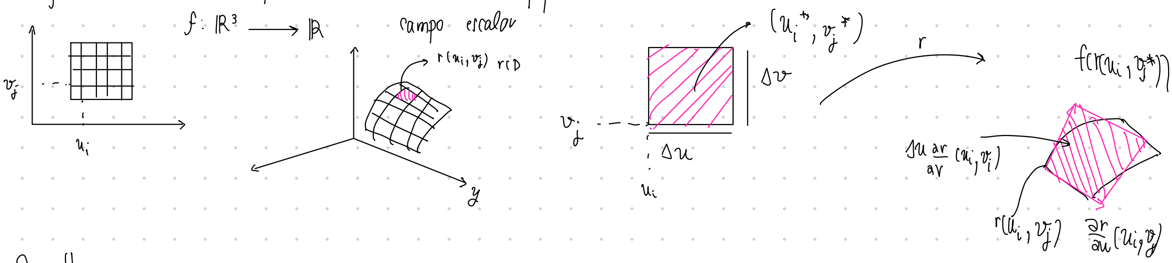
Clase 24

Integral de un campo escalar sobre una superficie

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ campo escalar



$$\approx \left\| \Delta v \frac{\partial r}{\partial v}(u_i, v_j) \times \Delta u \frac{\partial r}{\partial u}(u_i, v_j) \right\| f(r(u_i^*, v_j))$$

$$= \left\| \frac{\partial r}{\partial v}(u_i, v_j) \times \frac{\partial r}{\partial u}(u_i, v_j) \right\| f(r(u_i^*, v_j^*))\, \Delta v \Delta u$$

$$\sum_i \sum_j \left\| \frac{\partial r}{\partial v}(u_i, v_i) \times \frac{\partial r}{\partial u}(u_i, v_i) \right\| f(r(u_i^*, v_i^*))\, \Delta u\, \Delta v \longrightarrow \iint \left\| \frac{\partial r}{\partial u}(u_i, v_j) \right\| f(u, v)\, du\, dv$$

Def: Sea $S \in \mathbb{R}^3$ una superficie suave dada por una parametrización $r: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $r(D) = S$

La integral del campo escalar $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ sobre la superficie S, se define como

$$\iint_S F\,ds = \boxed{\iint_D \left\| \frac{\partial r}{\partial u}(u, v) \times \frac{\partial r}{\partial v}(u, v) \right\| f(r(u, v))\, du\, dv}$$

Recuerdo:

$$\int_\Gamma F\, dr = \int_a^b \left\| \frac{dr(t)}{dt} \right\| f(r(t))\, dt$$

Obs: Si S es una lámina tq la densidad de masa en $r(u, v)$ está dada por $f(r(u, v))$ entonces $\iint_S f\,ds$ es la masa de la lámina

Obs: $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$

$(x, y, z) \longrightarrow f(x, y, z) = 1$

$$\iint_S 1\, ds = \iint_D \left\| \frac{\partial r}{\partial u}(u, v) \times \frac{\partial r}{\partial v}(u, v) \right\| \cdot 1\, du\, dv = A(S)$$

Ejemplo: Calcular $\iint_S x^2\, ds$, donde S es la esfera dada por la ecuación $x^2 + y^2 + z^2 = 1$

Sol: El campo escalar s $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$

$(x, y, z) \longrightarrow x^2$

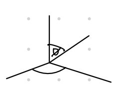
Parametrizando S

$r(\phi, \theta) = (\cos(\theta) \operatorname{sen} \phi, \operatorname{sen}(\theta) \operatorname{sen} \phi, \cos \phi)$

$\phi \in [0, \pi]$

$\theta \in [0, 2\pi]$

$$\left\| \frac{\partial r}{\partial \theta}(\phi, \theta) \times \frac{\partial r}{\partial \phi}(\phi, \theta) \right\| = \operatorname{sen} \phi$$



$$\iint_S x^2\, ds = \int_0^\pi \int_0^{2\pi} \operatorname{sen}(\phi)\, f(r(\phi, \theta))\, d\theta\, d\phi$$

$$f(r(\phi,\theta)) = f(\underbrace{\cos\theta \operatorname{sen}\phi}_{x}, \operatorname{sen}\theta \operatorname{sen}\phi, \cos(\theta)) = \cos^2 \operatorname{sen}^2\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \operatorname{sen}\phi \cos^2(\theta) \operatorname{sen}^2\phi \, d\theta \, d\phi = \int_0^{\pi}\int_0^{2\pi} \cos^2\theta \operatorname{sen}^3\phi \, d\theta \, d\phi = \frac{4\pi}{3}$$

Ejemplo: Calcular $\iint y \, ds$, donde S es la superficie dada por

$z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$

SOL: $f(x,y,z) = y$

Parametrizando S $\quad r(x,y) = (x, y, x+y^2)$ $\quad x \in [0,1]$ $\quad y \in [0,2]$

$$\frac{\partial r}{\partial x}(x,y) = (1,0,1)$$

$$\frac{\partial r}{\partial y}(x,y) = (0,1,2y)$$

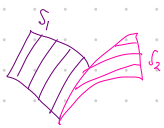
$$\frac{\partial r}{\partial x}(x,y) \times \frac{\partial r}{\partial y}(x,y) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 0 & 1 & 2y \end{vmatrix} = (-1,-2y,1)$$

$$\left\| \frac{\partial r}{\partial x}(x,y) \times \frac{\partial r}{\partial y}(x,y) \right\| = \sqrt{2+4y^2}$$

$$f(r(x,y)) = f(x,y,x+y^2) = y$$

$$\int_0^1\int_0^2 y\sqrt{2+4y^2} \, dy \, dx = 1\left(\int_0^2 \sqrt{2+4y^2} \, y \, dy\right) = \left.\frac{(2+4y^2)^{3/2}}{12}\right|_0^2 = \frac{13}{3}\sqrt{2}$$

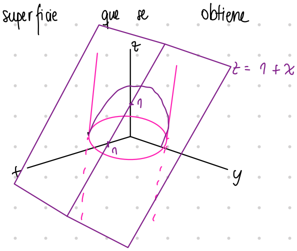
$$\frac{1}{12}\left[\frac{3}{2}(2+4y^2)^{1/2}\right) 8y$$

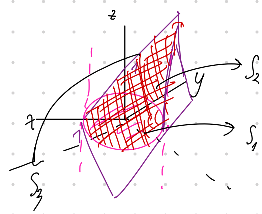


$s = S_1 \cup S_2$

$$\iint_S F \, ds = \iint_{S_1} f \, ds + \iint_{S_2} f \, ds$$

Ejemplo: Sea S la superficie que se obtiene al intersectar al cilindro $x^2+y^2=1$ con el plano $z=0$ y $z = 1+x$





Calcular $\iint z \, ds$

Sol: $$\iint_S z \, ds = \iint_{S_1} z \, ds + \iint_{S_2} z \, ds + \iint_{S_3} z \, ds$$

Parametrizar S_1, S_2 y S_3 de manera independiente, y luego calcular las integrales respectivas