Ayudantía 1

$A = A \cap S$

$A = A \cap (B \cup \bar{B})$ / ley distributiva

Sean dos eventos A y B contenidos en un espacio muestral Ω, muestre que:

(a) $\mathbb{P}(A \cap \overline{B}) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$

(b) $A \subset B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

(c) $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$

(d) $\mathbb{P}(\overline{A} \cap \overline{B}) = 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A \cap B)$

$P(A \cap \bar{B}) = P(A) - P(A \cap B)$

$A = A \cap B \quad \cup \quad A \cap \bar{B}$

a) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$ $\quad A = (A \cap B) \cup (A \cap \bar{B})$ / $P(\ )$

$P(A) = P[(A \cap B) \cup (A \cap \bar{B})] = P(A \cap \bar{B}) + P(A \cap B) - P((A \cap \bar{B}) \cap (A \cap B))$

$P(A) = P(A \cap B) + P(A \cap \bar{B}) - P(A \cap \cancel{B \cap \bar{B}}) = P(A \cap B) + P(A \cap \bar{B})$

$P(A \cap \bar{B}) = P(A) - P(A \cap B)$ //

b) $A \subset B \Rightarrow P(A) \leq P(B)$

↳ A contenido en B.

$B \cap A = A$

↳ del ítem anterior $\rightarrow P(B \cap \bar{A}) = P(B) - P(B \cap A)$

$P(B \cap \bar{A}) = P(B) - P(A)$

$P(B \cap \bar{A}) \geq 0 \iff P(B) - P(A) \geq 0 \iff \boxed{P(B) \geq P(A)}$

c) $P(A \cap B) = P(A) + P(B) - 1$

Por ley aditiva

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

del axioma 2: $\quad P(\ ) \leq 1$

$P(A \cup B) \leq 1$

$P(A) + P(B) - P(A \cap B) \leq 1$

$P(A) + P(B) - 1 \leq P(A \cap B)$

$P(A \cap B) = P(A) + P(B) - 1$

d) $P(\bar{A} \cap \bar{B}) = 1 - P(A) - P(B) + P(A \cap B)$

Por ley de Morgan:

$\bar{A} \cap \bar{B} = \overline{A \cup B}$ / $P(\ )$

$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$ / Ley complemento

$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$ / Ley aditiva

$P(\bar{A} \cap \bar{B}) = 1 - (P(A) + P(B) - P(A \cap B))$

$P(\bar{A} \cap \bar{B}) = 1 - P(A) - P(B) + P(A \cap B)$

$$\text{(a)} \int_0^{+\infty} \frac{\nu^k}{\Gamma(k)}\, x^{k-1}\, e^{-\nu x}\, dx = 1.$$

$$\frac{v^k}{\Gamma(k)} \int_0^\infty \left(\frac{z}{v}\right)^{k-1} e^{-z} \frac{dz}{v} = \frac{v^k}{\Gamma(k)} \int_0^\infty \frac{z^{k-1}}{\ }$$

Propiedades función $\Gamma(\cdot)$ y $B(\cdot,\cdot)$

(1) $\Gamma(k) = \int_0^\infty u^{k-1}\, e^{-u}\, du = \text{gamma}(k)$; (2) $\Gamma(a+1) = a\,\Gamma(a)$; (3) $\Gamma(n+1) = n!$, si $n \in \mathbb{N}_0$;

(4) $\Gamma(1/2) = \sqrt{\pi}$; (5) $B(q,r) = \int_0^1 x^{q-1}(1-x)^{r-1}\, dx$; (6) $B(q,r) = \dfrac{\Gamma(q)\,\Gamma(r)}{\Gamma(q+r)} = \text{beta}(q,r)$

2. $$\int_0^\infty \frac{v^k}{\Gamma(k)}\, x^{k-1} e^{-vx}\, dx = 1$$

$z = vx \Rightarrow x = z/v$

$dz = v\,dx \Rightarrow dx = dz/v$

$$\frac{v^k}{\Gamma(k)} \int_0^\infty \left(\frac{z}{v}\right)^{k-1} e^{-z} \frac{dz}{v} = \frac{\cancel{v^k}}{\Gamma(k)} \int_0^\infty \frac{z^{k-1}}{\cancel{v\, v^{k-1}}} e^{-z}\, dz = \frac{1}{\Gamma(k)} \underbrace{\int_0^\infty z^{k-1} e^{-z}\, dz}_{\Gamma(k)} = 1$$

$$\bullet \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$$

$z = \frac{x^2}{2} \longrightarrow x = \sqrt{2z}$

$dz = x\,dx \longrightarrow dx = \frac{dz}{x} = \frac{dz}{\sqrt{2z}}$

$$2 \int_0^\infty e^{-z} \frac{dz}{\sqrt{2z}} = \frac{2}{\sqrt{2}} \int_0^\infty z^{-1/2} e^{-z}\, dz$$

$$= \frac{2}{\sqrt{2}} \int_0^\infty z^{1/2 - 1} e^{-z}\, dz = \frac{2}{\sqrt{2}} \Gamma(1/2) = \frac{2}{\sqrt{2}} \sqrt{\pi} = \sqrt{2\pi}$$

3. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k\, b^{n-k}$$

$a = 1 \qquad b = 1$

$$\bullet \sum_{i=0}^{k} \frac{e^{-\alpha}\alpha^{k-i}}{(k-i)} \cdot \frac{e^{-\beta}\beta^{i}}{i!} = \frac{e^{-(\alpha+\beta)}(\alpha+\beta)^k}{k!}$$

$$e^{-\alpha}e^{-\beta} \sum_{i=0} \frac{\alpha^k \alpha^{-i}}{(k-i)!} \frac{\beta^i}{i!} = e^{-\alpha}e^{-\beta} \alpha^k \sum_{i=0}^{k} \frac{\alpha^{-i}}{(k-i)!} \cdot \frac{\beta^i}{i!} \cdot \frac{k!}{k!} = \frac{e^{-\alpha}e^{-\beta}\alpha^k}{k!} \sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!} \cdot \alpha^{-i}\beta^i = \frac{e^{-\alpha}e^{-\beta}\alpha^k}{k!} \sum_{i=0}^{k} \binom{k}{i} (\alpha^{-1}\beta)^i \cdot 1^{k-i}$$

$a = \alpha^{-1}\beta$

$b = 1$

$$\frac{e^{-\alpha}e^{-\beta}\alpha^k}{k!}(\alpha^{-1}\beta + 1)^k = \frac{e^{-\alpha}e^{-\beta}\alpha^k}{k!}[\alpha^{-1}\beta + 1]^k$$

$$\bullet \sum_{i=0}^{k} \frac{e^{-\alpha}\alpha^{k-i}}{(k-i)!} \cdot \frac{e^{-\beta}\beta^{i}}{i!} = \frac{e^{-(\alpha+\beta)}(\alpha+\beta)^k}{k!}$$

La idea de este problema es ver que constantes me sirven para utilizar alguna igualdad y eliminar la suma.

$$\sum_{i=0}^{k} \frac{e^{-\alpha}\alpha^{k-i}}{(k-i)!} \cdot \frac{e^{-\beta}\beta^{i}}{i!} = e^{-\alpha}e^{-\beta} \sum_{i=0}^{k} \frac{\alpha^k \alpha^{-i}}{(k-i)!} \cdot \frac{\beta^i}{i!}$$

$$e^{-\alpha}e^{-\beta}\alpha^k \sum_{i=0}^{k} \frac{\alpha^{-i}\beta^i}{(k-i)!\,i!} \cdot \frac{k!}{k!}$$, agrego un "1" conveniente

$$\frac{e^{-\alpha}e^{-\beta}\alpha^k}{k!} \sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!} \cdot (\alpha^{-1}\beta)^i = \frac{e^{-\alpha}e^{-\beta}\alpha^k}{k!} \sum_{i=0}^{k} \binom{k}{i} (\alpha^{-1}\beta)^i$$

Mediante la igualdad mostrada en el item anterior, se tiene:

$$\frac{e^{-\alpha}e^{-\beta}\alpha^k}{k!} \cdot (\alpha^{-1}\beta + 1)^k = \frac{e^{-\alpha}e^{-\beta}\alpha^k}{k!} [\alpha^{-1}(\beta+\alpha)]^k$$

$$\frac{e^{-\alpha}e^{-\beta}\cancel{\alpha^k}}{k!} \cdot \cancel{\alpha^{-k}} (\beta+\alpha)^k = \frac{e^{-(\alpha+\beta)}(\alpha+\beta)^k}{k!} \quad //$$

$$\bullet \sum_{y=x}^{\infty} \binom{y}{x} p^x (1-p)^{y-x} \frac{\nu^y e^{-\nu}}{y!} = \frac{(\nu p)^x e^{-\nu p}}{x!}$$

Expandiendo la combinación y realizando manipulaciones algebraicas:

$$\sum_{y=x}^{\infty} \binom{y}{x} p^x (1-p)^{y-x} \cdot \frac{\nu^y e^{-\nu}}{y!} = \sum_{y=x}^{\infty} \frac{\cancel{y!}}{x!\,(y-x)!} p^x (1-p)^{y-x} \frac{\nu^y e^{-\nu}}{\cancel{y!}}$$

$$\frac{p^x e^{-\nu}}{x!} \sum_{y=x}^{\infty} \frac{(1-p)^{y-x}}{(y-x)!} \cdot \nu^y \quad , \quad \text{sustitución: } z = y-x \longrightarrow y = z+x$$

$$y = x \rightarrow y-x=0 \rightarrow z=0$$

$$\frac{p^x e^{-\nu}}{x!} \sum_{z=0}^{\infty} \frac{(1-p)^z}{z!} \nu^{z+x} = \frac{p^x e^{-\nu}}{x!} \sum_{z=0}^{\infty} \frac{(1-p)^z \nu^z \nu^x}{z!}$$

$$\frac{p^x e^{-\nu} \nu^x}{x!} \sum_{z=0}^{\infty} \frac{[(1-p)\nu]^z}{z!}$$

De formulario se tiene la siguiente igualdad:

$$\sum_{z=0}^{\infty} \frac{\lambda^z}{z!} = e^{\lambda}$$

entonces:

$$\frac{p^x e^{-\nu} \nu^x}{x!} \sum_{z=0}^{\infty} \frac{[(1-p)\nu]^z}{z!} = \frac{p^x e^{-\nu} \nu^x}{x!} \cdot e^{(1-p)\nu}$$

$$\frac{p^x e^{-\nu} \nu^x}{x!} \cdot e^{\nu - \nu p} = \frac{p^x \cancel{e^{-\nu}} \nu^x}{x!} \cancel{e^{\nu}} e^{-\nu p}$$

$$\sum_{y=x}^{\infty} \binom{y}{x} p^x (1-p)^{y-x} \cdot \frac{\nu^y e^{-\nu}}{y!} = \frac{(p\nu)^x e^{-\nu p}}{x!} \quad //$$

Considere dos eventos cualquiera, A y B, tal que $\mathbb{P}(A) = 1/4, \mathbb{P}(B|A) = 1/2$ y $\mathbb{P}(A|B) = 1/4$, indique si las siguientes aseveraciones son verdaderas o falsas. Justifique en ambos casos:

(a) Los eventos A y B son mutuamente excluyentes.

(b) Los eventos A y B son mutuamente independientes.

(c) El evento A está contenido (subconjunto) en B.

(d) $\mathbb{P}(\bar{A}|\bar{B}) = 3/4$

(e) $\mathbb{P}(A|B) + \mathbb{P}(A|\bar{B}) = 1$

Problema 3

$$f(x) = \frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1} e^{-\left(\frac{x}{\eta}\right)^{\beta}} \longrightarrow \int_0^t f(x)\,dx = 1 - \exp\left[-\left(\frac{x}{\eta}\right)^{\beta}\right]$$

Esta integral se desarrolla mediante el método de sustitución:

$$\int_0^t \frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1} e^{-\left(\frac{x}{\eta}\right)^{\beta}} dx \quad , \text{ sustitución: } \quad u = \left(\frac{x}{\eta}\right)^{\beta}$$

$$du = \frac{\beta x^{\beta-1}}{\eta^{\beta}}\,dx \rightarrow dx = \frac{\eta^{\beta}}{\beta x^{\beta-1}}\,du$$

$$\int_0^{\left(\frac{t}{\eta}\right)^{\beta}} \frac{\cancel{\beta}}{\eta} \cdot \frac{\cancel{x^{\beta-1}}}{\eta^{\beta-1}}\, e^{-u}\, \frac{\eta^{\beta}}{\cancel{\beta x^{\beta-1}}}\,du = \int_0^{\left(\frac{t}{\eta}\right)^{\beta}} \frac{\cancel{\eta^{\beta}}}{\cancel{\eta^{\beta}}}\, e^{-u}\,du$$

$$\int_0^{\left(\frac{t}{\eta}\right)^{\beta}} e^{-u}\,du = -e^{-u}\Big|_0^{\left(\frac{t}{\eta}\right)^{\beta}} = -e^{-\left(\frac{t}{\eta}\right)^{\beta}} - (-e^{0})$$

$$-e^{-\left(\frac{t}{\eta}\right)^{\beta}} + 1 = 1 - \exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right] \,/\!/$$