Ayudantía 11

$\Sigma x_i = n\bar{x}$

Problema 1 (Casella & Berger, 2002)

Sea Y una variable aleatoria tal que $Y \sim N(\beta_0 + \beta_1 X, \sigma^2)$. Recuerde que el estimador de β_0 es $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{x}$.

(a) Muestre que el estimador $\hat{\beta}_0$ puede ser expresado como $\hat{\beta}_0 = \sum_{i=1}^{n} c_i Y_i$, con

$$c_i = \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}$$

(b) Verifique que

$$E[\hat{\beta}_0] = \beta_0 \quad \text{y} \quad \operatorname{Var}\hat{\beta}_0 = \sigma^2 \left[\frac{1}{nS_{xx}} \sum_{i=1}^{n} x_i^2 \right]$$

$y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$

$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$

PD: $\hat{\beta}_0 = \Sigma c_i y_i$

$c_i = \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}$

$\hat{\beta}_1 = \frac{1}{n}\Sigma y_i - \frac{S_{xy}}{S}\bar{x}$

$= \Sigma \frac{1}{n} y_i - \frac{\Sigma (x_i - \bar{x})(y_i - \bar{y})\bar{x}}{S_{xx}}$

$= \Sigma \left[\frac{1}{n} y_i - \frac{(x_i - \bar{x})(y_i - \bar{y})\bar{x}}{S_{xx}} \right]$

$= \Sigma \frac{(x_i\bar{x} - \bar{x}^2)(y_i - \bar{y})}{S_{xx}}$

$= \frac{\Sigma (x_i\bar{x} - \bar{x}^2) y_i - \Sigma (x_i\bar{x} - \bar{x}^2)\bar{y}}{S_{xx}}$

$= \Sigma(x_i\bar{x} - \bar{x}^2) y_i - (\underbrace{\bar{y}(\Sigma x_i \bar{x} - n\bar{x}^2)}_{\bar{y}(n\bar{x}^2 - n\bar{x}^2)})$

$= \frac{\Sigma (x_i - \bar{x})\bar{x} y_i}{S_{xx}}$

$= \Sigma \left[\frac{1}{n} y_i - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}} y_i \right]$

$= \Sigma c_i y_i$

b) PD: $E(\hat{\beta}_0) = \beta_0$

$\operatorname{Var}(\hat{\beta}_0) = \sigma^2 \left[\frac{1}{nS_{xx}} \sum_{i=1}^{n} x_i^2 \right]$

$\Sigma c_i = \Sigma \frac{1}{n} - \frac{\Sigma (x_i - \bar{x})\bar{x}}{n} = 1$

$E(\hat{\beta}_1) = E[\Sigma c_i y_i] = \Sigma c_i E(y_i)$

$= \Sigma c_i (\beta_0 + \beta_1 x_i)$

$= \underbrace{\Sigma c_i}_{1} \beta_0 + \beta_1 \Sigma c_i x_i$

$= \beta_0 + \beta_1 \Sigma c_i x_i$

$\Sigma c_i x_i = \Sigma \frac{x_i}{n} - \frac{\Sigma (x_i - \bar{x})\bar{x} x_i}{S_{xx}}$

$= \bar{x} - \frac{\bar{x}}{S_{xx}} \Sigma (x_i^2 - x_i \bar{x})$

$= \bar{x} - \frac{\bar{x}}{S_{xx}} \Sigma \left[(x_i - \bar{x})^2 + x_i\bar{x} - \bar{x}^2 \right]$

$= \bar{x} - \frac{\bar{x}}{S_{xx}} \left(S_{xx} + \bar{x}^2 n - n\bar{x}^2 \right)$

$= 0$

$\Rightarrow E(\hat{\beta}_0) = \beta_0$

$S_{xx} = \Sigma (x_i - \bar{x})^2$

$\operatorname{Var}(\hat{\beta}_0) = \Sigma c_i^2 \operatorname{Var}(x_i)$

$= \sigma^2 \Sigma c_i^2$

$\Sigma c_i^2 = \left[\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}} \right]^2$

$= \sum_{i=1}^{n} \left[\frac{1}{n^2} + \frac{(x_i - \bar{x})^2 \bar{x}^2}{S_{xx}^2} - 2\frac{1}{n} \frac{(x_i - \bar{x})\bar{x}}{S_{xx}} \right]$

$= \sum_{i=1}^{n} \frac{1}{n^2} + \frac{\bar{x}^2 \Sigma (x_i - \bar{x})^2}{S_{xx}^2}$

$= \frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} = \frac{1}{n} + \frac{\frac{1}{n}(\Sigma x_i)^2}{S_{xx}}$

Pendiente $= \frac{\Sigma x_i^2}{nS_{xx}}$

$\Rightarrow \operatorname{Var}(\hat{\beta}_1) = \sigma^2 \left(\frac{\Sigma x_i^2}{nS_{xx}} \right)$

$\beta_0 \sim N\left(\beta_1, \sigma^2 \left(\frac{\Sigma x_i^2}{nS_{xx}} \right) \right)$

Problema 3 (Jiménez & Olea, 2004)

Se ha observado que los campos de atracción se forman con mayor frecuencia si los núcleos están cercanos. En un experimento se colocaron 20 núcleos a distancias diferentes y se midió la incidencia de campos de atracción (Y) para las diferentes distancias (X). Lamentablemente se borro parte del análisis de regresión y se le solicita completarlo.

(a) Complete la tabla ANOVA que se entrega a continuación

20 datos

Fuente	Grados de Libertad	Sumas Cuadradas	Medias Cuadradas	Estadístico F_0
Predictor	1	2.0559 (1)	= 2,0559 (2)	301.08 (3)
Residuos	18	0,1224 (4)	0,0068 (5)	
Total	19 (n-1)	2,1783 (6)		

$18 + 1 = 19$

$(6) = (1) + (4)$

$\frac{SCE}{n/2} = \frac{(4)}{n-2}$

$\to MC\,Reg / MCE = \frac{2,0559}{301,08}$

$(5) = \frac{2,0559}{301,08}$

$(4) = 0,0068 \cdot 18$

(b) ¿Qué porcentaje de la variable total está siendo explicada por el modelo?

$R^2 = \frac{SC\,Reg}{SCT} = \frac{\Sigma(\hat{y}_i - \bar{y})^2}{\Sigma(\bar{y} - y_i)^2} = 1 - \frac{SCE}{SCT} = 1 - \frac{0,1224}{2,1783} = 94,36\%$

2] $y_1 = \theta_1 + \theta_2 + l_1$

$y_2 = 2\theta_2 + l_2$

$y_3 = -\theta_1 + \theta_2$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$

$\underset{\sim}{y} = \underset{\sim}{x}\theta + \underset{\sim}{l}$

$\hat{\theta} = \underbrace{(X^T X)^{-1}}_{} X^T y$

"$1/S_{xx}$" "S_{xy}^2"

$X^T X = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$

$(X^T X)^{-1} = \frac{1}{12} \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1/2 & 0 \\ 0 & 1/8 \end{pmatrix}$

$X^T y = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 - y_3 \\ y_1 + 2y_2 + y_3 \end{pmatrix}$

$(X^T X)^{-1} X^T y = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/6 \end{pmatrix} \begin{pmatrix} y_1 - y_3 \\ y_1 + 2y_2 + y_3 \end{pmatrix}$

$= \begin{pmatrix} (y_1 - y_3)/2 \\ \frac{x_1 + 2x_2 + y_3}{6} \end{pmatrix} = \hat{\theta}$

$\hat{\theta}_1 = \frac{y_1 - y_3}{2} \qquad \hat{\theta}_2 = \frac{y_1 + 2y_2 + y_3}{6}$