Nominalism and Immutability

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Can we do science without numbers? Quine (1961) says no. Numbers and other mathematical entities are indispensible. Field (1980) says yes. Numbers may be useful, but are not essential. With enough time and patience, we can do science without them.

Now consider a seemingly unrelated question: How much contingency is there? Spinoza says none. Whatever is true is necessary. Williamson (2013) says none with respect to what exists, but *some* with respect to how things are. The ordinary view, perhaps, is that there is some with respect to what exists, and some with respect to how things are.

These seemingly unrelated questions—one in the philosophy of math and science and the other in metaphysics—share an unexpectedly close connection. For as it turns out, a radical answer to the second leads to a breakthrough on the first.

The radical answer is immutabilism, a view endorsed by Leibniz. Immutabilism says that there is some contingency with respect to what exists, but *none* with respect to how things are. Immutabilism is thus a sort of converse of necessitism, which is the view endorsed by Williamson.

The breakthrough is a new strategy for doing science without numbers. Field shows how to do science without numbers in classical mechanics. His strategy, though, requires the existence of spacetime points. This is fine, so far it goes. But there are reasons you might want an alternative: You might be a relationalist, and so reject the existence of spacetime points. You might be concerned that the strategy will not generalize, especially to theories formulated in terms of state space. Or you might simply wonder whether we can get by with less. I think that we can, and immutabilism is the way to do it.

1 Nominalism

We can distinguish two views about scientific theories. **Scientific nominalism** is the view that the best scientific theories include a nominalist theory. A theory is nominalist when it only quantifies over concrete particulars.[1] Concrete particulars include things like particles.

Scientific Platonism, on the other hand, is the view that all of the best scientific theories are Platonist. A theory is Platonist when it is not nominalist. Thus, a Platonist theory will quantify over things like numbers or universals in addition to things like particles.[2]

Why might you be a scientific nominalist? Suppose that you are a **metaphysical nominalist**. You say that the only things that exists, fundamentally speaking, are concrete particulars. Suppose that you are also a **scientific fundamentalist**. You thus claim that fundamental reality is best described by at least one of the best scientific theories. In that case, scientific nominalism will follow.

Metaphysical nominalism, then, is one road to scientific nominalism. But there are also many others. For example, you might think that numbers exist, and that their existence is fundamental. Still, you might think that their role in science is merely *representational*. But if the role of numbers is merely representational, then we should be able to do science without them, given enough time and patience.[3]

Similarly, you might think that nominalist theories have certain virtues that Platonist theories lack. For example the most direct objects of scientific inquiry are concrete particulars—things like meter sticks and scales and particle accelerators. Nominalist theories are thus **intrinsic** in a way that Platonist theories are not. For nominalist theories explain the behavior of concrete particulars without appealing to anything *other* than concrete particulars. As such, you might think that nominalist theories

1. This is only a rough characterization. What it means for a theory to be nominalist depends on the background ideology. For example, suppose that we have a fundamental physical theory stated in terms of a feature-placing language of the sort suggested by Quine (1971). In that case, the theory quantifies over nothing, so *thereby* quantifies over nothing more than particle. But theories given in terms of a feature-placing language could fail to be nominalist. It might, for example, place mathematical features (like being an integer) alongside physical features (like having mass). Thus, I prefer to adopt a "we know them when we see them" approach towards identifying nominalist theories.

2. The views described here are *weak* scientific nominalism and *strong* scientific Platonism. Alternatively, you could be a *strong* scientific nominalist who thinks that all of the best physical theories are nominalist. You could also be a *weak* scientific Platonist who merely holds that some of the best physical theories are Platonist. Since it could be that some of the best theories are nominalist and some of the best theories are Platonist, the conjunction of weak nominalism and weak Platonism is consistent.

3. The distinction we are drawing here between metaphysical nominalism and scientific nominalism is similar to a distinction drawn by Field (1984). What we are calling scientific nominalism is roughly the disjunction of what he calls nominalism and lightweight Platonism. I say roughly, because a scientific nominalist might also reject substantial metaphysical questions altogether. But in that case, she might be a scientific nominalist while rejecting both nominalism (in Field's sense) and lightweight Platonism, since these are both substantial metaphysical views.

are both more satisfying and more illuminating.[4] This in turn suggests that the best scientific theories may include nominalist theories, since they have a unique profile when it comes to theoretical virtues.[5]

You can be a scientific nominalist, then, without being a metaphysical nominalist. The converse is also true: You can be a metaphysical nominalist without being a scientific nominalist.

For example, suppose that you are a logical atomist.[6] You say that the only fundamental facts are atomic facts about concrete material particulars. The best scientific *theories*, though, have strong and simple laws. Thus, the best scientific theories involve generality, which you take to be non-fundamental. Scientific fundamentalism is therefore false. But in that case, it may be that the best scientific theories are Platonist theories, despite the fact that the only things that exist, fundamentally speaking, are particles.[7] So you might be a scientific Platonist while also being a metaphysical nominalist.

For my own part, I am both a scientific nominalist and a metaphysical nominalist, though my commitment to the first is stronger than my commitment to the second.

Our focus in this paper will be scientific nominalism. What we want to know is: Can we build scientific theories while quantifying over nothing more than concrete particles? Can we do science without numbers? Thus, by nominalism, we will generally mean *scientific* nominalism. Likewise, by Platonism, we will generally mean *scientific* Platonism. Questions about metaphysical nominalism and metaphysical Platonism are also important, but will remain in the background, for the most part.

1.1 Quantities

The physical world is built using physical quantities like mass, charge, and distance. To fix on an example, suppose we perform a series of experiments and discover that the movement of particles is fully described by Newton's laws. These laws require there to be distance ratios between particles.[8] Thus, in order to state the laws, we need a

4. For example, synthetic geometries, like those proposed by Tarski (1952), would seem to be both more satisfying and more illuminating than the corresponding analytic geometries.

5. That one might prefer nominalist theories because they are more intrinsic is also a nice point originally made by Field (1984).

6. Following in the tradition of Russell (1918) and Wittgenstein (1922).

7. If there is no fundamental quantification, then how are we to characterize metaphysical nominalism? My preferred solution is to use a non-fundamental language with a fundamentality operator. Assuming that non-fundamental things are not fundamentally self-identical, we can then construe metaphysical nominalism as the claim that $\forall x(\mathsf{Fund}(x = x) \supset \mathsf{Concrete}(x))$.

8. Here is a simple case: Suppose the world is Newtonian with gravity the only force. There are three particles a, b, and c with b between a and c. The particles are at rest relative to one another and a is as

language that can describe distance ratios.

What are distance ratios? Suppose we use a meter stick to determine that a and b are two meters apart and c and d are one meter apart. Thus, a and b are twice as far apart as c and d. This is a distance ratio. Others include being three times as far apart, being half as far apart, and so on.

A Platonist can easily describe distance ratios in her fundamental theory. She could, for example, describe them using a distance ratio function from particles to real numbers. Thus, to express the idea that a and b are twice as far apart as c and d, she could write:

$$\delta(a, b, c, d) = 2 \tag{1}$$

If you like, you might think of this as a definite description:

The distance ratio of a and b to c and $d = 2$.

Such descriptions let the Platonist describe particle configurations, apply the dynamical laws, and predict how things move.

Suppose, though, that we are scientific nominalists. We are thus committed to theorizing about the physical world *without* using things like distance functions. In that case, how are we going to express facts involving distance ratios? This problem, as applied to physical quantities in general, is what Field (1984) calls **the problem of quantities**.

One strategy for solving the problem is **simple nominalism**. In the case of distance ratios, simple nominalism requires nothing more than particles and a pair of relations. Those relations are **congruence** and **betweenness**.

$\mathsf{Cong}(a, b, c, d)$	a and b are the same distance apart as c and d
$\mathsf{Bet}(a, b, c)$	b is on a straight line between a and c

In each case, the gloss on the right is merely intuitive. The congruence relation is naturally described by quantifying over distances and the between relations by quantifying over lines. But these are both just basic relations. Thus, they come with no commitment to the existence of things like distances or lines. There are no strings attached.

To see how a simple nominalism might account for distance ratios, consider a world

massive as b and c put together. In that case, by the law of universal gravitation, the collision of a and b will be simultaneous with the collision of b and c just in case the distance ratio of a and b to b and c is $\sqrt{2}$. But if there is no determinate distance ratio, the laws will fail to determine whether the collisions will be simultaneous. We thus get an unwanted failure of determinism.

in which there are exactly four point particles that are arranged as follows:

The betweenness and congruence relations are as illustrated, with the particles a and b twice as far apart as b and c. This distance ratio is the one that we want to explain.

The strategy, in this case, is straightforward. The simple nominalist says that a and b are twice as far apart as b and c, in the sense that there is an x between a and b, such that a and c are congruent with x and b, and x and b are congruent with b and c.

More generally, we can define the notion of an equally spaced line of particles using betweenness and congruence.[9] This gives us a defined polyadic predicate.

$\mathsf{Line}(x_1, \dots, x_n)$ $\quad$ $x_1, \dots, x_n$ form an equally spaced line

We then express the claim that a and b are twice as far apart as b and c using:

$$\exists x \ \mathsf{Line}(a, x, b, c) \tag{2}$$

Looking at the previous diagram, you can see why this is adequate, at an intuitive level. Since we have an equally spaced line, the distance between each pair can be thought of as a unit. There are two units between a and b, but there is only one unit between b and c. So a and b are twice as far apart as b and c.

The problem is that simple nominalism only works if there happens to be enough particles and they happen to be in the right place. For example, consider a world just like the last, except that the second particle from the left has been deleted.

This is a world in which a and b are twice as far apart as b and c. There is, however, no way to explain this distance ratio using betweenness and congruence relations. After all, consider a world in which a and b are *half* as far apart as b and c instead.

These worlds have *exactly the same* betweenness and congruence relations, but different

9. Particles a and b are *colocated* when there is some x such that $\mathsf{Cong}(a, b, x, x)$. The particles $a_1, \dots, a_n$ form a *line* when no two of them are colocated and a_k is between a_j and a_h whenever $j \leq k \leq h$. The line is *equally spaced* if a_j and a_{j+1} are congruent with a_k and a_{k+1} for all j and k such that $0 \leq j < n$ and $0 \leq k < n$.

distance ratios. Thus, the distance ratios cannot be explained by the betweenness and congruence relations. Such worlds are **sparse**.

Field's solution is to accept the existence of spacetime points. Spacetime points, like particles, can stand in betweenness and congruence relations. Unlike particles, though, spacetime points are always numerous and well-organized. You can always count on them being exactly where they need to be.

For example, suppose that we have laws guaranteeing that whenever there are two things, there is a spacetime point halfway between them. This means that in particular, there will be a spacetime point x halfway between a and b in the first of our two sparse worlds.

But now, since there is once again *something* halfway between a and b, we can explain the distance ratio in basically the same way as before. That is, a and b are twice as far apart as b and c because:

$$\exists x \ \mathsf{Line}(a, x, b, c) \tag{3}$$

The only difference is that now, x is a spacetime point instead of a particle. But the basic structure of the explanation is exactly the same.[10]

There is much to be said for the substantivalist strategy. Spacetime points are concrete particulars, and so accepting their existence is consistent with nominalism. Fields are also naturally thought of as properties of spacetime points, and so there is some reason to think that we will need spacetime points *anyway* to account for fields. In that case, using spacetime points to explain distance ratios is no further cost.

On the other hand, there are also reasons to be wary. After all, spacetime faces its own slate of challenges, ranging from shift arguments to hole arguments.[11] There is also the concern that the strategy will not generalize. Some of our best science describes quantities using things like state space. But while it is reasonably clear that a nominalist can accept the points of *ordinary* spacetime as concrete particulars, it is far from clear that a nominalist can accept the points of *state space* as concrete particulars. But in that case, how is a nominalist going to nominalize state space theories? We will return to this issue in §5.

Another strategy for expressing distance ratios is to accept distances rather than

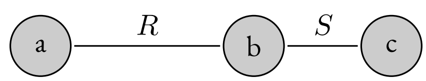
10. Early axiomatizations of Euclidean space using congruence and betweenness include those from Veblen (1904) and Pieri (1908). The project was later advanced by Alfred Tarski and his students, who gave increasingly simple axioms in (Tarski 1952), (Tarski 1959), and (Gupta 1965).

11. See for example Dasgupta (2016) and [...]

spacetime points. What are distances? Brent Mundy (1987) suggests a view on which distances are binary relations between particles. Particles thus *have* distances by *instantiating* them. Axioms governing distance relations are then given using second-order quantification into predicate position and a pair of second-order predicates.

$X \geq Y$	X is at least as great as Y
$\mathsf{Sum}(X, Y, Z)$	X and Y sum to Z

With appropriate axioms in place, we can then explain distance ratios, even in sparse worlds. For example, here is our three-particle world again.



The particles that a and b are then twice as far apart as b and c because a and b have the distance R and b and c have the distance S. Moreover, S and S sums to R. Thus, the distance ratio can be expressed with:

$$R(a, b) \wedge S(b, c) \wedge \mathsf{Sum}(S, S, R) \tag{4}$$

This solution, though, is not available to nominalists, since it requires higher-order quantification over relations.[12] But scientific nominalism is precisely the view that we can give scientific theories without quantifying over such things.

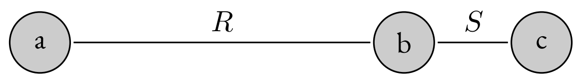
Mundy can explain distance ratios in sparse worlds because he is a Platonist. Thus, even if the particles are sparse, his *distance relations* are plentiful.[13] Like spacetime points, distance relations are numerous and well-organized. You can always count on them being where they need to be.

On the other hand, if there are distance relations, but they are not plentiful, we may once again find ourselves unable to express the needed distance ratios. For example, suppose you are an Aristotelian. You thus deny that distance relations exist when not instantiated. Now consider the following sparse world. There are exactly three particles.

12. A view that you might call *easygoing nominalism* accepts higher-order quantification over universals, while rejecting first-order quantification over universals. In contrast, *serious nominalism* rejects all quantification over universals *whatsoever*. There is an ongoing dispute about whether easygoing nominalism is compatible with the best reasons for being a nominalist. Arthur Prior Prior (1971) says yes. I am inclined to say no. For present purposes, we can just stipulate that our interest is in the question of whether we can do science as serious nominalists.

13. In particular, Mundy's axioms guarantee the distance relations form an *ordered semigroup*.

This time, the particles a and b are *three* times as far apart as b and c.



Mundy will say that the distance ratio is explained by the fact that a and b stand in distance relation R and b and c stand in distance relation S. Moreover, there is a distance relation X such that S and S sum to X, and S and X sum to R.

$$\exists X(R(a,b) \wedge S(b,c) \wedge \mathsf{Sum}(S,S,X) \wedge \mathsf{Sum}(S,X,R)) \tag{5}$$

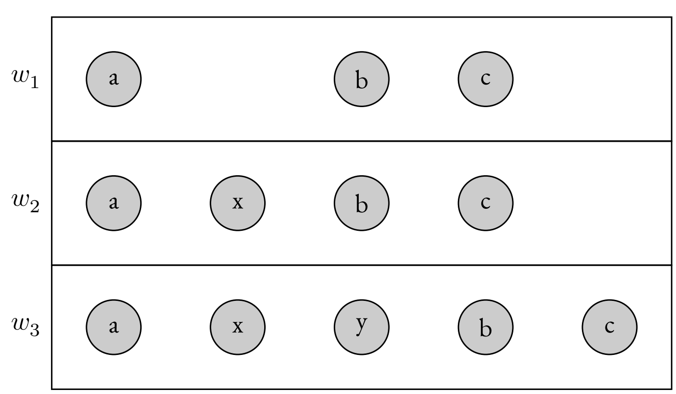
This requires the existence of a distance relation X that is the sum of S and S. But if the only distance relations that exist are the ones that are instantiated, there is no such X. Thus, the Aristotelian will find herself unable to explain the relevant distance ratio.

1.2 Uniqueness

We can give a theory of distance ratios without numbers, then, if we quantify over spacetime points or distance relations. You might wonder, though, whether these are the only options.

A natural thought, at this point, is that perhaps we could give a modal theory of distance ratios. For example, you might think that a and b are twice as far apart as b and c in a sparse world, not because there is actually a particle x halfway between a and b, but because there *could* have been.

As compelling as the basic idea might be, modal strategies face a serious challenge, which we will call the **problem of uniqueness**. For suppose that the actual world w_1 is as illustrated below. The particles a and b are twice as far apart as b and c, and so this is the distance ratio we need to explain. The modal proposal, then, is that this is so



because there is a possible world w_2 in which the three particles are exactly the same, but in which there is an additional particle x standing in the appropriate betweenness and congruence relations.

So far so good. The question is: What does it mean for a, b, and c to be *exactly the same* in worlds w_1 and w_2?

We could say that particles are exactly the same when they are *intrinsically* the same. Thus, the three particles in w_1 are exactly the same in w_2 because they stand in the same betweenness and congruence relations in w_2 that they do in w_1.[14]

But now observe: World w_3 is possible. Moreover, the particles in w_1 stand in the same betweenness and congruence relation in w_3 that they do in w_1. Thus, they are intrinsically the same and so, on the present proposal, *exactly* the same.

The problem is that a and b are three times as far apart as b and c in w_3. So if the possibility of w_2 is enough to make a and b twice as far apart as b and c in w_1, then by parallel reasoning, the possibility of w_3 is enough to make a and b three times as far apart as b and c in w_1. But in that case, the *very same particles* in the *very same world* are both twice as far apart and three times as far apart. This is absurd. So the proposal fails.

For the modal strategy to work, there has to be a relevant difference between world w_2 and world w_3. But if we only have congruence and betweenness relations *within* worlds, there is no relevant difference. This because the three particles from world w_1 stand in exactly the same betweenness and congruence within each world.

This leads to a natural suggestion. Maybe the worlds can be distinguished if we allow for relations not only *within* worlds, but *across* worlds.

Looking back at the above diagram, suppose that the congruence relations across worlds are as they appear. Thus, a and b at w_1 are congruent with a and b at w_2, but not congruent with a and b at w_3. In that case, we *do* have a relevant difference between w_2 and w_3. Thus, comparisons of congruence across worlds give us a natural strategy for solving the problem.

The question is: Can a nominalist express such comparisons across worlds? This will be our focus in the next section.

1.3 Cross-modal Comparisons

Comparisons across worlds are are ordinary and familiar. We might say that Socrates could have been taller than he is, or that the Athenians could have been happier than

14. Say that two particles a and b are *colocated* when $\mathsf{Bet}(a, b, a)$. The only non-trivial facts about betweenness and congruence in w_1 are that none of the three particles are colocated and that b is between a and c. Thus, all it takes for a, b, and c to be intrinsically the same in w_2 as they are in w_1 is for none of them to be colocated and for b to be between a and c.

they are. In the first case, we are saying that there is a possible world in which Socrates is taller than he is in the actual world. And in the second, we are saying that there is a possible world in which the Athenians are happier than they are in the actual world. Such comparisons are **cross-modal comparisons**.

There are a variety of approaches to the metaphysics of modality. For our purposes, we can focus on two, which will be used to illustrate the general challenges in expressing cross-modal comparisons as a nominalist.

Modal realism is the view that the most basic modal facts should be understood in terms of quantification over a pluriverse of island universes. These island universes are spatiotemporally disconnected from our own, but no less real or concrete.

Modalism is the view that modal facts are basic, and so should be understood in terms of modal operators. The modalist, unlike the modal realist, is at best agnostic about whether there are any other island universes.

The modal realist and the modalist, then, have very different views. They also have different languages: The modal realist has a language with quantifiers ranging over the pluriverse. The modalist has a language with modal operators.

Now consider how each might express cross-modal comparisons. Suppose that we want to explain the fact that:

Socrates could have been taller than he actually is. (6)

A modal realist can do this using nothing more than a pair of relations.

$$\exists x(\mathsf{Ctp}(x, s) \land \mathsf{Taller}(x, s)) \tag{7}$$

Thus, according to the modal realist, Socrates could have been taller than he actually is because there is a counterpart of Socrates, somewhere in the pluriverse, who is taller than Socrates.

The question is whether the modalist can do the same. She might try saying that Socrates could have been taller than he actually is by writing:

$$\Diamond(\mathsf{Taller}(s, s)) \tag{8}$$

But this says that it could have been that Socrates was taller than himself, not that it could have been that Socrates was taller than he actually is. She could try adding something like an actuality operator to her language. This would let her say:

$$\Diamond @(\mathsf{Taller}(s, s)) \tag{9}$$

But this is equivalent to saying that Socrates is *actually* taller than himself, which is also not what we wanted. These exhaust the most obvious syntactic possibilities. So the modalist has no clear strategy for expressing cross-modal comparisons. Call this the **problem of cross-modal comparisons**.

The standard solution is to use quantification over universals. In the case of height comparisons, this means quantifying over *height* properties. On this approach, a modalist will express (6) with:

$$\exists X(X(s) \wedge \Diamond(\exists Y(Y(s) \wedge Y > X))) \tag{10}$$

This says that there is a height that Socrates has, and it could have been that there was a height that Socrates had that was greater.[15]

Now return to the question of nominalism. The nominalist would like to give a theory of distance ratios by using cross-modal comparisons. She could try to do this as a modalist. But the standard solution for expressing cross-modal comparisons with modal operators is to use quantification over universals, which is not available to the nominalist.[16]

Could a nominalist be a modal realist? I think this question deserves some care. Suppose you are a nominalist and one day, you visit the oracle. She has a long track record of answering questions truthfully, so you ask her whether there is a pluriverse. She says yes. Moreover, she tells you that the pluriverse is built entirely out of concrete particulars. There are no numbers, universals, or any other such things.

Should you give up your nominalism? The answer, it seems to me, is clearly not. How could learning that there are *more* concrete particulars be inconsistent with the view that everything is a concrete particular? Giving up nominalism, on the grounds that there are many concrete universes, would be like giving up theism, on the grounds that there are many gods.

Nominalism and modal realism, then, are broadly consistent. The question is whether there is any coherent *grounds* for being both a nominalist and a modal realist, given our actual evidence. I say no. For as Lewis points out, insofar as we have reasons to accept the pluriverse, those reasons are similar to the ones we have for accepting numbers.[17] That is, in both cases, the reasons for accepting either the existence of the pluriverse or the existence of numbers are broadly a priori. But the nominalist denies that we can accept the existence of numbers on broadly a priori grounds. So she must

15. Variations of the standard solution have been endorsed by Morton (1984), Cresswell (1990), Milne (1992), and Kemp (2000).

16. This is another point at which easygoing nominalism is relevant. For if the nominalist can be an *easygoing* nominalist, then she can use the standard solution. Whether easygoing nominalism is compatible with the best motivations for nominalism is an interesting question. As in footnote 12, we can just stipulate that our interest here is in solutions that do not use higher-order quantification over universals.

17. See (Lewis 2001, Chapter 1).

also do the same for the pluriverse.

2 Compossible Immutabilism

We saw in §1 that, in order to do science without numbers, the nominalist needs to give a theory of quantities. One natural strategy is to use modality. But to use modality, she needs some way of ensuring that distance ratios are unique. This can be done if the nominalist can use cross-modal comparisons. The problem, though, is that there is no clear strategy for making cross-modal comparisons as a nominalist.

In this section, we are going to describe a new view about modality called compossible immutabilism. This new view will let nominalists make cross-modal comparisons which, in turn, will give them a new strategy for doing science without numbers.

2.1 The View

We ordinarily think that things are **mutable**, in the sense that they could have had different properties and could have stood in different relations. For example: My coffee mug is blue and sitting on my desk. But while that may be, it could have been a different color. It could have been green instead of blue. It also could have had a different location. Instead of being on my desk, it could have been in the kitchen. Thus, my coffee mug would seem to be mutable.

Things that are not mutable are **immutable**. Whether there are any immutable things is a matter of controversy, so there are no uncontroversial examples. Maybe God is immutable. Maybe numbers are.

Immutabalism is the view that necessarily, everything is immutable. Thus, while there may be contingency with respect to what exists, there is *no* contingency with respect to how things are. When things have properties and stand in relations, they have those properties and stand in those relations necessarily. The opposing view is **mutabilism**, which says that possibly, something is mutable.

Immutabilism has some precedence in the history of philosophy. Spinoza is an immutabilist in virtue of being a necessitarian. That is, because he holds that whatever is true is also necessary, it follows that necessarily, everything is immutable.

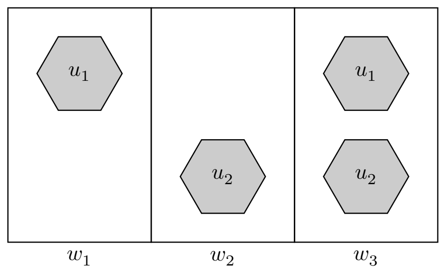
Leibniz is also an immutabilist, but of a less trivial variety. Unlike Spinoza, he accepts that there is contingency with respect to what exist. Still, he denies is that there is contingency with respect to how things are.

What we are going to call **Leibnizian immutabilism** is the view that necessarily, everything is worldbound. An individual is worldbound when necessarily, had things been any different, that individual would not have existed. Thus, according to the

Leibnizian, nothing exists at more than one possible world.[18]

Compossible immutabilism is a form of immutabilism. But unlike the Leibnizian immutabilist,the compossible immutabilist denies that things exist at only one possible world. In fact, quite the opposite: For the compossible immutabilist maintains that whenever there are two possible worlds, those worlds can be composed to form a third possible world, where everything from the first two worlds exists together. Thus, she endorses what you might think of as a paste-together theory of modality.

So for example, suppose that we have two possible worlds w_1 and w_2 below. Each contains a single universe. Compossible immutabilism says that whenever we have two



worlds, they can be composed to form a third world, in which everything from the first two possible worlds exists together. Thus, in this case, we have a third world w_3 that contains both of the universes u_1 and u_2.

What we are going to do now is make all of this more precise. Thus, we will describe the language of immutabilism in §2.2. We will then say more about various principles that the compossible immutabilist might accept in §2.3.

2.2 Language

We are going to think of the immutabilist as having a fundamental language $\mathcal{L}_I$. This language is a first-order plural language. It thus includes:

Singular variables $x, y, z, \ldots$
Plural variables $xx, yy, zz, \ldots$
Singular quantifiers $\exists x$ and $\forall x$
Plural quantifiers $\exists xx$ and $\forall xx$
Truth-functional operators $\wedge, \vee, \supset, \neg$
Modal operators $\Box, \Diamond$

18. Whether Leibniz himself accepted that necessarily. everything is worldbound is controversial. The standard view, though, is that he did. For more on these interpretive questions, see Mates (1989), Adams (1994), and Cover and Hawthorne (1990, 1999).

Singular variables range over **individuals**. Plural variables range over **pluralities**.

What are pluralities? A plurality is not a *thing*, like a particle or a spacetime point. A plurality is simply *things* taken together.

To borrow an example from George Boolos (1984), suppose you have a bowl of Cheerios. You consider each individually: Here is one Cheerio. There is another. Besides each individual Cheerio, though, you might also consider the *Cheerios*, taken together. For the Cheerios have properties that are not had by any Cheerio. The Cheerios fill the bowl, but no *Cheerio* fills the bowl. The Cheerios weigh 50 grams, but no *Cheerio* weighs 50 grams.

You might then wonder, what are the Cheerios? Maybe the Cheerios are a *set*. Maybe the Cheerios are a *fusion*.

Maybe. But many of us think that Cheerios taken together are *thin* in a way that sets or fusions are not. After all, if you accept that there is a set of Cheerios, then you are committed to sets. And if you accept that there is a fusion of Cheerios, then you are committed to fusions. But if you accept that there are Cheerios taken together, you are committed to nothing more than each Cheerio, taken one by one. The Cheerios taken together are what we are calling a plurality.

This point is worth belaboring because, in the present context, we want to build a version of immutabilism that is consistent with nominalism. Thus, it is important that pluralities are not abstract things, like sets.

Predicates in the immutabilist language are sorted. This means that each argument place can take either terms for individuals or terms for pluralities, but not both.[19] Predicates are allowed to have any finite arity. The logical predicates are:

$x =_s y$	x is identical y
$xx =_p yy$	the xx are identical to the yy
$x \prec yy$	x is among the yy
$x \sim y$	x is connected to y

The first is a singular identity predicate. The second is a plural identity predicate. We can thus identify individuals with individuals and pluralities with pluralities.[20] But we cannot identify individuals with pluralities or pluralities with individuals. The third logical predicate is the **among** predicate. This lets us say which individuals are among with pluralities. The fourth is a **connectedness** predicate. Two things are connected,

19. In other words, there are no complex sorts, including disjunctive sorts. If there were disjunctive sorts, then there could be argument places that could take *either* singular or plural terms. For my own part, I can see no use for disjunctive sorts, but would not be opposed to adding them.

20. Context will generally determine which identity predicate we have in mind. Thus, in most cases, we will drop the singular and plural subscripts.

intuitively speaking, when they are in the same universe.

Once the immutabilist has a language, there is the question of what logic she should accept. There are many systems that could be built. What we are going to do is build a system called **CI**. This system is the one that I accept. It includes the core principles of compossible immutabilism, along with others that help to fill out the view. While building this system, though, there will be various choice points. So we will also discuss other ways in which a compossible immutabilist system might be constructed.[21]

The background logic for **CI** includes a free logic, along with the propositional modal logic **S5**. The free logic ensures individuals and pluralities can both exist contingently. Names can only be assigned to individual and pluralities that actually exist. There are also axioms guaranteeing that pluralities are rigid and that connectedness is an equivalence relation.

Our background logic is a plural logic, and so there is the question of which things form pluralities. Compossible immutabilism *as such* is compatible with a wide variety of answers. The system **CI** that we are constructing, though, will include all the instances of Plural Comprehension.[22]

Plural Comprehension $\exists xx \forall y(y \prec xx \equiv \phi(y))$

This says that for any condition, there is a plurality consisting of all and only the things that satisfy that condition. Plural Comprehension also entails two further principles, which we will call Everything and Nothing.

Everything $\exists xx \forall y(y \prec xx)$
Nothing $\exists xx \forall y(y \not\prec xx)$

Everything says that there are things such that everything is one of them. Nothing says that there are things such that *nothing* is among them. Thus, Nothing guarantees the existence of an empty plurality.

We can now define several useful notions. A plurality xx is **closed under conectedness** when for all y and z, if y is among the xx and y is connected to z, then z is among the xx. A plurality xx is a **subplurality** of another plurality yy when every x that is among the xx is also among the yy. A **universe** is a non-empty plurality that is closed under connectedness and in which every pair of individuals is connected. A **multiverse** is a plurality that is closed under connectedness in which some pair of individuals is *not* connected.

In what follows, it will be helpful to talk about pluralities of universes. But this raises

21. A full model-theoretic description of **CI** is included in the appendix.

22. The formula ϕ is allowed to be an open formula with with both individual and plural free variables, though xx is not allowed to occur free in ϕ.

an immediate complication. For universes themselves are pluralities of individuals, so pluralities of universes would have to be pluralities *of pluralities*. That is, pluralities of universes would have to be *plupluralities*. But our immutabilist language has no syntax for talking about plupluralities. So it would seem that we have no way to talk about pluralities of universes.

Our solution will be to identify pluralities of universes with the plurality of individuals *in* those universes. Thus, suppose that we have two universes u_1 and u_2. What we are going to call the plurality of u_1 and u_2 is then just the plurality of individuals that are in either u_1 or u_2. Or putting it another way, what we are going to call a **plurality of universes** is just a plurality of individuals closed under connectedness. We will then say that a universe xx is *among* a plurality yy of universes when the xx are a subplurality of the yy.

2.3 Principles

There are many ways to be a compossible immutabilist. The core of the view, though, consists of two principles.

Immutability $\Box\forall x_1 \dots \Box\forall x_n \Box(R(x_1, \dots, x_n) \supset \Box(\mathsf{Exists}(x_1) \wedge \dots \wedge \mathsf{Exists}(x_n) \supset R(x_1, \dots, x_n)))$

Compossibility $\Box\forall xx \Box\forall yy \Diamond(\mathsf{Exists}(xx) \wedge \mathsf{Exists}(yy))$

Immutability says that if things stand in a relation then, necessarily, they stand in that relation, so long as they all exist.[23,24] Thinking of properties as one-place relations, this entails that if something has a property, then it has that property necessarily, so long as it exists.

Compossibility says that given any two possible pluralities, it could have been that those pluralities existed together. Our background logic includes Everything. Thus, in terms of worlds, Compossibility entails that given any two worlds, there is a third world

23. Immutability is a schema. Thus, R can be replaced by any *basic* predicate in the language. These predicates may include predicates with argument places taking plural terms. In that case, the singular variables and singular quantifiers will need to be replaced with plural variables and plural quantifiers. The same will apply for any other schemas given throughout the rest of this paper.

Note that our language does not have lambda abstraction, which would let us convert arbitrary conditions into complex predicates. Even if it did, lambda abstracts are not basic predicates, and so will not be substitution instances.

24. Singular and plural existence predicates are defined in the usual way.

$\mathsf{Exists}(x) \equiv_{def} \exists y(x = y)$
$\mathsf{Exist}(xx) \equiv_{def} \exists yy(xx = yy)$

in which everything from the first two exists together.

There is a sense in which Immutability, on its own, does not capture the full immutabilist picture. For the immutabilist says that there is no contingency with respect to how things are, only contingency with respect to what exists. The problem is that while Immutability guarantees that things are immutable across worlds in which they *exist*, it does not guarantee that things are immutable across worlds in which they *fail* to exist.

My own preference is to solve the problem by adopting serious actualism. This is the view that things have properties and stand in relations only when they exist.[25]

Actuality $\Box\forall x_1 \ldots \Box\forall x_n \Box(R(x_1, \ldots, x_n) \supset \mathsf{Exists}(x_1) \wedge \ldots \wedge \mathsf{Exists}(x_n))$

Actuality guarantees that non-existing things never have properties nor stand in relation. Thus, there is no concern that non-existent things could have had *different* properties or stood in *different* relations.

For those who are not serious actualists, an alternative would be to replace Immutability with a stronger principle.

Strong Immutability $\Box\forall x_1 \ldots \Box\forall x_n \Box(R(x_1, \ldots, x_n) \supset \Box(R(x_1, \ldots, x_n)))$

Strong Immutability says that if things stand in a relation, then necessarily, they stand in that relation *regardless* of whether they exist. In contrast, Immutability only says that if things stand in a relation then, necessarily, they stand in that relation *assuming* that they exist.

Compossibility tells us that two possible worlds can always be pasted together. There is still the question, though, of when worlds can be cut apart. For my own part, I accept the following two principles:

Coexistence $\Box\forall x \Box\forall y \Box(x \sim y \equiv \Box(\mathsf{Exists}(x) \equiv \mathsf{Exists}(y)))$
Separability $\Box\forall xx \Diamond(\mathsf{Exists}(xx) \wedge \forall y \exists x(x \prec xx \wedge x \sim y))$

Coexistence says necessarily, if two things are connected, then they are necessarily coexistent. Thus, existence is a holistic matter. When one thing from a possible universe exists, then everything else from that possible universe exists as well. Separability says

25. We are going to restrict the substitution instances of R to *non-logical* predicates. Thus, things are allowed to stand in logical relations, even when they fail to exist. The view that things fail to stand in even logical relations when they fail to exists is a view that you might call *very* serious actualism. I have no particular opposition to very serious actualism, but it creates additional complications, which would be a distraction from the task at hand.

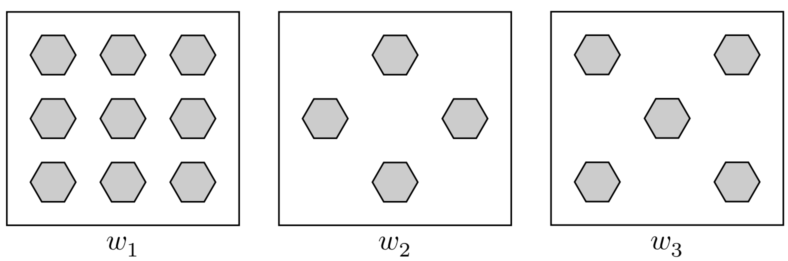
that necessarily, given any things, it could have been that those things existed, and that every thing was in the same universe as at least one of them. Given Coexistence, you can think of this as the claim that given a *plurality* of universes, those universes could have existed without any others.

Note that given Coexistence, the immutabilist could drop connectedness as a basic predicate. This is because she can define connectedness with:

$$x \sim y \equiv_{def} \Box(\mathsf{Exists}(x) \equiv \mathsf{Exists}(y)) \tag{11}$$

Thus, *what it is* for two things to be in the same universe is for them to be necessarily coexistent. There are several advantages to this approach. One is that that there is no need to treat Coexistence as a basic axiom. Another is that there is no longer any need for axioms saying that connectedness is an equivalence relation, since these will follow from the background logic, given (11).

Our system **CI** includes Immutability, Compossibility, Actuality, Coexistence, and Separability. It thus represents what you might think of as *cut-and-paste* theory of modality. For example, suppose that we start with world w_1 below, with the hexes being universes. Coexistence and Separability then tells us that we can cut w_1 apart,



so long as we are careful to cut around universes, and not through them. Thus, the possibility of w_1 entails the existence of w_2 and w_3. On the other hand, suppose that we start with w_2 and w_3. Compossibility then tells us that we can paste these worlds together to get w_1. Immutability and Coexistence ensure that when worlds are cut apart or pasted together, universes remain the same intrinsically. Actuality ensures that there are no changes with respect to non-existing universes. Thus, the only differences across worlds are with respect to which universes there are.

Compossibility says that two worlds can always be pasted together. Alternatively, though, you could accept something stronger. Namely:

Possible Pluriverse $\Diamond\exists xx\Box\forall x(x \prec xx)$

This says that there could have been a pluriverse, where a pluriverse is a plurality that includes every possible individual. Thus, where Compossibility says that two worlds can always be pasted together, Possible Pluriverse says that all the worlds *whatsoever* can be pasted together.

I am undecided about Possible Pluriverse. In fact, I am somewhat inclined to deny it. Denying Possible Pluriverse is equivalent to accepting:

Extensibility $\Box \forall xx \Diamond \exists x (x \not\prec xx)$

This says that necessarily, given any plurality of things, there could have been something that was not among them. Thus, modality turns out to be indefinitely extensible, in a certain sense.

Now that we have immutabilism fully on the table, it can be distinguished from other views in the literature.

Necessitarianism $\Box \forall x_1 \ldots \Box \forall x_n \Box(\phi(x_1, \ldots, x_n) \supset \Box(\phi(x_1, \ldots, x_n)))$
Necessitism $\Box \forall x \Box(\mathsf{Exists}(x))$

The first is necessitarianism. This is the view that necessarily, whenever a condition is satisfied, it is necessarily satisfied. There is thus no contingency whatsoever. The second is necessitism. This is the view that necessarily, everything necessarily exists. There is thus no contingency with respect to what exists, though there may be contingency with respect to how things are.

An immutabilist can accept that there is contingency with respect to what exists. Thus, an immutabilist need not be a necessitist. But necessitarianism is equivalent to the conjunction of necessitism and immutabilism. Thus, an immutabilist need not be a necessitarian either.

In fact, our own system **CI** includes Plural Comprehension, which entails Nothing. But it also includes Separability. These together entail that there could have been nothing. Given the background modal logic, this in turn entails:

Strong Contingentism $\Box \forall x \Diamond(\neg \mathsf{Exists}(x))$

Strong Contingentism says that necessarily, everything is contingent. Thus, there are no necessarily existing things, nor could there have been.

As we noted earlier, compossible immutabilism is one immutabilist view. Another is Leibnizian immutabilism. Leibnizian immutabilism says that individuals are always worldbound. It thus accepts:

Worldbound $\Box \forall x_1 \ldots \Box \forall x_n (\phi(x_1, \ldots, x_n) \supset \Box(\mathsf{Exists}(x_n) \supset \phi(x_1, \ldots, x_n)))$

This says that necessarily, every individual is such that necessarily, had anything been any different, it would not have existed. Worldbound entails Immutability, and so Leibnizian immutabilism is a genuine form of immutabilism. A compossible immutabilist, though, cannot accept Worldbound, since it is inconsistent with Compossibility, given minimal assumptions.[26] Thus, compossible immutabilism and Leibniz immutabilism are incompatible.

2.4 De Res Contingency

The most obvious objection to immutabilism is that it simply gets the modal facts wrong. The mug on my desk is blue, but could have been a different color. Socrates is a philosopher, but could have been a mathematician instead. Thus, immutabilism fails because it is simply false that things always have their properties and relations necessarily.

There are various responses an immutabilist could make. Here, we will consider two. The first is to distinguish between fundamental and non-fundamental individuals. The fundamental individuals are the basic building blocks of reality. These might include things like particles. Everything else is non-fundamental. These include things like cars, trees, and coffee mugs. A similar distinction can be drawn between fundamental and non-fundamental properties and relations.

An immutabilist might then restrict her immutabilism to the fundamental. That is, as a matter of necessarily, when *fundamental* things stand in fundamental relation, then they necessarily stand in that relation, whenever they exist. This allows ordinary things, which are not fundamental, to be mutable.

You might object that this still gets the modal facts wrong, since fundamental things are also mutable. But why think that? Maybe common sense suggests that they are. But even if so, there is no reason to expect common sense to be a reliable guide to the metaphysics of fundamental things, like particles.

The second response is to distinguish between two kinds of modality. On the one hand, there is fundamental modality, which is what the immutabilist uses when doing physics. On the other hand, there is ordinary modality, which is the stuff of common sense.

The immutabilist, then, might claim that her immutabilism is restricted to fundamental modality. Thus, her view is that necessarily (in the fundamental sense), if individuals stand in a relation, it is necessary (in the fundamental sense) that they stand in that relation, assuming they exist. But this is compatible with the view that there are individuals that could (in the ordinary sense) have stood in different relations.

The challenge for the immutabilist, then, is to show that she can explain the ordinary modal facts, which are mutabilist, in terms of the fundamental modal facts, which are immutabilist.

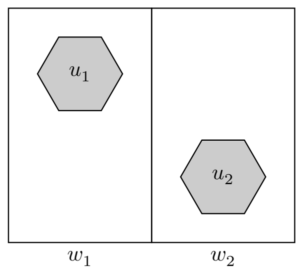
One way to do this would be to use counterpart theory. To see how this goes,

26. Those minimal assumptions being $\Diamond\exists x\Diamond\exists y(\neg\mathsf{Exists}(x))$.

suppose that an immutabilist wants to explain the fact that:

Socrates could have been a mathematician. (12)

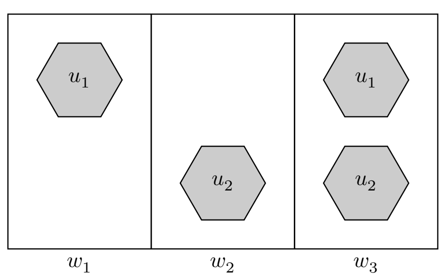
Thinking in terms of worlds, the actual world w_1 is a world in which Socrates is a philosopher and not a mathematician. This world, we can suppose, has exactly one universe u_1. It also has no non-trivial counterparts of Socrates.[27] An immutabilist



might then say that Socrates could have been a mathematician in w_1 because there is a possible world w_2 in which there is a counterpart of Socrates who is a mathematician. Call this counterpart Archimedes. This world, we can suppose, has exactly one universe u_2.

This explanation almost works. The problem is that it requires a cross-modal counterpart relation between Archimedes in w_2 and Socrates in w_1. But how are such cross-modal relations to be understood?

The immutabilist solution is to reduce relations *across* worlds to relations *within* worlds. Here is how that works in the present case: Given Compossibility, there is a third world w_3 in which the universe u_1 and u_2 exist together in a multiverse. This is illustrated on the next page. Socrates and Archimedes thus exist together in w_3. What



the immutabilist claims, then, is that Archimedes in w_2 is a counterpart of Socrates in w_1 because Archimedes w_3 is a counterpart of Socrates in w_3. Putting this all into the

27. A non-trivial counterpart of Socrates is a counterpart who is also non-identical.

official language, the proposal is to explain (12) with:

$$\Diamond\exists x(\mathsf{Math}(s) \wedge \wedge\Diamond(\mathsf{Cpt}(x,s))) \tag{13}$$

Thus, it could (in the ordinary sense) have been that Socrates was a mathematician because it could (in the fundamental sense) have been that there was someone who was a mathematician who could (in the fundamental sense) have been a counterpart of Socrates.[28]

Now hold on, you might say. Maybe Socrates and Archimedes are different in world w_3 than they are in world w_1 and w_2. Thus, they may be counterparts in world w_3. But this tells us nothing about whether Archimedes in world w_2 is a counterpart of Socrates in world w_1. Thus, we cannot use the fact that Archimedes in w_3 is a counterpart of Socrates in w_3 as a proxy for the fact that Archimedes in w_2 is a counterpart of Socrates in w_1.

The immutabilist, though, has a ready response. Necessarily, everything is immutable. Thus, Socrates and Archimedes do not differ across worlds because *nothing* differs across worlds. Thus, if Archimedes in w_3 is a counterpart of Socrates in w_3, this is enough to ensure that Archimedes in w_2 is a counterpart of Socrates in w_1.

Once again, this almost works, but there is still a concern. Immutability guarantees that Socrates and Archimedes are held fixed across worlds. With the help of Coexistence, it also guarantees that the universes they inhabit are held fixed across worlds. But still, the counterpart relation might be sensitive to facts about which other universes exist, and *those* facts are not held fixed across worlds.[29]

To illustrate the general problem: Suppose that Socrates in w_1 is the smartest person in the world. Moreover, suppose that being the smartest person in the world is necessary and sufficient for being a counterpart of Socrates. Next, suppose that Archimedes in w_2 is the smartest person in the world though, as it turns out, Archimedes in w_2 is somewhat less keen than Socrates in w_1. In that case, Archimedes in w_2 may be a counterpart of Socrates in w_1. But Archimedes in w_3 is not a counterpart of Socrates in

28. There is a complication here. Socrates is not fundamental. Neither is the property of being a mathematician nor the relation of being a counterpart. Thus, the fundamental language will not have any corresponding names or predicates. But then how are we explaining ordinary modal facts in terms of fundamental modal facts?

While this is left implicit in the main text, the proposed explanation goes in stages. First, we explain the *immutabilist* modal facts about non-fundamental individuals (like Socrates) and non-fundamental properties and relations (like *being a mathematician* and *being a counterpart*) in terms of fundamental immutabilist modal facts involving fundamental individuals (like particles) and fundamental relations (like betweenness and congruence). We then use these mid-level immutabilist modal facts to explain the ordinary mutabilist modal facts using counterpart theory.

29. Thanks to Ted Sider for raising this objection.

w_3. After all, Socrates is the smartest person in world w_3, and so is his only counterpart. Thus, there is no way to reduce the counterpart relation spanning w_2 and w_1 with a counterpart relation within w_3.

Putting the point more generally, we can distinguish between two kinds of relations. **Steady** relations are ones that hold across possible universes, regardless of whether those universes are in separate worlds or the same world, so long as the intrinsic character of each universe is held fixed. **Unsteady** relations are relations that do not. The objection, then, is that the counterpart relation may be unsteady. But the proposed strategy for reducing counterpart relations across worlds only works if counterpart relations are steady.

We could respond by insisting that counterpart relations are always steady. For my own part, I suspect that this might be. Still, it would be nice to have a less dogmatic solution.

One approach is to to use a counterpart relation with **plural indexing**. Thus, facts involving the ordinary two-place counterpart relation are explained using a four-place counterpart relation.

$$\mathsf{Ctp}(x, xx, y, yy) \tag{14}$$

Read this as saying that x relative to the xx is a counterpart of y relative to the yy. Or somewhat more intuitively, x is a counterpart of y, thinking of the xx as the entire world that x inhabits and the yy as the entire world that y inhabits.

We can now return to our problematic case. What we want to explain is why Archenemies in w_2 is a counterpart of Socrates in w_1. The only universe in w_1 is u_1 and the only universe in w_2 is u_2. What we claim, then, is that Archenemies in w_2 is a counterpart of Socrates in w_1 because in world w_3, Socrates is a counterpart of Archimedes, thinking of u_1 as the entire world that Socrates inhabits and u_2 as the entire world that Archimedes inhabits. This is true, despite the fact that in w_3, Socrates is *not* a counterpart of Archimedes, thinking of the plurality that includes both u_1 and u_2 as the world that they both inhabit.

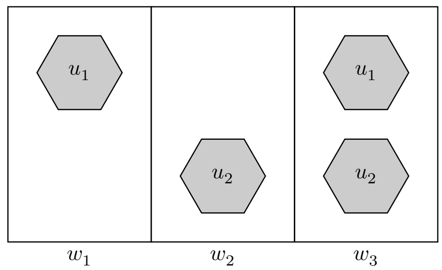
Summing up: The immutabilist can reduce ordinary *de res* modal claims using counterpart theory. This would seem to require counterpart relations across worlds. But those counterpart relations across worlds can be reduced to counterpart relations within worlds. This can be done because the immutabilist accepts both Compossibility and Immutability.[30]

30. There are further potential complications here. The counterpart theory we briefly sketched is broadly Lewisian. Lewisian counterpart theory, though, is materially inadequate in certain respects. Solutions generally involve quantifying over things like functions, sequences, or relations. But in that

2.5 Cross-Modal Comparisons

We saw in the last section that the immutabilist can explain cross-modal counterpart relationship. The basic strategy was to reduce relations across worlds to relations within worlds. That same strategy can also be used to explain cross-modal comparisons.

Suppose that we want to explain the fact that Socrates could have been taller than he actually is. The actual world is w_1 and contains a single universe u_1. Socrates, we



can suppose, is the only non-trivial counterpart of Socrates in w_1. We then claim that Socrates could have been taller than he actually is in w_1 because there is a world w_2 in which there is someone who is a counterpart of Socrates in w_1 and taller than Socrates is in w_1. Call this counterpart Archimedes.

This almost works. The problem is that we need two relations across worlds. First, we need a counterpart relation between Archimedes in w_2 and Socrates in w_1. Second, we need a *taller than* relation between Archimedes in w_2 and Socrates in w_1.

Fortunately, both of these relations *across* worlds can be reduced to relations *within* worlds. By Compossibility, there is a third world w_3 in which both u_1 and u_2 exist. Hence, this is a world in which Socrates and Archimedes both exist. Moreover, this is a world in which Archimedes is both a counterpart of Socrates and taller than Socrates. Hence, Archimedes in w_2 is both a counterpart of Socrates in w_1 and taller than Socrates in w_1. Putting this all into the official immutabilist language, Socrates could have been taller than he actually is because:

$$\Diamond(\exists x \Diamond(\mathsf{Cpt}(x, s) \wedge \mathsf{Taller}(x, s))) \tag{15}$$

This says that it could have been that there was someone who could have been both a counterpart of Socrates and taller than Socrates.

You might worry that (15) misses the mark. After all, suppose that Archimedes

case, how is a *nominalist* going to do counterpart theory? My own preference is to use a conventional permissibility operator in place of such quantification. Another option would be for a nominalist to do counterpart theory within a mathematical fiction. The result will be a fictionalism about ordinary modal claims that roughly matches her fictionalism about mathematics. This is not the place, though, to get into the details of either proposal.

in world w_3 is taller than Socrates. Still, it might be that Socrates and Archimedes are different heights in w_3 than they are in w_1 and w_2. Thus, from the fact that Archimedes is taller than Socrates in w_3, it does not follow that Archimedes in w_2 is taller than Socrates in w_1.

The immutabilist, though, will insist that it does. Her view is precisely that nothing ever differs across worlds. As such, the height of Socrates and Archimedes does not differ across worlds. The *taller than* relation is a steady relation. So the fact that Archimedes in w_3 is taller than Socrates in w_3 is enough to guarantee that Archimedes in w_2 is taller than Socrates in w_1.

Another concern you might have is that while the immutabilist can explain some cross-modal comparisons, she cannot explain those in which the existence of a multiverse is explicitly denied. For example, consider the claim that:

> Socrates could have been taller than he actually is *without having been in a multiverse*. (16)

An immutabilist, though, can explain these sorts of facts as well. Thinking in terms of our above diagram, suppose that Socrates in w_1 could have been taller than he actually is, without having been in a multiverse. This is true, the immutabilist claims, because there is a world w_2 in which there is someone who is not in a multiverse. Call this person Archimedes. Archimedes in w_2 is then a counterpart of Socrates in w_1 and taller than Socrates in w_1. This is because there is a third world w_3 in which Archimedes and Socrates both exist. Moreover, Archimedes in w_3 is both a counterpart of Socrates in w_3 and taller than Socrates in w_3. Putting this all in terms of the official language, (16) is explained with:

$$\Diamond \exists x (\forall y (x \sim y) \land \Diamond (\mathsf{Cpt}(x, s) \land \mathsf{Taller}(x, s))) \tag{17}$$

Thus, Socrates could have been taller than he actually is, without having been in a multiverse, because it could have been that there was someone, who was not in a multiverse, who could have been both a counterpart of Socrates and taller than Socrates.

3 Fictionalism

We described compossible immutabilism in the §2. This lets us solve the problem of cross-modal comparisons, which was posed in §1.3. We are now going to move towards giving a theory of distance ratios in §4. That project will be made easier, though, if we have an additional tool.

That tool is **pluriverse fictionalism**. According to the pluriverse fiction, there is vast plurality of concrete universes that includes every possible universe. Because we are interested in nominalism, our pluriverse fiction will be a *nominalist* pluriverse fiction.

According to the fiction, there are no numbers or universals or other things of that sort. Thus, according to the fiction, modal realism is true, but Platonism is not.

What we are going to show in the appendix is that fictionalist talk about the pluriverse is structurally equivalent to immutabilist talk in terms of operators. Thus, any fictionalist theory T_F can be directly translated into an immutabilist theory T_I with the same logical structure.

This suggests a natural approach to building immutabilist scientific theories. First, we build a physical theory T_F in the pluriverse fiction. To show that this theory is empirically adequate, we expand the fiction to include an appropriate set theory. This gives us a combined fiction in which we can prove the necessary representation theorems. Once we have those, we prove conservativeness. This establishes that T_F is empirically adequate. We then translate the fictionalist theory T_F to the immutabilist theory T_I. T_I is structurally equivalent to T_F. Structural equivalence preserves empirical adequacy. Thus, since the fictionalist theory T_F is empirically adequate, so is the immutabilist theory T_I.

The advantage of this procedure is that as an immutabilist, you can do science within the pluriverse fiction. You can also build immutabilist theory directly, if you want. But for my own part, I find it easier to work in the pluriverse fiction, and then translate the results.

In the rest of this section, we are going to build a fictionalist language and logic that an immutabilist can use to build scientific theories. Those readers who trust that this can be done should feel free to skip ahead to §4.

3.1 Language

We are going to think of the fictionalist as having plural language $\mathcal{L}_F$. This has much of the same syntax as the immutabilist language from §2.2.

Singular variables	$x, y, z, \ldots$
Plural variables	$xx, yy, zz, \ldots$
Singular quantifiers	$\exists x$ and $\forall x$
Plural quantifiers	$\exists xx$ and $\forall xx$
Truth-functional operators	$\wedge, \vee, \supset, \neg$
Logical predicates	$=_s, =_p, \prec, \sim$

There are also important differences, though. Unlike the immutabilist language, the fictionalist language has no modal operators. Instead, it has an actuality predicate:

$\mathsf{Act}(x)$ x is actual

Thus, the fictionalist draws a basic distinction between actual and non-actual individuals. The immutabilist has no need for such a distinction, since her quantifiers only range over actual individuals.

These are all the basic resources of the fictionalist language. It will be helpful, though, to also have a slate of defined plural predicates.

$\mathsf{Act}(xx)$	the xx are actual
$\mathsf{Clo}(xx)$	the xx are closed under connectedness
$\mathsf{Uni}(xx)$	the xx are a universe
$\mathsf{Multi}(xx)$	the xx are a multiverse

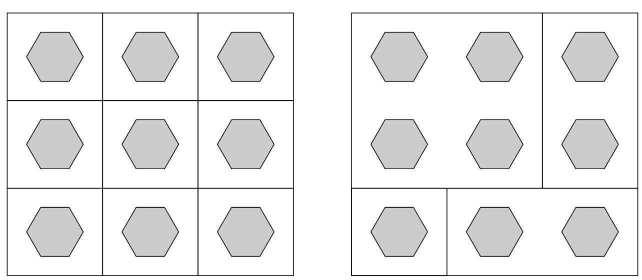
The plural actuality predicate is defined using the singular actuality predicate.[31] The rest are defined using the connectedness relation, as they were in §2.2. As before, we will talk about pluralities of universes, but these pluralities of universes are in fact just pluralities of individuals, which are closed under connectedness.

3.2 Pluralism about Worlds

We have a fictionalist language then. This language has the resources for talking about universes, along with which of those universes are actual. This brings us to a natural question: What are possible worlds?

When it comes to either fictionalism or modal realism, we can distinguish between two views. The **singularist** says that a possible world is an individual universe. The **pluralist** says that a possible world is any *plurality* of universes whatsoever.

In visual terms, the singularist says that the pluriverse is as illustrated on the left. The hexes are universes and the boxes are worlds. The pluralist, on the other hand, says that the pluriverse is as illustrated on the right. For the pluralist, any plurality of



universes *whatsoever* counts as a possible world. This is hard to illustrate, though, so we have included only a selection of worlds. Some have only one universe. Some have many.

Following David Lewis (2001), fictionalists and modal realists have generally been singularists. But singularism faces two serious problems. The first is that there is no

31. $\mathsf{Act}(xx) \equiv_{def} \forall x(x \prec xx \supset \mathsf{Act}(x))$

possible world in which there is a multiverse. The second is that there is no possible world in which there is nothing. But there *could* have been a multiverse and there *could* have been nothing. So singularism would seem to be materially inadequate.[32]

Pluralism avoids both problems. There can be pluralities of universes that contain many universes. Hence, there can be possible worlds in which there is a multiverse. There can also be an empty plurality. The empty plurality is trivially closed under connectedness. So there can also be a possible world in which there is nothing.

Our fictionalist is going to be a pluralist about possible worlds. This means that we will also have a defined plural world predicate.[33]

$\mathsf{World}(xx)$ the xx are a possible world

Thus, when the fictionalist says that there is a possible world, what she really means is that there are things that together form a possible world.

3.3 Principles

We have both a fictionalist language and a view about possible worlds. What we want to do now is find principles that guarantee that fictionalist talk can always be translated as immutabilist talk, and visa-versa.

More precisely: We want to identify a minimal immutabilist system **I** and a minimal fictionalist system **F** such that any sentence in the immutabilist language $\mathcal{L}_I$ can be translated as a sentence of the fictionalist language $\mathcal{L}_F$, and visa-versa, with these translations preserving logical entailment.

Two such systems are fully specified in the appendix. It will be useful, though, to say bit about how these systems work in general terms.

The immutabilist system **I** is strictly weaker than the system **CI** we built earlier. One is that Separability and Compossibility are dropped. The other is that Plural Comprehension is replaced with Everything. Thus, as a plural logic, **I** is quite weak. The only plurality whose existence it guarantees is the universal plurality.

The matching fictionalist system **F** is classical. Thus, unlike **I** or **CI**, the system has classical rules for the quantifiers and axioms guaranteeing the connectedness is an equivalence relation. There are then other various basic principles, the first three of

32. Lewis raises both objections himself on pp. 72-3 of his (2001). His response is to just bite the bullet. There could not have been a multiverse, nor could there have been nothing. For other potential solutions, see Yablo (1999), Sider (2003), Bricker (2001), and Parsons (2007).

33. $\mathsf{World}(xx) \equiv_{def} \mathsf{Clo}(xx)$

which have to do with the actuality predicate.

Actual World	$\exists xx \forall y(y \prec xx \equiv \mathsf{Act}(y))$
Actual Closure	$\forall x \forall y(\mathsf{Act}(x) \wedge x \sim y \supset \mathsf{Act}(y))$
Actual Names	$\mathsf{Act}(t)$ when t is a name

The first says that there is a plurality of all the things that are actual. The second says that if one thing in a universe is actual, then everything else in that universe is actual. The third says that individuals and pluralities with names are actual.

Finally, we have a fourth principle governing how basic relations interact with possible worlds. This is easiest to state if we have a defined subplurality predicate.[34]

$xx \prec^* yy$ the xx are a subplurality yy

When thinking in terms of individuals, the principle says that whenever things stand in basic relations, there is some possible world in which they all exist.

Relation World $\forall x_1 \ldots \forall x_n(R(x_1, \ldots, x_n) \supset \exists yy(\mathsf{World}(yy) \wedge (x_1 \prec yy \wedge \cdots \wedge x_n \prec yy)$

This is a schema, though, and we are allowing predicates with plural arguments as substitution instances. When doing so, the corresponding variables and quantifiers are replaced with plural variables and quantifiers. The corresponding $\prec$ predicates are also replaced with $\prec^*$ predicates.

Two further principles follow from this one. Individuals and pluralities always stand in identity relations to themselves. Thus, Relation World gives us :

Possible Individuals	$\forall x \exists yy(\mathsf{World}(yy) \wedge x \prec yy)$
Possible Pluralities	$\forall xx \exists yy(\mathsf{World}(yy) \wedge xx \prec^* yy))$

The first says that every possible individual is in a world. The second says that every possible *plurality* is also in a world.

This gives us the systems we wanted. Any fictionalist theory in **F** can be translated as an immutabilist theory in **I**, and visa-versa. The translations are provided in the appendix. The basic idea, though, is that fictional quantification over the pluriverse is translated as modalized quantification. Quantification over worlds is translated using modal operators. There is some complexity involving the actuality predicate but, otherwise, everything else is left the same. Going the other way, immutabilist modal operators are translated using quantification over worlds. Immutabilist quantifiers are

34. $xx \prec^* yy \equiv_{def} \forall z(z \prec xx \supset z \prec yy)$

translated as restricted quantifiers.

Once we have our translations, there are certain striking connections between the principle we might accept on the fictionalist side and the principles we might accept on the immutabilist side. For example, suppose we accept full Plural Comprehension on the fictionalist side. When translated, this corresponds to accepting not only Plural Comprehension, but also Possible Pluriverse and Separability on the immutabilist side. Or going the other way, suppose we accept Possible Pluriverse on the immutabilist side. This corresponds to accepting Everything on the fictionalist side.

Finally, it should be pointed that the variety of modal fictionalism I support is rather different than the modal fictionalism described by Rosen (1990, 1995). Rosen aims to reduce modal facts to facts about the fiction. It could have been that ϕ because, according to the fiction, there is a possible world at which ϕ. My own view is exactly the reverse: What is true in fiction is true in the fiction because it appropriately represents the modal facts.[35] The pluriverse fiction is thus a useful tool for reasoning about the modal facts, not a strategy for reducing or eliminating them.

4 Distance Ratios

We are now going to give an immutabilist theory of distance ratios. That theory will be nominalist, in the sense that it will only quantify over actual particles. The theory requires comparisons across worlds, so will a *compossible* immutabilist theory. This solves the problem of quantities raised in §1.

Our basic approach will be the one suggested at the beginning of §3. That is, we are going to start by giving a fictionalist theory of distance ratios in §4.1. Thus, we will be showing how a nominalist could give a theory of distance ratios, if modal realism *were* true. But of course, modal realism is not true. So we will translate the fictionalist theory to an immutabilist theory in §4.2.

4.1 Fictionalist Distance Ratios

Our fictionalist theory will use the fictionalist language from §3.1, extended to include the betweenness and congruence predicates.[36] There are different ways to go here, but we will think of the fictionalist as accepting a pluriverse of all physically possible

35. Thus, the brand of fictionalism I support has more in common with the non-proxy reduction strategy from Fine (2005) or the ersatz pluriverse from Sider (2002).

36. In fact, to state the Archimedean axiom, we need a language with some sort of device for saying *there are finitely many*. My preferred approach is to add a one-place logical plural predicate $\mathsf{Fin}(xx)$ saying that the xx are finite.

universes, rather than a pluriverse of all metaphysically possible universes.

The general strategy will be to give a theory of distance ratios by using congruence relations across worlds. But this means that our fictionalist will be departing from Lewis's modal realism in yet another important respect.

Lewis says two things about spatiotemporal relations and universes. First, he says that two things are in the same universe when there is *some distance* between them. Second, he says that two things are in the same universe when there are *spatiotemporal relations* between them.[37]

The second condition is problematic because it rules out congruence relations across universes. After all, congruence relations are spatiotemporal relations. Thus, if there are no spatiotemporal relations across worlds, there are no congruence relations across worlds.

Lewis seems to think of his two conditions as equivalent. But in fact, they are very different. The second is much stronger. Our strategy, then, will be to accept the first while rejecting the second. Thus, two things are in the same universe when there is some distance between them. But two things can stand in spatiotemporal relations without being in the same universe.

Filling in the details, say that a and b are **self-congruent** when a and b are congruent with a and b. Our fictionalist thus claims that two things are in the same universe if and only if they are self-congruent:

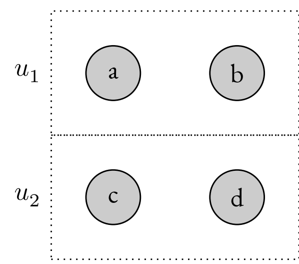
$$\forall a \forall b (a \sim b \equiv \mathsf{Cong}(a, b, a, b)) \tag{18}$$

There is some distance between two things if and only if they are self-congruent. So this can also be read as expressing the Lewisian idea that two things are in the same universe if and only if there is some distance between them.

This allows for congruence relations across universes. For example, consider the two universes illustrated below. Particles a and b are in the same universe because they are self-congruent. Likewise for c and d. Particles a and b are also congruent with c and d. We can thus use congruence to compare two particles from the first universe with two particles from the second. What we cannot do is use congruence to relate two particles from *different* universes with any other two particles. For example, a and c cannot be congruent with b and d, nor can a and c be congruent with c and d.[38] Thus,

37. See Lewis (2001, pp.2).

38. Suppose for reduction that a and b are in different universes, but that a and b are congruent with some x and y. By Cong-Symmetry and Cong-Transitivity (which are given below), a and b are then congruent with a and b. But then by (18), a and b are in the same universe, which is contrary to assumption.



we are allowing certain congruence relations across worlds, but not others.

Next, we want to rule out betweenness relations across universes. These are not needed and, moreover, would be problematic.[39] Thus:

$$\forall a \forall b \forall c (\mathsf{Bet}(a, b, c) \supset \mathsf{Cong}(a, b, a, b) \wedge \mathsf{Cong}(b, c, b, c)) \tag{19}$$

You can read this as saying that b is between a and c only if there is some distance between a and b and some distance between b and c. Given (18), this entails that if b is between a and c, then all three particles are in the same universe. There are thus no betweenness relations across universes.

Putting these things together, the basic picture is one on which there is a pluriverse in which universes are pluralities of spatiotemporally connected particles. For the most part, there are no basic relations across universes. The only exceptions are certain congruence relations.

When giving a substantivalist theory of distance ratios, there are two kinds of axioms. There are **existence axioms**, which entail that certain spacetime points exist. The other axioms are what you might call **restriction axioms**, since they merely restrict how spacetime points can be arranged.

Our fictionalist theory will also have both existence axioms and restriction axioms. The difference is that where a substantivalist uses existence axioms to fill space with points, a fictionalist uses existence axioms to fill the pluriverse with universes.

We are not going to give a full slate of axioms here. For our purposes, though, it will be useful to have three restriction axioms.

39. This because they would leave the fictionalist vulnerable to shift arguments. For example, suppose that u_1 has exactly two particles a and b and that these particles are exactly one foot apart. If we have cross-universe betweenness relations, we can define cross-universe colocation relations. We can thus describe one universe u_2 that contains exactly one particle c and another universe u_3 that contains exactly one particle d, with c colocated with a and d colocated with b. Thus, u_2 is just like u_3, except that the location of everything is shifted by a foot.

Cong-Symmetry:	$\forall a \forall b \forall c \forall d(\mathsf{Cong}(a, b, c, d) \supset \mathsf{Cong}(c, d, a, b)$
Cong-Transitivity:	$\forall a \forall b \forall c \forall d \forall e \forall f(\mathsf{Cong}(a, b, c, d) \wedge \mathsf{Cong}(c, d, e, f)$ $\supset \mathsf{Cong}(a, b, e, f))$
Three-Segment:	$\forall a \forall b \forall c \forall d \forall e \forall f(\mathsf{Cong}(a, b, d, e) \wedge \mathsf{Bet}(a, b, c) \wedge$ $\mathsf{Bet}(d, e, f) \supset (\mathsf{Cong}(a, c, d, f) \equiv \mathsf{Cong}(b, c, e, f)))$

The first says that congruence is symmetric. The second says that congruence is transitive. The third is a simplified version of what is sometimes called the five-segment axiom.

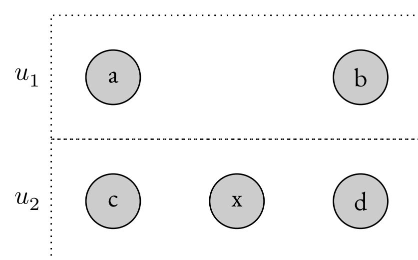
As an illustration of an existence axiom, suppose that the substantivalist has an axiom saying that for any two spacetime points, there is a third somewhere between them.

$$\forall a \forall b \exists x(\mathsf{Bet}(a, x, b)) \tag{20}$$

For our fictionalist, the analogous axiom will say that for any pair of particles, there is a pair of duplicates, somewhere in the pluriverse, that have a third particle somewhere between them.

$$\forall a \forall b \exists c \exists d \exists x(\mathsf{Cong}(a, b, c, d) \wedge \mathsf{Bet}(c, x, d)) \tag{21}$$

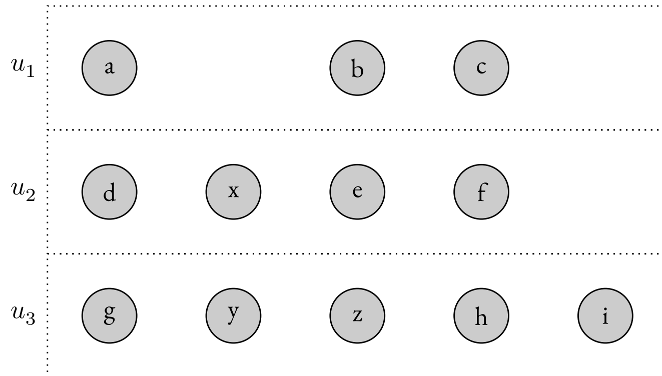
Now suppose that there is a universe u_1 with exactly two particles, as illustrated below. Since the duplicates required by (21) do not exist in u_1, the result will be a second



universe u_2 in which they do exist. Thus, existence axioms, like this one, are what is used to ensure that the pluriverse is sufficiently plentiful.

Once we have a slate of appropriate axioms, we want to show that the fictionalist can solve the problem of uniqueness. To simplify a bit, we can suppose that we only need to show that if a and b are twice as far apart as b and c, then they are not also three times as far apart.

Consider a pluriverse with exactly three universes, as pictured on the next page. Our universe is u_1 and has exactly three particles. We can suppose that a and b are twice as far apart as b and c, and so this is the distance ratio that we want to explain. The fictionalist proposal is that this distance ratio is explained by the existence of a universe like u_2. That is, a and b are twice as far apart as b and c in u_1 because there is a universe



u_2 in which the particles d, x, e, and f form an equally spaced line. Moreover, a and b are congruent with d and e, and b and c are congruent with e and f.

Now suppose for reductio that a and b are not only twice as far apart as b and c, but three times as far apart. For the fictionalist, this means that there is a universe u_3 with particles g, y, z, h, and i forming an equally spaced line. Moreover, a and b are congruent with g and h, and b and c are congruent with h and i.

Given these assumptions, we can now prove a contradiction using our three restriction axioms. First, we use symmetry and transitivity to show that d, x, e, and f are pairwise congruent with y, z, h, and i. By two applications of the three-segment axiom, we then have $\mathsf{Cong}(d, f, y, i)$. Since $\mathsf{Cong}(d, f, g, i)$ by symmetry and transitivity, another application of the three-segment axiom gives us $\mathsf{Cong}(d, d, g, y)$. So g and y are colocated. But g and y are part of an equally spaced line, so they are *not* colocated. So we have a contradiction. Thus, the very same particles in the very same world cannot be both twice as far apart and three times as far apart.

4.2 Immutabilist Distance Ratios

We are now going to sketch our immutabilist theory of distance ratios. The theory uses an immutabilist language that extends the fictionalist language with a pair of modal operators. The theory itself is a translation of the fictionalist theory we gave in §4.1 using the translation scheme from the appendix, which was briefly described at the end of §3.3.

To translate our fictionalist theory, we can start with restriction axioms. The fictionalist claims, for example, that congruence is symmetric. Applying the immutabilist translation scheme gives:

$$\Box\forall a\Box\forall b\Box\forall c\Box\forall d(\mathsf{Cong}(a,b,c,d) \supset \mathsf{Cong}(c,d,a,b)) \tag{22}$$

Given Actuality, this is equivalent to:

$$\Box(\forall a\forall b\forall c\forall d(\mathsf{Cong}(a,b,c,d) \supset \mathsf{Cong}(c,d,a,b)) \tag{23}$$

Thus, where the fictionalist says that congruence is symmetric, the immutabilist says that congruence is *necessarily* symmetric. Likewise for other restriction axioms.

Now for the existence axioms. Suppose that the fictionalist accepts an existence axiom saying that, for any pair of particles, there is a duplicate pair of particles with a third somewhere between them. The immutabilist translation then gives:

$$\Box\forall a \Box\forall b \Diamond\exists c \Diamond\exists d \Diamond\exists x(\mathsf{Cong}(a, b, c, d) \wedge \mathsf{Bet}(c, x, d)) \tag{24}$$

By Actuality and Immutability, this is equivalent to:

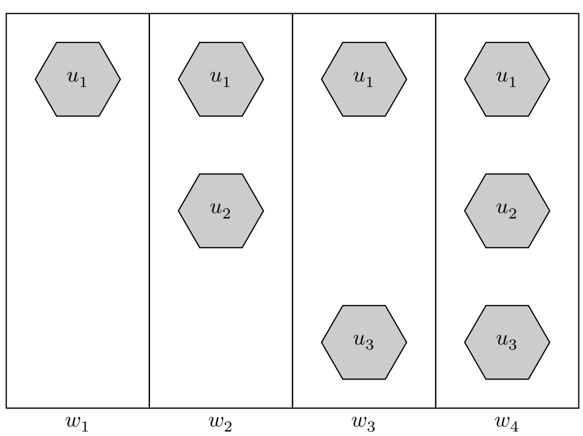
$$\Box\forall a \forall b \Diamond\exists c \exists d \exists x(\mathsf{Cong}(a, b, c, d) \wedge \mathsf{Bet}(c, x, d)) \tag{25}$$

Thus, where the fictionalist says that for any pair of particles, *there is* a duplicate pair with a third somewhere between them, the immutabilist says that necessarily, for any pair of particles, there *could have been* a duplicate pair with a third somewhere between them, somewhere in the multiverse.

Putting it another way: The substantivalist uses existence axioms to fill out spacetime and the fictionalist uses them to fill out the pluriverse. The immutabilist, on the other hand, uses them to fill out the possibilities.

Alright. What we are going to do now is show that an immutabilist can solve the problem of uniqueness. As before, we can illustrate the basic strategy by showing how to prove that the very same particles in the very same world cannot be both twice as far apart and three times as far apart.

Suppose that we have three merely possible universes u_1, u_2, and u_3. The betweenness and congruence relations within those universes are as illustrated on the preceding page. The actual world is w_1 and has exactly one universe, which is u_1. This is illustrated below. Now suppose that a and b are twice as far apart as b and c in w_1. For the



immutabilist, this is explained by the fact that there is a possible world w_2 that contains both u_1 and u_2. The particles in these universes stand in the cross-universe congruence relations illustrated on page 34.

So far so good. Now suppose for reductio that besides being twice as far apart, a and b are also three times as far apart as b and c at w_1. For the immutabilist, this means that there is a possible world w_3 that contains u_1 and u_3. The congruence relations between those universes at w_3 are not, however, as illustrated on page 34. Rather, a and b are congruent with g and h, and b and c are congruent with h and i.

We can now derive a contradiction. First, we note that if w_2 and w_3 are both possible worlds then, by Compossibility, there is a fourth possible world w_4 in which u_1, u_2, and u_3 all exist. Immutability then tells us that the congruence and betweenness relations in w_2 and w_3 carry over to w_4. The immutabilist then claims that the three restriction axioms from §4.1 are not just true, but necessary. But in that case, w_4 is in fact impossible, since we can prove a contradiction using the same proof we used in the case of modal realism. Thus, a and b are not both twice as far apart and three times as far apart as b and c in w_1.

This gives the nominalist a solution to the problem of uniqueness. With a bit more work, she can give a full slate of axioms for an empirically adequate theory of distance ratios.[40] This general strategy works for other quantities as well. And so the nominalist has a general solution to the problem of quantities, along with a new strategy for doing science without numbers.

5 State Space

We saw how to use immutabilism to give a theory of distance ratios in §4. One of the main advantages of the immutabilist approach, as compared to the existing substantivalist approach from Field (1980), is that it gives the nominalist a natural strategy for understanding state space. Thus, in this section, we are going to briefly sketch the problem, and show what an immutabilist theory of state space might look like, in very general terms. Most of the interesting details will have to be left for another time.

In his review of *Science Without Numbers*, David Malament (1982) raises what has become a serious challenge to nominalism.

The challenge goes like this: In classical physics, the dynamical laws are given using mathematical Euclidean spacetime.[41] This mathematical spacetime is just a mathematical object with certain formal features. Nominalists, of course, deny that

40. See my Berntson (2021).
41. Or Galilean spacetime or Maxwellean spacetime.

there are such things. But in that case, how is a nominalist going to state dynamical laws?

Field's solution is to trade mathematical spacetime for *concrete* spacetime. This concrete spacetime is built out of spacetime points, thought of as concrete material particulars. Thus, Field's solution is to be a substantivalist.

The problem, Malament points out, is that modern physics is often formulated in terms of not just mathematical spacetime, but also mathematical **state spaces**. A nominalist cannot accept mathematical state spaces, and so needs to find a concrete replacement. But what is that concrete replacement going to be? Call this **Malament's challenge**.

To illustrate the basic problem, Malament uses the case of Hamiltonian mechanics. His discussion is worth quoting at length:

> [I]t is simplest to identify Hamiltonian mechanics by its determination of a class of mathematical models. Each model is of form $\langle M, \Omega_{ab}, H\rangle$ where M is an even-dimensional manifold, Ω_{ab} is a symplectic form on M, and H is a smooth, real-valued ("Hamiltonian") scalar field on M. The points of M represent "possible dynamical states" of a given mechanical system. (Ω_{ab} and H jointly determine a "Hamiltonian vector field" which characterizes the dynamic evolution of the system.) Now Field can certainly try to trade Ω_{ab} and H in favor of "qualitative relations" they induce on M. If successful, he can reformulate the theory so that its subject matter is the set of "possible dynamical states" (of particular physical systems) and various relations into which they enter. But this is no victory at all! Even a generous nominalist like Field cannot feel entitled to quantify over *possible dynamical states*.

Hamiltonian mechanics can be thought of as a class of mathematical state spaces. Each mathematical state space is built using mathematical states, which are just points in a mathematical structure. These points are characterized using things like sets, functions, and real numbers.

This means that in order to give a corresponding nominalist theory, we need to do two things. First, we need to replace the sets, functions, and real numbers used to characterize mathematical states with intrinsic relations between them. Second, she needs to replace mathematical states with something concrete.

If we were Platonists, we could replace mathematical states with universals. We could, for example, replace mathematical states with **state relations**. These are binary relations between universes and times. Each such relation is a maximally specific way for a universe to be at a time. A universe then evolves over time because it stands in different state relations to different times. But nothing like this strategy is available to the nominalist.

One approach to the problem would be to look for empirically equivalent theories that use something like a concrete four-dimensional spacetime in place of mathematical state space. Perhaps this can be done, and the project deserves more attention.[42] But while this may work for some theories, it may not work in others. Thus, the nominalist has reason to want a more direct—and general—approach to the metaphysics of state space.

This is not the place to show that we can nominalize any state space theory that might come along. Nor is it the place to show, in any detail, that we can nominalize even a relatively simple state space theory like Hamiltonian mechanics. What we want to do, though, is motivate the idea that a immutabilist could, with some time and effort, give a nominalist theory of Hamiltonian mechanics.

The basic strategy is the same as before. First, we give a theory of Hamiltonian mechanics within the pluriverse fiction. Next, we use the combined fiction to prove empirical adequacy. Once we have that, we can directly translate the original fictionalist theory into an immutabilist theory. That translation procedure preserves empirical adequacy. So the resulting immutabilist theory is also empirically adequate.

To start, then, we will have a fictional pluriverse populated with **slices**. These are what you might think of as time-slices of universes. Slices are much larger than ordinary concrete particulars like cars, trees, and coffee mugs. Their existence is also much briefer. Still, neither of these are serious grounds for denying that slices are concrete particulars. Thus, the nominalist can use them to build a fictional pluriverse.

The character of each slice is fixed by what we will call **structural relations**. So for example, we might fix the number of particles in each slice by using an *at least as many particles* relation between slices. Slices with zero particles are slices such that there are no slices with strictly fewer particles. Slices with exactly one particle are slices such that the only slices with strictly fewer particles are slices with zero particles. And so on.

Suppose that after some time, we have selected enough structural relations to fix the positions of particles, along with their momentum.[43] We then need to add temporal relations to our fictional pluriverse. For these, we could use temporal betweenness and temporal congruence.

$$\mathsf{TempBet}(\mathsf{a}, \mathsf{b}, \mathsf{c})$$
$$\mathsf{TempCong}(\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d})$$

The first says that b is temporally between a and c. The second is temporal congruence. It says that the duration between a and b is the same as the duration between c and

42. See for example Chen (2017).
43. To see one way in which this might be done, see Schroeren (2020).

d. These can be used to fix duration ratios between slices in much the same way that spatial betweenness and congruence can be used to fix distance ratios between particles.

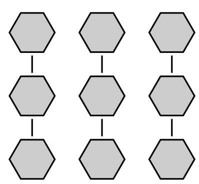
Once we have both structural relations and temporal relations, we need universes. That is, we need to say which slices are connected. One answer is that they are connected when they are temporally self-congruent.

$$a \sim b \equiv \mathsf{TempCong}(a, b, a, b) \tag{26}$$

Or perhaps more intuitively, two slices are in the same universe when there is some temporal duration between them.

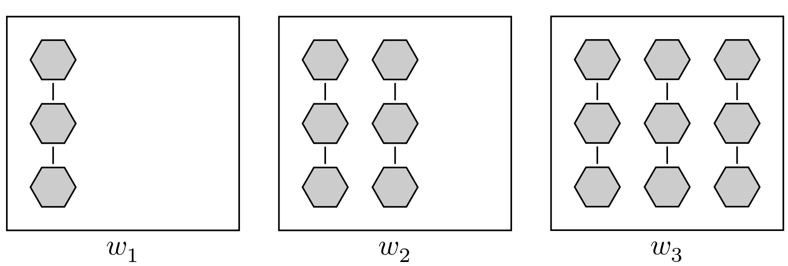
Once we have our basic relations between states, we need laws governing them. There will be laws governing the structural relations. There will also be laws governing the temporal relations. Importantly, there will also be dynamical laws telling us how temporal relations are determined, given the structural relations. Thus, the dynamical laws will tell us which slices are connected, and so which slices form universes.

This gives us a pluriverse like the one illustrated below. Hexes represent slices. There are lines between slices that are temporally connected. This gives us three strands of



slices. Each strand is a universe. The dynamical laws determine which slices stand in temporal relations and, so, which slices form universes. Slices that are not connected with lines do not form universes. Thus, take the three slices on either diagonal. These slices are not temporally connected, and so do not form a universe

There is good reason to think, then, that we can give a theory of Hamiltonian mechanics *within* the pluriverse fiction. But once we have such a theory, it can be directly translated into an immutabilist theory. That immutabilist theory will be one in which the pluriverse has been "cut up" and distributed across various worlds. For example, perhaps the actual world is w_1, with our universe being the only universe. The



character of our universe is then fixed, not by comparisons of states in our universe with states in other universes—for there are no other universes. Rather, the character is fixed by how the states in our universe could have compared to the states in other universes.

6 Objections

Immutabilism depends on the possibility of a multiverse. This, you might say, is a substantial claim about physical reality. After all, a multiverse might simply be physically impossible. Thus, the immutabilist strategy cannot succeed.

In response: I prefer immutabilist theories that require only physical possibility and necessity. But the immutabilist strategy itself is quite general, and compatible with other views. Thus, if you think that a multiverse is physically impossible, you could use a broader notion of possibility instead.

That said, I also maintain that a multiverse is physically possible. Moreover, our reason for thinking so are roughly the same as our reason for thinking that anything is physically possible.

By way of analogy: Is it physically possible for the stars to have spelled out the first line a Shakespearean sonnet? Surely yes. But what reason do we have for thinking so? The reason, it seems to me, is that such scenarios are consistent with laws as we know them. Moreover, there are no clear theoretical advantages to denying that such scenarios are possible.

The same is true when it comes to the possibility of a multiverse. A world in which there are many universes governed would seem to be compatible with the laws as we know them. Moreover, there is no serious theoretical advantage to denying such possibilities.[44] In fact, quite the opposite, given our present discussion of immutabilism.

This brings us to a second objection. Immutabilism describes the world using a language with modal operators. Thus, you might object to immutabilism on the grounds that it requires fundamental modality. But fundamental reality is not filled with threats and promises, or so you might say.

In response: We started by distinguishing between scientific nominalism and metaphysical nominalism. We then set out to defend *scientific* nominalism, which meant showing that we can construct certain kinds of scientific theories. The theories we built have modal operators, which are not defined in other terms within the theory. That much is true.

But all of this is compatible with a wide range of views about how fundamental

44. There is an interesting question about whether monism gives us reason to deny the possibility of a multiverse. See Schaffer (2013), Baron and Tallant (2016), and Siegel (2016). My own view is that it does not, but this is a question that will have to be left for another time.

scientific theories relate to fundamental reality. You could think that fundamental scientific theories describe a non-fundamental level of metaphysical reality. You could think that talk about fundamental reality is confused or misguided. Thus, nothing immediately follows regarding whether or not modality is fundamental.

In my own case, I think modality probably is fundamental. The question, though, is why we should think this is a serious cost?

Many of us are inclined to accept basic modal notions for other reasons. I myself am anti-Humean about laws. In particular, I say that when a proposition is a law, it is a law in virtue of being a metaphysical necessity. But in that case, I already have a fundamental notions of physical necessity. Thus, using physical necessity to also do science without numbers comes at no additional cost.

7 Conclusion

We started in §1 by describing scientific nominalism and one of the main challenges that it faces, which is what Field calls the problem of quantities. One natural idea is to solve the problem by using modality. But this solution, it turns out, would seem to depend on having a nominalist strategy for expressing cross-modal comparisons.

We thus described a new view about modality in §2 called compossible immutabilism. That new view allows nominalists to make cross-modal comparisons, and so solve the main problem standing in the way of a nominalist theory of quantities.

§3 developed a fictionalist language for building theories that can be converted into immutabilist theories. We then used this new language to help build a nominalist theory of distance ratios in §4. In §5, we quickly described how compossible immutabilism might help the nominalist understand state space theories. And in §6, we considered a range of objections.

Compossible immutabilism is gives the nominalist new tools for doing science without numbers. Now, we just need to put those tools to work, and see what sorts of theories we can built.

Appendix: Models, Translations, and Structural Equivalence

In this appendix, we are going to build frames and models, and then use them to give a model-theoretic specification of the systems **CI**, **I**, and **F** from the main text. We will then provide recursive translation from the sentence of $\mathcal{L}_I$ to the sentences of $\mathcal{L}_F$, and visa-versa, that preserve logical entailment, relative to the system **I** and **F**. Thus, given any immutabilist theory T_I in system **I**, there is a structurally equivalent fictionalist theory T_F in **F**, and visa-versa.

Definition 7.1: A **frame** is a tuple $\langle W, @, D, P\rangle$. This includes a set of worlds $W \subset P$, an actual world $@ \in W$, a singular domain D, and a plural domain $P \subset \mathcal{P}(D)$. Moreover, every frame also meets the following conditions:

For every $x \in D$, there is some $w \in W$ such that $x \in w$.
For every $x \in P$, there is some $w \in W$ such that $x \subseteq w$.

Each of the worlds in a frame is a set of individuals. Thus, you might think of each world as being identified with its domain. Each world is also in the plural domain. Thus, each world is also a plurality. The two indented conditions tell us that every individual is in a world, and so every individual is a possible individual. Likewise, every plurality is in a world. Thus, every plurality is also a possible plurality. Finally, given any frame, we can define a plural domain function $p : w \mapsto \mathcal{P}(w) \cap P$.

As we go on, it will be helpful to establish a notational convention. Suppose that we have a world w and a predicate R. Predicates have sorted argument places. Worlds also have both a singular domain and a plural domain. Thus, it will be helpful to have some notation for pairing each argument place of a predicate with the appropriate domain for a world. For this, we will use δ_i^w, where $\delta_i^w = w$ when the i-th argument place of R is singular, and $\delta_i^w = p(w)$ when the i-th place of R is plural. When the superscript is dropped, δ_i will refer to the appropriate outer domain for each argument place. Thus, $\delta_i = W$ when the i-th argument place of R is singular, and $\delta_i = P$ when the i-th argument place is plural.

Definition 7.2: A **valuation** is a function $[\![\cdot]\!]$ assigning denotations to names and predicates. This is done in the following way:

$$
\begin{array}{lll}
[\![c]\!] & \in & @ \\
[\![cc]\!] & \in & p(@) \\
[\![R]\!] & \subseteq & \delta_1 \times \cdots \times \delta_n \text{ when } R \text{ is a non-logical predicate}
\end{array}
$$

Definition 7.3: An **immutabilist model** $\mathcal{M}_I = \langle W, @, D, P, [\![\cdot]\!]^I\rangle$ is a frame together with a valuation function for the names and predicates in $\mathcal{L}_I$.

Definition 7.4: A **fictionalist model** $\mathcal{M}_F = \langle W, @, D, P, [\![\cdot]\!]^F\rangle$ is a frame together with a valuation function for the names and predicates in $\mathcal{L}_F$.

Definition 7.5: Let $\mathcal{M}_I$ be an immutabilist model. When $w \vDash_\sigma \phi$, we say that ϕ is true at world w relative to variable assignment σ. This relation is defined recursively:

$$\begin{array}{lll}
w \vDash_\sigma R(t_1, \dots, t_n) & \text{iff} & \langle \sigma(t_1), \dots, \sigma(t_n) \rangle \in [\![R]\!] \cap \delta_1^w \times \dots \times \delta_n^w \\
w \vDash_\sigma t_1 \prec t_2 & \text{iff} & \sigma(t_1) \in \sigma(t_2) \\
w \vDash_\sigma t_1 =_s t_2 & \text{iff} & \sigma(t_1) = \sigma(t_2) \\
w \vDash_\sigma t_1 =_p t_2 & \text{iff} & \sigma(t_1) = \sigma(t_2) \\
w \vDash_\sigma t_1 \sim t_2 & \text{iff} & \sigma(t_1) \in w \text{ iff } \sigma(t_2) \in w \text{ for all } w \in W \\
w \vDash_\sigma \neg\phi & \text{iff} & w \nvDash_\sigma \phi \\
w \vDash_\sigma \phi \land \psi & \text{iff} & w \vDash_\sigma \phi \text{ and } w \vDash_\sigma \psi \\
w \vDash_\sigma \forall x \phi & \text{iff} & w \vDash_{\sigma^*} \phi \text{ for all } \sigma^* \text{ such that } \sigma^*(x) \in w \\
w \vDash_\sigma \forall xx \phi & \text{iff} & w \vDash_{\sigma^*} \phi \text{ for all } \sigma^* \text{ such that } \sigma^*(xx) \in p(w) \\
w \vDash_\sigma \Box\phi & \text{iff} & v \vDash_\sigma \phi \text{ for all } v \in W
\end{array}$$

A sentence ϕ is true in $\mathcal{M}_I$ when true at all worlds relative to all variables assignments. When it is, we write $\mathcal{M}_I \vDash \phi$.

Definition 7.6: Let $\mathcal{M}_F$ be a fictionalist model. When $\vDash_\sigma \phi$, we say that ϕ is true relative to variable assignment σ. This relation is defined recursively:

$$\begin{array}{lll}
\vDash_\sigma R(t_1, \dots, t_n) & \text{iff} & \langle \sigma(t_1), \dots, \sigma(t_n) \rangle \in [\![R]\!] \\
\vDash_\sigma t_1 \prec t_2 & \text{iff} & \sigma(t_1) \in \sigma(t_2) \\
\vDash_\sigma t_1 =_s t_2 & \text{iff} & \sigma(t_1) = \sigma(t_2) \\
\vDash_\sigma t_1 =_p t_2 & \text{iff} & \sigma(t_1) = \sigma(t_2) \\
\vDash_\sigma t_1 \sim t_2 & \text{iff} & \sigma(t_1) \in w \text{ iff } \sigma(t_2) \in w \text{ for all } w \in W \\
\vDash_\sigma \neg\phi & \text{iff} & w \nvDash_\sigma \phi \\
\vDash_\sigma \phi \land \psi & \text{iff} & w \vDash_\sigma \phi \text{ and } w \vDash_\sigma \psi \\
\vDash_\sigma \forall x \phi & \text{iff} & w \vDash_{\sigma^*} \phi \text{ for all } \sigma^* \\
\vDash_\sigma \forall xx \phi & \text{iff} & w \vDash_{\sigma^*} \phi \text{ for all } \sigma^* \\
\vDash_\sigma \mathsf{Act}(t) & \text{iff} & \sigma(t) \in @
\end{array}$$

A sentence ϕ is true in $\mathcal{M}_F$ when true relative to all variable assignments. When it is, we write $\mathcal{M}_F \vDash \phi$.

Definition 7.7: A model $\mathcal{M}$ is **regular** if for every basic non-logical predicate:

> $\langle a_1, \dots, a_n \rangle \in [\![R]\!]$ only if there is some $w \in W$ such that $\langle a_1, \dots, a_n \rangle \in \delta_1^w \times \dots \times \delta_n^w$.

Somewhat more intuitively, a model is regular when individuals stand in non-logical relations only when they exist together in some worlds. The above formulation generalizes this basic thought to include pluralities.

Definition 7.8: A frame is a **compossible** when for every $w, v \in W$, there is a $u \in W$ such that $w \cup v \in u$.

Definition 7.9: A frame is a **separable** when for every $S \subset W$, if $\cup(S) \in P$, then $\cup(S) \in W$.

As you would expect, when a model is based on a compossible frame, we will say that the model is compossible and, when a model is based on a separable frame, we will say that the model is separable.

Observation 7.10: Every compossible model is regular.

Definition 7.11: A model is a **standard** when it is compossible and separable and makes every instance of Plural Comprehension true.

We can now characterize the three systems from the main text. **CI** consists of those sentences of $\mathcal{L}_I$ that are true in all standard models. **I** consists of those sentence of $\mathcal{L}_I$ that are true in all *regular* models. Finally, **F** consists of those sentences of $\mathcal{L}_F$ that are true in all regular models.

What we want to show now is that the languages $\mathcal{L}_I$ and $\mathcal{L}_F$ are structurally equivalent, modulo the use of **I** and **F** as the relevant background logics.

We will say that an $\mathcal{L}_I$ language and $\mathcal{L}_F$ language **correspond** when they have the same names and non-logical predicates. Given any two such languages, two models $\mathcal{M}_I$ and $\mathcal{M}_F$ **correspond** when the valuation functions assign the same names and the same predicates the same denotations.

Theorem 7.12: *Let $\mathcal{L}_I$ and $\mathcal{L}_F$ be corresponding languages. There is then a recursive translation g from the sentences of $\mathcal{L}_I$ to the sentences of $\mathcal{L}_F$ such that $\mathcal{M}_I \vDash \phi$ iff $\mathcal{M}_F \vDash f(\phi)$ whenever $\mathcal{M}_I$ and $\mathcal{M}_F$ are corresponding models.*

Proof. We will first divide the plural variables of $\mathcal{L}_I$ into two infinite stocks xx_i and yy_i. Now take any sentence ϕ. We can suppose that none of the variables in the second stock appear in ϕ. For if they do, we rewrite ϕ as ϕ^* using relettering, where ϕ^* does not include any such variables. We then translate ϕ using a family g_i of recursive function as follows:

$$g[\phi] = g_0[\phi]$$
$$g_n[R(t_1, \ldots, t_n)] = R(t_1, \ldots, t_n)$$
$$g_n[\neg\phi] = \neg g_n[\phi]$$
$$g_n[\phi \wedge \psi] = g_n[\phi] \wedge g_n[\psi]$$
$$g_n[\forall x\phi] = \forall x(x \prec yy_n \supset \phi)$$

$$g_n[\forall xx\phi] = \forall xx(xx \prec yy_n \supset \phi)$$
$$g_n[\Box(\phi)] = \forall yy_{n+1}(\mathsf{World}(yy_{n+1}) \supset g_{n+1}[\phi])$$

This gives us our translation. We could then easily verify, by induction on complexity, that $\mathcal{M}_I \vDash \phi$ iff $\mathcal{M}_F \vDash f(\phi)$ whenever M_I and M_F are corresponding models. ∎

Theorem 7.13: *Let $\mathcal{L}_F$ and $\mathcal{L}_I$ be corresponding languages. There is then a recursive translation f from the sentences of $\mathcal{L}_F$ to the sentences of $\mathcal{L}_I$ such that $\mathcal{M}_F \vDash \phi$ iff $\mathcal{M}_I \vDash g(\phi)$ whenever $\mathcal{M}_F$ and $\mathcal{M}_I$ are corresponding models.*

Proof. We start by dividing the plural variables of $\mathcal{L}_F$ into an infinite stock xx_i and a single yy. Now take any sentence ϕ of $\mathcal{L}_F$. We can suppose that the plural variable yy appears nowhere in ϕ. Because if it did, we could use relettering to derive a logically equivalent ϕ^*, and then use that for our translation. We then recursively translate ϕ as follows:

$$f[\phi] = \exists yy(f^*[\phi])$$
$$f^*[R(t_1, \ldots, t_n)] = R(t_1, \ldots, t_n)$$
$$f^*[\neg\phi] = \neg f^*[\phi]$$
$$f^*[\phi \wedge \psi] = f^*[\phi] \wedge f^*[\psi]$$
$$f^*[\forall x\phi] = \Box(\forall x\phi)$$
$$f^*[\forall xx\phi] = \Box(\forall xx\phi)$$
$$f^*[\mathsf{Act}(t)] = t \prec yy$$

As before, it remains to be shown that $\mathcal{M}_F \vDash \phi$ iff $\mathcal{M}_I \vDash g(\phi)$ whenever $\mathcal{M}_F$ and $\mathcal{M}_I$ are corresponding models. But once we have the translation scheme, this can easily be shown by induction. ∎

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