Clase 19 EDO

Valores propios complejos

$\vec{x}\,'(t) = A\,\vec{x}(t)$

con $A \in \mathbb{R}^{2\times 2}$ constante

y tenemos

$\lambda_1 = p + q\,i$

un valor propio

com $p, q \in \mathbb{R}$

y $q \neq 0$

y su vector propio asociado es:

$\vec{v}_1 = \vec{a} + \vec{b}\,i$

De álgebra lineal sabemos que el conjugado

$\lambda_2 = \overline{\lambda_1} = p - qi$

Tambien un valor propio

y $\vec{v}_2 = \overline{\vec{v}_1} = \vec{a} - \vec{b}i$

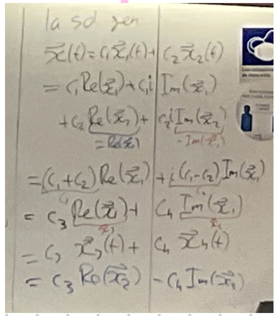
Sabemos que:

$\vec{x}_1(t) = e^{\lambda_1 \cdot t}\,\vec{v}_1$

Satisface la EDO pero es una solución compleja.

$$
\begin{aligned}
\vec{x}_1(t) &= e^{\lambda_1 \cdot t}\,\vec{v}_1 \\
&= e^{(p+qi)t}(\vec{a}+\vec{b}i) \\
&= e^{pt}(\cos(qt) + i\,\mathrm{sen}(qt))(a+bi) \\
&= \underbrace{e^{pt}(\cos(qt)\vec{a} - \mathrm{sen}(qt)\vec{b})}_{\mathrm{Re}\,(x_1)} \\
&\quad + i\,\underbrace{e^{pt}(\mathrm{sen}(qt)\vec{a} + \cos(qt)\vec{b})}_{\mathrm{Im}\,(x_1)}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda_2 \cdot t}\,\vec{v}_2 \\
&= e^{(p-qi)t}(\vec{a}-\vec{b}i) \\
&= e^{pt}(\cos(qt) - \mathrm{sen}(qt)\,i)(\vec{a}-\vec{b}i) \\
&= \underbrace{e^{pt}(\cos(qt)\vec{a} - \mathrm{sen}(qt)\vec{b})}_{\mathrm{Re}(x_2) = \mathrm{Re}(x_1)} + i\,\underbrace{e^{pt}(-\mathrm{sen}(qt)\vec{a} - \cos(qt)\vec{b})}_{\mathrm{Im}(\vec{x}_2) = \mathrm{Im}(\vec{x}_1)}
\end{aligned}
$$



No importa si tomas x_1 y x_2, sus partes reales e imaginarias.

$\Rightarrow$ $\boxed{\text{Sol gen.: } \vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) = c_3\,\mathrm{Re}(\vec{x}_2) - c_4\,\mathrm{Im}(\vec{x}_2)}$

Ejemplo 4

$$x'(t) = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} \vec{x}(t)$$

calcular los valores propios de A. $A\vec{v} = \lambda\vec{v}$

$(A - \lambda I)\vec{v} = \vec{0}$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -1 \\ -4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4$$

$-4 = (1-\lambda)^2$

$\pm\sqrt{4}\,i = 1-\lambda$

$\lambda_1 = 1+2i$

$\lambda_2 = 1-2i$

calcular vector propio λ_1

$A\vec{v}_1 = \lambda_1\vec{v}_1$

$(A - \lambda_1 I)\vec{v}_1 = \vec{0}$

$$\left[\begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} 1+2i & 0 \\ 0 & 1+2i \end{bmatrix}\right]\vec{v}_1 = 0$$

$$\begin{bmatrix} -2i & -1 \\ 4 & -2i \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

multiplicar 1era fila por 2i

$$\begin{bmatrix} 4 & -2i \\ 4 & -2i \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4v_{11} - 2i\,v_{11} = 0$

$2v_{11} - i = v_{21}$

elegimos $v_{11} = i$

entonces $v_{21} = 2$ $\Big\}$ $\begin{bmatrix} i \\ 2 \end{bmatrix}$

La sol gen.

$$
\begin{aligned}
\vec{x}(t) &= c_1\,\mathrm{Re}(e^{\lambda_1 t}\vec{v}_1) + c_2\,\mathrm{Im}(e^{\lambda_1 t}\vec{v}_1) \\
&= c_1\,\mathrm{Re}\left(e^{(1+2i)t}\begin{bmatrix} i \\ 2 \end{bmatrix}\right) + c_2\,\mathrm{Im}\left(e^{(1+2i)t}\begin{bmatrix} i \\ 2 \end{bmatrix}\right) \\
&= c_1\,\mathrm{Re}\left[e^t(\cos(2t) + i\,\mathrm{sen}(2t))\begin{bmatrix} i \\ 2 \end{bmatrix}\right] + c_2\,\mathrm{Im}\left(e^t(\cos(2t) + i\,\mathrm{sen}(2t))\begin{bmatrix} i \\ 2 \end{bmatrix}\right) \\
&= c_1 e^t\,\mathrm{Re}\left(\begin{bmatrix} \cos(2t)i - \mathrm{sen}(2t) \\ 2\cos(2t) + 2i\,\mathrm{sen}(2t) \end{bmatrix}\right) + c_2 e^t\,\mathrm{Im}\left(\begin{bmatrix} \cos(2t)i - \mathrm{sen}(2t) \\ 2\cos(2t) + 2i\,\mathrm{sen}(2t) \end{bmatrix}\right) \\
&= c_1 e^t\begin{bmatrix} -\mathrm{sen}(2t) \\ 2\cos(2t) \end{bmatrix} + c_2 e^t\begin{bmatrix} \cos(2t) \\ 2\,\mathrm{sen}(2t) \end{bmatrix}
\end{aligned}
$$

Ejemplo

$$\vec{x}\,' = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}\vec{x}$$

$$\begin{vmatrix} 2-\lambda & 2 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} = 2-\lambda(2-\lambda)(3-\lambda) = 0$$

$\lambda_1 = \lambda_2 = 3$

$\lambda_3 = 3$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_{21} + v_{31} = 0$

elegimos $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ Además $\vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ un vector propio

$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Ejemplo 3

$\vec{x}\,'(t) = \begin{bmatrix}-2 & 1\\ -1 & -4\end{bmatrix}\vec{x}(t)$

$\begin{vmatrix}-2-\lambda & 1\\ -1 & -4-\lambda\end{vmatrix} = 0$

$(-2-\lambda)(-4-\lambda)+1 = 0$

$(\lambda+3)^2 = 0 \qquad \lambda_1 = \lambda_2 = -3$

$\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{21}\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

$v_{11} + v_{21} = 0$

Elegimos $v_{11} = 1$ $v_{21} = -1$

y $\vec{v}_1 = \begin{bmatrix}1\\ -1\end{bmatrix}$

Es un valor propio incompleto: no hay otro vector propio lineal.

Tenemos una solución:

$\vec{x}_1(t) = e^{\lambda_1 t}\vec{V}_1$

IDEA 1: Buscar la solución como: $\vec{x}_2(t) = t\, e^{\lambda_1 t}\vec{V}_1$

Sustituir la EDO:

Lado izq.

$\frac{d}{dt}\vec{x}_2 = \frac{d}{dt}\left(t\, e^{\lambda_1 t}\vec{V}_1\right)$

$= e^{\lambda_1 t}\vec{V}_1 + t\lambda_1 e^{\lambda_1 t}\vec{V}_1$

$= (1+\lambda_1 \cdot t)\, e^{\lambda_1 \cdot t}$

Lado der

$A\vec{x}_2(t) = A\vec{v}_1 t e^{\lambda_1 t}$

$= \lambda_1 \vec{v}_1 t e^{\lambda_1 t}$

$(1+\lambda_1 t)\, e^{\lambda_1 t}\vec{v}_1 - \lambda_1 t e^{\lambda_1 t}$

$\underbrace{e^{\lambda_1 t}}_{\neq 0}\ \underbrace{\vec{c}}_{\neq 0} = 0$

No sirve

IDEA

$\boxed{\text{Buscar en la forma: } \vec{x}_2 = (t\vec{v}_1 + \vec{u})e^{\lambda t}}$

$\frac{d}{dt}\vec{x}_2(t) = \vec{v}_1 e^{\lambda t} + (t\vec{v}_1 + \vec{u})\lambda_1 e^{\lambda_1 t}$

$\|$

$A\vec{x}_p(t) = (tA\vec{V}_1 + A\vec{u})\, e^{\lambda_1 t}$

$= (t\lambda_1\vec{v}_1 + A\vec{u})\, e^{\lambda_1 t}$

$\vec{V}_1 e^{\lambda_1 t} + (t\vec{v}_1 + \vec{u})\lambda_1 e^{\lambda_1 t}$

$-\left(t\lambda_1\vec{v}_1 + A(\vec{w})\right) e^{\lambda_1 t}$

$\vec{v}_1 + \vec{w}\lambda_1 = A\vec{w}$

$(A - \lambda_1 I)w = \vec{v}_1$