Clase 18

$\text{rot}(F) = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$

$\text{rot}(F): \mathbb{R}^3 \to \mathbb{R}^3$

$F = (F_1, F_2, F_3)$

$= \left(\frac{\partial}{\partial y} F_3 - \frac{\partial}{\partial z} F_2, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$

$F: U \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, F conservativo $\iff$ $\text{Rot}(F) = 0$

U: sin hoyos

Para $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad F = (F_1, F_2)$ definimos $\tilde{F}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x, y, z) \longrightarrow (F_1(x,y), F_2(x,y), 0)$

$\boxed{\text{Rot}(\tilde{F}) = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k}}$

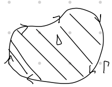
$= \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$

Reescribir Teorema de Green en términos del rotor:

$F: \mathbb{R}^2 \to \mathbb{R}^2$, $F = (F_1, F_2)$ definimos $\text{Rot}(F)$ como $\text{Rot}(\tilde{F})$. Con $\tilde{F}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x, y, z) \longrightarrow (F_1(x,y), F_2(x,y), 0)$

Teo Green

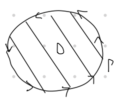


$F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\int_\Gamma F\,dr = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dx\,dy$$

Forma vectorial del teo de green

Sea $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ un campo vectorial con las condiciones para aplicar Teo de Green Γ curva cerrada, simple, suave por pedazos.



$\text{rot}(\tilde{F}) = \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$ Observar que $\text{rot}(\tilde{F}) \cdot (0,0,1) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \text{rot}(\tilde{F}) \cdot \hat{k}$

Teo de Green forma vectorial

$$\int_\Gamma F\,dr = \iint_D \text{rot}(F)\,\hat{k}\,dA \qquad dA = dx\,dy$$

Notación:

$F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $F = (P, Q, R) = P\hat{i} + Q\hat{j} + R\hat{z}$ ($P = F_1$, $Q = F_2$, $R = F_3$)

$r: [a, b] \longrightarrow \mathbb{R}^3$

$t \longrightarrow (x(t), y(t), z(t))$

parametrización de curva Γ.

$$\int_\Gamma F\,dr = \int_a^b \langle F(r(t)), r'(t)\rangle\,dt = \int_a^b \langle F(x(t), y(t), z(t)), (x'(t), y'(t), z'(t))\rangle\,dt =$$

$$= \int_a^b \langle (P(x(t), y(t), z(t)), Q(x(t), y(t), z(t)), R(x(t), y(t), z(t))), (x'(t), y'(t), z'(t))\rangle\,dt = \int_a^b P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)\,dt$$

$$= \int_a^b P(x(t), y(t), z(t))x'(t)\,dt + \int_a^b Q(x(t), y(t), z(t))y'(t)\,dt + \int_a^b R(x(t), y(t), z(t))z'(t)\,dt =$$

* $x'(t)\,dt = \frac{d(x(t))}{dt}\,dt = dx$

$P(x(t), y(t), z(t)) = P(x, y, z)$

$$\boxed{\int_\Gamma F\,dr = \int_a^b P(x,y,z)\,dx + \int_a^b Q(x,y,z)\,dy + \int_a^b R(x,y,z)\,dz = \int_a^b P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz}$$

Green

$F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$F: (P, Q)$

$$\oint_\Gamma F\,dr = \oint P\,dx + Q\,dy \overset{\text{Green}}{=} \iint_D \text{rot}(F)\,\hat{k}\,dA = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

Definimos la divergencia de F como: $\operatorname{div}(F): \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$(x,y,z) \longrightarrow \frac{\partial}{\partial x}P(x,y,z) + \frac{\partial Q}{\partial y}(x,y,z) + \frac{\partial R}{\partial z}(x,y,z)$$

$$\operatorname{div}(F) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \left\langle \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), (P, Q, R) \right\rangle = \nabla \cdot F$$

$$\boxed{\begin{array}{l} \operatorname{Rot}(F) = \nabla \times F \\ \operatorname{Div} = \nabla \cdot f \end{array}}$$

Ejercicio: $\operatorname{div}(\operatorname{Rot}(F)) = 0$

Forma normal del teo de Green

$F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$r: [a,b] \longrightarrow \mathbb{R}^2$

$t \to (x(t), y(t))$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \left(\frac{x'(t)}{\|r'(t)\|}, \frac{y'(t)}{\|r'(t)\|} \right)$$

como estamos en $\mathbb{R}^2$

$$n(t) = \left(\frac{y'(t)}{\|r'(t)\|}, \frac{-x'(t)}{\|r'(t)\|} \right)$$

$$\|r'(t)\| = \sqrt{x'(t)^2 + y'(t)^2}$$

$\Rightarrow \|n(t)\| = 1$

$\langle T(t), n(t) \rangle = 0 \Rightarrow n(t) \perp T(t)$

Queremos calcular la integral de $\underbrace{F \cdot n}_{\langle F, n\rangle}$ sobre Γ.

$$\int_\Gamma F \cdot n \, ds = \int_a^b \langle F(r(t)), n(t) \rangle \|r'(t)\| \, dt$$

$$\int_a^b \left\langle F(x(t), y(t)), \left(\frac{y'(t)}{\|r'(t)\|}, \frac{-x'(t)}{\|r'(t)\|} \right) \right\rangle \|r'(t)\| \, dt$$

$$= \int_a^b \left\langle (P(x,y), Q(x,y)), \left(\frac{y'(t)}{\|r'(t)\|}, -\frac{x(t)}{\|r'(t)\|} \right) \right\rangle \|r'(t)\| \, dt$$

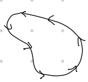
$$= \int_a^b \langle (P(x,y), Q(x,y)), (y'(t), -x'(t)) \rangle \, dt$$

$$= \int_a^b \left[P(x,y)\, y'(t) - Q(x,y)\, x'(t) \right] = \int_a^b \left[P(x,y)\, \underbrace{dy}_{\text{"Q"}} - Q(x,y)\, \underbrace{dx}_{\text{"P"}} \right]$$

$$= \int_\Gamma (-Q, P)\, dr = \iint \frac{\partial P}{\partial x} - \left(-\frac{\partial Q}{\partial y} \right) dA = \iint \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \, dA$$

$Q_x - P_y$

Forma normal de Green:



$F = (P, Q)$

$$\boxed{\int_\Gamma F \cdot n \, ds = \iint_R \operatorname{div}(F) \, dA}$$

$$\boxed{\int_\Gamma F \, dr = \iint \operatorname{rot}(F)\, k \, dA}$$