Problema 1

Considere una muestra aleatoria $\boldsymbol{X}$ proveniente de una distribución Beta(α, β), es decir,

$$f_{\boldsymbol{\theta}}(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \mathbb{1}_{[0,1]}(x)$$

Encuentre un estadístico completo

$$f_{\boldsymbol{\theta}}(x) = h(x)c(\boldsymbol{\theta}) \exp\left\{\sum_{j=1}^{k} \omega_j(\boldsymbol{\theta}) t_j(x)\right\},$$

$$\frac{1}{B(\alpha,\beta)} \exp\left\{ \ln \left\{ x^{\alpha-1} (1-x)^{\beta-1} \right\} \right\}$$

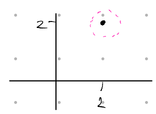
$$= \frac{1}{B(\alpha,\beta)} \exp\left\{ \log(x^{\alpha-1}) + \log((1-x)^{\beta-1}) \right\} I_{[0,1]}(x)$$

$$= \frac{1}{B(\alpha,\beta)} \exp\left\{ \underbrace{(\alpha-1)}_{w_1} \underbrace{\log(x)}_{t_1} + \underbrace{(\beta-1)}_{w_2} \underbrace{\log(1-x)}_{t_2} \right\}$$

$$T(\underset{\sim}{x}) = \left(\sum_{i=1}^{n} \log(x_i), \sum_{i=1}^{n} \log(1-X_i) \right) \rightarrow \text{suficiente,}$$

para que sea completo debe ser abierto

$$\left\{ (\alpha-1), (\beta-1) ; \alpha, \beta > 0 \right\}$$



$\alpha, \beta = 2 \quad (1,1) = \delta$

Bola

$$B_{0,1}(\delta) = \left\{ (x,y) \in \mathbb{R}^2 : \sqrt{(1-x)^2 + (1-y)^2} < 0{,}1 \right\}$$

Radio de 0,1

Problema 2 **(Casella y Berger, 2002)**

Sean $X_1, \ldots, X_n \overset{\text{iid}}{\sim} Unif(0,\theta), \theta > 0$. Muestre que $X_{(n)} = \max\{X_1, \ldots, X_n\}$ es un estadístico completo.

$$F(y) = P(Y \leq y) = P(X_1 \leq y, \ldots, X_n \leq y) \quad \text{por iid}$$

$$= P(X_1 \leq y) \cdots P(X_n \leq y)$$

$$= [F_{(y)}]^n$$

distribuye uniforme

$$f(y) = F'(y)$$

$$= n\left[F_X(y)\right]^{n-1} f_X(y)$$

$$= n \left[\frac{y}{\theta}\right]^{n-1} \cdot \frac{1}{\theta}$$

$$= n\, y^{n-1} \theta^{-n}$$

Para ser completo

$$E_\theta[g(T)] = 0 \iff P(g(t) = 0) = 1$$

función g,

$$\mathrm{E}_{\boldsymbol{\theta}}(g(T(\boldsymbol{X}))) = 0, \quad \forall \boldsymbol{\theta} \in \Theta$$

ssi

$$P_{\boldsymbol{\theta}}(g(T(\boldsymbol{X})) = 0) = 1, \quad \forall \boldsymbol{\theta} \in \Theta.$$

$$\mathsf{E}[g(X)] = \begin{cases} \sum_{x \in \Theta_X} g(x) \cdot p_X(x) \\ \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx \end{cases}$$

$$f_\theta(t) = n t^{n-1} \theta^{-n}$$

$$E_\theta[g(t)] = 0$$

$$\frac{d}{d\theta} \int g(t)\, t^{n-1} \theta^{-n} n = 0$$

truco

$$\frac{d}{d\theta}\left(\theta^{-n} \int_0^\theta g(t)\, n\, t^{n-1}\, dt \right)$$

$$\Rightarrow \underbrace{n\theta^{-n-1} \int g(t)\, n\, t^{n-1}\, dt}_{} + \theta^{-n} \frac{d}{d\theta} \int_0^\theta g(t) n t^{n-1} dt$$

$$-n\theta^{-1} \underbrace{\int^\theta \theta^{-n} g(t)\, n\, t^{n-1}\, dt}_{\text{Esperanza que por hipótesis es 0}} = 0$$

$$= \theta^{-n} n\, g(\theta)\, \theta^{n-1} = 0$$

como $\theta \neq 0$ y $n \neq 0 \Rightarrow g(\theta) = 0 \quad \forall \theta \Rightarrow P(g(\theta) = 0) = 1$

Problema 3 (Casella y Berger, 2002)

$E(X^k) = E\left[\left(\frac{X}{Y}\right)^k Y^k\right] = E\left[\left(\frac{X}{Y}\right)^k\right] \cdot E[Y^k]$

$\Rightarrow \frac{E[X^k]}{E(Y^k)} = E\left[\left(\frac{X}{Y}\right)^k\right]$

a) Muestre que si X/Y e Y son variables aleatorias independientes, entonces

$$\mathbb{E}\left[\left(\frac{X}{Y}\right)^k\right] = \frac{\mathbb{E}[X^k]}{\mathbb{E}[Y^k]}$$

b) Use este resultado y el teorema de Basu para mostrar que si $X_1, \ldots, X_n \overset{\text{iid}}{\sim} Gamma(\alpha, \beta)$, con α conocido, entonces para $T = \sum_{j=1}^n X_j$

$$\mathbb{E}[X_{(i)}|T] = \mathbb{E}\left[\frac{X_{(i)}}{T}T|T\right] = T\frac{\mathbb{E}[X_i]}{\mathbb{E}[T]}$$

$$\underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)}}_{c(\beta)} \exp\{ \underbrace{(\alpha-1)\ \log(x)}\}\ e^{\underbrace{-\beta}_{w_1} \underbrace{x}_{t_1(x)}}\ \underbrace{\mathbb{I}_{(0,\infty)}(x)}_{h(x)}$$

$F\left(\frac{x}{\theta}\right)$

$\frac{1}{\theta} f\left(\frac{x}{\theta}\right)$ } familia escala

$T = \sum_{i=1}^n X_i$

completo

$\mathcal{H} = (-\infty, 0) \Rightarrow$ contiene abiertos

Con $B = 1$ $z \sim$ Gamma $(\alpha, 1)$

$f(z) = \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z}\, \mathbb{I}_{(0,\infty)}(z)$

$x = \frac{1}{B} \cdot z$ ↝ No depende de Beta

$\frac{X_{(1)}}{\sum_{i=1}^n X_i} = \frac{\frac{1}{B} z_{(1)}}{\frac{1}{B}\sum_{i=1}^n z_i} = \frac{z_{(1)}}{\sum_{i=1}^n z_i}$

Como z no depende de β, $\frac{X_{(1)}}{\sum X_i}$ es ancilar

$E(X_{(1)} | T) = E\left[\underbrace{\frac{X_{(1)}}{T}}_{\text{Ancilar}} \overbrace{T}^{\text{completo}} \Big| T\right]$

x teo basú $E\left[\frac{X_{(1)}}{T} | T\right] \cdot T$