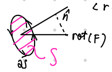
Clase 27

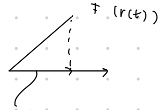
Intuición teorema de Stokes

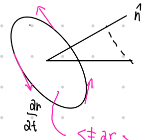
$$\int_{\partial S} F \, dr = \iint_S \operatorname{rot}(F)\, dS = \iint_S \langle \operatorname{rot}(F), n \rangle \, dS$$

$F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ campo



$$\int_{\partial S} F\, dr = \int_a^b \langle F(r(t)), \frac{dr(t)}{dt} \rangle \, dt \qquad r: [a,b] \longrightarrow \partial S$$



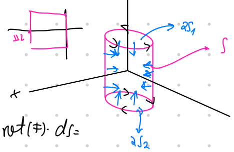


$\operatorname{rot}(F)(p) \approx \operatorname{rot}(F)(x)$ para $x \in S$

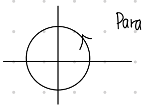
$$\iint_S \langle \operatorname{rot}(F), \hat{n} \rangle \, dS \approx \langle \operatorname{rot}(F), \hat{n}(p) \rangle \text{ Área } (S)$$

$$\langle \operatorname{rot}(F)(p), \hat{n}(p) \rangle \approx \frac{\int_{\partial S} F \, dr}{\text{Área } (S)}$$

Ejemplo Sea $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $(x, y, z) \longrightarrow (x^2 y, 3x^3 z, y z^2)$ y sea S el cilindro dado por $x^2+y^2=1$ con $z \in [-1,1]$. Verificar el teo de Stokes en S, orientando el cilindro hacia su interior



Verificar que $\iint_S \operatorname{rot}(F) \cdot dS = \int_{\partial S} F \cdot dr$



Parametrización S tenemos

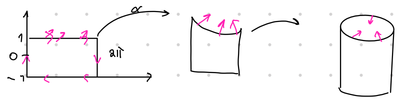
$\alpha: (\theta, z) \longrightarrow (\cos\theta, \operatorname{sen}\theta, z)$ $\quad 0 \leq \theta \leq 2\pi$, $\; -1 \leq z \leq 1$

Parametrizando ∂S_1

$\alpha_1(\theta) = (\cos\theta, \operatorname{sen}\theta, 1)$ $\quad \theta \in [0, 2\pi]$

Parametrizando ∂S_2:

$\alpha_2(\theta) = (\cos\theta, \operatorname{sen}\theta, -1)$



$$\operatorname{rot}(F)(x, y, z) = (z^2 - 3x^3, 0, 9x^2 z - x^2)$$

$$\iint_S \operatorname{rot}(F) \cdot dS = \int_0^{2\pi}\int_{-1}^{1} \langle \operatorname{rot}(F)(\alpha(\theta, z)), \hat{n} \rangle \, dz \, d\theta$$

$$\frac{d\alpha}{d\theta}(\theta, z) = (-\operatorname{sen}\theta, \cos\theta, 0) \qquad \frac{d\alpha}{dz}(\theta, z) = (0, 0, 1)$$

$$\frac{d\alpha}{d\theta}(\theta, z) \times \frac{d\alpha}{dz}(\theta, z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\operatorname{sen}\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta, \operatorname{sen}\theta, 0)$$

La normal apunta hacia el interior es $(-\cos\theta, -\operatorname{sen}\theta, 0)$

$$\iint_S \operatorname{rot}(F) \cdot dS = \int_0^{2\pi}\int_0^1 \langle \operatorname{rot}(F)(\alpha(\theta, z)), (-\cos\theta, -\operatorname{sen}\theta, 0) \rangle \, dz \, d\theta = \int_0^{2\pi}\int_{-1}^{1} \langle \underbrace{\operatorname{rot}(F)(\cos\theta, \operatorname{sen}\theta, z)}_{z^2 - 3\cos^3\theta,\ 0,\ 9\cos^2\theta z - \cos^2\theta}, (-\cos\theta, -\operatorname{sen}\theta, 0) \rangle \, dz$$

$$= \int_0^{2\pi}\int_{-1}^{1} -z^2\cos\theta + 3\cos^4(\theta) \, dz \, d\theta = \int_0^{2\pi} \cos\theta \left(\frac{-z^3}{3}\right)\Big|_{-1}^{1} d\theta + \int_0^{2\pi} 3\cos^4(\theta)\, z \Big|_{-1}^{1} d\theta$$

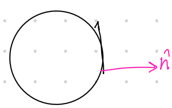
$$- \frac{2}{3}\int_0^{2\pi} \cancel{\cos\theta \, d\theta} + 6\int_0^{2\pi} \cos^4\theta \, d\theta = 6\int_0^{2\pi} \cos^4(\theta)\, d\theta = \iint_S \operatorname{rot}(F) \cdot dS$$

$$\int_{\partial S_1} F \cdot dr = \int_0^{2\pi} \langle F(\alpha_1(\theta)), \frac{d\alpha_1(\theta)}{d\theta} \rangle \, d\theta = \int_0^{2\pi} \langle F(\cos(\theta), \operatorname{sen}\theta, 1), (-\operatorname{sen}\theta, \cos\theta, 0) \rangle \, d\theta = \int_0^{2\pi} \langle (\cos^2\theta \operatorname{sen}\theta, 3\cos^3\theta, \operatorname{sen}\theta), (-\operatorname{sen}\theta, \cos\theta, 0) \rangle \, d\theta$$

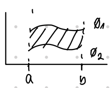
$$= \int_0^{2\pi} (-\cos^2\theta \operatorname{sen}^2\theta + 3\cos^4(\theta))\, d\theta = \int_0^{2\pi} -\cos^2(\theta)(1 - \cos^2\theta) + 3\cos^4(\theta)\, d\theta = -\int_0^{2\pi} \cos^2\theta + \int_0^{2\pi} 4\cos^4\theta \, d\theta = \int_{\partial S_1} F \cdot dr$$

$$\int_{\partial S_2^-} F \cdot dr = \int_0^{2\pi} \langle F(\alpha_2(\theta)), (-\operatorname{sen}\theta, \cos\theta, 0) \rangle \, d\theta = \int_0^{2\pi} \langle F(\cos\theta, \operatorname{sen}\theta, -1), (-\operatorname{sen}\theta, \cos\theta, 0) \rangle \, d\theta = \int_0^{2\pi} \langle \cos^2\theta \operatorname{sen}\theta, -3\cos^3\theta, \operatorname{sen}\theta), (-\operatorname{sen}\theta, \cos\theta, 0) \rangle \, d\theta$$

$$= \int_0^{2\pi} (-\cos^2\theta \operatorname{sen}^2\theta - 3\cos^4\theta)\, d\theta = \int_0^{2\pi} \cos^2\theta \, d\theta + \int_0^{2\pi} 2\cos^4\theta \, d\theta =$$



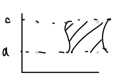
Regiones de tipo 1 y 2

① 

D: $\{(u, v): a \leq u \leq b, \; \phi(u) \leq v \leq \phi_2(u)\}$

$\phi_1, \phi_2 \; [a, b] \longrightarrow \mathbb{R}$ continuas

* Si mira hacia la superficie el pulgar es positivo. Sino, es negativo (este caso)

② 

W $\{(u, v) \quad c \leq v \leq d, \; y_1(v) \leq u \leq y_2(v)\}$

$r_1 \; r_2 \; [c, d] \longrightarrow \mathbb{R}$ continuas