$f(t)$	$F(s)$	$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\frac{1}{s}$	$\cosh(kt)$	$\frac{s}{s^2-k^2}$	$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
t^n	$\frac{n!}{s^{n+1}}$	$e^{at}f(t)$	$F(s-a)$	$f(t+T)=f(t)$	$\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$
e^{at}	$\frac{1}{s-a}$	$f(t-a)U(t-a)$	$e^{-as}F(s)$	$t^n f$	$(-1)^n F^{(n)}(s)$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$f(t)U(t-a)$	$e^{-as}\mathcal{L}(f(t+a))$	$\delta(t-t_0)$	e^{-st_0}
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$f'(t)$	$sF(s)-f(0)$	f	$\int_0^\infty e^{-st}f(t)dt$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$		

Pontificia Universidad Católica De Chile
Facultad De Matemáticas
Departamento De Matemática
Primer Semestre 2022

Ecuaciones Diferenciales - MAT1640
Ayudantía 13

Estabilidad de puntos críticos

1. Determine el tipo de punto crítico que es $(0,0)$ en los siguientes sistemas lineales indicando si es estable, inestable o asintóticamente estable:

   (a) $\dfrac{dx}{dt} = 3x + y, \quad \dfrac{dy}{dt} = 5x - y$

   (b) $\dfrac{dx}{dt} = x - 2y, \quad \dfrac{dy}{dt} = 2x - 3y$

2. Considere el sistema lineal

$$\frac{dx}{dt} = \epsilon x - y, \quad \frac{dy}{dt} = x + \epsilon y$$

   Muestre que el punto crítico $(0,0)$ es:

   (a) un punto espiral estable si $\epsilon < 0$

   (b) un centro si $\epsilon = 0$

   (c) un punto espiral inestable si $\epsilon > 0$

3. Determine la naturaleza del punto crítico $(0,0)$ en los siguientes sistemas casi lineales:

   (a)
$$\frac{dx}{dt} = x + 2y + x^2 + y^2, \quad \frac{dy}{dt} = 2x - 2y - 3xy$$

   (b)
$$\frac{dx}{dt} = 3x - y + x^3 + y^3, \quad \frac{dy}{dt} = 13x - 3y + 3xy$$

   (c)
$$\frac{dx}{dt} = 5x - 3y + y\left(x^2 + y^2\right), \quad \frac{dy}{dt} = 5x + y\left(x^2 + y^2\right)$$

Transformada de Laplace

4. Determine la transformada de Laplace de las siguientes funciones:

   (a) $f(t) = \sinh t$

   (b) $f(t) = \cos^2 t$

   (c) $f(t) = 3t^{5/2} - 4t^3$

   (d) $f(t) = te^t$

5. Encuentre la transformada de Laplace inversa de las siguientes funciones:

   (a) $F(s) = s^{-3/2}$

   (b) $F(s) = \dfrac{3}{s-4}$

   (c) $F(s) = \dfrac{10s-3}{25-s^2}$

   (d) $F(s) = 2s^{-1}e^{-3s}$

Estabilidad de puntos críticos

1. Determine el tipo de punto crítico que es $(0,0)$ en los siguientes sistemas lineales indicando si es estable, inestable o asintóticamente estable:

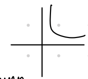
(a) $\dfrac{dx}{dt} = 3x + y, \quad \dfrac{dy}{dt} = 5x - y$

Punto crítico $\longrightarrow$ Valores propios

$$\begin{vmatrix} 3-\lambda & 1 \\ 5 & -1-\lambda \end{vmatrix} = (\lambda-3)(\lambda+1)-5 = 0$$

$$\lambda^2 - 2\lambda - 8 = 0$$

$$(\lambda-4)(\lambda+2) =$$



Reales con signos puntos

Punto silla

$e^{4x} \cdot C_1 + e^{-2x}$

uno que se va al ∞ y otro a 0.

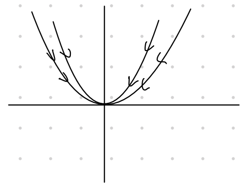
(b) $\dfrac{ax}{dt} = x - 2y, \quad \dfrac{ay}{dt} = 2x - 3y$

$$\begin{vmatrix} 1-\lambda & -2 \\ 2 & -3-\lambda \end{vmatrix} = (\lambda+3)(\lambda-1) + 4 = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda+1)^2 = 0$$

$$\lambda = -1 \quad (\times 2)$$



convergente e impropio estable

2. Considere el sistema lineal

$$\frac{dx}{dt} = \epsilon x - y, \quad \frac{dy}{dt} = x + \epsilon y$$

Muestre que el punto crítico $(0,0)$ es:

(a) un punto espiral estable si $\epsilon < 0$

(b) un centro si $\epsilon = 0$

(c) un punto espiral inestable si $\epsilon > 0$

$$\begin{vmatrix} \varepsilon-\lambda & -1 \\ 1 & \varepsilon-\lambda \end{vmatrix} = (\lambda-\varepsilon)^2 + 1 = 0$$

$$\lambda = \varepsilon \pm i$$

$\varepsilon > 0 \longrightarrow$ Espiral inestable

$\varepsilon = 0 \longrightarrow$ centro

$\varepsilon < 0 \longrightarrow$ espiral estable

3. Determine la naturaleza del punto crítico $(0,0)$ en los siguientes sistemas casi lineales:

(a)

$$\frac{dx}{dt} = x + 2y + x^2 + y^2, \quad \frac{dy}{dt} = 2x - 2y - 3xy$$

$$J = \begin{pmatrix} \frac{d}{dx}\left(\frac{dx}{dt}\right) & \frac{d}{dx}\left(\frac{dx}{dt}\right) \\ \frac{d}{dx}\left(\frac{dy}{dt}\right) & \frac{d}{dy}\left(\frac{dy}{dt}\right) \end{pmatrix} = \begin{pmatrix} 1+2x & 2+2y \\ 2-3y & -2-3x \end{pmatrix} \qquad J(0,0) = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (\lambda+2)(\lambda-1) - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda+3)(\lambda-2) = 0$$

$$\lambda = -3 \quad \lambda = 2$$

punto silla

(b)

$$\frac{dx}{dt} = 3x - y + x^3 + y^3, \quad \frac{dy}{dt} = 13x - 3y + 3xy$$

$$J = \begin{pmatrix} 3+3x^2 & -1+3y^2 \\ 13+3y & -3+3x \end{pmatrix} \qquad J(0,0) = \begin{pmatrix} 3 & -1 \\ 13 & -3 \end{pmatrix}$$

$$\begin{vmatrix} 3-\lambda & -1 \\ 13 & -3-\lambda \end{vmatrix} = (\lambda-3)(\lambda+3) + 13 = 0$$

$$\lambda^2 + 4 = 0$$

$$\lambda = \pm 2i$$

centro o espiral

(c)

$$\frac{dx}{dt} = 5x - 3y + y\left(x^2 + y^2\right), \quad \frac{dy}{dt} = 5x + y\left(x^2 + y^2\right)$$

$$J = \begin{pmatrix} 5+2xy & -3+(x^2+y^2)+2y^2 \\ 5+2xy & (x^2+y^2)+2y^2 \end{pmatrix} \Rightarrow J(0,0) = \begin{pmatrix} 5 & -3 \\ 5 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 5-\lambda & -3 \\ 5 & -\lambda \end{vmatrix} = 0 \qquad \lambda = \frac{5 \pm \sqrt{25-60}}{2}$$

$$\lambda^2 - 5\lambda + 15$$

espiral inestable

Transformada de Laplace

4. Determine la transformada de Laplace de las siguientes funciones:

(a) $f(t) = \sinh t$

(b) $f(t) = \cos^2 t$

(c) $f(t) = 3t^{5/2} - 4t^3$

(d) $f(t) = te^t$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} \qquad \Gamma(1/2) = \sqrt{\pi}$$

$$\Gamma(n+1) = n\,\Gamma(n)$$

$f(t)$	$F(s)$	$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\frac{1}{s}$	$\cosh(kt)$	$\frac{s}{s^2-k^2}$	$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
t^n	$\frac{n!}{s^{n+1}}$	$e^{at}f(t)$	$F(s-a)$	$f(t+T)=f(t)$	$\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$
e^{at}	$\frac{1}{s-a}$	$f(t-a)U(t-a)$	$e^{-as}F(s)$	$t^n f$	$(-1)^n F^{(n)}(s)$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$f(t)U(t-a)$	$e^{-as}\mathcal{L}(f(t+a))$	$\delta(t-t_0)$	e^{-st_0}
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$f'(t)$	$sF(s)-f(0)$	f	$\int_0^\infty e^{-st}f(t)dt$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$		

a) $f(t) = \sinh(t) = \dfrac{e^t - e^{-t}}{2}$

$$L\{\sinh(t)\} = \int_0^\infty e^{-st}\left(\frac{e^t - e^{-t}}{2}\right)dt = \frac{1}{2}\int_0^\infty e^{t(1-s)} - e^{-t(1+s)}\,dt = \frac{1}{2}\left(\frac{e^{t\overbrace{(1-s)}^{1-s<0,\ 1<s}}}{1-s} - \frac{e^{\overbrace{-t(1+s)}^{\text{esto converge por negativo}}}}{-(1+s)}\right)\Bigg|_0^\infty =$$

$$\frac{1}{2}\left(0 - 0 - \left(\frac{1}{1-s} + \frac{1}{s+1}\right)\right) = \frac{1}{2}\,\frac{2}{s^2-1} = \frac{1}{s^2-1}$$

$\rightarrow 1 < s$

$1+s > 0 \Rightarrow s > -1$

b) $f(t) = \cos^2(t)$

$$\mathcal{L}\{\cos^2(t)\} = L\left\{\frac{1}{2} + \frac{1}{2}\cos(2t)\right\} = \frac{1}{2}\underbrace{\mathcal{L}\{1\}}_{1/s} + \frac{1}{2}\underbrace{\mathcal{L}\{\cos(2t)\}}_{s/s^2+4} = \frac{1}{2s} + \frac{1}{2}\cdot\frac{s}{s^2+4}$$

$a\,\mathcal{L}\{f(t)\} + b\{g(t)\} = \mathcal{L}\{af(t) + bg(t)\}$

c) $f(t) = 3t^{5/2} - 4t^3$

$3\mathcal{L}\{t^{5/2}\} - 4\,\mathcal{L}\{t^3\}$

$\mathcal{L}\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}}$

$$3\,\frac{\overbrace{\Gamma(5/2+1)}^{45/8\cdot\sqrt{\pi}}}{s^{5/2+1}} - 4\,\frac{\overbrace{\Gamma(3+1)}^{3!}}{s^4}$$

$3\,\Gamma(\overset{2+1}{3}) = 3\cdot 2$

$$\frac{45}{8}\frac{\sqrt{\pi}}{s^{7/2}} - \frac{24}{s^4}$$

$\Gamma(5/2+1) = \frac{5}{2}\Gamma(5/2)$

$\frac{5}{2}\Gamma\left(\frac{3}{2}+1\right)$

$$\frac{5}{2}\cdot\frac{3}{2}\,\Gamma(3/2) = \underbrace{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}}_{\frac{45}{8}\cdot\sqrt{\pi}}\,\overbrace{\Gamma(1/2)}^{\sqrt{\pi}}$$

d) $\mathcal{L}\{te^t\}$ $\qquad \mathcal{L}\{e^{at}f(t)\} = F(s-a)$

$\mathcal{L}\{f(t)\} = F(s)$

$f(t) = t \longrightarrow \mathcal{L}\{t\} = \dfrac{1}{s^2} = F(s)$

$$F(s-1) = \frac{1}{(s-1)^2}$$

5. Encuentre la transformada de Laplace inversa de las siguientes funciones:

(a) $F(s) = s^{-3/2}$

(b) $F(s) = \dfrac{3}{s-4}$

(c) $F(s) = \dfrac{10s-3}{25-s^2}$

(d) $F(s) = 2s^{-1}e^{-3s}$

a) $F(s) = s^{-3/2}$

$$\mathcal{L}^{-1}\{s^{-3/2}\} = \mathcal{L}^{-1}\left\{\frac{1}{s^{3/2}}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^{1/2+1}}\right\}$$

Se parece pero hay que arreglar const

$f(t) = t^{1/2}$

$$f(t) = \frac{t^{1/2}}{\frac{1}{2}\sqrt{\pi}}$$

$$\mathcal{L}\{t^{1/2}\} = \frac{\Gamma(1/2+1)}{s^{3/2}} = \frac{1/2\,\Gamma(1/2)}{s^{3/2}} = \frac{1/2\cdot\sqrt{\pi}}{s^{3/2}}$$

$f(t)$	$F(s)$	$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\frac{1}{s}$	$\cosh(kt)$	$\frac{s}{s^2-k^2}$	$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
t^n	$\boxed{\frac{n!}{s^{n+1}}}$	$e^{at}f(t)$	$F(s-a)$	$f(t+T)=f(t)$	$\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$
e^{at}	$\frac{1}{s-a}$	$f(t-a)U(t-a)$	$e^{-as}F(s)$	$t^n f$	$(-1)^n F^{(n)}(s)$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$f(t)U(t-a)$	$e^{-as}\mathcal{L}(f(t+a))$	$\delta(t-t_0)$	e^{-st_0}
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$f'(t)$	$sF(s)-f(0)$	f	$\int_0^\infty e^{-st}f(t)dt$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$		

b) $F(s) = \dfrac{3}{s-4}$

$$F(s) = \frac{3}{s} \rightarrow F(s-4) = \frac{3}{s-4} \longrightarrow e^{4t}f(t)$$

$$\mathcal{L}^{-1}\left\{\frac{3}{s^{n+1}}\right\} = 3t^0 = 3 \longrightarrow f(t) = 3e^{4t}$$

c) $$F(s) = \frac{10s-3}{25-s^2} = \frac{10s}{25-s^2} - \frac{3s}{25-s^2} = \frac{3}{s^2-25} - \frac{10s}{s^2-25}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{1}{s^2-5^2}\right\} - 10\,\mathcal{L}^{-1}\left\{\frac{s}{s^2-5^2}\right\}$$

$$= \frac{3\sinh(5t)}{5} - 10\cosh(5t)$$

$\mathcal{L}\{\operatorname{sen}(kt)\} = \dfrac{k}{s^2+k^2}$

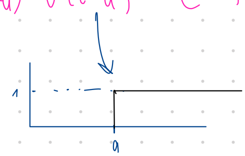
$\mathcal{L}\{\cos(kt)\} = \dfrac{s}{s^2+k^2}$

$$\boxed{\mathcal{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2}}$$

no está en la tabla

d) $F(s) = e^{-3s}\cdot\underbrace{\frac{2}{s}}_{F(s)}$

$\mathcal{L}\{f(t-a)\,U(t-a)\} = e^{-as}F(s)$



$$\mathcal{L}^{-1}\{e^{-3s}F(s)\} = f(t-3)\,U(t-3)$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2}{s}\right\} = 2\,t^3$$

$$\mathcal{L}^{-1}\left\{e^{3s}\frac{2}{s}\right\} = \frac{2}{s^3}\,U(t-3)$$

$2(t-2)^2 \, U(t-3)$

d) $\mathcal{L}^{-1}\{\frac{2}{s}e^{-3s}\} = e^{-3s}\cdot \underbrace{\frac{2}{s}}_{F(s)}$ $\qquad \mathcal{L}\{f(t-a)\,U(t-a)\} = e^{-as}F$

$\mathcal{L}^{-1}\{e^{-3s}F(s)\} = f(t-3)\cdot U(t-3)$

$\mathcal{L}^{-1}\{\frac{2}{s}\} = 2$

$\mathcal{L}^{-1}\{e^{-3s}\cdot\frac{2}{s}\} = 2\mathcal{U}(t-3)$