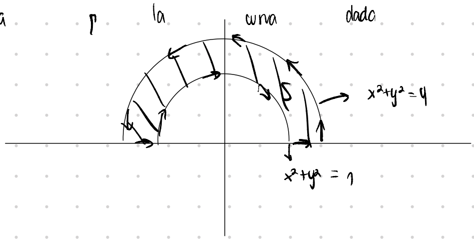
Clase 16

Ejemplo: Sea $F: \mathbb{R}^2 \to \mathbb{R}^2$

$(x, y) \to (y^2, 3xy)$

Sea ρ la curva dada por el borde de la región achurada.



Calcular:

$\int_\rho F$

Sol:

$S = \{(r,\theta) : 0 \leq \theta \leq \pi, \quad 1 \leq r \leq 2\}$

$F_1(x,y) = y^2 \Rightarrow \frac{\partial F_1}{\partial y} = 2y$

$F_2(x,y) = 3xy \Rightarrow \frac{\partial F_2}{\partial x} = 3y$

$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 3y - 2y = y \Rightarrow$ Aplicando Greene:

$$\iint_S y \, dy \, dx = \int_\rho F$$

$$\iint_S y \, dy \, dx = \int_0^\pi \int_1^2 \underbrace{r \operatorname{sen}(\theta)}_{y} \cdot \underbrace{r}_{\text{Jac}} \, dr \, d\theta = \int_0^\pi \int_1^2 r^2 \operatorname{sen}\theta \, dr \, d\theta = \int_0^\pi \operatorname{sen}\theta \left.\frac{r^3}{3}\right|_1^2 d\theta =$$

$$\int_0^\pi \operatorname{sen}(\theta) \left(\frac{8}{3} - \frac{1}{3}\right) = \frac{7}{3} \int_0^\pi \operatorname{sen}\theta \, d\theta = \frac{7}{3} (-\cos(\theta)) \Big|_0^\pi = \frac{14}{3}$$

Recordar ➡ T.C.V

$T: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$, biyectiva, diferenciable

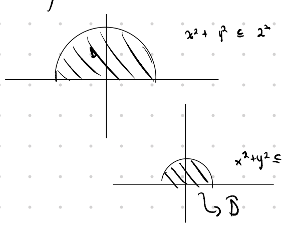
$f: T(\Omega) \to \mathbb{R}$. Entonces

$$\int_{T(\Omega)} f = \int_\Omega (f \circ T) \, |\det J(T)|$$

Caso coordenadas polares

$(r,\theta) \to (r\cos(\theta), r \operatorname{sen}(\theta)) \qquad |J(T)| = r$

Otra forma: sin polares



$$\iint y \, dy \, dx = \iint_D y \, dy \, dx - \iint_{\tilde{D}} y \, dy \, dx = \int_{-2}^2 \int_0^{\sqrt{4-x^2}} y \, dy \, dx - \int_{-1}^1 \int_0^{\sqrt{1-x^2}} y \, dy \, dx$$

$$= \int_{-2}^2 \frac{y^2}{2} \Big|_0^{\sqrt{4-x^2}} dx - \int_{-1}^1 \frac{y^2}{2} \Big|_0^{\sqrt{1-x^2}} = \int_{-2}^2 \frac{(4-x^2)}{2} dx - \int_{-1}^1 \frac{(1-x^2)}{2} dx$$

$$= \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$$

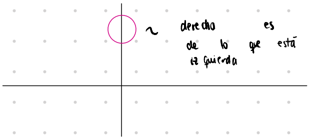
$D = \{(x,y) : -2 \leq x \leq 2, \ 0 \leq y \leq \sqrt{4-x^2}\}$

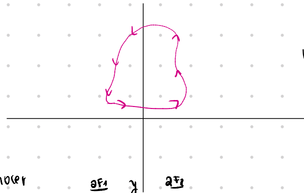
$\tilde{D} = \{(x,y) : -1 \leq x \leq 1, \ 0 \leq y \leq \sqrt{1-x^2}\}$

Ejemplo: Sea $F: \mathbb{R}^2 \to \mathbb{R}^2$

$(x, y) \to \left(2xy^2 + y,\ 2x^2y + x + \frac{x^2}{2}\right)$

Probar que $\int_\Gamma F = 0$, donde Γ es cualquier curva cerrada, simple, (biyectiva) simétrica respecto al eje y





$$\iint Q_x - P_y \, dx\,dy$$

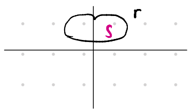
Para usar Green, necesitamos conocer $\frac{\partial F_1}{\partial y}$ y $\frac{\partial F_2}{\partial x}$

$$\frac{\partial F_1}{\partial y} = 4xy + 1 \qquad \frac{\partial F_2}{\partial x} = 4xy + 1 + x$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 4xy + 1 + x - 4xy - 1 = x$$

Aplicando Green, hay que mostrar que

$$\iint_S x \, dx\,dy = 0$$



S simétrica respecto al eje y

$(x, y) \in S_1 \iff (-x, y) \in S_2$ ($T_1\ T_2$)

$$\iint_S x \, dx\,dy = \iint_{S_1} x \, dx\,dy + \iint_{S_2} x \, dx\,dy$$

$T: S_2 \longrightarrow S_1$ biyectiva

$(x, y) \longrightarrow (-x, y)$

$T_1 = -x$

$$\begin{vmatrix} \frac{\partial x}{\partial T_1} & \frac{\partial x}{\partial T_2} \\ \frac{\partial y}{\partial T_1} & \frac{\partial y}{\partial T_2} \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = |-1| = 1$$

$\boxed{\text{Indicación: Teorema del cambio de variable}}$

Cuando $\iint 1 \, dx\,dy \to$ área

T.C.V $\qquad f(x, y) = x$

$$\iint_{S_1 = T(S_2)} x \, dx\,dy = \iint_{S_2} f(T(x,y)) \underbrace{|\det(J T)|}_{1 = |-1|} \, dx\,dy$$

$$= \iint_{S_2} f(-x, y) \, dx\,dy = \iint_{S_2} -x \, dx\,dy \Rightarrow \underbrace{\iint_S x\,dx\,dy}_{= \int_\Gamma F} = \iint_{S_1} x\,dx\,dy + \iint_{S_2} x\,dx\,dy$$

$$= -\iint x\,dx\,dy + \iint x\,dx\,dy$$