Senior Project - Scratch Work

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To incorporate vaccinations into the SIR model,

S = S(t) is the number of susceptible individuals
I = I(t) is the number of infected individuals
R = R(t) is the number of recovered individuals
V = V(t) is the number of vaccinated individuals

If N is the total population (7,900,000 in example), we define the following fractions:

$$s(t) = \frac{S(t)}{N}$$

This represents the susceptible fraction of the population, where $S(t)$ is the number of susceptible individuals.

$$i(t) = \frac{I(t)}{N}$$

This represents the infected fraction of the population, where $I(t)$ is the number of infected individuals.

$$r(t) = \frac{R(t)}{N}$$

This represents the recovered fraction of the population, where $R(t)$ is the number of recovered individuals.

Additionally, to account for vaccinations, we introduce a new fraction:

$$v(t) = \frac{V(t)}{N}$$

This represents the vaccinated fraction of the population, where $V(t)$ is the number of vaccinated individuals.

These fractions are convenient for analysis as they provide normalized measures of the prevalence of each group within the total population. They allow us to track changes in the susceptible, infected, recovered, and vaccinated populations relative to the entire community size.

Working with population counts may seem more natural, but using the fractions will simplify several of our computations. We may obtain identical information on the pandemic's progression from any of the two sets of dependent variables since they are proportionate to one another.

Now, let's address the questions:

1. Under the assumptions we have made, how do you think s(t) should vary with time? How should r(t) vary with time? How should i(t) vary with time?

2. Sketch on a piece of paper what you think the graph of each of these functions looks like.

3. Explain why, at each time t, s(t) + i(t) + r(t) = 1.

Variation with Time:

- $s(t)$: The susceptible fraction of the population should decrease over time due to new infections ($\beta \cdot s \cdot i$) and vaccinations ($\gamma_v \cdot s \cdot v$). Initially, $s(t)$ will experience a rapid decline as the epidemic spreads. As more individuals get infected or vaccinated, $s(t)$ will stabilize at a lower value.

- $r(t)$: The recovered fraction ($r(t)$) should increase over time due to recoveries ($\gamma \cdot i$) and vaccinations ($\gamma_v \cdot v$). Initially, $r(t)$ will be small, but as individuals recover or get vaccinated, $r(t)$ will rise, indicating immunity in the population.

- $i(t)$: The infected fraction will vary based on the balance between new infections and recoveries. The infected fraction ($i(t)$) will initially rise as the epidemic spreads ($\beta \cdot s \cdot i$). However, with recoveries ($\gamma \cdot i$) and vaccinations ($\gamma_v \cdot v$), $i(t)$ will eventually decrease. The overall shape of $i(t)$ depends on the balance between new infections and recoveries/vaccinations.

Sketch/Graphical Representation: - A sketch would show $s(t)$ decreasing, $r(t)$ increasing, and $i(t)$ varying depending on the epidemic dynamics.

- $s(t)$: Expect a decreasing curve that levels off over time.

- $r(t)$**:** Anticipate an increasing curve that eventually levels off as the epidemic progresses.

- $i(t)$**:** Predict an initially increasing curve that peaks and then decreases due to recoveries and vaccinations.

Explanation for $s(t) + i(t) + r(t) = 1$**:** The sum $s(t) + i(t) + r(t)$ represents the entire population. At any given time t, an individual is either susceptible, infected, or recovered. The total population is constant, so the sum of the fractions of individuals in each category must equal 1. As individuals move from being susceptible ($s(t)$) to infected ($i(t)$) and eventually to recovered ($r(t)$), the sum remains constant at 1, reflecting the entire population at that particular time. The inclusion of vaccination ($v(t)$) in the equations ensures that the total population is accounted for, maintaining the balance of the system.

Assumptions: No one is added to the susceptible group due to the exclusion of births and immigration. The time-rate of change of $S(t)$, the number of susceptibles, depends on the number already susceptible, the number of individuals already infected, and the amount of contact between susceptibles and infecteds. Each infected individual generates $b \cdot s(t)$ new infected individuals per day on average, where b is a fixed number of contacts. A fixed fraction k of the infected group will recover during any given day.

Equations: To express the equations below in terms of the fraction of the total population, we can divide each equation by N and introduce the fractions $s(t)$, $i(t)$, $r(t)$, and $v(t)$:

$$\frac{ds}{dt} = -\beta \cdot s \cdot i - \gamma_v \cdot s \cdot v$$

$$\frac{di}{dt} = \beta \cdot s \cdot i - \gamma \cdot i$$

$$\frac{dr}{dt} = \gamma \cdot i + \gamma_v \cdot v$$

$$\frac{dv}{dt} = \delta \cdot (1 - s - i - r)$$

The number of susceptible individuals, S(t), evolves based on the rate of new infections and vaccinations:

$$\frac{dS}{dt} = -\beta \cdot S \cdot I - \gamma_v \cdot S \cdot V$$

Here, β represents the transmission rate, I is the number of infected individuals, γ_v is the vaccination rate, and V is the number of vaccinated individuals.

The rate of change of the susceptible fraction $s(t)$ is influenced by new infections and vaccinations:

$$\frac{ds}{dt} = -b \cdot s \cdot i - \gamma_v \cdot s \cdot v$$

Here, $i(t)$ is the infected fraction of the population, γ_v is the vaccination rate, and $v(t)$ is the vaccinated fraction of the population.

The number of infected individuals, I(t), changes due to new infections and recoveries:

$$\frac{dI}{dt} = \beta \cdot S \cdot I - \gamma \cdot I$$

Here, γ is the recovery rate.

The rate of change of the infected fraction $i(t)$ is determined by new infections and recoveries:

$$\frac{di}{dt} = b \cdot s \cdot i - k \cdot i$$

Here, k is the fraction of the infected group that recovers each day.

The number of recovered individuals, R(t), increases as a result of recoveries:

$$\frac{dR}{dt} = \gamma \cdot I + \gamma_v \cdot V$$

The rate of change of the recovered fraction $r(t)$ is influenced by recoveries:

$$\frac{dr}{dt} = k \cdot i$$

The number of vaccinated individuals, V(t), changes due to the vaccination rate:

$$\frac{dV}{dt} = \delta \cdot (N - S - I - R)$$

Here, δ is the rate of vaccination, and N is the total population.

The rate of change of the vaccinated fraction $v(t)$ is determined by the vaccination rate:

$$\frac{dv}{dt} = \gamma_v \cdot s \cdot v$$

The Susceptible Equation: The Susceptible Equation is given by:

$$\frac{ds}{dt} = -b \cdot s \cdot i - \gamma_v \cdot s \cdot v$$

Explanation: - The presence of the $I(t)$ term reflects the assumption that the rate of new infections is proportional to the product of the susceptible and infected fractions, $s(t)$ and $i(t)$, respectively. This assumption arises from the concept that each infected individual makes b contacts per day, and a fraction $s(t)$ of these contacts is with susceptible individuals.

- The negative sign indicates that the susceptible fraction decreases over time due to new infections and vaccinations.

The Recovered Equation: The Recovered Equation is given by:

$$\frac{dr}{dt} = k \cdot i$$

Explanation: - This equation follows from the assumption that a fixed fraction k of the infected group recovers each day. The recovered fraction $r(t)$ increases as a result of recoveries.

The Infected Equation: The Infected Equation is given by:

$$\frac{di}{dt} = b \cdot s \cdot i - k \cdot i$$

Explanation: - The assumption underlying this equation is that the rate of change of the infected fraction is determined by the balance between new infections ($b \cdot s \cdot i$) and recoveries ($k \cdot i$).

- This reflects the assumption that a fixed fraction k of the infected group recovers each day.

Now, let's explain the components of the equation for the rate of change of the infected fraction:

$$\frac{di}{dt} = b \cdot s \cdot i - k \cdot i$$

- $b \cdot s \cdot i$ represents the rate of new infections, where b is the fixed number of contacts per day, $s(t)$ is the susceptible fraction, and $i(t)$ is the infected fraction.

- $k \cdot i$ represents the rate of recoveries, where k is the fraction of the infected group that recovers each day.

The provided information gives the complete SIR model for the COVID-19 pandemic in mid-2020, considering scaled variables and initial conditions. Let's summarize the model and the initial conditions:

Scaled SIR Model:

$$\frac{ds}{dt} = -b \cdot s \cdot i - \gamma_v \cdot s \cdot v, \quad s(0) = 1$$

$$\frac{di}{dt} = b \cdot s \cdot i - k \cdot i, \quad i(0) = 1.27 \times 10^{-6}$$

$$\frac{dr}{dt} = k \cdot i, \quad r(0) = 0$$

Initial Conditions:

$$S(0) = 7,900,000$$

$$I(0) = 10$$

$$R(0) = 0$$

Scaled Initial Conditions:

$$s(0) = 1$$

$$i(0) = 1.27 \times 10^{-6}$$

$$r(0) = 0$$

$$s_n = s_{n-1} + s\text{-slope}_{n-1}\Delta_t,$$

$$i_n = i_{n-1} + i\text{-slope}_{n-1}\Delta_t,$$

$$r_n = r_{n-1} + r\text{-slope}_{n-1}\Delta_t,$$

$$v_n = v_{n-1} + v\text{-slope}_{n-1}\Delta_t,$$

More specifically, given the SIR model with vaccinations:

$$\frac{ds}{dt} = -\beta si - \gamma_v sv, \quad s(0) = 1$$

$$\frac{di}{dt} = \beta si - \gamma i, \quad i(0) = 1.27 \times 10^{-6}$$

$$\frac{dr}{dt} = \gamma i, \quad r(0) = 0$$

$$\frac{dv}{dt} = \delta(1 - s - i - r), \quad v(0) = 0$$

The Euler formulas for each of these equations become:

$$s_n = s_{n-1} + (-\beta si - \gamma_v sv)\,\text{-slope}_{n-1}\Delta_t,$$

$$i_n = i_{n-1} + (\beta si - \gamma i)\,\text{-slope}_{n-1}\Delta_t,$$

$$r_n = r_{n-1} + (\gamma i)\,\text{-slope}_{n-1}\Delta_t,$$

$$v_n = v_{n-1} + (\delta(1 - s - i - r))\,\text{-slope}_{n-1}\Delta_t.$$

Introduction and Parameter Experimentation:

The infectious period for COVID-19 is estimated to be about five days, suggesting $k = \frac{1}{5}$. However, b is uncertain, and the "mixing rate" likely depends on various population characteristics like density. In this exploration, we focus on the infected fraction $i(t)$ to understand the epidemic's progress.

Experimentation with Changes in b: Procedure: - Keep k fixed at $\frac{1}{5}$. - Plot $i(t)$ with different values of b between 0.5 and 2.0.

Observations: - Describe how changes in b affect the graph of $i(t)$. - Stay alert for automatic changes in the vertical scale.

Explanations: - Briefly explain the observed changes based on your intuitive understanding of the epidemic model.

Experimentation with Changes in k: Procedure: - Return b to $\frac{1}{2}$. - Experiment with different values of k between 0.1 and 0.6.

Observations: - Describe changes in the graph of $i(t)$ with varying k. - Be alert for automatic changes in the vertical scale.

Explanations: - Explain the observed changes in terms of your intuitive understanding of the model.

Change in the Character of the Graph: - Identify the change in the graph of $i(t)$ near one end of the suggested range for k (0.1 to 0.6).

Predicting the Change Using the Differential Equation:** - Use the infected-fraction differential equation to explain how the change in the graph could have been predicted for a specific k value.

Model Comparison with Data: Graph Comparison: - Repeat the graph of the data alongside $i(t)$ with $k = \frac{1}{5}$ and $b = \frac{6}{10}$.

Model Evaluation: - Does the model seem reasonable or not? Explain your conclusion.

Note: - It's crucial to interpret the experimental results and model comparison in the context of real-world implications and potential limitations of the chosen model.