Pauta I1 - EYP2114

Inferencia Estadística

Problema 1.

$X_1 \dots X_n$ v.a iid $\exp\left(\frac{1}{\theta}\right)$.

(a)

$$\mathcal{L}(\theta; x_1 \dots x_n) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{1}{\theta} x_i} = \frac{1}{\theta^n} e^{-\frac{1}{\theta} \Sigma x_i}$$

$$\log \mathcal{L}(\theta; x_1 \dots, x_n) = -n \log \theta - \frac{1}{\theta} \Sigma x_i$$

1,0 pts.

$$\frac{\partial \log \mathcal{L}(\theta; x_1 \dots x_n)}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2} \Sigma x_i$$

$$\frac{1}{\hat{\theta}^2} \Sigma x_i = \frac{n}{\hat{\theta}} \Rightarrow \hat{\theta} = \bar{x}$$

0,5 pts

(b) $\hat{W}_n = \frac{1}{n} \sum_{i=1}^{n} X_i^2$

$$E(\hat{W}_n) = \frac{1}{n} \sum_{i=1}^{n} E(X_i^2) = \frac{1}{n} \sum_{i=1}^{n} \{Var(X_i) + E^2(X_i)\}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \{\theta^2 + \theta^2\} = \frac{2n\theta^2}{n} = 2\theta^2.$$

1,5

Luego, como $E(\hat{W}_n) = W_n$ es un estimador insesgado

(c) $\hat{W}_n = \bar{Y}$ donde $Y_i = X_i^2$.

Luego, por TCL.

$$\hat{W}_n = \bar{Y} \,\dot{\sim}\, N\left(E(Y_i) ; \frac{Var(Y_i)}{n}\right)$$

0,5

$$E(Y_i) = E(X_i^2) = Var(X_i) + E^2(X_i) = 2\theta^2$$
$$Var(Y_i) = Var(X_i^2) = E(X_i^4) - E^2(X_i^2) = E(X_i^4) - 4\theta^4$$
} 0,5

$$E(X_i^4) = \int_0^\infty x_i^4 \cdot \frac{1}{\theta} e^{-\frac{1}{\theta}x_i} dx_i = \theta^4 \Gamma(5) = 24\theta^4$$

Luego $Var(Y_i) = 24\theta^4 - 4\theta^4 = 20\theta^4$

} 0,5

Así $\hat{W}_n \sim N\left(2\theta^2; \frac{20\theta^4}{n}\right)$

(d) Si $\hat{\theta}_n = \sqrt{\frac{\hat{W}_n}{2}} = g(\hat{W}_n)$

$$g(\hat{W}_n) \sim N\left(g(W_n); [g'(W_n)]^2 Var(\hat{W}_n)\right)$$ } 1,0

$$g'(W_n) = \frac{1}{4}\sqrt{\frac{2}{W_n}} = \frac{1}{4}\sqrt{\frac{2}{2\theta^2}} = \frac{1}{4\theta}.$$

Así

$$g(\hat{W}_n) \sim N\left(\theta, \frac{1}{16\theta^2} \cdot \frac{20\cdot\theta^4}{n}\right)$$
$$\sim N\left(\theta; \frac{5\theta^2}{4n}\right)$$

} 0,5

+ 1 punto base

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Inferencia Estadística

Problema 2

Problema 1.

$$P(Y=y) = \left(\frac{\theta}{2}\right)^{|y|} (1-\theta)^{1-|y|} \qquad y = -1, 0, 1. \quad 0 \leq \theta \leq 1.$$

(a) $$W(Y) = \begin{cases} 2 & Y = 1 \\ 0 & Y = -1, 0. \end{cases}$$

$$E(\hat{\theta}) = E(W(Y)) = 2 \cdot P(Y=1) + 0\{P(Y=-1) + P(Y=0)\} \quad 1.0$$

$$= 2 \cdot \left(\frac{\theta}{2}\right)^1 (1-\theta)^0 = \theta$$

Luego $W(Y)$ es un estimador insesgado. 1.0

(b) $$P(Y=y) = \left(\frac{\theta}{2}\right)^{|y|} (1-\theta)^{1-|y|}$$

$$= \exp\left\{|y| \log\left(\frac{\theta}{2}\right) + (1-|y|) \log(1-\theta)\right\} \quad 1.0$$

$$= \exp\left\{|y| \log\left(\frac{\theta}{2}\right) + \log(1-\theta) - |y| \log(1-\theta)\right\}$$

$$= \exp\left\{|y| \left\{\log\left(\frac{\theta}{2}\right) - \log(1-\theta)\right\} + \log(1-\theta)\right\} \quad 0.5$$

Luego, $t(Y) = |Y|$ es un estadístico suficiente para θ. 0.5

(c) $\prod_{i=1}^{n} P(Y=y_i) = \prod_{i=1}^{n} \left(\frac{\theta}{2}\right)^{|y_i|} (1-\theta)^{1-|y_i|}$

$$= \left(\frac{\theta}{2}\right)^{\Sigma|y_i|} (1-\theta)^{n-\Sigma|y_i|}$$

$$\log \mathcal{L}(\theta, y_1 \ldots y_n) = \Sigma|y_i| \log\left(\frac{\theta}{2}\right) + (n-\Sigma|y_i|)\log(1-\theta)$$

$$\frac{\partial \log \mathcal{L}(\theta; y_1 \ldots y_n)}{\partial \theta} = \frac{2}{\theta}\Sigma|y_i| \cdot \frac{1}{2} + (n-\Sigma|y_i|)\frac{1}{1-\theta} * -1.$$

$$\frac{\Sigma|y_i|}{\hat\theta} - \frac{n-\Sigma|y_i|}{1-\hat\theta} = 0$$

con desarrollo

$$(1-\hat\theta)\Sigma|y_i| = (n-\Sigma|y_i|)\hat\theta$$

$$\Sigma|y_i| = \hat\theta(n-\cancel{\Sigma|y_i|}+\cancel{\Sigma|y_i|})$$

$$\hat\theta = \frac{\Sigma|y_i|}{n}$$ 1,0

$$E(\hat\theta) = \frac{1}{n}\sum_{i=1}^{n} E|y_i| = E(|Y_i|) = 0 \cdot P(|Y_i|=0) + 1 \cdot P(|Y_i|=1) \Big\} 0,3$$

$$= P(|Y_i|=1) = P(Y_i=-1) + P(Y_i=1)$$

$$= \left(\frac{\theta}{2}\right)^{|-1|}(1-\theta)^{1-|-1|} + \left(\frac{\theta}{2}\right)^{|1|}(1-\theta)^{1-|1|}$$

$$= \frac{\theta}{2} + \frac{\theta}{2} = \theta \quad \Big\} 0,2$$

$$E(W^2(Y)) = 4 \cdot P(Y=1) = 4 \cdot \frac{\theta}{2} = 2\theta$$

$$Var(W(Y)) = E(W^2(Y)) - E^2(W(Y))$$

$$= 2\theta - \theta^2$$

0,2

$$Var(\hat{\theta}_{EMV}) = Var\left(\frac{1}{n}\sum |y_i|\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var\,|Y_i|$$

$$= \frac{1}{n} Var(|Y_i|)$$

$$Var|Y_i| = E(|Y_i|^2) - E^2(|Y_i|) = E(|Y_i|^2) - \theta^2$$

$$E(|Y_i|^2) = 1^2 \cdot P(|Y_i| = 1) = \theta$$

Luego $Var(\hat{\theta}_{EMV}) = \theta - \theta^2$

0,2

$$Var(W(Y)) \geq Var(\hat{\theta}_{EMV})$$

$$2\theta - \cancel{\theta^2} \geq \theta - \cancel{\theta^2}$$

$$2\theta \geq \theta$$

0,1

Luego $\hat{\theta}_{EMV}$ es mejor que $\hat{\theta} = W(Y)$

+ 1 punto base

Parte a)

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Problema 3

1.0

$$P(X_1 = x_1, X_2 = x_2, X_3 = x_3) = \exp\left\{\log P(X_1 = x_1, X_2 = x_3, X_3 = x_3)\right\}$$

$$= \exp\left\{\log \frac{n!}{x_1!\,x_2!\,x_3!} + x_1 \cdot \log\left(\frac{1}{2} + \frac{\theta}{4}\right) + x_2 \log\left(\frac{1-\theta}{2}\right) + x_3 \cdot \log \frac{\theta}{4}\right\}$$

Luego, $t_i(x) = x_i$, pero basta con $t_1(x) = x_1$, $t_2(x) = x_2$

$c_1(\theta) = \log\left(\frac{1}{2} + \frac{\theta}{4}\right)$

$c_2(\theta) = \log\left(\frac{1-\theta}{2}\right)$

$c_3(\theta) = \log \frac{\theta}{4}$.

$S(x) = \log \frac{n!}{x_1!\,x_2!\,x_3!}$

1.0.

Luego, la distribución multinomial pertenece a la familia exponencial.

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Problema 3.

Parte b

$p_1 = \frac{1}{2} + \frac{\theta}{4}$ $\qquad p_2 = \frac{1-\theta}{2}$ $\qquad p_3 = 1 - (p_1 + p_2)$

$$= \cancel{1} - \cancel{\tfrac{1}{2}} - \frac{\theta}{4} - \cancel{\tfrac{1}{2}} + \frac{\theta}{2}$$

Sea $\vec{X} = (X_1, X_2, X_3)$ donde $\qquad = \frac{\theta}{4}$

X_i: nº de alumnos inscritos en la etapa i $\quad i = 1, 2, 3.$

0.5

$$P(X_1 = x_1, X_2 = x_2, X_3 = x_3) = \frac{n!}{x_1!\, x_2!\, x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

donde $x_1 + x_2 + x_3 = n.$

$$= \frac{n!}{x_1!\, x_2!\, x_3!} \left(\frac{1}{2} + \frac{\theta}{4}\right)^{x_1} \left(\frac{1-\theta}{2}\right)^{x_2} \left(\frac{\theta}{4}\right)^{x_3}$$

$$\log P(X_1 = x_1, X_2 = x_2, X_3 = x_3) = \log \frac{n!}{x_1!\, x_2!\, x_3!} + x_1 \cdot \log\left(\frac{1}{2} + \frac{\theta}{4}\right) + x_2 \log\left(\frac{1-\theta}{2}\right) + x_3 \log \frac{\theta}{4}$$

0.5

$$\frac{\partial \log P(\quad)}{\partial \theta} = x_1 \frac{1}{\frac{1}{2} + \frac{\theta}{4}} * \frac{1}{4} + \frac{x_2}{\frac{1-\theta}{2}} \cdot \frac{-1}{2} + \frac{x_3}{\frac{\theta}{4}} \cdot \frac{1}{4}$$

$$= \frac{x_1}{2+\theta} - \frac{x_2}{1-\theta} + \frac{x_3}{\theta}$$

Luego, la ecuación de estimación es:

0.5

$$n\hat{\theta}^2 + (n - 2X_1 + X_2)\hat{\theta} - 2X_3 = 0$$

$$\hat{\theta} = \frac{-(n - 2X_1 + X_2) \pm \sqrt{(n - 2X_1 + X_2)^2 + 8nX_3}}{2n}$$

reemplazando con los valores dados

0.5

$$\hat{\theta}_1 = 0.766$$

$$\hat{\theta}_2 = -0{,}113 \Rightarrow \text{este valor no es factible.}$$

por lo que la estimación máxima verosimil corresponde a $\hat{\theta}_2$

Parte c)

$$\frac{\partial^2 \log P(\,)}{\partial \theta^2} = \frac{-X_1}{(2+\theta)^2} \quad \frac{X_2}{(1-\theta)^2} - \frac{X_3}{\theta^2}$$ 0.5

$$-E\left(\frac{\partial^2 \log P(\,)}{\partial \theta^2}\right) = \frac{E(X_1)}{(2+\theta)^2} + \frac{E(X_2)}{(1-\theta)^2} + \frac{E(X_3)}{\theta^2}$$

$$= \frac{n\left(\frac{1}{2} + \frac{\theta}{4}\right)}{(2+\theta)^2} + \frac{n\cdot\left(\frac{1-\theta}{2}\right)}{(1-\theta)^2} + \frac{n\frac{\theta}{4}}{\theta^2}$$

$$= \frac{n}{4(2+\theta)} + \frac{n}{2(1-\theta)} + \frac{n}{4\theta}$$

1.0

Luego, la varianza asintótica es: $\left\{ n\left(\frac{1}{4(2+\theta)} + \frac{1}{2(1-\theta)} + \frac{1}{4\theta}\right)\right\}^{-1}$ 0.5

+1 pto base.