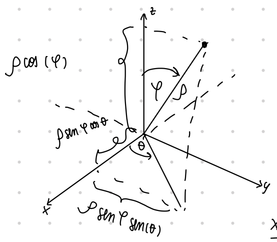
Problema 1

Parametrice las siguientes superficies:

(a) Esfera de ecuación $x^2 + y^2 + z^2 = R^2$, tal que $x \leq 0, z \geq 0$



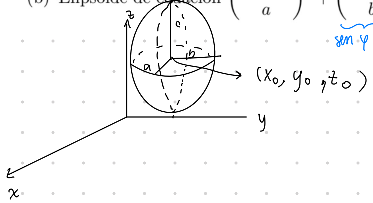
$x = \rho \operatorname{sen}\varphi \cos\theta \Rightarrow x = R \operatorname{sen}\varphi \cos\theta \qquad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$

$y = \rho \operatorname{sen}\varphi \operatorname{sen}(\theta) \qquad y = R \operatorname{sen}\varphi \operatorname{sen}\theta \qquad 0 \leq \varphi \leq \pi/2$

$z = \rho \cos(\varphi) \qquad z = R \cos\varphi$

$\frac{x - x_0}{a} = \operatorname{sen}\varphi \cos\theta$

(b) Elipsoide de ecuación $\left(\frac{x - x_0}{a}\right)^2 + \underbrace{\left(\frac{y - y_0}{b}\right)^2}_{\operatorname{sen}\varphi \operatorname{sen}\theta} + \overbrace{\left(\frac{z - z_0}{c}\right)^2}^{\cos\varphi} = 1$



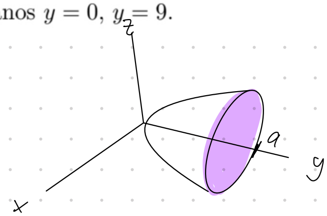
$x = a \operatorname{sen}\varphi \cos\theta + x_0 \qquad \varphi \in [0, \pi]$

$y = b \operatorname{sen}\varphi \operatorname{sen}\theta + y_0 \qquad \theta \in [0, 2\pi]$

$z = c \cos\varphi + z_0$

Problema 2

Un paraboloide es una superficie que nace de rotar una parábola en torno a su eje de simetría. Parametrice el paraboloide generado al rotar la función $y = x^2$ contenida en el plano XY en torno al eje y. A partir de la parametrización, calcule el area de la porción del paraboloide que queda entre los planos $y = 0$, $y = 9$.



Parametrización $= (t, t^2, 0)$

(L) rotando

$= (t\cos(\theta), t^2, t \operatorname{sen}(\theta))$

$\theta \in [0, 2\pi] \qquad t \in [0, 3]$

$\frac{\partial r}{\partial t} = (\cos(\theta), 2t, \operatorname{sen}(\theta))$

$\frac{\partial r}{\partial \theta} = (-t\operatorname{sen}(\theta), 0, t\cos\theta)$

$A(S) = \iint \left\| \frac{\partial r}{\partial u}(u,v) \times \frac{\partial r}{\partial v}(u,v) \right\| du\, dv$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos(\theta) & 2t & \operatorname{sen}(\theta) \\ -t\operatorname{sen}(\theta) & 0 & t\cos\theta \end{vmatrix} = \hat{i}\left(2t^2\cos\theta\right) - \hat{j}\left(t\cos^2\theta + t\operatorname{sen}^2\theta\right) + \hat{k}\left(2t^2\operatorname{sen}\theta\right)$$

$$= \left\| \begin{pmatrix} 2t^2\cos\theta \\ -t \\ 2t^2\operatorname{sen}\theta \end{pmatrix} \right\| = \sqrt{4t^4 + t^2}$$

$$\int_0^3 \int_0^{2\pi} \sqrt{4t^4 + t^2}\, d\theta\, dt = 2\pi \int_0^3 |t| \sqrt{4t^2 + 1}\, dt =$$

$u = 4t^2 + 1 \qquad du = 8t\, dt$

$$\frac{2\pi}{8} \int_1^{37} u^{1/2}\, du = \frac{\pi}{4} \left[\frac{u^{3/2}}{3/2}\right]_1^{37} = \frac{\pi}{6}\left[37^{3/2} - 1\right]$$

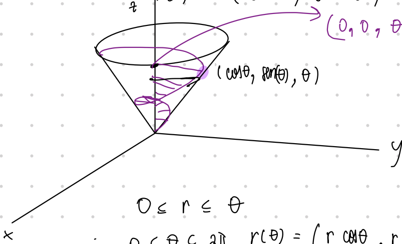
Problema 3

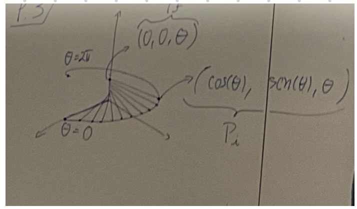
Considere la hélice $(\cos(\theta), \sin(\theta), \theta)$ con $0 \leq \theta \leq 2\pi$. Sea U la superficie que se obtiene al unir cada punto de la hélice horizontalmente con el eje z.

(a) Encuentre una parametrización para U

(b) Exprese el área de U como una integral de una variable

$r(t) = (\cos\theta, \operatorname{sen}\theta, \theta)$ $\qquad 0 \leq \theta \leq 2\pi$





$0 \leq r \leq \theta$

$0 \leq \theta \leq 2\pi \quad r(\theta) = (r\cos\theta, r\operatorname{sen}\theta, \theta)$

$r(\theta, t) = (1-t)\, P_i(\theta) + t\, P_f(\theta)$

$= (1-t)\left(\cos(t), \operatorname{sen}(t), \theta\right) + t(0, 0, \theta)$

$r(t) = \left((1-t)\operatorname{sen}\theta, (1-t)\operatorname{sen}\theta, \theta\right)$

$t \in [0, 1] \qquad \theta \in [0, 2\pi]$

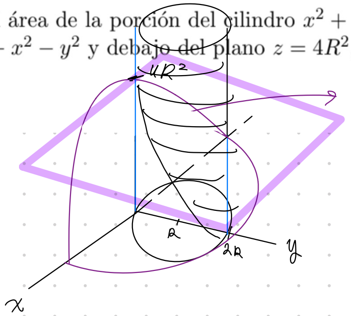
$\frac{\partial r}{\partial t} = (-\cos(\theta), -\operatorname{sen}(\theta), 0) \qquad \frac{\partial r}{\partial \theta} = \left(-(1-t)\operatorname{sen}\theta, (1-t)\cos\theta, 1\right)$

$\frac{\partial r}{\partial t} \times \frac{\partial r}{\partial \theta} = \hat{i}(-\operatorname{sen}\theta) - \hat{j}(-\cos\theta) + \hat{k}(-(1-t))$

$\left\| \frac{\partial r}{\partial t} \times \frac{\partial r}{\partial \theta} \right\| = \sqrt{1 + (1-t)^2} \qquad A = \int_0^{2\pi}\int_0^1 \sqrt{1+(1-t)^2}\, dt\, d\theta = 2\pi \int_0^1 \sqrt{1+(1-t)^2}\, dt$

Problema 4

Calcule el área de la porción del cilindro $x^2 + (y-R)^2 = R^2$ que queda por fuera del paraboloide $z = 4R^2 - x^2 - y^2$ y debajo del plano $z = 4R^2$, con $R > 0$.



$z = 0 \Rightarrow x^2 + y^2 = 4R^2$

$r = 2R$

$x = R\cos(t)$

$y = R + R\operatorname{sen}(t)$

$z = z$

$t \in [0, 2\pi]$

$z \in [4R^2 - x^2 - y^2, 4R^2]$

$$4R^2 - x^2 - y^2 = 4R^2 - (R\cos(t))^2 - (R\,sen(t) + R)^2 = 4R^2 - R^2 - 2R^2 sen(t) - R^2$$

$$= 2R^2 - 2R^2\, sen(t)$$

$$= 2R^2\ (1 - sen(t))$$

$$\Rightarrow z \in [2R^2(1 - sen(t)), 4R^2]$$

$$r_t = (-R\,sen(t), R\cos(t), 0)$$

$$r_z = (0, 0, 1)$$

$$r_t \times r_z = \hat{i}(R\cos(t)) - \hat{j}(-R\,sen(t)) + \hat{k}(0)$$

$$= \begin{pmatrix} R\cos(t) \\ -R\,sen(t) \\ 0 \end{pmatrix} \Rightarrow \| \| = \sqrt{R^2} = R$$

$$A = \int_0^{2\pi} \int_{2R^2(1-sen(t))}^{4R^2} R\ dz\,dt =$$

$$R \int_0^{2\pi} 4R^2 - 2R^2 + 2R^2\, sen(t)\ dt$$

$$R \int_0^{2\pi} 2R^2 + 2R^2 sen(t) dt = R \cdot 2R^2 \int_0^{2\pi} 1 + sen(t) dt$$

$$= R^3 \cdot 2 \cdot 2\pi = 4\pi R^3$$