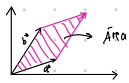
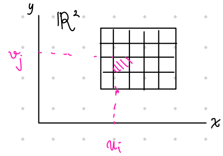
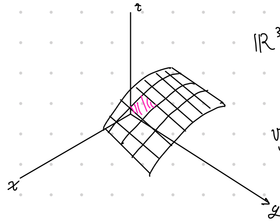
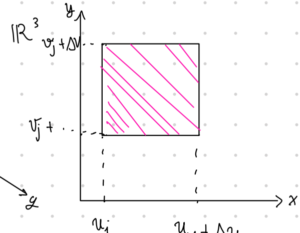
Clase 23



Área $= \| a \times \vec{b} \|$







$S = r(D)$

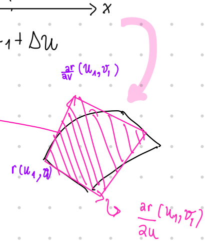
Def: Sea $S \subseteq \mathbb{R}^3$ una superficie suave de parametrización $r: D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ tal que $r(D) = S$ se define como

$$A(S) = \iint \left\| \frac{\partial r}{\partial u}(u, v) \times \frac{\partial r}{\partial v}(u, v) \right\| du\, dv$$

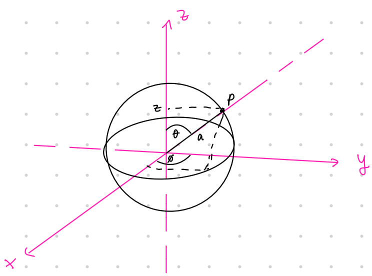
área: $\left\| \frac{\partial r}{\partial v}(u_1, v_1) \times \frac{\partial r}{\partial u}(u_1, v_1) \right\| \Delta u \Delta v$

área total $\approx \sum_i \sum_j \left\| \frac{\partial r}{\partial v}(u_1, v_1) \times \frac{\partial r}{\partial u}(u_1, v_1) \right\| \Delta u \; \Delta v$

$\Delta u, \Delta v \to 0$ $\quad \approx \iint \left\| \frac{\partial r}{\partial v}(u, v) \times \frac{\partial r}{\partial u}(u, v) \right\| du\, dv$



Ejemplo: Encontrar área de la esfera de $\mathbb{R}^3$ centrada en el origen de radio $a > 0$



$r: [0, \pi] \times [0, 2\pi] \longrightarrow \mathbb{R}^3$

$(\theta, \phi) \longrightarrow (a \operatorname{sen}(\theta) \cos(\phi),\ a \operatorname{sen}(\theta) \operatorname{sen}(\phi),\ a \cos(\theta))$

$\frac{\partial r}{\partial \phi}(\theta, \phi) = (-a \operatorname{sen}(\theta) \operatorname{sen}(\phi),\ a \operatorname{sen}(\theta) \cos(\phi),\ 0)$

$\frac{\partial r}{\partial \theta}(\theta, \phi) = (a \cos(\theta) \cos(\phi);\ a \cos\theta \operatorname{sen}\phi,\ -a \operatorname{sen}(\theta))$

$\frac{\partial r}{\partial \theta}(\theta, \phi) \times \frac{\partial r}{\partial \phi}(\theta, \phi) =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\theta \cos\phi & a\cos\theta \operatorname{sen}\phi & -a\operatorname{sen}(\theta) \\ -a\operatorname{sen}\theta \operatorname{sen}\phi & a\operatorname{sen}\theta\cos\phi & 0 \end{vmatrix} = (a^2 \operatorname{sen}^2(\theta)\cos(\phi),\ a^2 \operatorname{sen}^2(\theta) \operatorname{sen}\phi,\ a^2 \cos\theta \operatorname{sen}\theta)$$

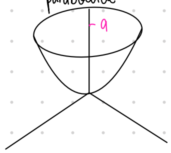
$\Rightarrow \left\| \frac{\partial r}{\partial \theta}(\theta, \phi) \times \frac{\partial r}{\partial \phi}(\theta, \phi) \right\| = \sqrt{a^4 \operatorname{sen}^4(\theta) \cos^2\phi + a^4 \operatorname{sen}^4(\theta) \operatorname{sen}^2\phi + a^4 \cos^2\theta \operatorname{sen}^2(\theta)}$

$\cdots = a^2 \operatorname{sen}(\theta)$

$$\int_0^{\pi}\int_0^{2\pi} a^2 \operatorname{sen}(\theta)\, d\phi\, d\theta = a^2\, 2\pi \int_0^{\pi} \operatorname{sen}(\theta)\, d\theta = 4\pi a^2$$

$-\cos(\theta) \Big|_0^{\pi} = --1 + 1 = 2$

Ejemplo: Calcular el área del paraboloide $z = x^2 + y^2$ con $z \leq 9$



$0 \leq z \leq 9$

$r(x, y) = (x, y, x^2 + y^2)$

$0 \leq x^2 + y^2 \leq 9$

$-3 \leq x \leq 3$

$-\sqrt{9 - x^2} \leq y \leq \sqrt{9 - x^2}$

$\frac{\partial r}{\partial x}(x, y) = (1, 0, 2x)$

$\frac{\partial r}{\partial y}(x, y) = (0, 1, 2y)$

$$\frac{\partial r}{\partial x}(x, y) \times \frac{\partial r}{\partial y}(x, y) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = (-2x, -2y, 1)$$

$\left\| \frac{\partial r}{\partial x}(x, y) \times \frac{\partial r}{\partial y}(x, y) \right\| = \sqrt{4x^2 + 4y^2 + 1}$

área: $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{4x^2 + 4y^2 + 1}\, dy\, dx$

cambiar a polares

$\theta \in [0, 2\pi]$

$r \in [0, 3]$

$x = r\cos\theta$

$y = r \operatorname{sen}(\theta)$

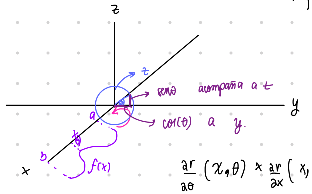
$$\int_0^{2\pi}\int_0^3 \sqrt{4r^2\cos^2(\theta)+4r^2\operatorname{sen}^2(\theta)+1}\; r\, dr\, d\theta = \int_0^{2\pi}\int_0^3 \sqrt{4r^2+1}\; r\, dr\, d\theta = 2\pi\int_0^3 \sqrt{4r^2+1}\; r\,dr = \dots = 2\pi\left(\frac{1}{8}\right)\frac{2}{3}(1+4r^2)^{3/2}\Big|_0^3$$

$u = 4r^2+1$

$du = 8r\,dr$

Ejemplo: Sea $f : [a,b] \longrightarrow (0,+\infty)$

Calcular el área de la superficie de revolución definida por f, en torno al eje x.



$$r(x,\theta) = (x,\, f(x)\cos(\theta),\, f(x)\operatorname{sen}(\theta))$$

$0 \le \theta \le 2\pi$

$a \le x \le b$

$$\frac{\partial r}{\partial \theta}(x,\theta) = (0,\, -f(x)\operatorname{sen}\theta,\, f(x)\cos(\theta))$$

$$\frac{\partial r}{\partial x}(x,\theta) = (1,\, f'(x)\cos\theta,\, f'(x)\operatorname{sen}(\theta))$$

$$\frac{\partial r}{\partial \theta}(x,\theta) \times \frac{\partial r}{\partial x}(x,\theta) = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 0 & -f(x)\operatorname{sen}\theta & f(x)\cos\theta \\ 1 & f'(x)\cos\theta & f'(x)\operatorname{sen}\theta \end{vmatrix} = (-f(x)f'(x),\, f(x)\cos\theta,\, f(x)\operatorname{sen}\theta) = \|\cdots\| = \sqrt{f^2(x)f'(x)^2 + f(x)^2\cos^2(\theta) + f(x)^2\operatorname{sen}^2(\theta)}$$

$$= f(x)\sqrt{f'(x)^2+1}$$

$$\int_a^b\int_0^{2\pi} f(x)\sqrt{f'(x)^2+1}\; d\theta\, dx = 2\pi\int_a^b f(x)\sqrt{f'(x)^2+1}\; dx$$