Draft 05/13/21

Near Closeness and Conditionals

Daniel Berntson

This paper presents a new systems of conditional logic **B2**, which is strictly intermediate in strength between the existing systems **B1** and **B3** from Burgess (1981) and Lewis (1973a). After presenting and motivating the new system, we will show that it is characterized by a natural classes of frames. These frames correspond to the idea that conditionals are about the *near* closeness, rather than the exact closeness, of various possibilities. Along the way, we will also give new characterization results for **B1** and **B3**, along with two other new systems **B1.1** and **B1.2**.

1. Systems

$\mathcal{L}$ is the language of propositional logic extended with the two-place sentential operator $\Box\!\!\rightarrow$. We will generally think of this operator as expressing the subjunctive conditional, so read $A \Box\!\!\rightarrow B$ as saying that had it been that A, it would have been that B. Most of what follows, though, applies equally well to the indicative reading.

Besides this first operator, we will also have a second defined operator $\Diamond\!\!\rightarrow$ that is its dual.

$$A \Diamond\!\!\rightarrow B \equiv \neg(A \Box\!\!\rightarrow \neg B)$$

Because we are reading $\Box\!\!\rightarrow$ as expressing the subjunctive conditional, we will read $A \Diamond\!\!\rightarrow B$ as saying that had it been that A, it might have been that B.[1]

1. This is not meant to beg any questions about the duality of would and might counterfactuals. Those who deny duality have two options. They can either (a) read $A \Diamond\!\!\rightarrow B$ as saying that it is false that had it been that A, it would not have been that B or (b) read $A \Box\!\!\rightarrow B$ as saying that it is false that had it been that A, it might not have been that B. If duality is denied, I prefer the second option. Both options preserve the duality of $\Box\!\!\rightarrow$ and $\Diamond\!\!\rightarrow$.

What we are going to call **B1** is a Hilbert-style counterfactual logic from Burgess. That system has two rules of inference:

MP $A, A \supset B \vdash B$
SLE $\vdash (A \boxright B) \supset (A^* \boxright B^*)$ when $\vdash A \equiv A^*$ and $\vdash B \equiv B^*$

The first is Modus Ponens for the material conditional. The second is the Substitution of Logical Equivalents, which lets us substitute logically equivalent expressions in either the antecedent or consequent of a counterfactual. The basic axiom schemas are then:

PL A when $\vdash_{PL} A$
ID $A \boxright A$
IM $(A \boxright B \wedge C) \supset (A \wedge B \boxright C)$
CL $(A \boxright B \wedge C) \supset (A \boxright B)$
CR $(A \boxright B) \wedge (A \boxright C) \supset (A \boxright B \wedge C)$
DR $(A \boxright C) \wedge (B \boxright C) \supset (A \vee B \boxright C)$

The first says that every theorem of propositional logic is a theorem of **B1**.[2] The second is Identity, which says that every sentence counterfactually entails itself. The third is Import, which lets us move a conjunct from the consequent of a counterfactual to its antecedent. The fourth is Conjunction Left, which lets us drop a conjunct from the consequent. The last two are Conjunction Right and Disjunction Right. These rules are symmetric. Where CR lets us *conjoin* the consequents of counterfactuals that share an *antecedent*, DR lets us *disjoin* the antecedents of counterfactuals that share a *consequent*.

Other systems can be formed by extending **B1** with various axioms. Our main

2. This is the least essential of the axioms. We could instead requires **B1** to include all the theorems of a weaker non-classical system. Field (2016), for example, uses a non-classical variant of **B1** to build an indicative conditional that is compatible with naive truth. If we were to deny classicality, we would want to use a sequent calculus for our proof theory. Since we are here assuming classicality, we are using a simpler Hilbert-style system.

interest will be in the following:

DL	$(A \vee B \mathrel{\Box\!\!\rightarrow} C) \supset (A \mathrel{\Box\!\!\rightarrow} C) \vee (B \mathrel{\Box\!\!\rightarrow} C)$
DM	$(A \vee B \mathrel{\Box\!\!\rightarrow} B) \wedge (B \vee C \mathrel{\Box\!\!\rightarrow} C) \supset (A \vee C \mathrel{\Box\!\!\rightarrow} C) \vee (C \vee D \mathrel{\Box\!\!\rightarrow} D)$ when A, B, C, D are pairwise disjoint
RM	$(A \mathrel{\Box\!\!\rightarrow} C) \wedge (A \mathrel{\Diamond\!\!\rightarrow} B) \supset (A \wedge B \mathrel{\Box\!\!\rightarrow} C)$
CEM	$(A \mathrel{\Box\!\!\rightarrow} B) \vee (A \mathrel{\Box\!\!\rightarrow} \neg B)$
ST	$(A \mathrel{\Box\!\!\rightarrow} C) \supset (A \wedge B \mathrel{\Box\!\!\rightarrow} C)$

The first is Disjunction Left. It says that when you have a counterfactual with a disjunctive antecedent, that disjunction can be distributed over the entire counterfactual. The second is the Diamond axiom, which places a certain constraint on pairwise disjoint sentences, where A and B are pairwise disjoint when $A \vdash \neg B$ and $B \vdash \neg A$. The third is Rational Monotonicity, which lets us strengthen antecedents under certain restricted circumstances. The fourth is the Counterfactual Law of Excluded Middle, which is a sort of counterfactual analogue of the Law of Excluded Middle. The fifth is Strengthening, which lets us strengthen antecedents. Adding these axioms to **B1** gives us the following taxonomy of systems:

System	Axioms
B1	PL, ID, IM, CL, CR, DR
B1.1	**B1**, DL
B1.2	**B1**, DM
B2	**B1**, DL, DM
B3	**B1**, RM
B4	**B1**, CEM
B5	**B1**, ST

These systems are generally numbered in order of strength, with two exceptions. **B1.1** and **B1.2** are strictly stronger than **B1**, but neither is stronger than the other. Similarly, **B4** and **B5** are both strictly stronger than **B1**-**B3**, but neither is stronger than the other.

Most of these systems have prominent defenders in the literature. Robert Stalnaker (1968) was an early defender of what he calls **C2**, which is the result of adding weak centering to **B4**:

WC $\quad (A \mathrel{\Box\!\!\rightarrow} B) \supset (A \supset B)$

The weakest system that David Lewis (1973a, 1973b) considers is **V**, which we are calling **B3**. That system is notable because it is the weakest system that can be modeled using his systems of spheres. Lewis ultimately endorses a stronger system **VC** that adds both weak centering and strong centering. Strong centering is the principle that:

SC $(A \wedge B) \supset (A \mathbin{\Box\!\!\rightarrow} B)$

John Pollock (1975, 1976a, 1976b) rejects RM, and so accepts a system that he calls **SS**.[3] That system is formed by adding both strong and weak centering to **B1**. Finally, defenders of the strict conditional analysis generally accept **B5**, together with both strong and weak centering.[4]

Setting aside strong and weak centering, then, we can think of the strict conditional analysis as accepting **B5**, Stalnaker as accepting **B4**, Lewis as accepting **B3**, and Pollock as accepting **B1**.

My own view is that either **B1.1** or **B2** is the right logic for counterfactuals.[5] My reasons for thinking this are, in broad outline, as follows: First, I accept the mainstream view that Strengthening is not a theorem. We cannot strengthen the antecedent of a counterfactual with any sentence whatsoever. This rules out **B5**.

Second, I side with Lewis against Stalnaker on Counterfactual Excluded Middle.[6] It is neither true that had I flipped a coin one minute ago, it would have landed heads, nor is it true that had I flipped a coin one minute ago, it would have landed tails. This rules out **B4**.

Third, I side with Pollock against Lewis on Rational Monotonicity, though I also reject Pollock's alleged counterexamples. I deny the the validity of RM because this

3. Angelika Kratzer (1981) also accepts **SS**. She suggests modeling counterfactuals with what she calls premise semantics. Her model theory is, on the face of it, very different from the modal theory used by Pollock. But as Lewis (1981) points out, the two approaches are in fact equivalent. Every Kratzer model determines a unique Pollock model, and visa-versa.

4. Some of the most recent defenders of the strict conditional analysis include von Fintel (2001) and Gillies (2007).

5. Again, setting aside the question of centering. I am inclined to accept modus ponens for counterfactuals, which entails weak centering, given a classical background logic. I am undecided about strong centering.

6. For present purposes, anyway. My considered view is that counterfactuals generally have both a Lewis reading and a Stalnaker reading. On the Lewis reading, CEM fails; on the Stalnaker reading, CEM holds. In the present context, I am interested in the Lewis reading, so reject CEM.

strikes me as the most natural way to resolve the paradox of counterfactual tolerance. This paradox was introduced in my (2021) and will be briefly sketched in §2. Since I reject RM, this rules out **B3**.

Fourth, while I side with Pollock against Lewis on the question of Rational Monotonicity, I also think that Disjunction Left is clearly valid. For example, suppose that:

Had it either rained or snowed, Naomi would have been pleased. (1)

From this it would seem to follow that at least one of the following claims is true:

Had it rained, Naomi would have been pleased. (2)

Had it snowed, Naomi would have been pleased. (3)

The problem for Pollock's is that his preferred **B1** does not validate DL. As a result, the truth of (1) is entirely consistent with the falsity of both (2) and (3). This strikes me as absurd, and so I reject **B1**.

It may be worth pointing out that Disjunction Left has a certain passing resemblance to a much more controversial principle called the Simplification of Disjunctive Antecedents.

SDA $(A \vee B \mathrel{\Box\!\!\rightarrow} C) \supset (A \mathrel{\Box\!\!\rightarrow} C) \wedge (B \mathrel{\Box\!\!\rightarrow} C)$

The difference between the two principles is that where SDA has a conjunction in the consequent, DL has a disjunction. This makes all the difference in the world. Suppose for example that (1) is true. In that case, the SDA tells us that *both* (2) and (3) are true. DL makes the much weaker claim that *at least one* of them is. As a further illustration of the difference, we might also note that while Disjunction Left is a theorem of Lewis's **B3**, the Simplification of Disjunctive Antecedents is not.

This leaves us to decide between **B1.1** and **B2**, the difference between which is that **B2** validates Diamond, but **B1.1** does not.

DM is long-winded, so its validity does not simply jump off the page. Nevertheless, I think DM is valid, so ultimately accept **B2**. Here is what the axiom is telling us. Suppose that both of the following are true:

Had you lived in either Seattle or Portland, you would have lived in Portland. (4)

Had you lived in either Portland or Los Angeles, you would have lived in Los Angeles. (5)

Now consider any fourth disjoint possibility—living in San Francisco, say. DM says that at least one of the following is true:

> Had you lived in either Seattle or San Fransisco, you would have lived in San Fransisco. (6)
>
> Had you lived in either San Fransisco or Los Angeles, you would have lived in Los Angeles. (7)

From a theoretical perspective, there are two reason to like DM. The first is that while it may not be valid, RM is still compelling. We would thus like to replace it with a *minimal* weakening. We would like to preserve as much of the content of RM as we can without generating paradox. DM is strictly weaker than RM and does not generate paradox. Thus, there is good reason to think that DM is entailed by the minimal weakening of RM.

The second reason has to do with models. There is a natural thought that counterfactuals can be modeled in terms of relative closeness. In fact, I think they should be modeled using *near* closeness rather than exact closeness. We will say more about this in §4. For present purposes, the main point is that near closeness would seem to form a semiorder. But if so, it follows that DM is valid, which gives us some reason to accept DM.

Here is the plan for the rest of this paper. In §2, we will sketch the paradox of counterfactual tolerance as a way of motivating our interest in in systems weaker tha **B3**. We will then describe Burgess accessibility models in §3. §4 introduces the idea of using near closeness to think about accessibility. §5 catalogues various useful properties of near closeness and exactl closeness relations. In §6, we will use these properties to give finite characterization results. §6.2 addresses difficulties that arise when dealing with infinite frames, which let us give full characterization results. The bulk of the the technical material is then in the last two sections. §7 shows how to prove exact soundness. §8 shows how to prove completeness and decidability using a new canonical models procedure.

2. The Paradox of Counterfactual Tolerance

Planck lengths are incredibly small. You would quite literally need a billion trillion of them just to span that diameter of a proton. Now suppose that Barack Obama is in fact h Planck lengths tall. It would then seem that the following claims are true:

Tolerance:	For all positive integers $n > h$, had Obama been at least n Planck lengths, he might have been at least $n+1$ Planck lengths.
Boundedness:	There are positive integers $k > j > h$ such that had Obama been at least j Planck lengths, he would not have been at least k Planck lengths.
Heights:	For all positive integers n, had Obama been at least $n+1$ Planck lengths, he would have been at least n Planck lengths.

Tolerance says that had Obama been at least seven feet, he might have been at least one Planck length taller, and likewise for other heights. Boundedness will be true if, for example, had Obama been at least seven feet, he would not have been at least a thousand feet. Heights says that had Obama been at least seven feet and one Planck length, he would thereby have been at least seven feet, and likewise for other heights.

Given these three attractive claims, we can prove a flat contradiction using any system extending **B3**. First, we observe that Limited Transitivity is valid in not only **B3**, but any system extending **B1**.[7]

LT $\quad (A \mathrel{\Box\!\!\rightarrow} B) \wedge (A \wedge B \mathrel{\Box\!\!\rightarrow} C) \supset (A \mathrel{\Box\!\!\rightarrow} C)$

Besides being derivable, the principle is also compelling in its own right, and so often taken as basic, even in systems that are not extension of **B1**.[8]

This gives us everything we need to sketch the paradox. Let p_n express the claim that Obama is at least n inches tall, p_{n+1} the claim that Obama is at least $n+1$ inches tall, and so on. We then reason as follows:

1.	$p_n \mathrel{\Box\!\!\rightarrow} \neg p_k$	boundedness
2.	$p_n \mathrel{\Diamond\!\!\rightarrow} p_{n+1}$	tolerance
3.	$p_{n+1} \mathrel{\Box\!\!\rightarrow} p_n$	heights
4.	$p_n \wedge p_{n+1} \mathrel{\Box\!\!\rightarrow} \neg p_k$	1,2, RM
5.	$p_{n+1} \wedge p_n \mathrel{\Box\!\!\rightarrow} \neg p_k$	4, SLE
6.	$p_{n+1} \mathrel{\Box\!\!\rightarrow} \neg p_k$	3, 5, LT

7. This is proved in lemma 8.7.
8. See for example Fine (2012).

This argument is paradoxical because it can be iterated. In particular, after $k - n - 1$ applications, we get:

$$p_{k-1} \mathrel{\Box\!\!\rightarrow} \neg p_k \tag{8}$$

But tolerance tells us that

$$p_{k-1} \mathrel{\Diamond\!\!\rightarrow} p_k \tag{9}$$

and so we have a flat contradiction. Since we have reasoned to paradox, something has to go: Either one of the claims we started with is false or **B3** is invalid.

There are many strategies for responding to the paradox, several of which I consider at length in my (2021). Rather than repeating that discussion here, I will simply note that denying RM is one natural solution and, in fact, my own preferred solution. We thus have good reason to be interested in systems strictly weaker than **B3**.

3. Burgess Models

A natural thought is that counterfactuals should be modeled in terms of the relative closeness of possibilities. $A \mathrel{\Box\!\!\rightarrow} B$ is true if and only if all of the closest possible worlds at which A is true are worlds at which B is true. Similarly, $A \mathrel{\Diamond\!\!\rightarrow} B$ is true if and only if some of the closest possible worlds at which A is true are worlds at which B is true.

This idea can be be filled out in different ways. Lewis (1973a) uses systems of spheres. Stalnaker (1968) uses selection functions. Here, we are going to use Burgess models, which are the most general.

Definition 3.1: A **frame** $\mathcal{F} = \langle W, f, \leq \rangle$ consists of a non-empty set of worlds W, a function f assigning every world x a **local domain** $W_x \subseteq W$, and a function $\leq$ assigning every world x an **accessibility relation** $\leq_x \subseteq W_x \times W_x$.

The worlds in the local domain W_x of x are the worlds that are **possible** relative to x. For each such world, there is then a corresponding two-place accessibility relation $\leq_x$ on that local domain. When $b \leq_x a$, we will say that b is **accessible** from a relative to x. We can then define other useful relations:

$$\begin{array}{lll} b <_x a & \text{iff} & b \leq_x a \text{ and not } a \leq_x b \\ b \approx_x a & \text{iff} & b \leq_x a \text{ and } a \leq_x b \\ b \sim_x a & \text{iff} & b \leq_x a \text{ or } a \leq_x b \end{array}$$

The first is the relation of b being **strictly accessible** from a relative to x. The second is the relation of b and a being **coaccessible** relative to x. The third is the relation of b and a being **connected** relative to x.

These various relations can all be thought of in terms of relative closeness. We will say more about this in §4 but, at a first pass, $b \leq a$ says that b is at least as close as a to x. $b < a$ says that b is strictly closer than a to x. $b \approx_x a$ says that b and a are equally close to x. $b \sim_x a$ says that the distance of b from x is commensurable with the distance of a from x. When $a \in N_x$, that means that there is some distance from a to x or, alternatively, that the distance of a from x is defined.

Definition 3.2: A **Burgess model** $\mathcal{M} = \langle \mathcal{F}, V \rangle$ consists of a frame $\mathcal{F}$ and a valuation function V assigning every atomic sentence p of $\mathcal{L}$ a denotation $V(p) \subseteq W$. A sentence A is **true at a world** x in a model $\mathcal{M}$ when $\mathcal{M}, x \models A$. This relation is defined recursively:

$\mathcal{M}, x \models p$	iff	$x \in V(p)$
$\mathcal{M}, x \models \neg A$	iff	$x \not\models A$
$\mathcal{M}, x \models A \vee B$	iff	either $x \models A$ or $\models B$
$\mathcal{M}, x \models A \wedge B$	iff	$x \models A$ and $\models B$
$\mathcal{M}, x \models A \supset B$	iff	either $x \not\models A$ or $\models B$
$\mathcal{M}, x \models A \mathrel{\Box\!\!\rightarrow} B$	iff	for every $a \models A$ such that $a \in N_x$, there is a $b \models A$ such that $b \leq_x a$ and, for all c such that $c \leq_x b$, if $c \models A$ then $c \models B$
$\mathcal{M}, x \models A \mathrel{\Diamond\!\!\rightarrow} B$	iff	there is an $a \models A$ such that $a \in N_x$ and, for all $b \models A$ such that $b \leq_x a$, there is a c such that $c \leq_x b$ with $c \models A$ and $c \models B$

Once we have our models, soundness and completeness are defined in the usual way. A system S is **sound** in a frame $\mathcal{F}$ when every theorem of S is true at every world in every model based on $\mathcal{F}$. A system S is sound in a class of frames Γ when S is sound in every frame in Γ. A system S is **complete** in a class of frames Γ when every S-consistent set of sentences is true at some world in some model based on some frame in Γ. A set of sentences Γ is S-consistent when no sentence of the form $A \wedge \neg A$ can be derived from Γ using S. Finally, we will say that a class of frames Γ **generates** a system S when S is sound and complete in Γ.

The class of all frames whatsoever generates a system that we might call **B0**. This system is of some technical interest, but is far too weak to be a good match for our ordinary counterfactual practice. It invalidates, for example, the axiom of

Identity, which says that $A \mathrel{\Box\!\!\rightarrow} A$. But surely, if there are any logical truths involving counterfactuals, this is one of them.

Because the logic of **B0** is so weak, we need to add restrictions on the accessibility relation to generate more useful systems. The conditions most commonly used in the literature are below.

Property	**Definition**
Reflexive	$a \leq a$
Pairwise Connected	$(a \leq b) \vee (b \leq a)$
Anti-Symmetric	$(b \leq a) \supset \neg(a \leq b)$
Symmetric	$(b \leq a) \supset (a \leq b)$
Fully Transitive	$(c \leq b) \wedge (b \leq a) \supset (c \leq a)$
Preorder	reflexive, fully transitive
Total Preorder	pairwise connected, fully transitive
Total Order	pairwise connected, fully transitive, anti-symmetric
Universal	pairwise connected, symmetric

The general consensus has been that at a minimum, we should require accessibility relations to be reflexive and transitive. That is, the consensus has been that we should require accessibility relations to form a *preorder*. The most common systems are then generated by various classes of preorders.[9]

Theorem 3.3 (Burgess)**:** ***B1*** *is generated by the class of all preorders.*

Theorem 3.4 (Lewis)**:** ***B3*** *is generated by the class of all total preorders.*

Theorem 3.5 (Stalnaker)**:** ***B4*** *is generated by the class of all wellfounded total orders.*

Theorem 3.6 (Kripke)**:** ***B5*** *is generated by the class of all universal relations.*

9. Here, and elsewhere, we will often run together the distinction between frames and accessibility relations. We will say, for example, that **B1** is generated by the class of all *preorders*, when what we really mean is that **B1** is generated by the class of all *frames* in which every world is assigned an accessibility relation that forms a preorder.

The problem is that while using preorders to model counterfactuals is natural and useful, it can also be misleading. For while certain systems may be generated by certain classes of preorders, they are not *characterized* by them. There is thus a clear sense in which the systems and the classes of frames are not an exact match.

Definition 3.7: A system S is **inverse sound** in a class of frames Γ when S is not sound in any frame not in Γ.

Definition 3.8: A system S is **exactly sound** in a class of frames Γ when (a) S is sound in Γ and (b) S is inverse sound in Γ.

Definition 3.9: A system S **characterizes** a class of frames Γ relative to class of frames Δ when (a) S is exactly sound in $\Delta \cap \Gamma$ and (b) S is complete in $\Delta \cap \Gamma$.

Definition 3.10: A system S **characterizes** a class of frames Γ when S characterizes Γ relative to the class of all frames.

We will generally treat characterization as a symmetric relation. So, when a system characterizes a class of frames, we will also say that the class of frames characterizes the system.

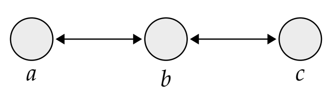
Characterization is a stronger requirement than soundness and completeness. To show that a system is characterized by a class, we need to show that it is not just sound and complete, but *exactly* sound and complete. This means we need to prove inverse soundness. We need to show that if a frame is *not* in the class, then the system is *not* sound. This lets us make several observations.

Proposition 3.11: ***B1*** *is not characterized by the class of all preorders.*

Proposition 3.12: ***B3*** *is not characterized by the class of all wellfounded total preorders.*

Proposition 3.13: ***B5*** *is not characterized by the class of all universal relations.*

Proof. These three propositions can be demonstrated by considering the following frame. Accessibility relations are represented by arrows.



Worlds c and b are coaccessible and b and a are coaccessible, but c and a are not connected. As a result, this frame is not fully transitive. Still, **B1**, **B3**, and **B5** are all sound in this frame by theorem 7.20. Thus, none of these systems is characterized by any fully transitive class of frames. Preorders are fully transitive. So none of these systems is characterized by any class of preorders. ■

Proposition 3.14: ***B4*** *is characterized by the class of all total orders that are wellfounded.*

Proof. Soundness and completeness are by theorem 3.5. The inverse of soundness is straightforward, so left to the reader. ■

There is nothing wrong with using a class of models that merely generates a system. We have soundness and completeness after all! Still, proving characterization results can be illuminating.

When a class of frames merely generates a system, that class has unnecessary structure. This unnecessary structure may be useful and is generally harmless, but is also not needed to support the theorems. When a class of frames *characterizes* a class of frames, on the other hand, that class has all *and only* the structure needed to support the theorems. The system and the class of frames exactly match.

4. Near and Exact

We are going to introduce several new frame conditions in the next two sections, many of which are non-transitive. To get a feel for how these relations work, and how they might be motivated, it will be helpful to think about accessibility in terms of **near closeness** and **exact closeness**.

Imagine that you are standing in the middle of a grassy field with a number of brightly colored balls arranged at various distances. One question you might be interested in is which balls are at least as close as others in the strictest sense. In the strictest sense, if b is even one Planck length farther than a, it is not as close as b, since it is not *exactly* as close.

While you might be interested in which balls are exactly as close, you might also be interested in which balls are *nearly* as close. A ball b is nearly as close as a when it is no more than r farther away than a. Here, we will call r the **tolerance margin**. To make things more concrete, you might think of r as being exactly one foot. So, a ball b is nearly as close as a iff it is no more than one foot farther away.

In ordinary conversation, when we say that b is nearly as close as a, that often implies that b is strictly farther than a. As we are using the term here, there is no such implication. If a ball b is nearly as close as a, this is compatible with it also being strictly closer.

Once we have the relation of near closeness, we can use it to define others. A ball b is **significantly closer** than a when b is nearly as close as a, but a is not nearly as close as b. Using our chosen tolerance margin, this means that a ball is significantly closer when it is more than one foot closer. We can then say that b is **significantly farther** than a when a is significantly closer than b.

The distance of two balls is **roughly equal** when a is nearly as close as b and b is nearly as close as a. Using a tolerance margin of one foot, the distance of two balls is roughly equivalent if neither is more than one foot closer than the other.

Where exact closeness is fully transitive, near closeness is not. Suppose that a is three feet away, b is four feet away, and c is five feet away. In that case, using a one foot tolerance margin, b is nearly as close as a and c is nearly as close as b, but c is not nearly as close as a.

5. Properties of Closeness Relations

There is by now a long tradition of thinking about counterfactuals in terms of exact closeness. The usual thought is that $A \mathrel{\Box\!\!\rightarrow} B$ is true iff all of the *exactly* closest A worlds are B worlds. Similarly, $A \mathrel{\Diamond\!\!\rightarrow} B$ is true iff some of the exactly closest A worlds are B worlds.

This is a natural way to think about counterfactuals if you accept a system that is at least as strong as Lewis's **B3**. For many of us who accept weaker systems, though, it makes more sense to think about closeness in terms of near closeness. $A \mathrel{\Box\!\!\rightarrow} B$ is true iff all of the *nearly* closest A worlds are B worlds. $A \mathrel{\Diamond\!\!\rightarrow} B$ is true iff some of the nearly closest A worlds are B worlds.

If we are going to model counterfactuals in terms of near closeness, then we need to impose appropriate constraints on accessibility. But which constraints are those? For example, near closeness is not fully transitive, and so not a preorder. But while that may be, we would be making a mistake if we tried to model near closeness by just *deleting* full transitivity from the definition of a preorder. There is more to near closeness than just reflexivity.

In this section, we are going to catalogue several properties of near closeness and exact closeness relations. These can be broadly divided into three kinds: transitivity properties, connectedness properties, and directedness properties. Once we have a suitable catalogue of properties, we will use them to give characterization results in

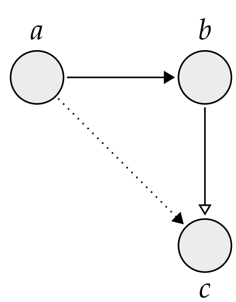
§6 and §6.2.

5.1. Transitivity Properties

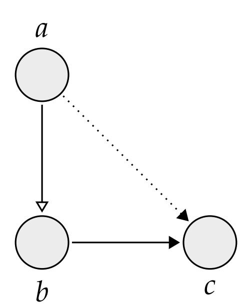
A **transitivity property** is a non-trivial property that is entailed by full transitivity. Thus, full transitivity is itself on example of a transitivity property, but not the only one. There are many others.

Property	**Definition**
Left Transitive	$(c < b) \wedge (b \leq a) \supset (c \leq a)$
Right Transitive	$(c \leq b) \wedge (b < a) \supset (c \leq a)$
Exactly Left Transitive	$(c < b) \wedge (b \leq a) \supset (c < a)$
Exactly Right Transitive	$(c \leq b) \wedge (b < a) \supset (c < a)$
Strict Transitive	$(c < b) \wedge (b < a) \supset (c < a)$
Zigzag Transitive	$(d < c) \wedge (c \leq b) \wedge (b < a) \supset (d < a)$
Double Left Transitive	$(d < c) \wedge (c < b) \wedge (b \leq a) \supset (d < a)$
Double Right Transitive	$(d \leq c) \wedge (c < b) \wedge (b < a) \supset (d < a)$
Simply Transitive	left transitive, right transitive
Double Transitive	zigzag transitive, double left transitive, double right transitive
Nearly Transitive	simply transitive, double transitive
Exactly Transitive	exactly left transitive, exactly right transitive

The first pair of transitivity properties are left transitivity and right transitivity. These are illustrated below:



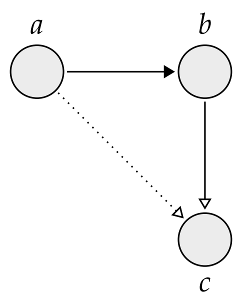
Left Transitivity



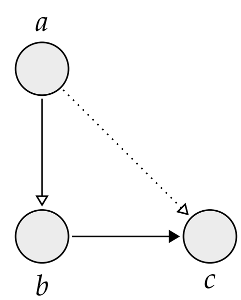
Right Transitivity

Solid arrows represent accessibility relations that appear in the antecedent of the corresponding definition. Dotted arrows represents accessibility relations that appear in the consequent. Arrows with open tips represents strict accessibility relations. Arrows with closed tips represent accessibility relations that may or may not be strict. In terms of near closeness, left transitivity says that if *c* is significantly closer than *b* and *b* is nearly as close as *a*, then *c* is nearly as close as *a*. Right transitivity says that if *c* is nearly as close as *b* and *b* is significantly closer that *a*, then *c* is nearly as close as *a*. A relation is simply transitive when it is both left transitive and right transitive.

The second pair of properties are exact left transitivity and exact right transitivity. These are properties of exact closeness, but not near closeness.



Exactly Left Transitive



Exactly Right Transitive

In terms of exact closeness, exact left transitivity says that if *c* is strictly closer than *b* and *b* is at least as close as *a*, then *c* is strictly closer than *a*. Exact right transitivity says that if *c* is at least as close as *b* and *b* is strictly closer than *a*, then *c* is strictly closer than *a*. When a relation is both exactly left transitive and exactly right right transitive, we will say that it is exactly transitive.

That exact transitivity fails for near closeness can be shown as follows. Suppose that *a* is four feet away, *b* is five feet away, and *c* is three feet away. Using a one-foot tolerance margin, *b* is nearly as close as *a* and *c* is significantly closer than *b*, but *c* is not significantly closer than *a*.

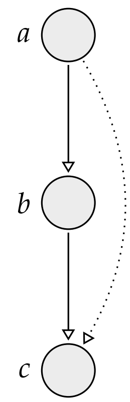
Observation 5.1: Every exactly left transitive relation is left transitive.[10]

Observation 5.2: Every exactly right transitive relation is right transitive.

Observation 5.3: Every exactly transitive relation is simply transitive.

10. Here and elsewhere, I will call especially straightforward propositions *observations*. Proofs of observations are left to the reader.

The fifth property on our table is strict transitivity. This property is called strict transitivity because it takes two strict accessibility relation as input, then gives back a third strict accessibility relation as output.



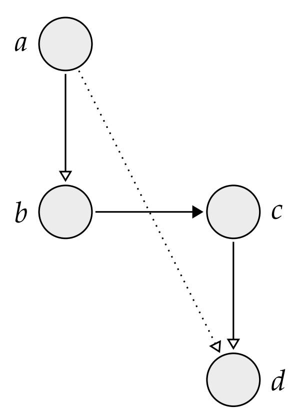
Strictly Transitive

Strict transitivity says that if c is significantly closer than b and b is significantly closer than a, then c is significantly closer than a.

Observation 5.4: If a relation is either right transitive or left transitive, then it is strictly transitive.

Observation 5.5: If a relation is pairwise connected and strictly transitive, then it is also simply transitive.

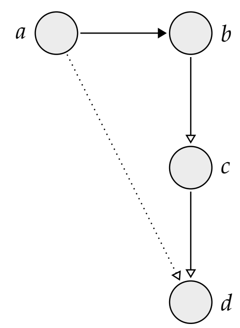
The sixth property is zigzag transitivity. Unlike the other properties consider so far, zigzag transitivity takes three accessibility relations as input, then gives us a strict accessibility relation as output.



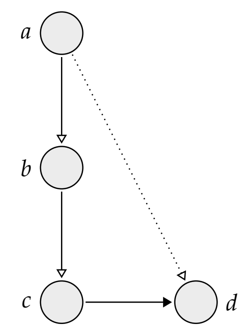
Zigzag Transitive

In terms of near closeness, zigzag transitivity says that if d is significantly closer than c and c is nearly as close as b and b is significantly closer thousand a, then d is significantly closer than a.

The last two basic properties on the list are double left transitivity and double right transitivity. These are like zigzag transitivity, but permute the patterns of strict and non-strict accessibility relations in the antecedent.



Double Left Transitive



Double Right Transitive

Double left transitivity says that if d is significantly closer than c and c is significantly closer than b and b is nearly as close as a, then d is significantly closer than a. Double right transitivity says that if d is nearly as close as c and c is significantly closer than b and b is significantly closer than a, then d is significantly closer than a. When a relation is zigzag transitive, double left transitive, and double right transitive, we will say that it is double transitive.

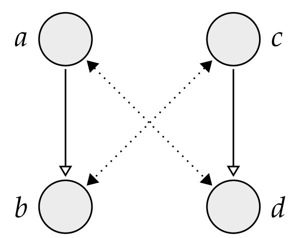
Observation 5.6: If a relation is reflexive, then it is strictly transitive if it is either zigzag transitive or double left transitive or double right transitive.

5.2. Connectedness Properties

A property is a **connectedness property** when it is a non-trivial property entailed by pairwise connectedness. Pairwise connectedness is a connectedness property then, but not the only one. There are many others.

Property	Definition
Bowtie Connected	$(d < c) \wedge (b < a) \supset (d \sim a) \vee (b \sim c)$
Diamond Connected	$(c < b) \wedge (b < a) \supset (c \sim d) \vee (d \sim a)$
Triangle Connected	$(b < a) \supset (b \sim c) \vee (c \sim a)$

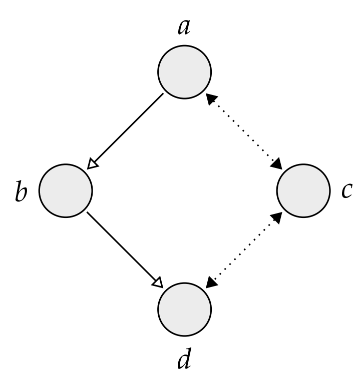
Bowtie connectedness says that given any pair of strict accessibility relations, one of the crossing pairs is connected. As such, it might also be called *cross connectedness*.



Bowtie Connected

Bowtie connectedness says that if d is significantly closer than c and b is significantly closer than a, then either d and a are commensurable or b and c are commensurable.

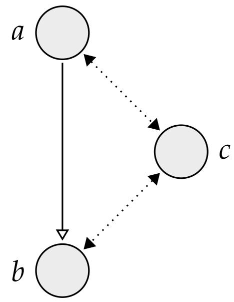
The second property is diamond connectedness. It says that if b is significantly closer than a and that d is significantly closer than b, then either a and c are commensurable or c and d are commensurable.



Diamond Connected

Observation 5.7: If a relation is diamond connected, then it is double right transitive iff it is double left transitive.

The last connectedness property is triangle connectedness. Triangle connectedness says that if b is significantly closer than a , then either c and a are commensurable or b and c are commensurable.



Triangle Connected

Observation 5.8: If a relation is triangle connected, then it is also diamond connected and bowtie connected.

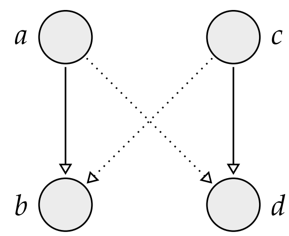
Observation 5.9: If a relation is triangle connected, then it is strictly left transitive iff it is strictly right transitive.

5.3. Directedness Properties

What we are going to call **directedness properties** are hybrid properties, which generally result from conjoining a connectedness properties and an appropriate transitivity properties. Directedness properties are important in their own right because they play an especially prominent role in characterizing counterfactuals. Roughly speaking, our model theory is sensitive to directedness properties even when it is not sensitive to the underlying connectedness or transitivity properties taken individually.

Property	**Definition**
Bowtie Directed	$(d < c) \wedge (b < a) \supset (d < a) \vee (b < c)$
Diamond Directed	$(c < b) \wedge (b < a) \supset (c < d) \vee (d < a)$
Triangle Directed	$(b < a) \supset (b < c) \vee (c < a)$
Pairwise Directed	$(b < a) \vee (a < b)$

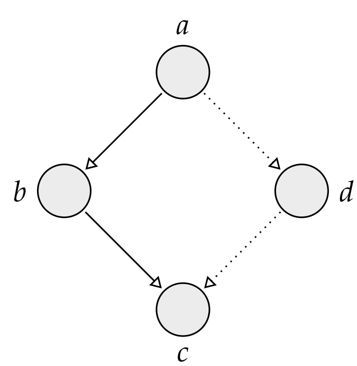
The first property is bowtie directedness. It says that if b is significantly closer than a and d is significantly closer than c, then either b is significantly closer than c or d is significantly closer than a.



Bowtie Directedness

Observation 5.10: A relation is bowtie directed iff it is bowtie connected and zigzag transitive.

The second property is diamond directedness. It says that if *b* is significantly closer than *a* and *c* is significantly closer than *b*, then either *d* is significantly closer than *a* or *c* is significantly closer than *d*.



Diamond Directedness

Observation 5.11: A relation is diamond directed iff it is diamond connected and double transitive.

Semiorders are strictly weaker than preorders. They are generally attributed to Luce (1956), who introduced them to model intransitive preferences.[11] A relation forms a **semiorder** when it is pairwise connected, zigzag transitive, and diamond directed.

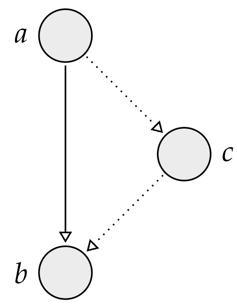
Observation 5.12: A relation is a semiorder iff it is pairwise connected and double transitive.

11. While semiorders are generally attributed to Luce, they were in fact first introduced almost forty years earlier by Wiener (1914), a math prodigy who studied under Bertrand Russell and completed his Ph.D. from Harvard at the age of eighteen. See Fishburn and Monjardet (1992).

Observation 5.13: Every semiorder is simply transitive.

Observation 5.14: A relation is a semiorder iff it is pairwise connected and nearly transitive.

The third relation is triangle directedness, which is sometimes called *almost connectedness*. This is a property of exact closeness relations, but not near closeness relations.



Triangle Directedness

In terms of exact closeness, triangle directedness says that if b is strictly closer than a, then either c is strictly closer than a or b is strictly closer than c.

Observation 5.15: A relation is triangle directed iff it is triangle connected and exactly transitive.

The easiest way to see that near closeness does not satisfy triangle directedness is to note that by the above observation, triangle directedness entails exact transitivity. But we already saw that exact transitivity is not a property of near closeness. So triangle directedness is not a property of near closeness.

The last directedness property on our list is pairwise directedness. This says that for any pair of worlds b and a, either b is strictly closer than a or a is strictly closer than b. Unlike our other directedness properties, pairwise directedness is not the result of conjoining a connectedness property with a transitivity property. Instead, it is the result of conjoining pairwise connectedness with antisymmetry.

Observation 5.16: A relation is pairwise directed iff it is pairwise connected and antisymmetric.

6. Characterization Results

The last section introduced several properties of near and exact closeness relations. In this section, we are going to use those properties to give characterization results.

This will be done in two steps. First, we will give finite characterization results. These are important because, among other things, they entail decidability. We will then deal with certain difficulties raised by infinite frames, which will let us give full characterization results.

6.1. Finite Frames

The final section of this paper will give a canonical models construction for proving completeness. That procedure if different from, but in many way complementary to, the step-by-step procedure given by Burgess (1981).

Our canonical modals procedure will build pairwise connected models for any set of sentences that are consistent in **B1.1**. The Burgess step-by-step procedure, on the other hand, builds fully transitive models for any set of sentences that are consistent in **B1**. As a result, it is helpful to be able to switch back and forth between the two. For classes of frames that are pairwise connected, but not fully transitive, we need the canonical models procedure. For classes of frames that are fully transitive, but not pairwise connected, we need the step-by-step procedure.

We are going to start by describing what we can show using our canonical models procedure. The results for **B2**-**B4** are already known. The results for **B1.1** and **B2** are new.

Theorem 6.1: *When restricting to the class of finite frames that are pairwise connected, each of the systems on the left is characterized by the class listed in the center. The class or pairwise connected relations meeting this condition is also known by the condition listed on the right.*

System	Added Condition	AKA
B1.1	zigzag transitive	
B2	double transitive	semiorder
B3	fully transitive	total partial order
B4	fully transitive, asymmetric	total order
B5	fully transitive, symmetric	universal relation

Proof. Exact soundness is by theorem 7.20, since the conditions listed here are stronger than the conditions listed there. Completeness is by theorem 8.16. ∎

Corollary 6.2: ***B1.1*** *and* ***B2-B5*** *are decidable.*

Proof. By theorem 6.1. ■

The above theorem can be be thought of as answering a certain question. Suppose that we assume from the start that relative closeness is pairwise connected. Which systems can be generated and which conditions can be used to generate them? The above table provides an answer. If we require relative closeness to be zigzag transitive, we get **B1.1**. If we require relative closeness to be double transitive, we get **B2**. If we require relative closeness to be full transitivity, we get **B3**.

Theorem 6.1 also demonstrates the usefulness of the canonical models procedure. If we could only build models that were fully transitive, the only total zigzag orders or semiorders we could build would be total *preorders*. But there are consistent sentences of **B1.1** and **B2** whose only models are not total preorders. Thus, we need the new canonical models procedure for the first two lines of the table.

Theorem 6.3: *When restricting to the class of finite preorders, each of the systems on the left is characterized by the class listed in the center. The class of preorders meeting this condition is also known by the condition listed on the right.*

System	Added Condition	AKA
B1		preorder
B1.1	bowtie connected	
B1.2	diamond connected	
B2	bowtie connected, diamond connected	
B3	triangle connected	
B4	pairwise connected, antisymmetric	total order
B5	symmetric	universal relation

Proof. Exact soundness is by theorem 7.20, since the conditions listed here are stronger than the conditions listed there. Completeness can be shown using the Burgess step-by-step procedure. ■

Corollary 6.4: ***B1-B5*** *are decidable.*

Proof. By theorem 6.3. ■

Suppose that we assume from the start that the relative closeness of possibilities if fully transitive. Which systems can we then generate and what conditions must we use to generate them? The table in theorem 6.3 provides an answer. By adding increasingly stronger connectedness principles, we can generate **B1**-**B5**.

This second theorem demonstrates the usefulness of the Burgess step-by-step procedure. If we only had the canonical models construction, we would only be able to build pairwise connected models. But in that case, we would have no way of showing the result for **B1**, since the only preorders we would be able to build would be total preorders. But there are consistent sentences of **B1** whose only models are not total preorders. So we cannot use the canonical models procedure for the first line of the table.

Theorem 6.1 and theorem 6.3 tell us which systems can be characterized, given that we have finite domains and either pairwise connectedness or full transitivity as a background assumption. But what if we have finite domains, but neither pairwise connectedness nor full transitivity as a background assumption? In that case, **B1**-**B5** are characterized using the frame conditions listed in theorem 6.6.

Definition 6.5: A relation is a **left order** when it is reflexive and left transitive.

Theorem 6.6 (Finite Characterization)**:** *When restricting to the class of finite frames, each of the systems on the left is characterized by the class of left orders listed in the center. The class of left orders meeting that condition is also known by the condition listed on the right.*

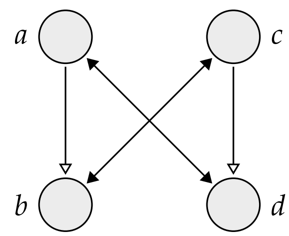
System	Added Condition	AKA
B1	zigzag transitive	
B1.1	bowtie directed	
B1.2	diamond directed	
B1.1	bowtie directed, diamond directed	
B2	triangle directed	
B3	pairwise directed, strictly transitive	total order
B4	symmetric	

Proof. Since finite frames are wellfounded, exact soundness follows from theorem 7.20. Completeness is by theorem 6.3, given that the conditions listed here are weaker than the conditions listed there. ∎

6.2. Infinite Frames

We have shown that **B1**-**B5** are finitely characterized by various frame conditions. While this is a good start, we would like to allow for infinite frames. Doing this requires additional work.

Definition 6.7: The worlds a, b, c, d form a **bowtie cycle** when $(d < c) \wedge (b < a) \wedge (d \approx a) \wedge (b \approx c)$.



Four worlds form a bowtie cycle when d is significantly closer than c and b is significantly closer than a but, nevertheless, the distance of d and a is roughly equal and the distance of b and c is roughly equal.

An important step in finitely characterizing **B1** is ruling out bowtie cycles. In the finite case, we ruled out bowtie cycles by accepting zigzag transitivity. The problem is that in the infinite case, there can be infinite sequences of worlds that *look* just like bowtie cycles, at least so far as the model theory is concerned. Thus, once we allow for infinite frames, **B1** is no longer sound with respect to the class of all zigzag orders.

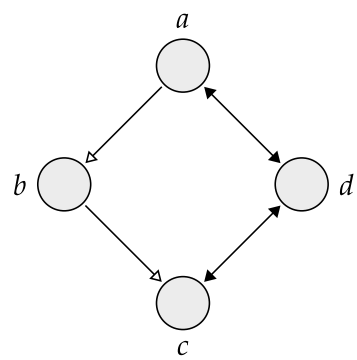
Definition 6.8: A **bowtie sequence** $\langle a_1, b_1, c_1, d_1, a_2, b_2 \ldots \rangle$ is any countably infinite sequence meeting all of the following conditions for some world x and all n and m:

1. $d_n <_x c_n$ and $b_n <_x a_n$
2. $d_n \approx_x a_n$ and $b_n \approx_x c_n$ and $d_n \approx_x a_{n+1}$
3. $\neg(a_n \leq_x b_m)$ and $\neg(c_n \leq_x d_m)$

A relation is **bowtie wellfounded** (**bwf**) when it does not have any bowtie sequences.

There is a similar challenge when characterizing **B2**. In particular, if we allow for infinite frames, **B2** is no longer sound with respect to the class of all diamond orders.

Definition 6.9: The worlds a, b, c and d form a **diamond cycle** when $(c < b) \wedge (b < a) \wedge (c \approx d) \wedge (d \approx a)$.



Four worlds form a diamond cycle when b is significantly closer than a and c is significantly closer than b but, nevertheless, the distance of a and d is roughly equal and the distance of d and c is roughly equal.

Diamond directedness rules out diamond cycles, which is needed to ensure that **B2** is sound. The problem is that once we have infinite frames, there can be certain infinite sequences that *look* just like diamond cycles, at least so far as the model theory is concerned.

Definition 6.10: A **diamond sequence** $\langle a_1, b_1, c_1, d_1, a_2, b_2 \ldots \rangle$ is any countably infinite sequence meeting all of the following conditions for some world x and all n and m:

1. $c_n <_x a_n$
2. $c_n \approx_x d_n$ and $d_n \approx_x a_n$ and $d_n \approx_x a_{n+1}$
3. $\neg(a_n \leq_x b_m)$ and $\neg(b_n \leq_x c_m)$

A relation is **diamond wellfounded** (**dwf**) when it does not have any diamond sequences.

Observation 6.11: Every relation that is pairwise connected and zigzag transitive is bowtie wellfounded and diamond wellfounded.

Observation 6.12: Every preorder is bowtie wellfounded and diamond wellfounded.

Observation 6.13: Every finite frame is bowtie wellfounded, diamond wellfounded, and fully wellfounded.

Besides **B1**-**B2**, we also need to add a wellfoundedness condition to characterize **B5**. For while **B5** is sound in the class of all finite total orders, it is not sound in the class of all total orders. The problem is that non-wellfounded sequences of strict accessibility relations *look* just like symmetric accessibility relations from the perspective of the model theory. Thus, in order to fully characterize **B5**, we need to add full wellfoundedness as an additional constraint.

We are now in a position to give results analogous to theorem 6.1, theorem 6.3, and theorem 6.6 while allowing for infinite frames.

Theorem 6.14: *When restricting to the class of frames that are pairwise connected, each of the systems on the left is characterized by the class listed in the center. This class is also known by the condition listed on the right.*

System	Added Condition	AKA
B1.1	zigzag transitive	
B2	double transitive	semiorder
B3	fully transitive	total partial order
B4	fully transitive, asymmetric, wf	wellfounded total order
B5	fully transitive, symmetric	universal relation

Proof. Exact soundness is by theorem 7.20 and observation 6.11. Completeness is by theorem 8.16 and observation 6.13. ∎

Theorem 6.15: *When restricting to the class of preorders, each of the systems on the left is characterized by the class listed in the center. This class is also known by the condition listed on the right.*

System	Added Condition	AKA
B1		preorder
B1.1	bowtie connected	
B1.2	diamond connected	
B2	bowtie connected, diamond connected	
B3	triangle connected	
B4	pairwise connected, antisymmetric, wf	wf total order
B5	symmetric	universal relation

Proof. Exact soundness is by theorem 7.20 and observation 6.12. Completeness is by theorem 6.3 and observation 6.13. ■

Theorem 6.16 (Characterization)**:** *Each of the systems on the left is characterized by the class of left orders listed in the center. That class is also known by the condition listed on the right.*

System	Added Condition	AKA
B1	zigzag transitive, bwf	
B1.1	bowtie directed, bwf	
B1.2	diamond directed, dwf	
B2	bowtie directed, diamond directed, bwf, dwf	
B3	triangle directed	
B4	pairwise directed, strictly transitive, wf	wf total order
B5	symmetric	

Proof. Exact soundness is by theorem 7.20. Completeness is by theorem 6.6 and observation 6.13. ■

Something that emerges from the above is that most of our systems are characterized by second-order frame conditions. A striking exception to this general rule is Lewis's **B3**, which is first-order characterizable, since there is no need to appeal to further wellfoundedness conditions.

Another observation is that while second-order frame conditions are needed to characterize **B1**-**B2**, they are not needed to *generate* those systems. Those classes can be generated using purely first-order frames. On the other hand, **B4** is neither characterized nor generated by any first-order frame conditions. So there is a sense in which **B4** is more deeply second-order than **B1**-**B2**.

7. Exact Soundness

Exact soundness proofs have two parts and, as such, this section is divided into two parts. The first subsection will prove soundness. The second subsection will prove inverse soundness.

As we go along, it will be helpful to have some further terminology. A world c is an **example** of $A \Box\!\!\rightarrow B$ when $c \models A \wedge B$. A world b is a **witness** of $A \Box\!\!\rightarrow B$ relative to x when $b \models A$ and every c such that $c \leq_x b$ and $c \models A$ is an example. A world a **confirms** $A \Box\!\!\rightarrow B$ relative to x when $a \models A$ and there is some b that is a witness relative to x such that $b \leq_x a$.

Similarly, we will say that a world c is an **example** of $A \Diamond\!\!\rightarrow B$ when $c \models A \wedge B$. A world b is a **supporter** of $A \Diamond\!\!\rightarrow B$ relative to x when $b \models A$ and there is some c such that that $c \leq b$ that is an example. A world a **confirms** $A \Diamond\!\!\rightarrow B$ relative to x when $a \models A$ and every b such that $b \leq_x a$ and $b \models A$ is a supporter.

Putting these definitions together, $A \Box\!\!\rightarrow B$ is true at a world iff every possible A world confirms it. Similarly, $A \Diamond\!\!\rightarrow B$ is true at a world iff some possible A world confirms it.

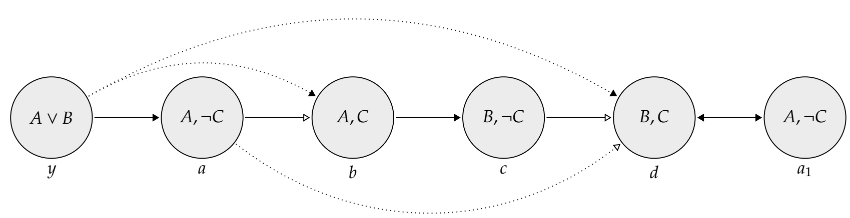
7.1. Soundness

Observation 7.1: PL, IM, CL are all valid in all frames.

Observation 7.2: If a frame is reflexive, ID is valid.

Proposition 7.3: *If a frame is reflexive, left transitive, zigzag transitive, and bowtie wellfounded, then DR is valid.*

Proof. Suppose that $x \models (A \Box\!\!\rightarrow C) \wedge (B \Box\!\!\rightarrow C)$ and $x \models (A \vee B) \Diamond\!\!\rightarrow \neg C$. There is thus some y that confirms $(A \vee B) \Diamond\!\!\rightarrow \neg C$ with an example a such that $a \leq y$. Either $a \models A$ or $a \models B$. The basic reasoning is the same either way, so suppose that $a \models A$. Since a confirms $A \Box\!\!\rightarrow C$, there is a witness b such that $b <_x a$, with $b \leq_x y$ by left transitivity. This gives us the first three worlds below.



Next, there must be an example c of $A \vee B \Diamond\!\!\rightarrow \neg C$ such that $c \leq_x b$. This world confirms $B \Box\!\!\rightarrow C$, so there is also a witness d such that $d <_x c$, from which it follows that $e <_x b$ by zigzag transitivity and $d \leq_x y$ by left transitivity.

This last observation means that we will need yet another example a_1 of $A \mathbin{\Diamond\!\!\rightarrow} B$, with this example being such that $a_1 \approx_x d$. We know that $a_1 \approx_x d$ because, if $a_1 <_x d$, it follows that $a_1 \leq_x b$ by left transitivity, which is a contradiction.

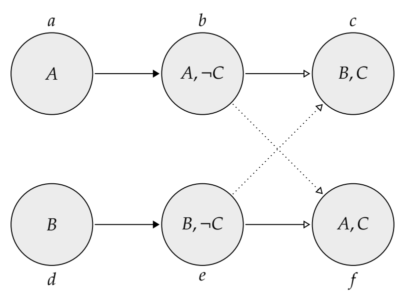
The problem now is that we cannot stop with a_1. We in fact need to add a full copy $\langle a_1, b_1, c_1, d_1 \rangle$ of the worlds $\langle a, b, c, d \rangle$ on the right of the diagram. Once we do, we will be in a similar position, so will need to add yet another copy $\langle a_2, b_2, c_2, d_2 \rangle$ to the right of that diagram, and so on and so forth for all n. The result is a sequence of worlds that is not bowtie wellfounded, which is contrary to assumption. ∎

Proposition 7.4: *If a frame is reflexive, left transitive, zigzag transitive, and bowtie wellfounded, then CR is valid.*

Proof. By reasoning similar to proposition 7.3. ∎

Proposition 7.5: *If a frame is reflexive, left transitive, and bowtie directed, then DL is valid.*

Proof. Suppose that $x \models (A \vee B \mathbin{\Diamond\!\!\rightarrow} C)$ and $x \models (A \mathbin{\Diamond\!\!\rightarrow} \neg C)$ and $x \models (B \mathbin{\Diamond\!\!\rightarrow} \neg C)$. We thus have a world a that confirms $A \mathbin{\Diamond\!\!\rightarrow} \neg C$ with an example b such that $b \leq_x a$. Since b confirms $A \vee B \mathbin{\Box\!\!\rightarrow} C$, there is also a witness c such that $c < b$, with $c \leq_x a$ by left transitivity. If it were the case that $c \models A$, there would need to be an example y of $A \mathbin{\Diamond\!\!\rightarrow} \neg C$ such that $y \leq_x c$. But this would contradict the fact that c witnesses $A \vee B \mathbin{\Box\!\!\rightarrow} C$. So $c \not\models A$. This gives us the worlds on the top row. Analogous reasoning gives us the worlds on the bottom, given that there must be a d that confirms $B \mathbin{\Diamond\!\!\rightarrow} \neg C$.



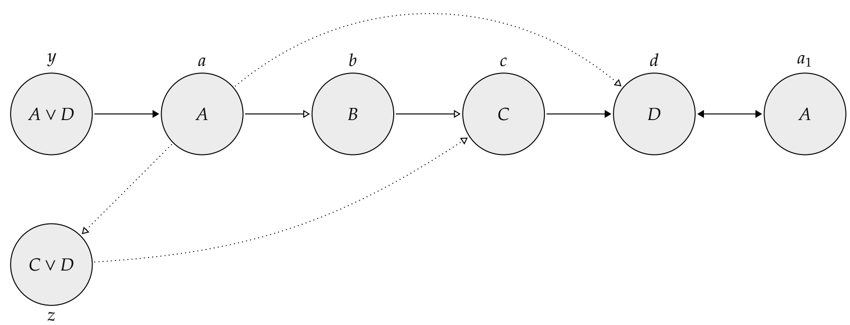
There are now two possibilities given bowtie connectedness. These are that $f <_x b$ or $c <_x e$. Each of these cases leads to contradiction.

First, suppose that $f <_x b$. This gives us $f \leq_x a$ by left transitivity, so there must be an example y of $A \mathbin{\Diamond\!\!\rightarrow} \neg C$ such that $y \leq_x f$. But this contradicts the fact that f

witnesses $A \vee B \mathrel{\Box\!\!\to} C$. If we instead suppose that $c <_x e$, we get a contradiction for similar reasons. So the proposition holds. ■

Proposition 7.6: *If a frame is reflexive, diamond directed, and diamond wellfounded, then DM is valid.*

Proof. Let A, B, C, D be pairwise disjoint. Suppose that $x \models (A \vee B \mathrel{\Box\!\!\to} A) \wedge (B \vee C \mathrel{\Box\!\!\to} B)$ and $x \models C \vee D \mathrel{\Diamond\!\!\to} C$ and $x \models A \vee D \mathrel{\Diamond\!\!\to} A$. There is thus a y that confirms $A \vee D \mathrel{\Diamond\!\!\to} A$ and so a witness a such that $a \leq_x y$. Since a confirms $A \vee B \mathrel{\Box\!\!\to} B$, there is a witness b such that $b <_x a$. And since b confirms $B \vee C \mathrel{\Box\!\!\to} C$, there is a witness c such that $c <_x b$.



We now observe that given our initial assumptions, there is a z that confirms $C \vee D \mathrel{\Diamond\!\!\to} D$. By diamond directedness, either $c <_x z$ or $a <_x z$.

Suppose then that $c <_x z$. From this it follows that there is an example d of $C \vee D \mathrel{\Diamond\!\!\to} D$ such that $d \leq_x c$. By diamond connectedness $d <_x a$ and so $d \leq_x y$ by left transitivity. Because $d \leq_x y$, we now need a further example a_1 of $A \vee D \mathrel{\Diamond\!\!\to} A$ such that $a_1 \leq_x d$. Moreover, it cannot be that $a_1 <_x d$, since this would entail that $a_1 \leq_x b$, which is a contradiction. So $a_1 \approx_x d$. Furthermore, it cannot be that $a_1 = a$ because in that case $d <_x a_1$, which contradicts the assumption that $a_1 \leq_x d$. So a_1 is distinct from all the previous worlds. But now by similar reasoning, we will need to add a fully copy $\langle a_n, b_n, c_n, d_n \rangle$ of $\langle a, b, c, d \rangle$ for all n on the right. But in that case, we get a sequence of worlds $\langle a, d, a_1, d_1, a_2, d_2 \ldots \rangle$ that violates diamond wellfoundedness, and so we have a contradiction.

Suppose instead that $a <_x z$. This results in $z \leq_x y$ by left transitivity. In that case, we will need to construct a similar infinite sequence of worlds for basically the same reasons. That sequence will also violate diamond wellfoundedness, so we will again have a contradiction. ■

Observation 7.7: If a frame is reflexive and triangle connected, then RM is valid.

Observation 7.8: If a frame is a total order and wellfounded, then CEM is valid.

Observation 7.9: If a frame is symmetric, then ST is valid.

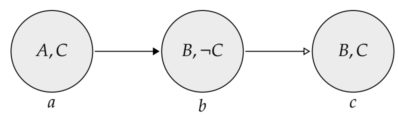
7.2. Inverse Soundness

Now that we have shown soundness, we turn to showing the inverse. This is done by providing several countermodels.

Observation 7.10: If a frame is not reflexive, then ID is not valid.

Proposition 7.11: *If a frame is reflexive, but not left transitive, then DR is not valid.*

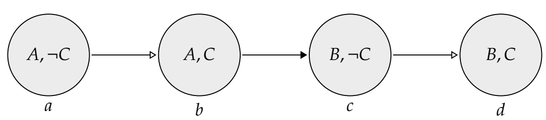
Proof.



■

Proposition 7.12: *If a frame is reflexive and left transitive, but not zigzag transitive, then DR is not valid.*

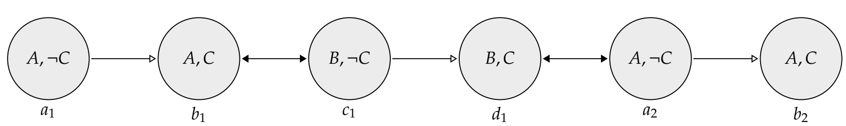
Proof. The following is a countermodel.



■

Proposition 7.13: *If a frame is reflexive, left transitive, and zigzag transitive, but not bowtie wellfounded, then DR is not valid.*

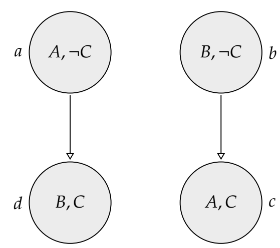
Proof. The following is a countermodel, with the bowtie sequence extending infinitely to the right.



The countermodel is just like the one used for proposition 7.12, except that zigzag transitivity holds and there is a copy $\langle a_n, b_n, c_n, d_n \rangle$ of $\langle a, b, c, d \rangle$ for each n pasted on the right. To simplify the diagram, the zigzag transitive arrows have been left implicit. ■

Proposition 7.14: *If a frame is reflexive, but not bowtie directed, then DL is not valid.*

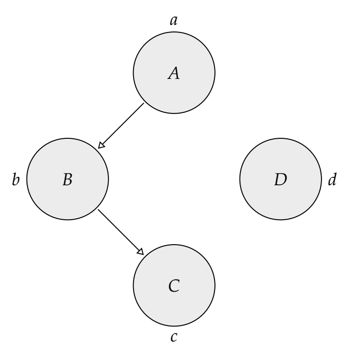
Proof. The following is a countermodel.



■

Proposition 7.15: *If a frame is reflexive, but not diamond directed, then DM is not valid.*

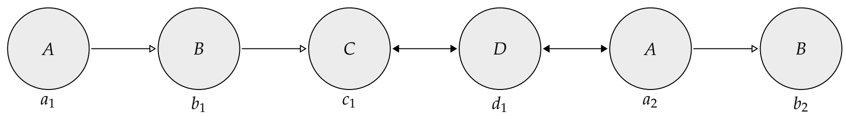
Proof. The following is a countermodel.



There may be further accessibility relations between the illustrated worlds. It can be easily verified, though, that so long as we have neither $d <_x a$ nor $c <_x d$, this is in fact a countermodel. ■

Proposition 7.16: *If a frame is reflexive and diamond directed, but not diamond wellfounded, then DM is not valid.*

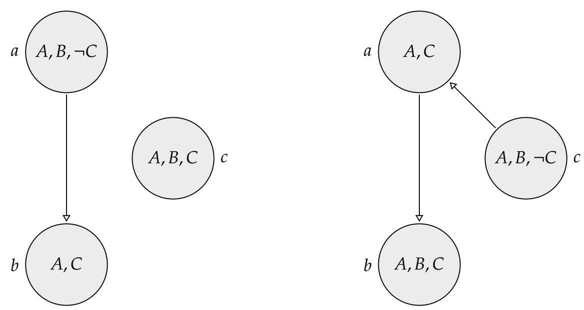
Proof. The following is a countermodel, with the diamond sequence extending infinitely to the right.



Diamond directedness holds in the model, so the model is double left and double left transitive. To simplify the diagram, those accessibility relations have been left implicit. ■

Proposition 7.17: *If a frame is reflexive, but not triangle directed, then RM is not valid.*

Proof. Consider and frame that is reflexive but not triangle connected. There are thus words a and b and c such that $b < a$ but neither $c <_x a$ nor $c < b$. There are then two cases to consider. If not $a <_x c$, we can use the countermodel on the left. If $a <_x c$, on the other hand, we can use the countermodel on the right instead.



■

Observation 7.18: If a frame is not a total ordering and wellfounded, then CEM is not valid.

Observation 7.19: If a frame is not symmetric, then ST is not valid.

Theorem 7.20 (Exact Soundness)**:** *Each system listed below is exactly sound in the corresponding class of frames.*

System	Frames
B1	zigzag order, bwf
B1.1	bowtie order, bwf
B2	diamond order, bwf, dwf
B3	triangle order
B4	total order, wf
B5	symmetric order

Proof. By the preceding. ∎

8. Completeness

The most familiar canonical constructions, like those from Henkin (1949), build a single canonical model for every consistent sentence of the target language. What we are going to do instead is assign each sentence of $\mathcal{L}$ a type and then, for each type, build a corresponding canonical model.

In what follows, we are going to show that every set of sentences that is consistent in **B1.1** has a finite model. The basic procedure, though, can be easily extended to any system extending **B1.1**. This includes my preferred system **B2**, along with **B3**-**B5**.

Definition 8.1: Fix an enumeration $p_1, p_2, \ldots$ of the atomic sentence of $\mathcal{L}$. The **atomic type** of a sentence A is the smallest n such that $p_1, \ldots, p_n$ includes all the atomic sentences in A.

Definition 8.2: The **modal depth** of a sentence A is given by $f(A)$, where this is defined recursively with:

$f(A) = 0$ when A is an atom.
$f(\neg A) = f(A)$
$f(A \wedge B) = f(A \vee B) = f(A \supset B) = max(f(A), f(B))$
$f(A \mathrel{\Box\!\!\rightarrow} B) = max(f(A), f(B)) + 1$

Definition 8.3: The **type** of a sentence is $t = \langle n, m \rangle$, where n is the atomic type and m is the modal depth.

Definition 8.4: The **states** of type $\langle n, m \rangle$ are the members of $Y^{n,m}$, where this set is defined recursively with:

$$
\begin{array}{lcl}
X^{n,0} & = & \text{the set of atomic sentence of type } \langle n, 0\rangle \\
Y^{n,m} & = & \text{the set of consistent conjunctions } A_1 \wedge \ldots \wedge A_n \text{ with } A_i \text{ being} \\
 & & \text{either } B_i \text{ or } \neg B_i \text{ for the enumerated } B_i \in X^{n,m} \\
X^{n,m+1} & = & \text{the union of the } X^{n,m} \text{ and all sentences of the form } A \vee B \mathrel{\Box\!\!\rightarrow} \\
 & & \neg B \text{ for } A, B \in Y^{n,m}
\end{array}
$$

Note that in the above construction, we always fix an enumeration of the relative atoms in $X^{n,m}$ in order to form the states in $Y^{n,m}$. This is important because it ensures that numerically distinct states are always logically inconsistent.

We now have almost everything needed to build our canonical models. As a final bit of preamble, we are going to fix a function mapping each state x to a maximal consistent set x^* such that $x \in x^*$. We then institute the following shorthand, where a and b are also states.

$$
\begin{array}{lll}
a \trianglelefteq_x b & \text{iff} & x^* \vdash \neg(a \vee b \mathrel{\Box\!\!\rightarrow} \neg a) \\
a \triangleleft_x b & \text{iff} & a \trianglelefteq_x b \text{ and } a \ntrianglerighteq_x b
\end{array}
$$

When the x is arbitrary or clear from context, we will drop the corresponding subscript, and so just write $a \trianglelefteq b$ and $a \triangleleft b$.

Definition 8.5: For every type t, the corresponding canonical model $\mathcal{M}^t$ is constructed as follows:

$$
\begin{array}{lcl}
W & = & \{x \mid x \text{ is a state of type } t\} \\
N_x & = & \{y \mid y \trianglelefteq_x z \text{ for some } z\} \\
a \leq_x b & \text{iff} & a \trianglelefteq_x b \\
V(p) & = & \{x \in W \mid p \in x^*\}
\end{array}
$$

Proposition 8.6 (Deduction Theorem)**:** $A \vdash B$ *iff* $\vdash A \supset B$.

Proof. The proof is the same as in the propositional case, and so left to the reader. ■

Lemma 8.7: *The following schemas are all valid in any system extending* ***B1****:*

LT	$(A \mathbin{\Box\!\!\rightarrow} B) \wedge (A \wedge B \mathbin{\Box\!\!\rightarrow} C) \supset (A \mathbin{\Box\!\!\rightarrow} C)$
RN	$A \mathbin{\Box\!\!\rightarrow} B$ *when* $\vdash A \supset B$
CW	$(A \mathbin{\Box\!\!\rightarrow} B) \supset (A \mathbin{\Box\!\!\rightarrow} C)$ *when* $\vdash B \supset C$
A1	$(A \mathbin{\Box\!\!\rightarrow} C) \supset (A \vee B \mathbin{\Box\!\!\rightarrow} C \vee B)$
A2	$(A \vee B \vee C \mathbin{\Box\!\!\rightarrow} \neg B \wedge \neg C) \supset (A \vee B \mathbin{\Box\!\!\rightarrow} \neg B)$
A3	$(A \vee B \mathbin{\Box\!\!\rightarrow} \neg B) \wedge (A \vee C \mathbin{\Box\!\!\rightarrow} \neg C) \supset (A \vee B \vee C \mathbin{\Box\!\!\rightarrow} \neg B \wedge \neg C)$ *when* $B \vdash \neg C$
A4	$(B \mathbin{\Box\!\!\rightarrow} C) \supset (A \vee B \mathbin{\Box\!\!\rightarrow} C)$ *when* $A \vdash C$
A5	$(A \mathbin{\Box\!\!\rightarrow} C) \wedge (B \mathbin{\Box\!\!\rightarrow} D) \supset (A \vee B \mathbin{\Box\!\!\rightarrow} C \vee D)$

Proof. To show CW, let $\vdash B \supset C$. Then:

1.	$A \mathbin{\Box\!\!\rightarrow} B$	
2.	$A \mathbin{\Box\!\!\rightarrow} C \wedge B$	1, SLE
3.	$A \mathbin{\Box\!\!\rightarrow} C$	2, CL

To show RN:

1.	$A \mathbin{\Box\!\!\rightarrow} A$	ID
2.	$A \mathbin{\Box\!\!\rightarrow} B$	1, CW

To show A1:

1.	$A \mathbin{\Box\!\!\rightarrow} B$	
2.	$A \mathbin{\Box\!\!\rightarrow} B \vee C$	1, CW
3.	$C \mathbin{\Box\!\!\rightarrow} B \vee C$	RN
4.	$A \vee C \mathbin{\Box\!\!\rightarrow} B \vee C$	2, 3, A2

To show LT:

1.	$A \mathbin{\Box\!\!\rightarrow} B$	
2.	$A \wedge B \mathbin{\Box\!\!\rightarrow} C$	
3.	$(A \wedge B) \vee (A \wedge \neg B) \mathbin{\Box\!\!\rightarrow} C \vee (A \wedge \neg B)$	2, A1
4.	$A \mathbin{\Box\!\!\rightarrow} C \vee (A \wedge \neg B)$	3, SLE
5.	$A \mathbin{\Box\!\!\rightarrow} B \wedge (C \vee (A \wedge \neg B))$	1,4, CR
6.	$A \mathbin{\Box\!\!\rightarrow} C$	5, SLE

To show A3, let $B \vdash \neg C$. Then:

1. $A \vee B \boxright \neg B$
2. $A \vee C \boxright \neg C$
3. $A \vee B \vee C \boxright \neg B \vee C$ 1, CW
4. $A \vee B \vee C \boxright B \vee \neg C$ 2, CW
5. $A \vee B \vee C \boxright (\neg B \vee C) \wedge (\neg C \vee B)$ 3, 4, CR
6. $A \vee B \vee C \boxright \neg B \wedge \neg C$ 5, SLE

To show A4, let $A \vdash C$. Then:

1. $B \boxright C$
2. $A \boxright C$ RN
3. $A \vee B \boxright C$ 2, 1, DR

To show A5:

1. $A \boxright C$
2. $B \boxright D$
3. $A \boxright C \vee D$ 1, CW
4. $B \boxright C \vee D$ 2, CW
5. $A \vee B \boxright C \vee D$ 3,4, DR

■

Lemma 8.8: *Every $\trianglelefteq$ is a total zigzag order.*

Proof. To show that $\trianglelefteq$ is a total zigzag order, we need to show that it is pairwise connected and zigzag transitive. These follow by proposition 8.9 and proposition 8.10.

■

Proposition 8.9: *$\trianglelefteq$ is pairwise connected.*

Proof. We need to show that if $a \trianglelefteq c$ for some c and $b \trianglelefteq d$ for some d, then either $a \trianglelefteq b$ or $b \trianglelefteq a$.

1. $(a \vee b \boxright \neg a) \wedge (a \vee b \boxright \neg b)$
2. $a \vee b \boxright \neg a \wedge \neg b$ 1, CR
3. $(a \boxright \neg a \wedge \neg b) \vee (b \boxright \neg a \wedge \neg b)$ 2, DL
4. $(a \boxright \neg a) \vee (b \boxright \neg b)$ 3, CL, PL
5. $c \boxright \neg a$ RN
6. $d \boxright \neg b$ RN
7. $(a \vee c \boxright \neg a) \vee (b \vee d \boxright \neg b)$ 6, DR, PL

Pairwise connectedness follows by contraposition. ∎

Proposition 8.10: *$\trianglelefteq$ is zigzag connected.*

Proof. We need to show that if $a \trianglelefteq e$ for some e and $b \triangleleft a$ and $c \trianglelefteq b$ and $d \triangleleft c$, then $d \triangleleft a$.

1.	$a \vee b \mathrel{\Box\!\!\rightarrow} \neg a$	
2.	$\neg(b \vee c \mathrel{\Box\!\!\rightarrow} \neg c)$	
3.	$c \vee d \mathrel{\Box\!\!\rightarrow} \neg c$	
4.	$a \vee b \vee c \vee d \mathrel{\Box\!\!\rightarrow} \neg a \vee \neg c$	1, 3 A5
5.	$a \vee b \vee c \vee d \mathrel{\Box\!\!\rightarrow} a \vee b \vee c \vee d$	ID
6.	$(a \vee d) \vee (b \vee c) \mathrel{\Box\!\!\rightarrow} b \vee d$	4, 5, CR, SLE
7.	$(a \vee d \mathrel{\Box\!\!\rightarrow} b \vee d) \vee (b \vee c \mathrel{\Box\!\!\rightarrow} b \vee d)$	6, DL
8.	$(a \vee d \mathrel{\Box\!\!\rightarrow} \neg a) \vee (b \vee c \mathrel{\Box\!\!\rightarrow} \neg c)$	7, CW
9.	$a \vee d \mathrel{\Box\!\!\rightarrow} \neg a$	2, 8, PL

∎

Proposition 8.11: *Every sentence A of type $\langle n, m\rangle$ is logically equivalent to a sentence B of type $\langle i, j\rangle$ whenever $i \geq n$ and $j \geq j$.*

Proof. Given any A of type $\langle n, m\rangle$, we can find a logically equivalent B_1 of type $\langle n+1, m\rangle$ by using $A \wedge (A \vee p_{n+1})$, and a logically equivalent B_2 of type $\langle n, m+1\rangle$ by using $A \wedge (A \vee (A \mathrel{\Box\!\!\rightarrow} A))$. ∎

Lemma 8.12: *Let A and B be sentences of type t. Then $A \mathrel{\Box\!\!\rightarrow} B$ is logically equivalent to $\bigvee_i \bigwedge_j \left(a_i \vee b_j \mathrel{\Box\!\!\rightarrow} \neg b_j\right)$, where the a_i and b_j are the states of type t such that $a_i \vdash A \wedge B$ and $b_j \vdash A \wedge \neg B$ respectively.*

Proof. The proof is by induction. For the base case, suppose A and B are both of type $\langle m, 0\rangle$ and consider the sentence $A \mathrel{\Box\!\!\rightarrow} B$. This is equivalent to

$$(A \wedge B) \vee (A \wedge \neg B) \mathrel{\Box\!\!\rightarrow} \neg(A \wedge \neg B) \tag{10}$$

by substitution. Every sentence of type $\langle m, 0\rangle$ is equivalent to a sentence of type $\langle m, 0\rangle$ in disjunctive normal form in which every conjunction is maximal with respect to sentences of type $\langle m, 0\rangle$. As such, (10) is equivalent to

$$\bigvee_i (a_i) \vee \bigvee_j (b_j) \mathrel{\Box\!\!\rightarrow} \bigwedge_j \neg(b_j) \tag{11}$$

where the a_i and b_j are as described. The only thing left to show is that this is equivalent to the target sentence:

$$\bigvee_i \bigwedge_j \left(a_i \vee b_j \mathrel{\Box\!\!\rightarrow} \neg b_j\right) \tag{12}$$

The proof from (11) to (12) uses repeated applications of DL and A2. The other direction uses disjunctive syllogism and repeated applications of A3 and A4. This gives us the base case. The induction step is essentially the same, with the exception that we use the induction hypothesis when showing that (10) is equivalent to (11). So the full result follows. ■

Observation 8.13: If A is a state of type t and x is a state of type t, then $x \vdash A$ iff $x^* \vdash A$.

Lemma 8.14: *Let A and B be sentences of type t. Then the following are equivalent:*

$$x \vdash A \mathrel{\Box\!\!\rightarrow} B \tag{13}$$

For every state $a \vdash A$ of type t such that $a \trianglelefteq g$ for some state g of type t, there is a state $b \vdash A$ of type t such that $b \trianglelefteq a$ and, for every state $c \vdash A$ of type t, if $c \trianglelefteq b$, then $c \vdash B$. (14)

Proof. Suppose (13). By lemma 8.12 and the fact that x^* is maximal, there is some $d \vdash A \wedge B$ such that $d \triangleleft e$ for all $e \vdash A \wedge \neg B$. Now consider any a of the type described. If $d \trianglelefteq a$, then d is the needed b, so suppose otherwise. In that case, $a \triangleleft d$ (because $\trianglelefteq$ is pairwise connected), and so $a \triangleleft e$ (because $\triangleleft$ is transitive). But then a is itself is the requisite b (since $\trianglelefteq$ is reflexive), and so (14).

For the other direction, suppose (14) and consider any $a \vdash A$. What we are going to show is that there is always a $b \vdash A$ such that $x^* \vdash a \vee b \mathrel{\Box\!\!\rightarrow} B$. But in that case, we can use DR to disjoin the antecedents of all such counterfactuals, with the result being a disjunctive antecedent that is logically equivalent to A. So $x^* \vdash A \mathrel{\Box\!\!\rightarrow} B$ by substitution, and therefore (13) by observation 8.13.

Suppose then that $a \vdash A$ and that there is no d such that $a \trianglelefteq d$. In that case, we have $x^* \vdash a \vee a \mathrel{\Box\!\!\rightarrow} \neg a$ and so $x^* \vdash a \vee a \mathrel{\Box\!\!\rightarrow} B$ by (B8), and so a itself can be the requisite b. Now suppose instead that $a \vdash A$ and that there is some d such that $a \trianglelefteq d$. It thus follows that there is some b as descried in (14) and, furthermore, $b \models B$ because $\trianglelefteq$ is reflexive. There are then two cases. If $a \trianglelefteq b$, then $a \vdash B$, and so $x^* \vdash a \vee b \mathrel{\Box\!\!\rightarrow} B$ by ID and CW. On the other hand, if $a \not\trianglelefteq b$, then $x^* \vdash a \vee b \mathrel{\Box\!\!\rightarrow} b$ and so $x^* \vdash a \vee b \mathrel{\Box\!\!\rightarrow} B$.

The upshot is that for any $a \models A$, there is a $b \vdash A$ such that $x^* \vdash a \vee b \mathrel{\Box\!\!\rightarrow} B$, as claimed. ■

Lemma 8.15 (Truth Lemma)**:** *Let A be a sentence of type t and $\mathcal{M}^t$ the canonical model of that same type. Then for all $x \in W^{m,n}$:*

$$\mathcal{M}^t, x \models A \text{ iff } x \vdash A$$

Proof. The proposition holds for atomic sentences by construction and, whenever it holds for a set of sentences, it also holds for the truth functional compounds of those sentences. The proof thus reduces to the case in which A has the form $B \mathrel{\Box\!\!\rightarrow} C$. That it holds in this case follows from proposition 8.11 and lemma 8.14 by induction and the construction of $\mathcal{M}^t$. ■

Theorem 8.16 (Completeness)**:** *Given a consistent set of sentences of any of the systems on the left, there is a finite pairwise connected model meeting the added conditions on the right.*

System	**Added Condition**
B1.1	zigzag transitive
B2	double transitive
B3	fully transitive
B4	fully transitive, asymmetric
B5	fully transitive, symmetric

Proof. By the preceding. ■

References

Berntson, Daniel. 2021. "The Paradox of Counterfactual Tolerance." April 15.

Burgess, John P. 1981. "Quick Completeness Proofs for Some Logics of Conditionals." *Notre Dame Journal of Formal Logic* 22, no. 1 (January): 76–84.

Field, Hartry. 2016. "Indicative Conditionals, Restricted Quantification, and Naive Truth." *The Review of Symbolic Logic* 9, no. 1 (March): 181–208.

Fine, Kit. 2012. "Counterfactuals without Possible Worlds." *The Journal of Philosophy* 109 (3): 221–246.

Fishburn, Peter, and Bernard Monjardet. 1992. "Norbert Wiener on the Theory of Measurement 1914 1915 1921." *Journal of Mathematical Psychology* 36, no. 2 (June): 165–184.

Gillies, Anthony S. 2007. "Counterfactual Scorekeeping." *Linguistics and Philosophy* 30 (3): 329–360.

Henkin, Leon. 1949. "The Completeness of the First-Order Functional Calculus." *Journal of Symbolic Logic* 14, no. 3 (September): 159–166.

Kratzer, Angelika. 1981. "Partition and Revision: The Semantics of Counterfactuals." *Journal of Philosophical Logic* 10, no. 2 (May).

Lewis, David K. 1973a. *Counterfactuals.* 278–281. 2. Blackwell.

———. 1973b. "Counterfactuals and Comparative Possibility." In *IFS,* edited by William L. Harper, Robert C. Stalnaker, and Glenn Pearce, 57–85. Dordrecht: Springer Netherlands.

———. 1981. "Ordering Semantics and Premise Semantics for Counterfactuals." *Journal of Philosophical Logic* 10 (2): 217–234.

Luce, R. Duncan. 1956. "Semiorders and a Theory of Utility Discrimination." *Econometrica* 24, no. 2 (April): 178.

Pollock, John L. 1975. "Four Kinds of Conditionals." *American Philosophical Quarterly* 12 (1): 51–59.

———. 1976a. *Subjunctive Reasoning.* Reidel.

Pollock, John L. 1976b. “The ‘Possible Worlds’ Analysis of Counterfactuals.” *Philosophical Studies* 29, no. 6 (June): 469–476.

Stalnaker, Robert C. 1968. “A Theory of Conditionals.” In *Studies in Logical Theory (American Philosophical Quarterly Monographs 2),* edited by Nicholas Rescher, 98–112. Blackwell.

———. 1975. “Indicative Conditionals.” *Philosophia* 5 (3): 269–286.

———. 1981. “A Defense of Conditional Excluded Middle.” In *Ifs,* edited by William Harper, Robert C. Stalnaker, and Glenn Pearce, 87–104. Reidel.

Stalnaker, Robert C., and Richmond Thomason. 1970. “A Semantic Analysis of Conditional Logic.” *Theoria* 36 (1): 23–42.

Von Fintel, Kai. 2001. “Counterfactuals in a Dynamic Context.” In *Ken Hale: A Life in Language,* edited by Michael Kenstowicz, 123–152. MIT Press.

Wiener, Norbert. 1914. “A Contribution to the Theory of Relative Position.” *Proceedings of the Cambridge Philosophical Society* 17:441–449.