Pontificia Universidad Católica de Chile
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Métodos Estadísticos
EYP2405
Pauta I1

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1) a) (5*ptos.*) Si $Y \sim U(0,\theta)$, entonces $E(Y) = \frac{\theta}{2}$ y $Var(Y) = \frac{\theta^2}{12}$. Igualando los momentos muestrales y poblacionales: $\bar{Y} = \frac{\theta}{2} \Rightarrow$ que el Estimador de Momentos de θ es $\tilde{\theta} = 2\bar{Y}$.

i) $E(\tilde{\theta}) = 2E(\bar{Y}) = 2 \cdot \frac{\theta}{2} = \theta$.

ii) $Var(\tilde{\theta}) = 4Var(\bar{Y}) = 4 \cdot \frac{\theta^2}{12n} = \frac{\theta^2}{3n}$.

b) (5*ptos.*)

$$\begin{aligned} L(\theta) &= \frac{1}{\theta^n}\prod_{i=1}^{n} I_{[0,\theta]}(y_i) \\ &= \frac{1}{\theta^n} I_{[Y_{(n)},\infty](\theta)} \end{aligned}$$

$\Rightarrow Y_{(n)}$ es el EMV de θ.

Sabemos que:

$$\begin{aligned} g_n(y) &= n[F(y)]^{n-1}f(y), \quad y \leq \theta \\ &= n\left(\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta}, \quad 0 \leq y \leq \theta \end{aligned}$$

Finalmente calcularemos su media:

$$\begin{aligned} E(Y_{(n)}) &= \frac{n}{\theta^n}\int_0^{\theta} y^n dy = \frac{n}{\theta^n}\frac{\theta^{n+1}}{n+1} \\ &= \left(\frac{n}{n+1}\right)\theta \end{aligned}$$

c) (5*ptos.*) Sabemos que:

$$\begin{aligned} g_{1n}(y_1,y_n) &= n(n-1)[F(y_n)-F(y_1)]^{n-2}f(y_1)f(y_n), \quad y_1 \leq y_n \\ &= n(n-1)(y_n-y_1)^{n-2}, \quad 0 \leq y_1 \leq y_n \leq 1 \end{aligned}$$

Ya que ahora $\theta = 1; U(0,1)$.

d) (5*ptos.*) Tenemos que $R = Y_n - Y_1$ y

$$g_{1n}(y_1, y_n) = n(n-1)(y_n - y_1)^{n-2}, \qquad y_1 \leq y_n$$

Sea

$$\begin{aligned} S &= Y_{(n)} \\ r &= y_n - y_1 \\ s &= y_n \end{aligned}$$

de donde

$$\begin{aligned} y_1 &= s - r \\ y_n &= s \end{aligned}$$

$$\Rightarrow J = \begin{vmatrix} \frac{\partial y_1}{\partial r} & \frac{\partial y_1}{\partial s} \\ \\ \frac{\partial y_n}{\partial r} & \frac{\partial y_n}{\partial s} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ \\ 0 & 1 \end{vmatrix} = 1$$

$\therefore f_{r,s} = n(n-1)r^{n-2}, \quad 0 \leq r \leq s \leq 1$

Finalmente:

$$\begin{aligned} f_R(r) &= \int_r^1 n(n-1)r^{n-2}ds \\ &= n(n-1)r^{n-2}(1-r), \qquad 0 \leq r \leq 1 \end{aligned}$$

2) a) (5*ptos.*) El modelo estadistico esta dado por:

$$\mathfrak{F} = \{f(\cdot, \theta), \theta \in \Omega\}$$

con:

$$\begin{aligned} f(y, \theta) &= f(y_{11}, ..., y_{1n_1}, y_{21}, ..., y_{2n_2}; \theta) \\ &= \prod_{i=1}^{2} \prod_{j=1}^{n_i} f(y_{ij}; \theta) \\ &= \prod_{i=1}^{2} \prod_{j=1}^{n_i} \frac{1}{2\pi\sigma^2} exp\left\{-\frac{1}{2\sigma^2}\left(y_{ij} - \mu_i\right)^2\right\} \end{aligned}$$

b) (5*ptos.*) Sabemos que:

i) $\frac{(n_1-1)S_1^2}{\sigma^2} \sim \chi^2(n_1 - 1)$

ii) $\frac{(n_2-1)S_2^2}{\sigma^2} \sim \chi^2(n_2-1)$

donde $i)$ y $ii)$ son independientes.

Luego la *v.a.*

$$\begin{aligned} F &= \frac{\frac{(n_1-1)S_1^2}{\sigma^2(n_1-1)}}{\frac{(n_2-1)S_2^2}{\sigma^2(n_2-1)}} \\ &= \frac{S_1^2}{S_2^2} \sim F(n_1-1, n_2-1) \end{aligned}$$

$\therefore$

$$\begin{aligned} P\left(\frac{S_1^2}{S_2^2} > c\right) &= P\left(F > c\right) \\ &= 1 - P\left(F \leq c\right) \end{aligned}$$

c) (5*ptos.*) La verosimilitud y la log-verosimilitud estan dadas por:

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{2}\prod_{j=1}^{n_i} \frac{1}{\sqrt{2\pi\sigma^2}} exp\left\{-\frac{1}{2\sigma^2}(y_{ij}-\mu_i)^2\right\} \\ l(\theta) &= \sum_{i=1}^{2}\sum_{j=1}^{n_i}\left\{-\frac{1}{2}log(2\pi) - \frac{1}{2}log(\sigma^2) - \frac{1}{2\sigma^2}(y_{ij}-\mu_i)^2\right\} \end{aligned}$$

de donde:

$$\begin{aligned} \frac{\partial l(\theta)}{\partial \mu_1} &= \sum_{j=1}^{n_1}\left\{-\frac{2}{2\sigma^2}\left(y_{1j}-\mu_1\right)(-1)\right\} \\ &= \frac{1}{\sigma^2}\sum_{j=1}^{n_1}\left(y_{1j}-\mu_1\right) \\ \frac{\partial l(\theta)}{\partial \mu_2} &= \frac{1}{\sigma^2}\sum_{j=1}^{n_2}\left(y_{2j}-\mu_2\right) \\ \frac{\partial l(\theta)}{\partial \sigma^2} &= \sum_{i=1}^{2}\sum_{j=1}^{n_i}\left\{-\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}\left(y_{ij}-\mu_i\right)^2\right\} \\ &= \frac{1}{2\sigma^2}\left\{-n + \frac{1}{\sigma^2}\sum_{i=1}^{2}\sum_{j=1}^{n_i}\left(y_{ij}-\mu_i\right)^2\right\} \end{aligned}$$

con $n = n_1 + n_2$.

Finalmente:

$$\frac{\partial l(\theta)}{\partial \mu_1} = 0 \quad \Rightarrow \quad \hat{\mu_1} = \bar{Y_1}$$

$$\frac{\partial l(\theta)}{\partial \mu_2} = 0 \quad \Rightarrow \quad \hat{\mu_2} = \bar{Y_2}$$

$$\frac{\partial l(\theta)}{\partial \sigma_2} = 0 \quad \Rightarrow \quad \hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{2}\sum_{j=1}^{n_i}\left(Y_{ij} - \bar{Y_i}\right)^2$$

d) ($5ptos.$) Se tiene que:

$$\begin{aligned} \frac{n\hat{\sigma^2}}{\sigma^2} &= \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{\sigma^2} \\ &= \underbrace{\frac{(n_1-1)S_1^2}{\sigma^2}}_{\chi^2(n_1-1)} + \underbrace{\frac{(n_2-1)S_2^2}{\sigma^2}}_{\chi^2(n_2-1)} \end{aligned}$$

$\Rightarrow \frac{n\hat{\sigma^2}}{\sigma^2} \sim \chi^2(n_1+n_2-2)$

3) a) ($12ptos.$) Aqui tenemos:

$$\Omega = \{2,3\} \times \left\{\frac{1}{2}, \frac{1}{3}\right\} = \left\{\left(2, \frac{1}{2}\right), \left(2, \frac{1}{3}\right), \left(3, \frac{1}{2}\right), \left(3, \frac{1}{3}\right)\right\}$$

donde:

$$L(\theta) = L(n, \pi) = \begin{pmatrix} n \\ y \end{pmatrix} \pi^y (1-\pi)^{n-y}$$

Debemos analizar $L \quad \forall \ \theta \in \Omega$. Consideremos la siguiente tabla:

y	(2, 1/2)	(2, 1/3)	(3, 1/2)	(3, 1/3)
0	1/4	4/9	1/8	8/27
1	1/2	4/9	3/8	12/27
2	1/4	1/9	3/8	6/27
3	0	0	1/8	1/27

Luego:

$$\hat{\theta} = (\hat{n}, \hat{\pi}) = \begin{cases} (2, 1/3), & \text{si y=0;} \\ (2, 1/2), & \text{si y=1;} \\ (3, 1/2), & \text{si y=2;} \\ (3, 1/3), & \text{si y=3.} \end{cases}$$

b) (8$ptos.$) Tenemos que:

$$E(Y) = \theta y_0^{\theta} \int_{y_0}^{\infty} y^{-\theta} dy = \frac{\theta y_0}{\theta - 1}$$

de donde $\bar{Y} = \frac{\theta y_0}{\theta - 1}$

$\Rightarrow$ que el estimador de momentos de θ es:

$$\tilde{\theta} = \frac{\bar{Y}}{\bar{Y} - y_0}$$