{DU}]^T \mathbf{R}^{-1} [\mathbf{DU}]$ is square and full rank => invertible

$\boldsymbol{\lambda} = -2 ([\mathbf{DU}]^T \mathbf{R}^{-1} [\mathbf{DU}])^{-1} [\mathbf{1}_{px1}^T \mathbf{0}_{qx1}^T]^T$

F. SOLVE FOR $\mathbf{w}$ BY SUBSTITUTING IN EXPRESSION FOR λ

$\mathbf{w} = -\frac{1}{2} \mathbf{R}^{-1} [\mathbf{DU}] \boldsymbol{\lambda}$

$\mathbf{w}^{TCIMF} = \mathbf{R}^{-1} [\mathbf{DU}] ([\mathbf{DU}]^T \mathbf{R}^{-1} [\mathbf{DU}])^{-1} [\mathbf{1}_{px1}^T \mathbf{0}_{qx1}^T]^T$

REFERENCES

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